# MATH431 Modern Particle Physics Solutions 9

1a The case in the absence of electric source was solved in problem set 5 question 3. From the Euler–Lagrange eq. of motion the second term gives

$$\frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial A_{\mu,\nu}}\right) - \frac{\partial L}{\partial A_\mu} = 0$$

$$\frac{\partial L}{\partial A_\mu} = j^\mu \Rightarrow \frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial A_{\mu,\nu}}\right) = \partial_\nu F^{\mu\nu} = j^\mu$$

$$\partial_\mu \partial_\nu F^{\mu\nu} = 0 \rightarrow \partial_\mu j^\mu = 0$$

1b In the Lorentz gauge we impose $\partial_\mu A^\mu = 0$. The derivative of the mass term $\frac{1}{2}m^2 A_\mu A^\mu$ with respect to $A^\nu$ gives $m^2 A^\nu$. Hence

$$(\partial_\mu \partial^\mu A^\nu + m^2 A^\nu) = j^\nu$$

.

2

$$L = \frac{1}{2}(\partial_\mu \phi_i)^2 - \frac{1}{2}\mu^2\phi_i^2 - \frac{1}{4}\lambda(\phi_i^2)^2$$

with $\mu^2 < 0$ and $\lambda > 0$

$$\frac{1}{2}\left((\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2 + (\partial_\mu\phi_3)^2\right) - \frac{1}{2}\mu^2(\phi_1^2+\phi_2^2+\phi_3^2) - \frac{1}{4}\lambda(\phi_1^2+\phi_2^2+\phi_3^2)^2$$

$$V = \frac{1}{2}\mu(\phi_1^2+\phi_2^2+\phi_3^2) + \frac{1}{4}\lambda(\phi_1^2+\phi_2^2+\phi_3^2)^2$$

$$\frac{\partial V}{\partial \phi_i} = (\mu^2 + \lambda(\phi_1^2+\phi_2^2+\phi_3^2))\phi_i = 0$$

$$\mu^2 + \lambda(\phi_1^2+\phi_2^2+\phi_3^2) = 0$$

Take

$$\langle \phi_1 \rangle = \sqrt{-\frac{\mu^2}{\lambda}} = v$$

Expand

$$\phi(x) = (v + h(x)) \quad ; \quad \phi_2(x) \quad ; \quad \phi_3(x)$$

Inserting into the Lagrangian we have

$$\frac{1}{2}\left((\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2 + (\partial_\mu\phi_3)^2\right) - \frac{1}{2}\mu^2((v+h(x))^2+\phi_2^2+\phi_3^2) - \frac{1}{4}\lambda((v+h(x))^2+\phi_2^2+\phi_3^2)^2$$

with $-\mu^2 = v^2\lambda$ we get

$$\begin{aligned}
&\frac{1}{2}((\partial_\mu h)^2 + (\partial_\mu \phi_2)^2 + (\partial_\mu \phi_3)^2) + \\
&\frac{v^2\lambda}{2}(h^2(x) + v^2 + 2vh(x) + \phi_2^2 + \phi_3^2) - \frac{\lambda}{4}(6v^2h^2(x) + \cdots) = \\
&\frac{1}{2}\left((\partial_\mu h)^2 + (\partial_\mu \phi_2)^2 + (\partial_\mu \phi_3)^2\right) - \frac{1}{2}\lambda v^2 h^2(x) + \\
&\qquad\qquad\text{cubic and quartic interaction terms + constant}
\end{aligned}$$

hence the Lagrangian describes one massive scalar field and two massless Goldstone bosons.

3 The Lagrangians of the two massive fields are:

$$\mathcal{L}_1 = (\partial_\mu \Phi_1)^\dagger(\partial^\mu \Phi_1) - m_1^2 \Phi_1^\dagger \Phi_1$$
$$\mathcal{L}_2 = (\partial_\mu \Phi_2)^\dagger(\partial^\mu \Phi_2) - m_2^2 \Phi_2^\dagger \Phi_2$$

the total Lagrangian is $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$. Hence $\mathcal{L}$ is not invariant under the exchange

$$\Phi_1 \leftrightarrow \Phi_2$$

Combining $\Phi = (\Phi_1, \Phi_2)$ the Lagrangian

$$\mathcal{L} = (\partial_\mu \Phi)^\dagger(\partial^\mu \Phi) - \mu^2 \Phi^\dagger \Phi - \lambda(\Phi^\dagger \Phi)^2$$

preserves the symmetry which is spontenousely broken in the vacuum for $\mu^2 < 0$.

**4.**

4a. The Lagrangian density is invariant under the global transformation

$$\phi \to e^{i\alpha}\phi$$

4b. In terms of $\phi_1$ and $\phi_2$ $\mathcal{L}$ can be written as:

$$\mathcal{L} = \frac{1}{2}\left((\partial_\mu \phi_1)^2 + (\partial_\mu \phi_2)^2\right) - \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) - \frac{1}{4}\lambda(\phi_1^2 + \phi_2^2)^2$$

$$V(\phi_1, \phi_2) = \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) + \frac{1}{4}\lambda(\phi_1^2 + \phi_2^2)^2$$

$$\frac{\partial V}{\partial \phi_i} = \mu^2 \phi_i + \lambda(\phi_1^2 + \phi_2^2)\phi_i = \phi_i\left(\mu^2 + \lambda(\phi_1^2 + \phi_2^2)\right) = 0$$

$$\Rightarrow \mu^2 + \lambda(\phi_1^2 + \phi_2^2) = 0$$

take

$$\langle \phi_1 \rangle = \sqrt{\frac{-\mu^2}{\lambda}} = v \langle \phi_2 \rangle = 0$$

Expand the Lagrangian density about the vacuum with

$$\phi(x) = (v + \eta(x) + i\zeta(x))$$

$$\mathcal{L}' = \frac{1}{2}\left((\partial_\mu \eta)^2 + (\partial_\mu \zeta)^2\right) + \mu^2\eta^2 + \text{constant} + (\text{cubic \& quartic terms in } \eta \text{ \& } \zeta)$$

The term $\mu^2\eta^2$ is a mass term for the field $\eta$ with $m_\eta = \sqrt{-2\mu^2}$. There is no corresponding mass term for the field $\zeta$ which is a massless scalar field and hence a Goldstone Boson.