MATH328 Modern Particle Physics Solutions 8

1a

$$3 = (2, 1/3) + (1, -2/3)$$

under $SU(2) \times U(1)$. The proton and neutron form an isospin doublet with $SU(2)_I$ charge $+1/2$ and $-1/2$, repectively.Then,

$$+1 = \alpha\frac{1}{2} + \beta\frac{1}{3}$$
$$0 = \alpha(-\frac{1}{2}) + \beta\frac{1}{3}$$

which gives $\alpha = +1$, $\beta = +3/2$

1b The decomposition of the sextet, octet and decuplet of $SU(3)$ in terms of $SU(2) \times U(1)$ is:

$$6 = \{(3, 2/3) + (2, -1/3) + (1, -4/3)\}$$
$$8 = \{(2, +1) + (3, 0) + (1, 0) + (2, -1)\}$$
$$10 = \{(4, +1) + (3, 0) + (2, -1) + (1, -2)\}$$

The electric charges of the states are:

$$6 = \{(2, 1, 0) + (0, -1) + (-2)\}$$
$$8 = \{(2, 1) + (1, 0, -1) + (0) + (-1, -2)\}$$
$$10 = \{(3, 2, 1, 0) + (1, 0, -1) + (-1, -2) + (-3)\}$$

2a. 4 diagonal generators.

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} , \quad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} ,$$
$$\lambda_{15} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} , \quad \lambda_{24} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix} .$$

This basis corresponds to the decomposition $SU(5) \to SU(4) \times U(1)$. Another basis

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad ,$$

$$\lambda_{15} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \quad ,$$

$$\lambda_{24} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix} \quad ,$$

which corresponds to the decomposition $SU(5) \to SU(3) \times SU(2) \times U(1)$

2b. $D = 24$.

2c.

$$5 = (3, 1, 1/3) + (1, 2, -1/2)$$

under $SU(3) \times SU(2) \times U(1)$.

2d.

$$\bar{5} = (\bar{3}, 1, -1/3) + (1, 2, 1/2)$$

$$\begin{aligned} 5 \times \bar{5} &= \{(3, 1, 1/3) + (1, 2, -1/2)\} \times \{(\bar{3}, 1, -1/3) + (1, 2, 1/2)\} = \\ 24 + 1 &= \{(8, 1, 0) + (1, 3, 0) + (1, 1, 0) + (3, 2, 5/6) + (\bar{3}, 2, -5/6)\} + (1, 0) \end{aligned}$$

3. The solution of problem 3 is in the file set17sols8q3.pdf

4a. To show that orbital angular momentum is a constant of the motion we have to show that it commutes with the Hamiltonian, *i.e.*

$$\left[\hat{L}_i, \hat{H}\right] = 0,$$

where the $\hat{L}_i$'s are the components of the angular momentum operator in the x, y and z directions. For a free particle the Hamiltonian is given by $H = \vec{p}^2/(2m)$ and the orbital angular momentum is given by $\vec{L} = \vec{r} \times \vec{p}$. Hence, for example, for L_z (we drop the hats from now on) we have

$$
\begin{aligned}
[L_z, p^2] &= [xp_y - yp_z, p_x^2 + p_y^2 + p_z^2] \\
&= [x, p_x^2]p_y - [y, p_y^2]p_x \\
&= (p_x[x, p_x] + [x, p_x]p_x)p_y - (p_y[y, p_y] + [y, p_y]p_y)p_x \\
&= (2i\hbar p_x p_y - 2i\hbar p_y p_x) = 0
\end{aligned}
$$

where we used the commutation relations $[x_i, p_j] = i\hbar\delta_{ij}$. Similar results are obtained for L_x and L_y. Hence, the orbital angular momentum commutes with the Hamiltonian and is a contant of the motion.

4b. Similarly to show that the orbital angular momentum is not a constant of the motion for a Dirac particle, we have to show that it does not commute with the Dirac Hamiltonian,

$$H = \vec{\alpha} \cdot \vec{p} + \beta m = \alpha_i p_i + \beta m$$

where summation over i is assumed.

$$[L_z, H] = [x, H]p_y - [y, H]p_x = i\hbar(\alpha_x p_y - \alpha_y p_x) = i\hbar(\vec{\alpha} \times \vec{p})_z$$

hence

$$[\vec{L}, H] = i\hbar\vec{\alpha} \times \vec{p} \neq 0$$

and consequently orbital aangular momentum does not commute with the Hamiltonian and is not a constant of the motion.

4c. The total angular momentum for a Dirac particle is

$$\vec{J} = \vec{L} + \vec{S}$$

where $\vec{L}$ is the orbital angular momentum and $\vec{S}$ is the spin angular momentum. we saw in part b that $[\vec{L}, H] = i\hbar\vec{\alpha} \times \vec{p}$ hence we need to find $\vec{S}$ such that

$$[\vec{S}, H] = -i\hbar\vec{\alpha} \times \vec{p}$$

We take

$$\vec{S} = \frac{1}{2}\vec{\Sigma}$$

with

$$\Sigma_j = \begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}$$

where σ_j are the Pauli matrices and the 0 entries are 2×2 zero matrices. It is easy to verify that

$$[\frac{1}{2}\vec{\Sigma}, H] = -i\hbar\vec{\alpha} \times \vec{p}$$

and therefore $[\vec{J}, H] = 0$ and the total angular momentum $\vec{J}$ is a constant of the motion.