MATH431 Modern Particle Physics Solutions 7

1.

$$\bar{u}_f(\not{p}_f - m)\gamma^\mu u_i = \bar{u}_f\gamma^\mu(\not{p}_i - m)u_i = 0 \qquad \text{(Dirac eq.)}$$

$$\Rightarrow 2m\bar{u}_f\gamma^\mu u_i = \bar{u}_f(\not{p}_f\gamma^\mu + \gamma^\mu\not{p}_i)u_i$$

$$\not{p}_f\gamma^\mu + \gamma^\mu\not{p}_i = \gamma^\nu\gamma^\mu p_{f_\nu} + \gamma^\mu\gamma^\nu p_{i_\nu}$$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$
$$\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu = -2i\sigma^{\mu\nu}$$

Hence
$\gamma^\mu\gamma^\nu = g^{\mu\nu} - i\sigma^{\mu\nu}; \gamma^\nu\gamma^\mu = g^{\mu\nu} + i\sigma^{\mu\nu}$

$$\Rightarrow \not{p}_f\gamma^\mu + \gamma^\mu\not{p}_i = g^{\mu\nu}(p_f + p_i)_\nu + i\sigma^{\mu\nu}(p_f - p_i)_\nu = (p_f + p_i)^\mu + i\sigma^{\mu\nu}(p_f - p_i)_\nu$$

$$\Rightarrow \bar{u}_f\gamma^\mu U_i = \frac{1}{2m}\bar{u}_f[(p_f + p_i)^\mu + i\sigma^{\mu\nu}(p_f - p_i)_\nu]u_i$$

2. We consider an electron in a constant magentic field $\vec{B} = (0, 0, B)$ with $B > 0$.
(a.)
The vector potential

$$A^\mu = (0, 0, Bx, 0)$$

(b.)

$$(i\partial_0 - m)\phi = \vec{\sigma}\cdot(\vec{p} - e\vec{A})\chi$$
$$(i\partial_0 + m)\chi = \vec{\sigma}\cdot(\vec{p} - e\vec{A})\phi$$

where, as usual, $\vec{p} = -i\nabla$.
(c.) Assuming a solution of the form

$$\phi(x) = \phi(\vec{x})e^{-iEt}, \chi(x) = \chi(\vec{x})e^{-iEt}$$

Inserting into the equations from (b.) these equations become

$$(E - m)\phi(\vec{x}) = \vec{\sigma}\cdot(\vec{p} - e\vec{A})\chi(\vec{x})$$
$$(E + m)\chi(\vec{x}) = \vec{\sigma}\cdot(\vec{p} - e\vec{A})\phi(\vec{x})$$

Substituting $\chi(\vec{x})$ from the second equation into the first and repeating the steps that we too in class when deriving the gyromagnetic factor from the Dirac equation, we get

$$\begin{aligned}(E^2 - m^2)\phi(\vec{x}) &= [(\vec{p} - e\vec{A})^2 - e\vec{\sigma}\cdot\vec{B}]\phi(\vec{x}) \\ &= [\vec{p}^2 + e^2B^2x^2 - 2ep_yBx - e\sigma_zB]\phi(\vec{x})\end{aligned}$$

Since p_x, p_y commute with x, we can seach for solutions of the form

$$\phi(\vec{x}) = e^{i(p_y y + p_z z)} f(x)$$

where p_y and p_z are c-numbers and $f(x)$, as $\phi(\vec{x})$, is a two component spinor. The equation for $f(x)$ becomes

$$[-\frac{d^2}{dx^2} + (p_y - eBx)^2 - eB\sigma_z] f(x) = (E^2 - m^2 - p_z^2) f(x)$$

$f(x)$ can be taken to be an eigenfunction of σ_z with eigenvalues $\sigma = \pm 1$, $\sigma_z f = \sigma f$. Then

$$[-\frac{d^2}{dx^2} + \frac{1}{2}(2e^2B^2)(x - \frac{p_y}{eB})^2] f(x) = (E^2 - m^2 - p_z^2 + eB\sigma) f(x)$$

This is formally identical to the Schrödinger equation of an harmonic oscillator with frequency $2|e|B$. The energy levels are therefore given by

$$E^2 - m^2 - p_z^2 + eB\sigma = (n + \frac{1}{2}) 2|e|B$$

or

$$E = [m^2 + p_z^2 + (2n + 1 + \sigma)|e|B]^{\frac{1}{2}}$$

Observe that there is a continuous degeneracy in p_x and p_y, as well as a discrete degeneracy

$$E(n, p_z, \sigma = +1) = E(n+1, p_z, \sigma = -1).$$

In the nonrelativistic limit $p_z << m^2$, $(2n+1)|e|B << m^2$ the nonrelativistic limit therefore gives

$$E(n, p_z, \sigma) \simeq m + \frac{p_z^2}{2m} + \left(n + \frac{1+\sigma}{2}\right)\omega_B$$

with $\omega_B = |e|B/m$. These are the Landau levels of nonrelativistic quantum mechanics.

3.

$$UU^\dagger = 1 \Rightarrow U = \mathrm{e}^{iH} \quad ; \quad U^\dagger = \mathrm{e}^{-iH^\dagger} = \mathrm{e}^{-iH}$$

Hence we must have $H = H^\dagger$. H must be hermitian.

4a. 3 diagonal generators.

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_{15} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

4b. $D = 15$. Three diagonal generators of part $(2a)$ plus:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_9 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_{10} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\lambda_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad , \quad \lambda_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

4c.

$$4 = (3, 1/3) + (1, -1)$$

under the maximal subgroup $SU(3) \times U(1)$

4d.

see separate figure

4e.

$$\begin{aligned} 4 \times \bar{4} = \{(3, 1/3) + (1, -1)\} \times \{(\bar{3}, -1/3) + (1, +1)\} = \\ 15 + 1 = \{(3, +4/3) + (8, 0) + (1, 0) + (\bar{3}, -4/3)\} + (1, 0) \end{aligned}$$

4f.

$$\begin{aligned} 4 \times 4 = \{(3, 1/3) + (1, -1)\} \times \{(3, 1/3) + (1, -1)\} = \\ 6 + 10 = \{(6, 2/3) + (3, -2/3) + (1, -2)\} + \{(\bar{3}, 2/3) + (3, -2/3)\} \end{aligned}$$

6g.

$SU(4)$ decomposes as $SU(3) \times U(1)$ under the Standard Model, where $SU(3)$ corresponds to the gauge symmetry of the strong interactions and the $U(1)$ can be indentified as baryon minus lepton number. The fundamental representation decomposes as a triplet and a singlet of $SU(3)$ with the triplet corresponding to a quark, with baryon number $1/3$, and the singlet to a lepton, with lepton number -1. The Cartan generators correspond to the two diagonal generators of $SU(3)$ and the $U(1)_{B-L}$ generator.