MATH431 Modern Particle Physics Solutions 6

1.

$$\begin{aligned}\gamma^{5\dagger} =&(i\gamma^0\gamma^1\gamma^2\gamma^3)^\dagger\\ =&-i\gamma^{3\dagger}\gamma^{2\dagger}\gamma^{1\dagger}\gamma^{0\dagger}\\ =&-i(\gamma^0\gamma^3\gamma^0)(\gamma^0\gamma^2\gamma^0)(\gamma^0\gamma^1\gamma^0)\gamma^0\\ =&-i\gamma^0\gamma^3\gamma^2\gamma^1 = i\gamma^0\gamma^2\gamma^3\gamma^1 = -i\gamma^0\gamma^2\gamma^1\gamma^3 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5.\end{aligned}$$

For the second part, it's best to do for each μ in turn:

$$\begin{aligned}\gamma^5\gamma^0 =&i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 = -i\gamma^0\gamma^1\gamma^2\gamma^0\gamma^3 = i\gamma^0\gamma^1\gamma^0\gamma^2\gamma^3\\ =&-i\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^0\gamma^5 \Rightarrow \{\gamma^5,\gamma^0\} = 0,\\ \gamma^5\gamma^1 =&i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1 = -i\gamma^0\gamma^1\gamma^2\gamma^1\gamma^3 = i\gamma^0\gamma^1\gamma^1\gamma^2\gamma^3\\ =&-i\gamma^1\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^1\gamma^5 \Rightarrow \{\gamma^5,\gamma^1\} = 0.\end{aligned}$$

It is clear that the other two calculations will be similar.

2. We have

$$(\gamma^0)^2 = 1, \quad (\gamma^i)^2 = -1, \quad i = 1,2,3.$$

$$\begin{aligned}\gamma^0\gamma^1\gamma^2 =&-\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3 = i\gamma^5\gamma^3\\ \gamma^0\gamma^1\gamma^3 =&-\gamma^0\gamma^1\gamma^2\gamma^2\gamma^3 = \gamma^0\gamma^1\gamma^2\gamma^3\gamma^2 = -i\gamma^5\gamma^2.\end{aligned}$$

3.

$$\{\gamma_\mu,\gamma_\nu\} = 2\eta_{\mu\nu}1_4 \Rightarrow \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}1_4,$$

where 1_4 is the 4-dimensional identity matrix, usually not written explicitly. Taking the trace, and using $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, $\mathrm{tr}1_4 = 4$, we get

$$\mathrm{tr}[\gamma_\mu\gamma_\nu] = 4\eta_{\mu\nu}.$$

Now

$$\begin{aligned}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma =&-\gamma_\nu\gamma_\mu\gamma_\rho\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma\\ =&\gamma_\nu\gamma_\rho\gamma_\mu\gamma_\sigma - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma\\ =&-\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu + 2\eta_{\mu\sigma}\gamma_\nu\gamma_\rho - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma\\ \Rightarrow \gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma + \gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu =&2\eta_{\mu\sigma}\gamma_\nu\gamma_\rho - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma.\end{aligned}$$

Taking the trace and using

$$\mathrm{tr}[\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu] = \mathrm{tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma],$$

together with $\mathrm{tr}[\gamma_\mu\gamma_\nu = 4\eta_{\mu\nu}$, we find

$$\mathrm{tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] = 4[\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}]$$

$$\psi(r,\theta,\phi) = R(r)\begin{pmatrix} 1 \\ 0 \\ ia\cos\theta \\ iae^{i\phi}\sin\theta \end{pmatrix}$$

Normalize

$$\int d^3\mathbf{r}\psi^\dagger\psi = 1 = \int 4\pi r^2 dr(|R|^2(1+a^2))$$

$$\Rightarrow \int_0^\infty r^2|R|^2 dr = [4\pi(1+a^2)]^{-1}$$

(a.)

$$L_z = -i\frac{\partial}{\partial\phi} \Rightarrow L_z\psi = R\begin{pmatrix} 0 \\ 0 \\ 0 \\ iae^{i\phi}\sin\theta \end{pmatrix} \not\propto \psi$$

so ψ is not an eigenstate of L_z.

(b.)

$$\langle L_z\rangle = \int d^3\mathbf{r}\psi^\dagger L_z\psi = \int 2\pi r^2 d\cos\theta|R|^2 a^2\sin^2\theta$$

$$\int_{-1}^{1} d\cos\theta(1-\cos^2\theta) = 2 - \frac{2}{3} = \frac{4}{3} \Rightarrow \langle L_z\rangle = \frac{8\pi}{3}a^2\cdot\frac{1}{4\pi(1+a^2)}$$

$$\Rightarrow \langle L_z\rangle = \frac{2a^2}{3(1+a^2)}$$

In H-atom , $v/c \sim \alpha \Rightarrow \langle L_z\rangle = O(v^2/c^2)$. This is a relativistic effect - spin–orbit interaction.

(c.)

$$S_z = \frac{1}{2}\hbar\begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\Rightarrow S_z\psi = \frac{1}{2}\hbar R\begin{pmatrix} 1 \\ 0 \\ ia\cos\theta \\ -iae^{i\phi}\sin\theta \end{pmatrix}$$

$$(L_z+S_z)\psi = R\begin{pmatrix} 1 \\ 0 \\ ia\cos\theta \\ iae^{i\phi}\sin\theta \end{pmatrix} = \frac{1}{2}\psi \Rightarrow J_z = +\frac{1}{2}$$

5. (a) Operating with $\gamma^\nu \partial_\nu$ from the left on the Dirac equation we have

$$\begin{aligned} \gamma^\nu \partial_\nu \left(i\gamma^\mu \partial_\mu - m\right) \psi(x) &= \\ i\frac{1}{2}\left(\gamma^\nu \gamma^\mu + \gamma^\mu \gamma^\nu\right) \partial_\nu \partial_\mu \psi - m\gamma^\nu \partial_\nu \psi &= \\ i\eta^{\mu\nu} \partial_\nu \partial_\mu \psi + im^2 \psi &= \\ i\left(\partial^\mu \partial_\mu + m^2\right) I\psi &= 0 \end{aligned}$$

where I is the 4×4 identity matrix.

(b)

$$\gamma^\mu \partial_\mu \psi + im\psi = 0 \quad , \quad (\partial\psi^\dagger)\gamma^{\mu^\dagger} - im\psi^\dagger = 0$$

$$\begin{aligned} &\Rightarrow (\partial\psi^\dagger)\gamma^0\gamma^\mu\gamma^0 - im\psi^\dagger = 0 \\ &\Rightarrow (\partial_\mu\bar{\psi})\gamma^\mu - im\bar{\psi} = 0 \end{aligned}$$

(c)

- $\partial_\mu(\bar{\psi}\gamma^\mu\psi) = (\partial_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu(\partial_\mu\psi)$
  $= (im\bar{\psi})\psi + \bar{\psi}(-im\psi) = 0$

- $\partial_\mu(\bar{\psi}\gamma^\mu\gamma^5\psi) = (\partial_\mu\bar{\psi})\gamma^\mu\gamma^5\psi + \bar{\psi}\gamma^\mu\gamma^5(\partial_\mu\psi)$
  $= (im\bar{\psi})\gamma^5\psi - \bar{\psi}\gamma^5(\gamma^\mu\partial_\mu\psi) = 2im\bar{\psi}\gamma^5\psi$