MATH328 Modern Particle Physics Solutions 4

1. Considering the Lagrangian

$$L(q(t), \dot{q}(t), t)$$

and the variation

$$\begin{aligned} q(t) &\rightarrow q(t) \ + \ \delta(q(t)) = q(t) \ + \ \epsilon h(q(t), t) \\ \dot{q}(t) &\rightarrow \dot{q}(t) \ + \ \frac{d}{dt}\left(\delta(q(t))\right) \end{aligned}$$

Variation of the action gives rise to the Euler-Lagrange equation of motion

$$S(t) = \int dt L(q(t), \dot{q}(t), t) \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

since L is invariant under the variation

$$L(q + \delta q, \dot{q} + \frac{d}{dt}\delta q, t) = L + \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}\delta q + \cdots = L$$

$$\Longrightarrow \quad \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}\delta q = 0$$

Now take

$$\begin{aligned} \epsilon\frac{dQ}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right) &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d\delta q}{dt} = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}\delta q = 0 \\ \Longrightarrow \frac{dQ}{dt} = 0, & \end{aligned}$$

where we have used the Euler–Lagrange equation to go from the second to third equality. Therefore, we have that

$$Q(t) = \frac{\partial L}{\partial \dot{q}} h(q(t), t)$$

is conserved.

2. The Lagrangian of the given two-dimensional potential is

$$L \ = \ \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{4}k(x^2 + y^2)^2 \, .$$

(ii) The Euler-Lagrange equations are

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= m\ddot{x} + k(x^2 + y^2)x \ = \ 0 \, , \\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} &= m\ddot{y} + k(x^2 + y^2)y \ = \ 0 \, . \end{aligned}$$

(iii) The Hamiltonian is

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{4}mw^2(x^2+y^2)^2 \quad \text{with} \quad w = \sqrt{\frac{k}{m}}\,.$$

(iv) In polar coordintes we derive:

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{1}{4}kr^4\,,$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}\,,$$
$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}\,.$$

Hence we get

$$\begin{aligned} H &= \sum_i p_i \dot{q}_i - L = m\dot{r}^2 + mr^2\dot{\phi}^2 - \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{4}mw^2r^4 \\ &= \frac{1}{2}m(\dot{r}^2 + mr^2\dot{\phi}^2) + \frac{1}{4}mw^2r^4 \\ &= \frac{{p_r}^2}{2m} + \frac{{p_\phi}^2}{2mr^2} + \frac{1}{4}mw^2r^4\,. \end{aligned}$$

(v) There are two constants of the motion.
Since the Hamiltonian does not depend explicitly on time, the energy is a constant of the motion with $E = H(q_0, p_0)$. Since it does not depend explicitly on ϕ, also p_ϕ is a constant of the motion, corresponding to conservation of the angular momentum w.r.t. the symmetry axis.

3. With $\rho = |\psi|^2 = \psi^*\psi$, we have

$$\frac{\partial \rho}{\partial t} = \frac{\partial \psi^*}{\partial t}\psi + \psi^*\frac{\partial \psi}{\partial t}.$$

Now from the Schrödinger equation,

$$i\hbar\frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x})\right)\psi, \tag{1}$$

and so, taking the complex conjugate

$$-i\hbar\frac{\partial \psi^*}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x})\right)\psi^* \tag{2}$$

(assuming that $V(\mathbf{x})$ is real.) Multiplying (1) by ψ^* and (2) by ψ and subtracting, we have

$$i\hbar\left(\psi^*\frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t}\psi\right) = -\frac{\hbar^2}{2m}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) = -\frac{\hbar^2}{2m}\nabla.\left[\psi^*\nabla\psi - \psi\nabla\psi^*\right],$$

and so

$$\frac{\partial \rho}{\partial t} = \frac{i\hbar}{2m}\nabla.\left[\psi^*\nabla\psi - \psi\nabla\psi^*\right],$$

i.e.

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\mathbf{j} = 0,$$

$$\text{where}\quad \mathbf{j} = -\frac{i\hbar}{2m}\left[\psi^*\nabla\psi - \psi\nabla\psi^*\right]$$

4. Inserting the given solution

$$\phi(x,y) = \sum_{n=1}^{\infty}\phi_n(x)\,\mathrm{cs}\left(\frac{n\pi y}{R}\right)$$

into the five-dimensional Klein-Gordon equation and carrying out $\partial^2/\partial y^2$, one obtains a four-dimensional Klein-Gordon equation for each of the Fourier coefficients $\phi_n(x)$:

$$\left(\partial_0^2 - \nabla^2 + m^2 + \left(\frac{n\pi}{R}\right)^2\right)\phi_n(x) = 0\,.$$

Therefore the given $\phi(x,y)$ is a solution of the five-dimensional Klein-Gordon equation *if* the $\phi_n(x)$ are solutions of the four-dimensional equations. The masses m_n of the fields ϕ_n are given by

$$m_n^2 = m^2 + \left(\frac{n\pi}{R}\right)^2\,.$$

If the five-dimensional mass m is zero, we get the equally spaced mass spectrum

$$m_n = \frac{n\pi}{R}\,.$$

The infinite set of particles are called Kaluza-Klein tower.