MATH431 Modern Particle Physics Solutions 3

1.

$$ds^2 = dt^2 - dx^2 - dy^2 \tag{1}$$

a.

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad , \quad g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

b. The line element is invariant under 3 translations (dt, dx and dy). 2 boosts ($dtdx$, $dtdy$) and 1 rotation ($dxdy$).

The generators associated with the transformations are: $i\partial_t = p_0$, $i\partial_x = p_1$ and $i\partial_y = p_2$, are the generators of translations. K_1 and K_2 are the boost generators and J_3 is the generator of rotations in the $x - y$ plane.

c.

$$W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma$$

$$W^0 = -\frac{1}{2}\epsilon^{0ijk} J_{ij} P_k = -J^k P_k$$

$$\begin{aligned} W^i = &-\frac{1}{2}\epsilon^{i0jk} J_{0j} P_k - \frac{1}{2}\epsilon^{ij0k} J_{j0} P_k - \frac{1}{2}\epsilon^{ijk0} J_{jk} P_0 = \\ &\frac{1}{2}\epsilon^{0ijk} J_{0j} P_k + \frac{1}{2}\epsilon^{0ijk} J_{0j} P_k + \frac{1}{2}\epsilon^{0ijk} J_{jk} P_0 = \\ &\epsilon^{ijk} K_j P_k + J^i P_0 \end{aligned}$$

$$\text{where} \quad K^i = J^{i0} = -J^{0i} \quad ; \quad J^i = \frac{1}{2}\epsilon^{ijk} J_{jk}$$

$$K_i = -J_{i0} = J_{0i} \quad ; \quad J_i = \frac{1}{2}\epsilon_{ijk} J^{jk} = -J^i$$

In the case of the two dimensional line element eq. (1)

$$\vec{K} = (K_1, K_2, 0) \qquad \vec{J} = (0, 0, J_3)$$

for $m^2 = 0 \rightarrow P^\mu = (p, 0, p, 0)$

$$\begin{aligned} W^0 &= 0 \\ W^1 &= J^1 P_0 + \epsilon^{1jk} K_j P_k = 0 + \epsilon^{123} K_2 P_3 + \epsilon^{132} K_3 P_2 = 0 \\ W^2 &= J^2 P_0 + \epsilon^{2jk} K_j P_k = 0 + \epsilon^{231} K_3 P_1 + \epsilon^{213} K_1 P_3 = 0 \\ W^3 &= J^3 P_0 + \epsilon^{3jk} K_j P_k = J_3 P_0 + \epsilon^{312} K_1 P_2 + \epsilon^{321} K_2 P_1 = (J^3 + \epsilon^{312} K_1) P_0 \end{aligned}$$

For $m^2 > 0 \rightarrow P = (m, 0, 0, 0)$

$$\begin{aligned} W^0 &= 0 \\ W^1 &= 0 \\ W^2 &= 0 \\ W^3 &= J^3 m \end{aligned}$$

2.

a.

$$\begin{aligned} W_\sigma P^\sigma &= -\frac{1}{2}\epsilon_{\sigma\mu\nu\lambda} J^{\mu\nu} P^\lambda P^\sigma \\ &= \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma} J^{\mu\nu} P^\lambda P^\sigma \\ &= \frac{1}{2}\epsilon_{\mu\nu\sigma\lambda} J^{\mu\nu} P^\sigma P^\lambda \\ &= -\frac{1}{2}\epsilon_{\mu\nu\lambda\sigma} J^{\mu\nu} P^\lambda P^\sigma, \end{aligned}$$

where in the second line we permuted σ three times across; in the third line we changed the dummy summation indices; in the fourth line we again permute λ and σ and exchange P^λ and P^σ. Hence, the second line is equal to minus the fourth line, which can only hold if they vanish.

b.

We use the identity

$$[A, BC] = [A, B]\, C + B\, [A, C]\,,$$

where A, B and C are operators. Hence, we have

$$\begin{aligned} [J_{\rho\sigma}, W_\mu W^\mu] &= \eta^{\mu\nu} [J_{\rho\sigma}, W_\nu W_\mu] \\ &= \eta^{\mu\nu} \left([J_{\rho\sigma}, W_\nu]\, W_\mu + W_\nu\, [J_{\rho\sigma}, W_\nu]\right) \\ &= \eta^{\mu\nu} \left(i\left(\eta_{\sigma\nu} W_\rho W_\mu - \eta_{\rho\nu} W_\sigma W_\mu + \eta_{\sigma\mu} W_\nu W_\rho - \eta_{\rho\mu} W_\nu W_\sigma\right)\right) \\ &= i\eta^{\mu\nu}\eta_{\sigma\nu} W_\rho W_\mu - i\eta^{\mu\nu}\eta_{\rho\mu} W_\nu W_\sigma + i\eta^{\mu\nu}\eta_{\sigma\mu} W_\nu W_\rho - i\eta^{\mu\nu}\eta_{\rho\nu} W_\sigma W_\mu \\ &= i\delta^\mu{}_\sigma W_\rho W_\mu - iW_\rho W_\mu + iW_\sigma W_\rho - iW_\sigma W_\rho = 0, \end{aligned}$$

where we used that

$$\eta^{\mu\nu}\eta_{\sigma\nu} = \delta^\mu{}_\sigma, \quad \text{etc}$$

That $W_\mu W^\mu$ commutes with P_λ is obvious because $[P_\nu, W_\rho] = 0$.

3. In polar coordinates

$$\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \end{aligned}$$

radial speed is therefore $\dot{r}$ and tangential speed is $r\dot{\phi}$. So the kinetic energy $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$ and the Lagrangian is given by

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - V$$

Hence

$$\begin{aligned} \frac{\partial L}{\partial \dot{r}} &= m\dot{r} \\ \frac{\partial L}{\partial r} &= mr\dot{\phi}^2 - \frac{\partial V}{\partial r} \end{aligned}$$

so the Euler–Lagrange equation for r is

$$m\frac{d\dot{r}}{dt} - mr\dot{\phi}^2 + \frac{\partial V}{\partial r} = 0$$

Similarly,

$$\frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}$$
$$\frac{\partial L}{\partial \phi} = \frac{\partial V}{\partial \phi}$$

So the Euler–Lagrange equation for ϕ is

$$m\frac{d}{dt}(r^2\dot{\phi}) + \frac{\partial V}{\partial \phi} = 0$$

If the potential is axisymmetric, $\partial V/\partial\phi = 0$. The last equation then states that the angular momentum $mr^2\dot{\phi}$ is constant. If the motion is circular $\dot{r} = 0 = \ddot{r}$ and the radial equation becomes

$$-m\frac{v^2}{r} = -\frac{\partial V}{\partial r},$$

where$v = r\dot{\phi}$ is the speed. The force $\partial V/\partial r$ is equal to m times the centripetal acceleration $-v^2/r$.

4.

$$\dot{p}_0 = -\frac{\partial H}{\partial q_0}$$

Therefore, p_0 is constant in time if H does not depend on q_0.

An axisymmetric potential does not depend on ϕ so p_ϕ is a constant of the motion.

In polar coordinates

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

The kinetic energy is given by

$$\frac{1}{2}m\left[\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\dot{\phi})^2\right]$$

and the potential energy is V

$$L = \frac{1}{2}m\left[\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\dot{\phi})^2\right] - V$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$
$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$
$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\dot{\phi}$$

$$H = \sum_i p_i q_i - L = m\dot{r}^2 + mr^2\dot{\theta}^2 + mr^2\sin^2\theta\dot{\phi}^2 - \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) + V$$
$$= \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) + V$$
$$= \frac{{p_r}^2}{2m} + \frac{{p_\theta}^2}{2mr^2} + \frac{{p_\phi}^2}{2mr^2\sin^2\theta} + V$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = -\frac{\partial V}{\partial \phi} = 0$$

$\Rightarrow$ angular momentum about the symmetry axis is conserved.