MATH431 Modern Particle Physics Solutions 10

1a The vacuum expectation value of the electroweak Higgs field ϕ:

$$\langle\phi_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix}$$

The Standard Model $SU(2)\times U(1)_Y$ quantum numbers of the component of the Higgs field with non–trivial VEV are:

$$\phi : T = \frac{1}{2} \quad , \quad T_3 = -\frac{1}{2} \quad , \quad Y = 1$$

and its electric charge

$$Q_{\rm e.m.}(\phi) = T_3 + \frac{1}{2}Y = -\frac{1}{2} + \frac{1}{2} = 0.$$

Hence, the electromagnetic symmetry remains unbroken by the VEV of the Higgs field.

The Lagrangian density of the Higgs field is given by:

$$\mathcal{L} = \left|\left(i\partial_\mu - ig\vec{T}\cdot\vec{W}_\mu - ig'\frac{Y}{2}B_\mu\right)\phi\right|^2 - V(\phi)$$

where

$$V(\phi) = \mu^2\phi^\dagger\phi + \lambda\left(\phi^\dagger\phi\right)^2$$

The relevant term for the gauge boson masses:

$$\left|\left(g\frac{\vec{\tau}}{2}\cdot\vec{W}_\mu + g'\frac{1}{2}B_\mu\right)\phi\right|^2 =$$

$$\left|\left(\frac{g}{2}\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\cdot W^1_\mu + \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\cdot W^2_\mu + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\cdot W^3_\mu\right] + \frac{g'}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\cdot B_\mu\right)\phi\right|^2 =$$

$$\frac{1}{4}\left|\left[g\begin{pmatrix}W^3_\mu & W^2_\mu - iW^1_\mu\\ W^2_\mu + iW^1_\mu & -W^3_\mu\end{pmatrix} + g'\begin{pmatrix}B_\mu & 0\\ 0 & B_\mu\end{pmatrix}\right]\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix}\right|^2 =$$

$$\frac{1}{8}\left|\begin{pmatrix}gW^3_\mu + g'B_\mu & g(W^2_\mu - iW^1_\mu)\\ g(W^2_\mu + iW^1_\mu) & -gW^3_\mu + g'B_\mu\end{pmatrix}\begin{pmatrix}0\\ v\end{pmatrix}\right|^2 =$$

$$\frac{1}{8}v^2\left(g\left(W^2_\mu + iW^1_\mu\right), \left(-gW^3_\mu + g'B_\mu\right)\right)\begin{pmatrix}g\left(W^2_\mu - iW^1_\mu\right)\\ -gW^3_\mu + g'B_\mu\end{pmatrix} =$$

$$\frac{1}{4}g^2v^2W^{\mu+}W^{\mu-} + \frac{1}{8}v^2(g^2+g'^2)\frac{\left(-gW^3_\mu + g'B_\mu\right)^2}{\left(\sqrt{g^2+g'^2}\right)^2} =$$

$$\left(\frac{1}{2}gv\right)^2 W^{\mu+}W^{\mu-} + \frac{1}{2}v^2\frac{(g^2+g'^2)}{4}\left(\frac{-gW^3_\mu + g'B_\mu}{\sqrt{g^2+g'^2}}\right)^2 =$$

$$M^2_{W^\pm}\quad W^{\mu+}W^{\mu-} \quad + \quad \frac{1}{2}M^2_Z Z^2$$

reading off from the last two lines we have

$$M_{W^\pm} = \frac{gv}{2} \quad ; \quad M_Z = \frac{1}{2} v \left(g^2 + g'^2\right)^{\frac{1}{2}}$$

hence

$$\frac{M_W}{M_Z} = \frac{g}{\left(g^2 + g'^2\right)^{\frac{1}{2}}} = \cos\theta_W$$

1b We will derive the general case for Higgs representations and then substitute. The general term for the gauge bosons masses has the form

$$\sum_j \left| \left(-ig\vec{T}_j \cdot \vec{W}_\mu - ig' \frac{Y_j}{2} B_\mu \right) \phi_j \right|^2 =$$

$$\sum_j \left| \left(-ig \left[T_j^1 W_\mu^1 + T_j^2 W_\mu^2 + T_j^3 W_\mu^3 \right] - ig' \frac{Y_j}{2} B_\mu \right) \phi_j \right|^2 =$$

$$\sum_j \left| \left(g \left[T_j^+ W_\mu^- + T_j^- W_\mu^+ \right] + \left[g T_j^3 W_\mu^3 + \frac{g'}{2} Y_j B_\mu \right] \right) \phi_j \right|^2$$

where in the last line we used

$$T_j^\pm = \frac{1}{\sqrt{2}} \left(T^1 \pm i T^2 \right) \quad , \quad W_\mu^\pm = \frac{1}{\sqrt{2}} \left(W_\mu^1 \pm i W_\mu^2 \right).$$

which yields

$$T^1 W_\mu^1 + T^2 W_\mu^2 = T^+ W_\mu^- + T^- W_\mu^+$$

M_W^2 is the coefficient of $W^+ W^-$,

$$M_W^2 = \left\langle \sum_j g^2 \phi_j^\dagger \left(T_j^+ T_j^- + T_j^- T_j^+ \right) \phi_j \right\rangle =$$

$$g^2 \left\langle \sum_j \phi_j^\dagger \left(\vec{T}_j^2 - T_{j3}^2 \right) \phi_j \right\rangle =$$

$$g^2 \left\langle \sum_j \phi_{j0}^\dagger \left(T_j(T_j + 1) - T_{j3}^2 \right) \phi_{j0} \right\rangle =$$

$$\frac{g^2}{2} \sum_j v_j^2 \left(T_j(T_j + 1) - T_{j3}^2 \right)$$

where we used

$$T_j^+ T_j^- + T_j^- T_j^+ = T_j^1 T_j^1 + T_j^2 T_j^2 = \vec{T}_j^2 - T_{j3}^2 = T_j(T_j + 1) - T_{j3}^2$$

and where in the last line correspond to the quantum eigenvalues of the electrically neutral component of the Higgs representations. We require that $Q(\phi_{j0}) = 0$, which ensures that electric charge remains unbroken by the vacuum expectation values of the Higgs fields.

$$\Rightarrow T_{j3} + \frac{Y_j}{2} = 0 \Rightarrow T_{j3} = -\frac{Y_j}{2}$$

$$\Rightarrow M_W^2 = \frac{g^2}{2} \sum_j \left(T_j(T_j+1) - \frac{Y_j^2}{2} \right) v_j^2$$

M_Z^2 is obtained from the matrix element of the second term

$$\sum_j \left| \left[g T_j^3 W_\mu^3 + \frac{g'}{2} Y_j B_\mu \right] \phi_j \right|^2$$

using $T_{j3} = -\frac{Y_j}{2}$ we get

$$\sum_j \left| \frac{Y_j}{2} \left[-g W_\mu^3 + g' B_\mu \right] \phi_j \right|^2 =$$

$$\frac{1}{8} \sum_j (y_j^2 v_j^2) \frac{\left| \left[-g W_\mu^3 + g' B_\mu \right] \right|^2}{(g'^2 + g^2)} (g'^2 + g^2)$$

$$\Rightarrow M_Z^2 = \frac{(g'^2 + g^2)}{4} \sum_j v_j^2 y_j^2$$

$$\Rightarrow \frac{M_W^2}{M_Z^2} = \frac{\frac{g^2}{2} \sum_j v_j^2 \left(T_j(T_j+1) - \frac{Y_j^2}{4} \right)}{\frac{(g^2+g'^2)}{4} \sum_j v_j^2 Y_j^2} =$$

$$\cos^2\theta_W \frac{\sum_j v_j^2 \left(T_j(T_j+1) - \frac{Y_j^2}{4} \right)}{\frac{1}{2} \sum_j v_j^2 Y_j^2}$$

$$\Rightarrow \rho^2 = \left(\frac{M_W}{M_Z \cos\theta_W} \right)^2 = \frac{\sum_j v_j^2 \left(T_j(T_j+1) - \frac{Y_j^2}{4} \right)}{\frac{1}{2} \sum_j v_j^2 Y_j^2}$$

For $T_j = 3$ and $Y = 4$ we have

$$\rho^2 = \frac{v^2}{v^2} \frac{(12-4)}{\frac{1}{2} 16} = 1$$

For any number of doublets $T_j = \frac{1}{2},\ Y_j = 1$

$$\rho^2 = \frac{\sum_j v_j^2 \left(\frac{3}{4} - \frac{1}{4} \right)}{\frac{1}{2} \sum_j v_j^2} = \frac{\sum_j v_j^2}{\sum_j v_j^2} = 1$$

For a triplet $T_j = 1$ $Y = 2$

$$\rho^2 = \frac{v^2}{\frac{1}{2}v^2}\frac{(2-1)}{4} \neq 1$$

Experimentally, $\rho \approx 1.0000....$ is highly constrained.

(2a) The fundamental and anti–fundamental representations of $SU(5)$ are

$$\begin{aligned} 5 &= (3,1)_{-\frac{2}{3}} + (1,2)_{+1} \\ \bar{5} &= (\bar{3},1)_{\frac{2}{3}} + (1,2)_{-1} \end{aligned}$$

The 24 representation is obtained by takingthe product $5 \times \bar{5}$

$$\begin{aligned} 5 \times \bar{5} &= \left[(3,1)_{-\frac{2}{3}} + (1,2)_{+1}\right] \times \left[(\bar{3},1)_{\frac{2}{3}} + (1,2)_{-1}\right] \\ &= (8,1)_0 + (1,1)_0 + (3,2)_{-\frac{5}{3}} + (\bar{3},2)_{+\frac{5}{3}} + (1,3)_0 + (1,1)_0 = 24 + 1 \end{aligned}$$

The leptoquarks in the 24 representation are:

$$(X,Y) + (\bar{X},\bar{Y}) = (\bar{3},2)_{\frac{5}{3}} \ + \ (3,2)_{-\frac{5}{3}}$$

with electric charges

$$\begin{aligned} Q(X) &= \ \frac{1}{2} + \frac{1}{2}\left(\frac{5}{3}\right) = \frac{4}{3} \\ Q(Y) &= -\frac{1}{2} + \frac{1}{2}\left(\frac{5}{3}\right) = \frac{1}{3} \end{aligned}$$

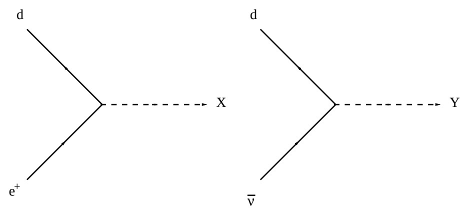
$$\begin{aligned} 5 \times 5 &= \left[(3,1)_{-\frac{2}{3}} + (1,2)_{+1}\right] \times \left[(3,1)_{-\frac{2}{3}} + (1,2)_{1}\right] \\ &= \left[(6,1)_{-\frac{4}{3}} + (1,3)_2 + (3,2)_{\frac{1}{3}}\right] + \left[(\bar{3},1)_{-\frac{4}{3}} + (3,2)_{\frac{1}{3}} + (1,1)_2\right] \\ &= 15_S + 10_A \end{aligned}$$

In $SU(5)$ the matter states are embedded in the $\bar{5}$ and 10_A representations as:

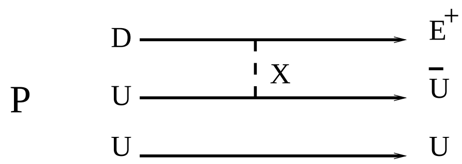
$$\begin{aligned} D_L^C &= (3,1)_{+\frac{2}{3}} \\ L_L &= (1,2)_{-1} \\ U_L^C &= (\bar{3},1)_{-\frac{4}{3}} \\ Q_L &= (3,2)_{\frac{1}{3}} \\ E_L^C &= (1,1)_2 \end{aligned}$$

Note that all the states are taken as left–handed fields.

The diagrams that couple fermions to the leptoquarks vector bosons



and a diagram leading to proton decay



(3a)

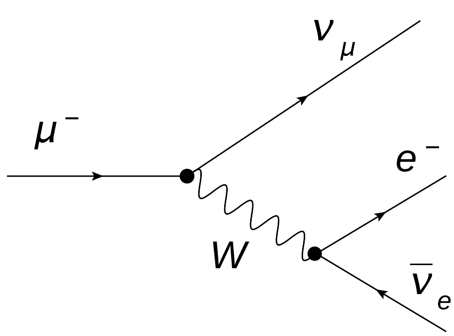
$$\bar{u}_e \gamma^\mu \tfrac{1}{2}\left(1-\gamma^5\right) u_\nu \;=\; \bar{u}_e \tfrac{1}{2}\left(1+\gamma^5\right)\gamma^\mu u_\nu \,,$$

which contains the left-handed electron field $\bar{u}_e^L$, as

$$\bar{u}_e^L \;=\; u_e^{L\,\dagger}\gamma^0 \;=\; u_e^\dagger \tfrac{1}{2}\left(1-\gamma^5\right)\gamma^0 \;=\; \bar{u}_e \tfrac{1}{2}\left(1+\gamma^5\right) \,.$$

(3b) The decay $\mu \to e\gamma$ is allowed by kinematics and w.r.t. charge conservation, but is forbidden in the Standard Model as it would violate the separate conservation of the lepton numbers N_μ and N_e. Empirically, $N_{e,\mu,\tau}$ are conserved and the decay $\mu \to e\gamma$ is not observed.

(3c) Feynman diagram for $\mu^- \to \nu_\mu \, e^- \, \bar{\nu}_e$ in the Standard Model to lowest order:



The solid ‘external’ lines stand for the incoming and outgoing spin-1/2 particles, the muon, electron, muon-neutrino and electron anti-neutrino. The wavy internal line is

the propagator of the W^- boson, the carrier of the weak interaction. It mediates the transition from the initial state μ^- to the final state ν_μ and creates the e^- and $\bar{\nu}_e$ in the final state. The solid dots denote the vertices of this weak interaction.
The algebraic expressions for the different elements are:

– incoming μ^-: spinor u

– outgoing ν_μ: spinor $\bar{u}$

– outgoing e^-: spinor $\bar{u}$

– outgoing $\bar{\nu}_e$: spinor v

– vertices: $-i\frac{g}{\sqrt{2}}\gamma^\alpha\frac{1}{2}(1-\gamma^5)$ and $-i\frac{g}{\sqrt{2}}\gamma^\beta\frac{1}{2}(1-\gamma^5)$, with weak coupling constant g

– propagator: $i\eta_{\alpha\beta}/(q^2-m_W^2)$, where q is the four-momentum transfer and m_W is the mass of the W boson

(3d) The momentum transfer squared, q^2, is of the order of (but limited by) m_μ^2, which is very small compared to m_W^2. Thefore the propagator is well approximated by $-i\eta_{\alpha\beta}/m_W^2$. This is a very strong suppression factor, which would not be present if the W would be massless.

(3e) The τ lepton is much heavier than the μ or the e, therefore it can decay into both:

$$\tau^- \longrightarrow \nu_\tau\, e^-\, \bar{\nu}_e\ , \quad \tau^- \longrightarrow \nu_\tau\, \mu^-\, \bar{\nu}_\mu\,.$$

The decays are very similar, with only small differences due to the different masses of the final state particles. In addition, as the τ is also heavier than light hadrons, it can decay in many hadronic final states like pions (the ν_τ must always be there due to N_τ conservation). In these cases the W intially couples to a quark pair which then hadronises. One example is the decay

$$\tau^- \longrightarrow \nu_\tau\, \pi^-\, \pi^0\,.$$

4. (a) *First step:*

$$i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so $i\sigma_2$ is unitary.

Second step: Prove by explicit matrix multiplication that $\sigma_2\,\sigma_a^\star\,\sigma_2 = -\sigma_a$ $(i=1,2,3)$. For $i=1$ we have e.g.

$$\sigma_2\,\sigma_1^\star\,\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -\sigma_1\,.$$

Third step: Choose $W = i\sigma_2$, and with $\sigma_2^\star \sigma_2^\star = 1\!\!1 = \sigma_2\sigma_2$ and the result of step two we write:

$$\begin{aligned} W^\dagger U^\star W &= -\,i\sigma_2^\dagger U^\star i\sigma_2 \;=\; \sigma_2 \exp\left(-\tfrac{i}{2}\theta_a\sigma_a^\star\right)\sigma_2 \\ &= \sigma_2 \left(1\!\!1 - \frac{i}{2}\theta_a\sigma_a^\star - \frac{1}{4}\frac{1}{2!}\theta_a\theta_a - \frac{i}{8}\frac{1}{3!}\theta_a\theta_a\theta_b\sigma_b^\star - \ldots\right)\sigma_2 \\ &= \left(1\!\!1 + \frac{i}{2}\theta_a\sigma_a - \frac{1}{4}\frac{1}{2!}\theta_a\theta_a + \frac{i}{8}\frac{1}{3!}\theta_a\theta_a\theta_b\sigma_b - \ldots\right) \;=\; U\,. \end{aligned}$$

Taking the complex conjugate of

$$W^\dagger U^\star W \;=\; U$$

we now also have

$$W^{\dagger\star} U W^\star \;=\; U^\star$$

and as $W = i\sigma_2 = W^\star$ we arrive at the desired relation

$$U^\star \;=\; W^\dagger\, U\, W\,.$$

(b) In the Standard Model, fermion and gauge boson masses are obtained in a gauge invariant way through electroweak symmetry breaking which is mediated by a Higgs potential. Because of the unitary equivalence between the fundamental and the complex conjugate representations of $SU(2)$, gauge invariant mass terms for both up- and down quarks (which are grouped together in $SU(2)$ doublets) can be constructed from only one complex Higgs doublet. In other words, it is due to this special property of $SU(2)$ that the Higgs sector in the Standard Model is the minimal one resulting in only one physical Higgs boson.