MATH431 Modern Particle Physics Solutions 1

1.

$$
\begin{aligned}
L &= \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{pmatrix} \\
\Rightarrow L^T &= L. \\
\text{So} \quad L^T \eta L &= \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{pmatrix} \\
&= \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{pmatrix} \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ \frac{\gamma v}{c} & -\gamma \end{pmatrix} \\
&= \begin{pmatrix} \gamma^2 - \left(\frac{\gamma v}{c}\right)^2 & -\gamma \frac{\gamma v}{c} + \gamma \frac{\gamma v}{c} \\ -\gamma \frac{\gamma v}{c} + \gamma \frac{\gamma v}{c} & \left(\frac{\gamma v}{c}\right)^2 - \gamma^2 \end{pmatrix} \\
&= \begin{pmatrix} \gamma^2 \left(1 - \frac{v^2}{c^2}\right) & 0 \\ 0 & -\gamma^2 \left(1 - \frac{v^2}{c^2}\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \eta,
\end{aligned}
$$

since

$$
\gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}}.
$$

2(i). Taking the inverse of (1), we have

$$(L^T\eta L)^{-1} = \eta^{-1} \Rightarrow L^{-1}\eta^{-1}(L^T)^{-1} = \eta^{-1}$$

Multiplying on the left by L and on the right by L^T, we find

$$\begin{aligned} LL^{-1}\eta^{-1}(L^T)^{-1}L^T =& L\eta^{-1}L^T \\ \eta^{-1} =& L\eta^{-1}L^T, \end{aligned}$$

as required.

(ii) Putting $\mu = \nu = 0$ in (3), we have

$$\begin{aligned} \eta_{\alpha\beta}L^\alpha{}_0 L^\beta{}_0 = 1, \quad \eta^{\alpha\beta}L^0{}_\alpha L^0{}_\beta = 1 \\ \text{i.e. } \left(L^0{}_0\right)^2 - \left(L^1{}_0\right)^2 - \left(L^2{}_0\right)^2 - \left(L^3{}_0\right)^2 = 1, \\ \left(\bar{L}^0{}_0\right)^2 - \left(\bar{L}^0{}_1\right)^2 - \left(\bar{L}^0{}_2\right)^2 - \left(\bar{L}^0{}_3\right)^2 = 1 \\ \Rightarrow \left(L^0{}_0\right)^2 - |\mathbf{l}|^2 = 1, \quad \left(\bar{L}^0{}_0\right)^2 - |\bar{\mathbf{l}}|^2 = 1 \\ \Rightarrow |\mathbf{l}| = \sqrt{(L^0{}_0)^2 - 1}, \quad |\bar{\mathbf{l}}| = \sqrt{\left(\bar{L}^0{}_0\right)^2 - 1}. \end{aligned}$$

(iii)

$$\begin{aligned} (\bar{L}L)^0{}_0 =& \bar{L}^0{}_\alpha L^\alpha{}_0 \\ =& \bar{L}^0{}_0 L^0{}_0 + \bar{L}^0{}_1 L^1{}_0 + \bar{L}^0{}_2 L^2{}_0 + \bar{L}^0{}_3 L^3{}_0 \\ =& \bar{L}^0{}_0 L^0{}_0 + \bar{\mathbf{l}}.\mathbf{l} \end{aligned}$$

(iv)

$$\begin{aligned} |\bar{\mathbf{l}}.\mathbf{l}| \leq |\mathbf{l}||\bar{\mathbf{l}}| \Rightarrow -|\mathbf{l}||\bar{\mathbf{l}}| \leq \bar{\mathbf{l}}.\mathbf{l} \leq |\mathbf{l}||\bar{\mathbf{l}}| \\ \Rightarrow \quad \text{(using (iii))} \quad (\bar{L}L)^0{}_0 - \bar{L}^0{}_0 L^0{}_0 \geq -|\mathbf{l}||\bar{\mathbf{l}}| \\ \Rightarrow (\bar{L}L)^0{}_0 \geq \bar{L}^0{}_0 L^0{}_0 - |\mathbf{l}||\bar{\mathbf{l}}| \\ \text{i.e. (using (ii))} \quad (\bar{L}L)^0{}_0 \geq \bar{L}^0{}_0 L^0{}_0 - \sqrt{(L^0{}_0)^2 - 1}\sqrt{\left(\bar{L}^0{}_0\right)^2 - 1}. \end{aligned}$$

(v)

$$\begin{aligned} (x-y)^2 \geq 0 \Rightarrow x^2 - 2xy + y^2 \geq 0 \Rightarrow -2xy \geq -x^2 - y^2 \\ \Rightarrow x^2y^2 - 2xy + 1 \geq x^2y^2 - x^2 - y^2 + 1 \\ \Rightarrow x^2y^2 - 2xy + 1 \geq (x^2-1)(y^2-1) \Rightarrow (xy-1)^2 \geq (x^2-1)(y^2-1). \\ \Rightarrow \text{either} \quad xy - 1 \geq \sqrt{x^2-1}\sqrt{y^2-1} \quad \text{or} \quad xy - 1 \leq -\sqrt{x^2-1}\sqrt{y^2-1}. \end{aligned}$$

If $x, y \geq 1$ then $xy - 1$ is positive, and we must have the first inequality, implying

$$xy - \sqrt{x^2-1}\sqrt{y^2-1} \geq 1.$$

Writing $L^0{}_0 = x$, $\bar{L}^0{}_0 = y$, we have from (iv)

$$(\bar{L}L)^0{}_0 \geq \bar{L}^0{}_0 L^0{}_0 - \sqrt{(L^0{}_0)^2 - 1}\sqrt{\left(\bar{L}^0{}_0\right)^2 - 1} \geq 1.$$

(vi) Obviously if $\det \bar{L} = \det L = 1$, then

$$\det(\bar{L}L) = \det \bar{L} \det L = 1.$$

(vii) We have now shown that if $L \in \mathcal{L}_+^\uparrow$, $L \in \mathcal{L}_+^\uparrow$, then $\bar{L}L \in \mathcal{L}_+^\uparrow$. It is clear that $1 \in \mathcal{L}_+^\uparrow$.

(viii) We can write (1) as

$$(\eta^{-1}L^T\eta)L = \eta^{-1}\eta = 1,$$

which shows that $L^{-1} = \eta^{-1}L^T\eta$, i.e. $(L^{-1})^\mu{}_\nu = \eta^{\mu\alpha}(L^T)_\alpha{}^\beta\eta_{\beta\nu} = \eta^{\mu\alpha}L^\beta{}_\alpha\eta_{\beta\nu}$. So $(L^{-1})^0{}_0 = \eta^{00}L^0{}_0\eta_{00} = L^0{}_0$. Moreover,

$$\det L^{-1} = \det \eta^{-1} \det L^T \det \eta = (-1) \det L(-1) = \det L = 1.$$

So $L^{-1} \in \mathcal{L}_+^\uparrow$. The remaining group property is associativity, $(L_1L_2)L_3 = L_1(L_2L_3)$, which is true for all matrices. So $\mathcal{L}_+^\uparrow$ *is* a group.

3.

The two inertial frames are moving relative to each other in the direction of the x axis at constant speed v. Denoting $\beta = v/c$, where c is the speed of light, and $\gamma = 1/\sqrt{1-\beta^2}$ the Lorentz transformations between the two frames are given by

$$\begin{aligned} a'^0 &= \gamma(a^0 - \beta a^1) \\ a'^1 &= \gamma(-\beta a^0 + a^1) \\ a'^2 &= a^2 \\ a'^3 &= a^3 \end{aligned}$$

or in matrix form

$$\begin{pmatrix} a'^0 \\ a'^1 \\ a'^2 \\ a'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix}$$

$$\begin{aligned} b'^0 &= \gamma(b^0 - \beta b^1) \\ b'^1 &= \gamma(-\beta b^0 + b^1) \\ b'^2 &= b^2 \\ b'^3 &= b^3 \end{aligned}$$

which can be written similarly in matrix form. Lowering the indices on the left–hand side changes the sign on the spatial components on the right–hand side

$$
\begin{aligned}
b'_0 &= \gamma(b^0 - \beta b^1) \\
b'_1 &= -\gamma(-\beta b^0 + b^1) \\
b'_2 &= -b^2 \\
b'_3 &= -b^3
\end{aligned}
$$

Then

$$a'^\mu b'_\mu = a'^0 b'_0 + a'^1 b'_1 + a'^2 b'_2 + a'^3 b'_3 +$$

$$
\begin{aligned}
a'^\mu b'_\mu &= \gamma(a^0 - \beta a^1)\gamma(b^0 - \beta b^1) - \gamma(-\beta a^0 + a^1)\gamma(-\beta b^0 + b^1) - a^2 b_2 - a^3 b_3 \\
&= \gamma^2 \left[a^0 b^0 (1 - \beta^2) - a^1 b^1 (1 - \beta^2) \right] - a^2 b_2 - a^3 b_3 \\
&= a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 = a^0 b_0 + a^1 b_1 + a^2 b_2 + a^3 b_3 \\
&= a^\mu b_\mu
\end{aligned}
$$

4.

Lorentz transformations, derivatives and quantum operators

a.

$$
\begin{aligned}
& a_0 = a^0,\ a_1 = -a^1,\ a_2 = -a^2,\ a_3 = -a^3 \\
& a'_0 = a'^0 = \gamma(a^0 - \beta a^1) = \gamma(a_0 + \beta a_1) \\
& a'_1 = -a'^1 = -\gamma(-\beta a^0 + a^1) = \gamma(\beta a_0 + a_1) \\
\text{and}\ \ & a'_2 = a_2,\ a'_3 = a_3
\end{aligned}
\tag{1}
$$

b.

Suppose we have a function $f(x^0, x^1, x^2, x^3)$ which we express as a function of $x'^0,\ x'^1,\ x'^2,\ x'^3$ by expressing x^μ as a function of x'^μ. The standard chain rule for partial differentiation says that

$$\frac{\partial f(x^\nu(x'^\mu))}{\partial x'^\mu} = \sum_{\nu=0}^{3} \frac{\partial f}{\partial x^\nu} \frac{\partial x^\nu}{\partial x'^\mu} \quad \text{for} \quad \mu = 0, 1, 2, 3$$

Using the summation convention and writing as an operator equation we get

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu}$$

We need x as a function of x', the inverse of the Lorentz transformation that gives x' as a function of x. For a boost along the x' axis, the inverse is a boost with the opposite speed, so

$$x^0 = \gamma(x'^0 + \beta x'^1),\ x^1 = \gamma(\beta x'^0 + x'^1),\ x^2 = x'^2 x^3 = x'^3$$

Hence

$$\frac{\partial x^0}{\partial x'^0} = \gamma, \quad \frac{\partial x^0}{\partial x'^1} = \gamma\beta, \quad \frac{\partial x^1}{\partial x'^0} = \gamma\beta, \quad \frac{\partial x^1}{\partial x'^1} = \gamma$$

and

$$\frac{\partial}{\partial x'^0} = \gamma(\frac{\partial}{\partial x^0} + \beta\frac{\partial}{\partial x^1}), \quad \frac{\partial}{\partial x'^1} = \gamma(\beta\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}), \quad \frac{\partial}{\partial x'^2} = \frac{\partial}{\partial x^2}, \quad \frac{\partial}{\partial x'^3} = \frac{\partial}{\partial x^3}$$

which is the same as as (1) with $a_\mu = \frac{\partial}{\partial x^\mu}$
c.

The operator for momentum $\vec{p}$ is $-i\hbar\vec{\nabla}$, *i.e.*

$$p^1 = -i\hbar\frac{\partial}{\partial x_1},\ p^2 = -i\hbar\frac{\partial}{\partial x_2},\ p^3 = -i\hbar\frac{\partial}{\partial x_3} \qquad (5)$$

and

$$p_1 = i\hbar\frac{\partial}{\partial x^1},\ p_2 = i\hbar\frac{\partial}{\partial x^2},\ p_3 = i\hbar\frac{\partial}{\partial x^3} \qquad (6)$$

The Schrödinger equation says

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi$$

where H is the energy operator. Since $p^0 = \frac{E}{c}$ and $x^0 = ct$, this can be written as

$$p_0 = i\hbar\frac{\partial}{\partial x^0} \qquad (7)$$

If we write (5) and (7) in terms of p_μ, we remove the sign difference between the 0 component and the others.

$$p_\mu = i\hbar\frac{\partial}{\partial x^\mu},$$

which transforms as a Lorentz covariant vector, as we have shown in (b).