

**MATH431 - Modern Particle Physics**  
**Set Work: Sheet 3;**

1. Consider the Poincare group in 1+2 dimensions, with the infinitesimal line element

$$ds^2 = dt^2 - dx^2 - dy^2$$

- a. Write in matrix form the metric for this line element and its inverse.
- b. Write all the transformations under which this line element is invariant. Write down the generators associated with each transformation.
- c. The Pauli-Lubanski vector in four dimensions is given by

$$W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\sigma\rho}J_{\nu\sigma}P_\rho$$

Write down the four components of the Pauli-Lubanski vector in the case of the 3 dimensional line element given above, for massless and massive particle states.

2. The Pauli-Lubanski vector in four dimensions is given by

$$W_\sigma = -\frac{1}{2}\epsilon_{\sigma\mu\nu\lambda}J^{\mu\nu}P^\lambda,$$

where  $\epsilon_{\sigma\mu\nu\lambda}$  is asymmetric under odd permutation of the indices;  $J^{\mu\nu}$  are the generator of Lorentz transformations; and  $P^\lambda$  is the momentum four vector.

- a. Show that  $W_\sigma P^\sigma = 0$
- b. The commutation relations of the Pauli-Lubanski with the generators of the Poincare group are given by

$$\begin{aligned}[J_{\mu\nu}, W_\rho] &= i(\eta_{\nu\rho}W_\mu - \eta_{\mu\rho}W_\nu), \\ [P_\nu, W_\rho] &= 0\end{aligned}$$

Show that the generators of the Poincare transformations commute with  $W_\mu W^\mu$ .

3. Write down the Lagrangian for a particle of mass  $m$  in a potential  $V(r, \phi)$  when referred to planar polar coordinates  $(r, \phi)$ . Hence show that the equations of motion are

$$m\ddot{r} - mr\dot{\phi}^2 = -\frac{\partial V}{\partial r} \quad mr\ddot{\phi} + 2m\dot{r}\dot{\phi} = -\frac{1}{r}\frac{\partial V}{\partial \phi}.$$

Hence derive the principle of conservation of angular momentum in the plane, and obtain the usual formula  $v^2/r$  for centripetal acceleration.

4. a. Show that if the Hamiltonian is independent of a generalized coordinate  $q_0$ , then the conjugate momentum  $p_0$  is a constant of the motion. Such coordinates are called **cyclic coordinates**. Give two examples of a physical system that has a cyclic coordinate.

b. Show that in 3 dimension spherical polar coordinates the Hamiltonian of a particle of mass  $m$  moving in a potential  $V(\vec{x})$  is

$$H = \frac{1}{2m} \left( p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta} \right) + V(\vec{x}).$$

show that  $p_\phi = \text{constant}$  when  $\partial V / \partial \phi \equiv 0$  and interpret this result physically.