MATH431 Modern Particle Physics Solutions 9

1a The case in the absence of electric source was solved in problem set 5 question 2. From the Euler–Lagrange eq. of motion the second term gives

$$\frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial A_{\mu,\nu}}\right)-\frac{\partial L}{\partial A_\mu}=0$$

$$\frac{\partial L}{\partial A_\mu}=j^\mu \Rightarrow \frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial A_{\mu,\nu}}\right)=\partial_\nu F^{\mu\nu}=j^\mu$$

$$\partial_\mu\partial_\nu F^{\mu\nu}=0\rightarrow\partial_\mu j^\mu=0$$

1b In the Lorentz gauge we impose $\partial_\mu A^\mu=0$. The derivative of the mass term $\frac{1}{2}m^2 A_\mu A^\mu$ with respect to A^ν gives m^2A^ν. Hence

$$(\partial_\mu\partial^\mu A^\nu+m^2A^\nu)=j^\nu$$

.

2

$$L=\frac{1}{2}(\partial_\mu\phi_i)^2-\frac{1}{2}\mu^2\phi_i^2-\frac{1}{4}\lambda(\phi_i^2)^2$$

with $\mu^2<0$ and $\lambda>0$

$$\frac{1}{2}\left((\partial_\mu\phi_1)^2+(\partial_\mu\phi_2)^2+(\partial_\mu\phi_3)^2\right)-\frac{1}{2}\mu^2(\phi_1^2+\phi_2^2+\phi_3^2)-\frac{1}{4}\lambda(\phi_1^2+\phi_2^2+\phi_3^2)^2$$

$$V=\frac{1}{2}\mu(\phi_1^2+\phi_2^2+\phi_3^2)+\frac{1}{4}\lambda(\phi_1^2+\phi_2^2+\phi_3^2)^2$$

$$\frac{\partial V}{\partial\phi_i}=(\mu^2+\lambda(\phi_1^2+\phi_2^2+\phi_3^2))\phi_i=0$$

$$\mu^2+\lambda(\phi_1^2+\phi_2^2+\phi_3^2)=0$$

Take

$$\langle\phi_1\rangle=\sqrt{-\frac{\mu^2}{\lambda}}=v$$

Expand

$$\phi(x)=(v+h(x))\quad;\quad\phi_2(x)\quad;\quad\phi_3(x)$$

Inserting into the Lagrangian we have

$$\frac{1}{2}\left((\partial_\mu\phi_1)^2+(\partial_\mu\phi_2)^2+(\partial_\mu\phi_3)^2\right)-\frac{1}{2}\mu^2((v+h(x))^2+\phi_2^2+\phi_3^2)-\frac{1}{4}\lambda((v+h(x))^2+\phi_2^2+\phi_3^2)^2$$

with $-\mu^2 = v^2\lambda$ we get

$$
\begin{aligned}
&\frac{1}{2}((\partial_\mu h)^2 + (\partial_\mu \phi_2)^2 + (\partial_\mu \phi_3)^2) + \\
&\frac{v^2\lambda}{2}(h^2(x) + v^2 + 2vh(x) + \phi_2^2 + \phi_3^2) - \frac{\lambda}{4}(6v^2h^2(x) + \cdots) = \\
&\frac{1}{2}\left((\partial_\mu h)^2 + (\partial_\mu \phi_2)^2 + (\partial_\mu \phi_3)^2\right) - \frac{1}{2}\lambda v^2 h^2(x) + \\
&\qquad\qquad \text{cubic and quartic interaction terms + constant}
\end{aligned}
$$

hence the Lagrangian describes one massive scalar field and two massless Goldstone bosons.

3 The Lagrangians of the two massive fields are:

$$
\begin{aligned}
\mathcal{L}_1 &= (\partial_\mu \Phi_1)^\dagger (\partial^\mu \Phi_1) - m_1^2 \Phi_1^\dagger \Phi_1 \\
\mathcal{L}_2 &= (\partial_\mu \Phi_2)^\dagger (\partial^\mu \Phi_2) - m_2^2 \Phi_2^\dagger \Phi_2
\end{aligned}
$$

the total Lagrangian is $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$. Hence $\mathcal{L}$ is not invariant under the exchange

$$\Phi_1 \leftrightarrow \Phi_2$$

Combining $\Phi = (\Phi_1, \Phi_2)$ the Lagrangian

$$\mathcal{L} = (\partial_\mu \Phi)^\dagger (\partial^\mu \Phi) - \mu^2 \Phi^\dagger \Phi - \lambda(\Phi^\dagger \Phi)^2$$

preserves the symmetry which is spontenousely broken in the vacuum for $\mu^2 < 0$.

4 Considering the Lagrangian

$$L(q(t), \dot{q}(t), t)$$

and the variation

$$
\begin{aligned}
q(t) &\rightarrow q(t) + \delta(q(t)) = q(t) + \epsilon h(q(t), t) \\
\dot{q}(t) &\rightarrow \dot{q}(t) + \frac{d}{dt}(\delta(q(t)))
\end{aligned}
$$

Variation of the action gives rise to the Euler-Lagrange equation of motion

$$S(t) = \int dt L(q(t), \dot{q}(t), t) \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

since L is invariant under the variation

$$L(q + \delta q, \dot{q} + \frac{d}{dt}\delta q, t) = L + \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}\delta q + \cdots = L$$

$$\implies \quad \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}\delta q = 0$$

Now take

$$\begin{aligned} \epsilon\frac{dQ}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right) &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d\delta q}{dt} &= \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}\delta q &= 0 \\ \implies \frac{dQ}{dt} = 0, \end{aligned}$$

where we have used the Euler–Lagrange equation to go from the second to third equality. Therefore, we have that

$$Q(t) = \frac{\partial L}{\partial \dot{q}}h(q(t), t)$$

is conserved.