MATH431 Modern Particle Physics Solutions 7

1

$$UU^\dagger = 1 \Rightarrow U = \mathrm{e}^{iH} \quad ; \quad U^\dagger = \mathrm{e}^{-iH^\dagger} = \mathrm{e}^{-iH}$$

Hence we must have $H = H^\dagger$. H must be hermitian.

2a. 3 diagonal generators.

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_{15} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

2b. $D = 15$. Three diagonal generators of part $(2a)$ plus:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_9 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_{10} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\lambda_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad , \quad \lambda_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

6c.

$$4 = (3, 1/3) + (1, -1)$$

under the maximal subgroup $SU(3) \times U(1)$

6d.

see separate figure

6e.

$$4 \times \bar{4} = \{(3, 1/3) + (1, -1)\} \times \{(\bar{3}, -1/3) + (1, +1)\} =$$
$$15 + 1 = \{(3, +4/3) + (8, 0) + (1, 0) + (\bar{3}, -4/3)\} + (1, 0)$$

6f.

$$4 \times 4 = \{(3, 1/3) + (1, -1)\} \times \{(3, 1/3) + (1, -1)\} =$$
$$6 + 10 = \{(6, 2/3) + (3, -2/3) + (1, -2)\} + \{(\bar{3}, 2/3) + (3, -2/3)\}$$

6g.

$SU(4)$ decomposes as $SU(3) \times U(1)$ under the Standard Model, where $SU(3)$ corresponds to the gauge symmetry of the strong interactions and the $U(1)$ can be indentified as baryon minus lepton number. The fundamental representation decomposes as a triplet and a singlet of $SU(3)$ with the triplet corresponding to a quark, with baryon number $1/3$, and the singlet to a lepton, with lepton number -1. The Cartan generators correspond to the two diagonal generators of $SU(3)$ and the $U(1)_{B-L}$ generator.