MATH431 Modern Particle Physics Solutions 6

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$$\psi(r,\theta,\phi) = R(r)\begin{pmatrix} 1 \\ 0 \\ ia\cos\theta \\ iae^{i\phi}\sin\theta \end{pmatrix}$$

Normalize

$$\int d^3\mathbf{r}\psi^\dagger\psi = 1 = \int 4\pi r^2 dr(|R|^2(1+a^2))$$

$$\Rightarrow \int_0^\infty r^2|R|^2 dr = [4\pi(1+a^2)]^{-1}$$

(a.)

$$L_z = -i\frac{\partial}{\partial\phi} \Rightarrow L_z\psi = R\begin{pmatrix} 0 \\ 0 \\ 0 \\ iae^{i\phi}\sin\theta \end{pmatrix} \not\propto \psi$$

so ψ is not an eigenstate of L_z.

(b.)

$$\langle L_z\rangle = \int d^3\mathbf{r}\psi^\dagger L_z\psi = \int 2\pi r^2 d\cos\theta|R|^2 a^2\sin^2\theta$$

$$\int_{-1}^{1} d\cos\theta(1-\cos^2\theta) = 2 - \frac{2}{3} = \frac{4}{3} \Rightarrow \langle L_z\rangle = \frac{8\pi}{3}a^2\cdot\frac{1}{4\pi(1+a^2)}$$

$$\Rightarrow \langle L_z\rangle = \frac{2a^2}{3(1+a^2)}$$

In H-atom , $v/c \sim \alpha \Rightarrow \langle L_z\rangle = O(v^2/c^2)$. This is a relativistic effect - spin–orbit interaction.

(c.)

$$S_z = \frac{1}{2}\hbar\begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\Rightarrow S_z\psi = \frac{1}{2}\hbar R\begin{pmatrix} 1 \\ 0 \\ ia\cos\theta \\ -iae^{i\phi}\sin\theta \end{pmatrix}$$

$$(L_z+S_z)\psi = R\begin{pmatrix} 1 \\ 0 \\ ia\cos\theta \\ iae^{i\phi}\sin\theta \end{pmatrix} = \frac{1}{2}\psi \Rightarrow J_z = +\frac{1}{2}$$

2. (a) Operating with $\gamma^\nu \partial_\nu$ from the left on the Dirac equation we have

$$\begin{aligned}
\gamma^\nu \partial_\nu \left(i\gamma^\mu \partial_\mu - m\right)\psi(x) &= \\
i\frac{1}{2}\left(\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu\right)\partial_\nu\partial_\mu\psi - m\gamma^\nu\partial_\nu\psi &= \\
i\eta^{\mu\nu}\partial_\nu\partial_\mu\psi + im^2\psi &= \\
i\left(\partial^\mu\partial_\mu + m^2\right)I\psi &= 0
\end{aligned}$$

where I is the 4×4 identity matrix.

(b)

$$\gamma^\mu\partial_\mu\psi + im\psi = 0 \quad , \quad (\partial\psi^\dagger){\gamma^\mu}^\dagger - im\psi^\dagger = 0$$

$$\begin{aligned}
&\Rightarrow (\partial\psi^\dagger)\gamma^0\gamma^\mu\gamma^0 - im\psi^\dagger = 0 \\
&\Rightarrow (\partial_\mu\bar{\psi})\gamma^\mu - im\bar{\psi} = 0
\end{aligned}$$

(c)

- $\partial_\mu(\bar{\psi}\gamma^\mu\psi) = (\partial_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu(\partial_\mu\psi)$
  $= (im\bar{\psi})\psi + \bar{\psi}(-im\psi) = 0$

- $\partial_\mu(\bar{\psi}\gamma^\mu\gamma^5\psi) = (\partial_\mu\bar{\psi})\gamma^\mu\gamma^5\psi + \bar{\psi}\gamma^\mu\gamma^5(\partial_\mu\psi)$
  $= (im\bar{\psi})\gamma^5\psi - \bar{\psi}\gamma^5(\gamma^\mu\partial_\mu\psi) = 2im\bar{\psi}\gamma^5\psi$

3.

$$\bar{u}_f(\not{p}_f - m)\gamma^\mu u_i = \bar{u}_f\gamma^\mu(\not{p}_i - m)u_i = 0 \qquad \text{(Dirac eq.)}$$

$$\Rightarrow 2m\bar{u}_f\gamma^\mu u_i = \bar{u}_f(\not{p}_f\gamma^\mu + \gamma^\mu\not{p}_i)u_i$$

$$\not{p}_f\gamma^\mu + \gamma^\mu\not{p}_i = \gamma^\nu\gamma^\mu p_{f_\nu} + \gamma^\mu\gamma^\nu p_{i_\nu}$$

$$\begin{aligned}
\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu &= 2g^{\mu\nu} \\
\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu &= -2i\sigma^{\mu\nu}
\end{aligned}$$

Hence
$\gamma^\mu\gamma^\nu = g^{\mu\nu} - i\sigma^{\mu\nu}; \gamma^\nu\gamma^\mu = g^{\mu\nu} + i\sigma^{\mu\nu}$

$$\Rightarrow \not{p}_f\gamma^\mu + \gamma^\mu\not{p}_i = g^{\mu\nu}(p_f + p_i)_\nu + i\sigma^{\mu\nu}(p_f - p_i)_\nu = (p_f + p_i)^\mu + i\sigma^{\mu\nu}(p_f - p_i)_\nu$$

$$\Rightarrow \bar{u}_f\gamma^\mu U_i = \frac{1}{2m}\bar{u}_f[(p_f + p_i)^\mu + i\sigma^{\mu\nu}(p_f - p_i)_\nu]u_i$$

4. We consider an electron in a constant magentic field $\vec{B} = (0, 0, B)$ with $B > 0$.
(a.)
The vector potential

$$A^\mu = (0, 0, Bx, 0)$$

(b.)

$$(i\partial_0 - m)\phi = \vec{\sigma} \cdot (\vec{p} - e\vec{A})\chi$$
$$(i\partial_0 + m)\chi = \vec{\sigma} \cdot (\vec{p} - e\vec{A})\phi$$

where, as usual, $\vec{p} = -i\nabla$.

(c.) Assuming a solution of the form

$$\phi(x) = \phi(\vec{x})e^{-iEt}, \chi(x) = \chi(\vec{x})e^{-iEt}$$

Inserting into the equations from (b.) these equations become

$$(E - m)\phi(\vec{x}) = \vec{\sigma} \cdot (\vec{p} - e\vec{A})\chi(\vec{x})$$
$$(E + m)\chi(\vec{x}) = \vec{\sigma} \cdot (\vec{p} - e\vec{A})\phi(\vec{x})$$

Substituting $\chi(\vec{x})$ from the second equation into the first and repeating the steps that we too in class when deriving the gyromagnetic factor from the Dirac equation, we get

$$\begin{aligned}(E^2 - m^2)\phi(\vec{x}) &= [(\vec{p} - e\vec{A})^2 - e\vec{\sigma} \cdot \vec{B}]\phi(\vec{x}) \\ &= [\vec{p}^2 + e^2B^2x^2 - 2ep_yBx - e\sigma_zB]\phi(\vec{x})\end{aligned}$$

Since p_x, p_y commute with x, we can seach for solutions of the form

$$\phi(\vec{x}) = e^{i(p_y y + p_z z)}f(x)$$

where p_y and p_z are c-numbers and $f(x)$, as $\phi(\vec{x})$, is a two component spinor. The equation for $f(x)$ becomes

$$[-\frac{d^2}{dx^2} + (p_y - eBx)^2 - eB\sigma_z]f(x) = (E^2 - m^2 - p_z^2)f(x)$$

$f(x)$ can be taken to be an eigenfunction of σ_z with eigenvalues $\sigma = \pm 1$, $\sigma_z f = \sigma f$. Then

$$[-\frac{d^2}{dx^2} + \frac{1}{2}(2e^2B^2)(x - \frac{p_y}{eB})^2]f(x) = (E^2 - m^2 - p_z^2 + eB\sigma)f(x)$$

This is formally identical to the Schrödinger equation of an harmonic oscillator with frequency $2|e|B$. The energy levels are therefore given by

$$E^2 - m^2 - p_z^2 + eB\sigma = (n + \frac{1}{2})2|e|B$$

or

$$E = [m^2 + p_z^2 + (2n+1+\sigma)|e|B]^{\frac{1}{2}}$$

Observe that there is a continuous degeneracy in p_x and p_y, as well as a discrete degeneracy

$$E(n, p_z, \sigma = +1) = E(n+1, p_z, \sigma = -1).$$

In the nonrelativistic limit $p_z << m^2$, $(2n+1)|e|B << m^2$ the nonrelativistic limit therefore gives

$$E(n, p_z, \sigma) \simeq m + \frac{p_z^2}{2m} + \left(n + \frac{1+\sigma}{2}\right)\omega_B$$

with $\omega_B = |e|B/m$. These are the Landau levels of nonrelativistic quantum mechanics.