

MATH431 Modern Particle Physics Solutions 5

1. The Lagrangian of the given two-dimensional potential is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{4}k(x^2 + y^2)^2.$$

(ii) The Euler-Lagrange equations are

$$\begin{aligned}\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= m\ddot{x} + k(x^2 + y^2)x = 0, \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} &= m\ddot{y} + k(x^2 + y^2)y = 0.\end{aligned}$$

(iii) The Hamiltonian is

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{4}mw^2(x^2 + y^2)^2 \quad \text{with} \quad w = \sqrt{\frac{k}{m}}.$$

(iv) In polar coordinates we derive:

$$\begin{aligned}L &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{1}{4}kr^4, \\ p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}.\end{aligned}$$

Hence we get

$$\begin{aligned}H &= \sum_i p_i \dot{q}_i - L = m\dot{r}^2 + mr^2\dot{\phi}^2 - \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{4}mw^2r^4 \\ &= \frac{1}{2}m(\dot{r}^2 + mr^2\dot{\phi}^2) + \frac{1}{4}mw^2r^4 \\ &= \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + \frac{1}{4}mw^2r^4.\end{aligned}$$

(v) There are two constants of the motion.

Since the Hamiltonian does not depend explicitly on time, the energy is a constant of the motion with $E = H(q_0, p_0)$. Since it does not depend explicitly on ϕ , also p_ϕ is a constant of the motion, corresponding to conservation of the angular momentum w.r.t. the symmetry axis.

2. Two-particle states are defined by

$$\begin{aligned}
|\mathbf{p}_1, \mathbf{p}_2\rangle &= a^\dagger(\mathbf{p}_1) a^\dagger(\mathbf{p}_2) |0\rangle \\
|\mathbf{p}_1, \mathbf{p}_2\rangle &= |\mathbf{p}_2, \mathbf{p}_1\rangle \quad \text{as} \quad [a(\mathbf{p}_1), a(\mathbf{p}_2)] = 0 \\
\langle \mathbf{p}'_1, \mathbf{p}'_2 | \mathbf{p}_1, \mathbf{p}_2 \rangle &= \langle 0 | a(\mathbf{p}'_1) a(\mathbf{p}'_2) a^\dagger(\mathbf{p}_1) a^\dagger(\mathbf{p}_2) | 0 \rangle \\
&= \langle 0 | a(\mathbf{p}'_1) \{ a^\dagger(\mathbf{p}_1) a(\mathbf{p}'_2) + (2\pi)^3 2p_1^0 \delta(\mathbf{p}_1 - \mathbf{p}'_2) \} a^\dagger(\mathbf{p}_2) | 0 \rangle \\
&= \langle 0 | a(\mathbf{p}'_1) a^\dagger(\mathbf{p}_1) \{ a^\dagger(\mathbf{p}_2) a(\mathbf{p}'_2) + (2\pi)^3 2p_1^0 \delta(\mathbf{p}_2 - \mathbf{p}'_2) \} | 0 \rangle \\
&\quad + (2\pi)^3 2p_1^0 \delta(\mathbf{p}_1 - \mathbf{p}'_2) \langle 0 | a(\mathbf{p}'_1) a^\dagger(\mathbf{p}_2) | 0 \rangle \\
&= (2\pi)^3 2p_2^0 \delta(\mathbf{p}_2 - \mathbf{p}'_2) \langle 0 | \{ a^\dagger(\mathbf{p}_1) a(\mathbf{p}'_1) + (2\pi)^3 2p_1^0 \delta(\mathbf{p}_1 - \mathbf{p}'_1) \} | 0 \rangle \\
&\quad + (2\pi)^3 2p_1^0 \delta(\mathbf{p}_1 - \mathbf{p}'_2) \langle 0 | \{ a^\dagger(\mathbf{p}_2) a(\mathbf{p}'_1) + (2\pi)^3 2p_2^0 \delta(\mathbf{p}_2 - \mathbf{p}'_1) \} | 0 \rangle \\
&= (2\pi)^6 (2p_1^0)(2p_2^0) \{ \delta(\mathbf{p}_1 - \mathbf{p}'_1) \delta(\mathbf{p}_2 - \mathbf{p}'_2) + \delta(\mathbf{p}_1 - \mathbf{p}'_2) \delta(\mathbf{p}_2 - \mathbf{p}'_1) \}.
\end{aligned}$$

3(a).

The electromagnetic field strength tensor,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

$$\begin{aligned}
L_{e.m.} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\
&= -\frac{1}{4} (A_{\mu,\nu} - A_{\nu,\mu}) (A^{\mu,\nu} - A^{\nu,\mu}) \\
&= -\frac{1}{4} \eta^{\alpha\mu} \eta^{\beta\nu} (A_{\beta,\alpha} - A_{\alpha,\beta}) (A_{\mu,\nu} - A_{\nu,\mu})
\end{aligned}$$

The Euler–Lagrange equations of motion are obtained from the Lagrangian

$$\frac{\partial}{\partial x^\nu} \left(\frac{\partial L}{\partial A_{\mu,\nu}} \right) - \frac{\partial L}{\partial A_\mu} = 0$$

$$\frac{\partial L}{\partial A_\mu} = 0 \Rightarrow \frac{\partial}{\partial x^\nu} \left(\frac{\partial L}{\partial A_{\mu,\nu}} \right) = 0$$

$$\begin{aligned}
\frac{\partial L}{\partial A_{p,q}} &= -\frac{1}{4} \eta^{\alpha\mu} \eta^{\beta\nu} (\delta_\alpha^p \delta_\beta^q - \delta_\beta^p \delta_\alpha^q) (A_{\mu,\nu} - A_{\nu,\mu}) + (A_{\alpha,\beta} - A_{\beta,\alpha}) (\delta_\mu^p \delta_\nu^q - \delta_\nu^p \delta_\mu^q) \\
&= -\frac{1}{4} (\eta^{p\mu} \eta^{q\nu} - \eta^{q\mu} \eta^{p\nu}) (A_{\mu,\nu} - A_{\nu,\mu}) + (\eta^{\alpha p} \eta^{\beta q} - \eta^{\alpha q} \eta^{\beta p}) (A_{\beta,\alpha} - A_{\alpha,\beta}) \\
&= -\frac{1}{4} (A^{q,p} - A^{p,q}) \\
&= -F^{pq}
\end{aligned}$$

$$\frac{\partial}{\partial x^\nu} \left(\frac{\partial L}{\partial A_{\mu,\nu}} \right) = -\frac{1}{4} \frac{\partial}{\partial x^\nu} F^{\mu\nu} = 0$$

which are Maxwell's equation in the absence of sources.

$$A_\mu \rightarrow \tilde{A}_\mu = A_\mu + \partial_\mu \Lambda$$

$$\begin{aligned} \tilde{F}_{\mu\nu} &= \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu = \partial_\mu (A_\nu - \partial_\nu \Lambda) - \partial_\nu (A_\mu - \partial_\mu \Lambda) \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - \partial_\mu \partial_\nu \Lambda + \partial_\nu \partial_\mu \Lambda = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu} \end{aligned}$$

Therefore $L_{e.m.}$ is also invariant under the transformation.

(b) the Lagrangian for a massive photon without sources is given by

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + m^2 A_\mu A^\mu$$

The first term is invariant under transformation. The second term transforms to

$$\rightarrow m^2 \tilde{A}_\mu \tilde{A}^\mu = m^2 A_\mu A^\mu + m^2 (\partial_\mu \Lambda \partial^\mu \Lambda - \partial_\mu \Lambda \partial^\mu \Lambda - A_\mu \partial^\mu \Lambda)$$

therefore the mass term does not vanish under the transformation. Requiring invariance imposes $m^2 = 0$.

4.

$$\begin{aligned} \gamma^{5\dagger} &= (i\gamma^0 \gamma^1 \gamma^2 \gamma^3)^\dagger \\ &= -i\gamma^{3\dagger} \gamma^{2\dagger} \gamma^{1\dagger} \gamma^{0\dagger} \\ &= -i(\gamma^0 \gamma^3 \gamma^0)(\gamma^0 \gamma^2 \gamma^0)(\gamma^0 \gamma^1 \gamma^0) \gamma^0 \\ &= -i\gamma^0 \gamma^3 \gamma^2 \gamma^1 = i\gamma^0 \gamma^2 \gamma^3 \gamma^1 = -i\gamma^0 \gamma^2 \gamma^1 \gamma^3 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \gamma^5. \end{aligned}$$

For the second part, it's best to do for each μ in turn:

$$\begin{aligned} \gamma^5 \gamma^0 &= i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^0 = -i\gamma^0 \gamma^1 \gamma^2 \gamma^0 \gamma^3 = i\gamma^0 \gamma^1 \gamma^0 \gamma^2 \gamma^3 \\ &= -i\gamma^0 \gamma^0 \gamma^1 \gamma^2 \gamma^3 = -\gamma^0 \gamma^5 \Rightarrow \{\gamma^5, \gamma^0\} = 0, \\ \gamma^5 \gamma^1 &= i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^1 = -i\gamma^0 \gamma^1 \gamma^2 \gamma^1 \gamma^3 = i\gamma^0 \gamma^1 \gamma^1 \gamma^2 \gamma^3 \\ &= -i\gamma^1 \gamma^0 \gamma^1 \gamma^2 \gamma^3 = -\gamma^1 \gamma^5 \Rightarrow \{\gamma^5, \gamma^1\} = 0. \end{aligned}$$

It is clear that the other two calculations will be similar.

5. N.B. Of course the question was wrong—should have said $(\gamma^0)^2 = 1$, $(\gamma^i)^2 = -1$, $i = 1, 2, 3$.

$$\begin{aligned} \gamma^0 \gamma^1 \gamma^2 &= -\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^3 = i\gamma^5 \gamma^3 \\ \gamma^0 \gamma^1 \gamma^3 &= -\gamma^0 \gamma^1 \gamma^2 \gamma^2 \gamma^3 = \gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^2 = -i\gamma^5 \gamma^2. \end{aligned}$$

6.

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}1_4 \Rightarrow \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}1_4,$$

where 1_4 is the 4-dimensional identity matrix, usually not written explicitly. Taking the trace, and using $\text{tr}(AB) = \text{tr}(BA)$, $\text{tr}1_4 = 4$, we get

$$\text{tr}[\gamma_\mu\gamma_\nu] = 4\eta_{\mu\nu}.$$

Now

$$\begin{aligned} \gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma &= -\gamma_\nu\gamma_\mu\gamma_\rho\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma \\ &= \gamma_\nu\gamma_\rho\gamma_\mu\gamma_\sigma - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma \\ &= -\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu + 2\eta_{\mu\sigma}\gamma_\nu\gamma_\rho - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma \\ \Rightarrow \gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma + \gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu &= 2\eta_{\mu\sigma}\gamma_\nu\gamma_\rho - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma. \end{aligned}$$

Taking the trace and using

$$\text{tr}[\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu] = \text{tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma],$$

together with $\text{tr}[\gamma_\mu\gamma_\nu] = 4\eta_{\mu\nu}$, we find

$$\text{tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] = 4[\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}]$$