MATH328 Modern Particle Physics Solutions 4

1. With $\rho = |\psi|^2 = \psi^*\psi$, we have

$$\frac{\partial \rho}{\partial t} = \frac{\partial \psi^*}{\partial t}\psi + \psi^* \frac{\partial \psi}{\partial t}.$$

Now from the Schrödinger equation,

$$i\hbar \frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x})\right)\psi, \tag{1}$$

and so, taking the complex conjugate

$$-i\hbar \frac{\partial \psi^*}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x})\right)\psi^* \tag{2}$$

(assuming that $V(\mathbf{x})$ is real.) Multiplying (1) by ψ^* and (2) by ψ and subtracting, we have

$$i\hbar\left(\psi^*\frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t}\psi\right) = -\frac{\hbar^2}{2m}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) = -\frac{\hbar^2}{2m}\nabla.\left[\psi^*\nabla\psi - \psi\nabla\psi^*\right],$$

and so

$$\frac{\partial \rho}{\partial t} = \frac{i\hbar}{2m}\nabla.\left[\psi^*\nabla\psi - \psi\nabla\psi^*\right],$$

i.e.

$$\frac{\partial \rho}{\partial t} + \text{div}\mathbf{j} = 0,$$

$$\text{where} \quad \mathbf{j} = -\frac{i\hbar}{2m}\left[\psi^*\nabla\psi - \psi\nabla\psi^*\right]$$

2. The generalised momentum

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}.$$

The Hamiltonian density is defined by

$$\mathcal{H} = \dot{\phi}(x)\pi(x) - \mathcal{L},$$

then

$$H = \int d^3\mathbf{x}\dot{\phi}(\mathbf{x},t)\pi(\mathbf{x},t) - L = \int \mathcal{H}d^3\mathbf{x}.$$

For the KG field $\pi = \dot{\phi}$ and

$$\mathcal{H} = \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m^2\phi^2.$$

$$[\phi(\mathbf{x},t),H] = [\phi(\mathbf{x},t),\int[\tfrac{1}{2}\pi(\mathbf{x}',t)^2+\tfrac{1}{2}(\nabla\phi(\mathbf{x}',t)^2+\tfrac{1}{2}m^2\phi(\mathbf{x}',t)^2]d^3x']$$

$$= \tfrac{1}{2}\int d^3\mathbf{x}'\left\{[\phi(\mathbf{x}),\pi(\mathbf{x}')^2]+[\phi(\mathbf{x}),(\nabla'\phi(\mathbf{x}'))^2]+m^2[\phi(\mathbf{x}),\phi(\mathbf{x}')^2]\right\}.$$

$$[\phi(\mathbf{x}),\phi(\mathbf{x}')]=0 \Rightarrow [\phi(\mathbf{x}),\nabla'\phi(\mathbf{x}')]=0.$$

$$\text{So}\quad [\phi(\mathbf{x}),H] = \tfrac{1}{2}\int\{\pi(\mathbf{x}')[\phi(\mathbf{x}),\pi(\mathbf{x}')]+[\phi(\mathbf{x}),\phi(\mathbf{x}')]\pi(\mathbf{x}')\}$$

$$= i\hbar\int\pi(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')d^3\mathbf{x}'$$

$$= i\hbar\pi(\mathbf{x}) = i\hbar\dot{\phi}(\mathbf{x})\quad \text{OK.}$$

$$[\pi(\mathbf{x},t),H] = -\left[\int d^3\mathbf{x}'\left\{\tfrac{1}{2}\pi(\mathbf{x}')^2+\tfrac{1}{2}(\nabla\phi(\mathbf{x}')^2+\tfrac{1}{2}m^2\phi(\mathbf{x}')^2\right\},\pi(\mathbf{x})\right]$$

$$= -\tfrac{1}{2}\int d^3\mathbf{x}'\{\nabla'\phi(\mathbf{x}')[\nabla'\phi(\mathbf{x}'),\pi(\mathbf{x})]+[\nabla'\phi(\mathbf{x}'),\pi(\mathbf{x})]\nabla'\phi(\mathbf{x}')\}$$

$$-\tfrac{1}{2}m^2\int d^3\mathbf{x}'\{\phi(\mathbf{x}')[\phi(\mathbf{x}'),\pi(\mathbf{x})]+[\phi(\mathbf{x}'),\pi(\mathbf{x})]\phi(\mathbf{x}')\}$$

$$= -i\hbar\int d^3\mathbf{x}'\left\{\nabla'\phi(\mathbf{x}').\nabla'\delta(\mathbf{x}-\mathbf{x}')+m^2\phi(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')\right\}$$

$$= i\hbar\int d^3\mathbf{x}'\left\{(\nabla')^2\phi(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')-m^2\phi(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')\right\}$$

$$= i\hbar\left\{\nabla^2\phi(\mathbf{x})-m^2\phi(\mathbf{x})\right\}$$

$$= i\hbar\ddot{\phi}(\mathbf{x})\quad \text{using K-G equation}$$

$$= i\hbar\dot{\pi}(\mathbf{x})$$

We write

$$(\partial\phi)^2 = \eta_{\rho\sigma}\partial^\rho\phi\partial^\sigma\phi,$$

then using

$$\frac{\partial(\partial^\rho)\phi}{\partial(\partial^\mu\phi)} = \delta^\rho{}_\mu,$$

we have

$$\frac{\partial}{\partial(\partial^\mu\phi)}(\partial\phi)^2 = \eta_{\rho\sigma}(\delta^\rho{}_\mu\partial^\sigma\phi+\partial^\rho\phi\delta^\sigma{}_\mu) = 2\partial_\mu\phi.$$

So equation of motion becomes

$$\partial^\mu(\partial_\mu\phi) = -m^2\phi-\frac{1}{2}\lambda_3\phi^2-\frac{1}{3!}\lambda_4\phi^3,$$

or

$$\partial^2\phi+m^2\phi+\frac{1}{2}\lambda_3\phi^2+\frac{1}{3!}\lambda_4\phi^3 = 0.$$

3. Inserting the given solution

$$\phi(x,y) \;=\; \sum_{n=1}^{\infty} \phi_n(x)\, \mathrm{cs}\left(\frac{n\pi y}{R}\right)$$

into the five-dimensional Klein-Gordon equation and carrying out $\partial^2/\partial y^2$, one obtains a four-dimensional Klein-Gordon equation for each of the Fourier coefficients $\phi_n(x)$:

$$\left(\partial_0^2 - \nabla^2 + m^2 + \left(\frac{n\pi}{R}\right)^2\right)\phi_n(x) = 0\,.$$

Therefore the given $\phi(x,y)$ is a solution of the five-dimensional Klein-Gordon equation *if* the $\phi_n(x)$ are solutions of the four-dimensional equations. The masses m_n of the fields ϕ_n are given by

$$m_n^2 = m^2 + \left(\frac{n\pi}{R}\right)^2\,.$$

If the five-dimensional mass m is zero, we get the equally spaced mass spectrum

$$m_n = \frac{n\pi}{R}\,.$$

The infinite set of particles are called Kaluza-Klein tower.