

MATH431 Modern Particle Physics Solutions 3

1.

$$ds^2 = dt^2 - dx^2 - dy^2 \quad (1)$$

a.

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad , \quad g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

b. The line element is invariant under 3 translations (dt , dx and dy). 2 boosts ($dt dx$, $dt dy$) and 1 rotation ($dx dy$).

The generators associated with the transformations are: $i\partial_t = p_0$, $i\partial_x = p_1$ and $i\partial_y = p_2$, are the generators of translations. K_1 and K_2 are the boost generators and J_3 is the generator of rotations in the $x - y$ plane.

c.

$$W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma$$

$$W^0 = -\frac{1}{2}\epsilon^{0ijk} J_{ij} P_k = -J^k P_k$$

$$\begin{aligned} W^i &= -\frac{1}{2}\epsilon^{i0jk} J_{0j} P_k - \frac{1}{2}\epsilon^{ij0k} J_{j0} P_k - \frac{1}{2}\epsilon^{ijk0} J_{jk} P_0 = \\ &= \frac{1}{2}\epsilon^{0ijk} J_{0j} P_k + \frac{1}{2}\epsilon^{0ijk} J_{0j} P_k + \frac{1}{2}\epsilon^{0ijk} J_{jk} P_0 = \\ &= \epsilon^{ijk} K_j P_k + J^i P_0 \end{aligned}$$

$$\begin{aligned} \text{where } K^i &= J^{i0} = -J^{0i} \quad ; \quad J^i = \frac{1}{2}\epsilon^{ijk} J_{jk} \\ K_i &= -J_{i0} = J_{0i} \quad ; \quad J_i = \frac{1}{2}\epsilon_{ijk} J^{jk} = -J^i \end{aligned}$$

In the case of the two dimensional line element eq. (1)

$$\vec{K} = (K_1, K_2, 0) \quad \vec{J} = (0, 0, J_3)$$

for $m^2 = 0 \rightarrow P^\mu = (p, 0, 0, 0)$

$$W^0 = 0$$

$$W^1 = J^1 P_0 + \epsilon^{1jk} K_j P_k = 0 + \epsilon^{123} K_2 P_3 + \epsilon^{132} K_3 P_2 = 0$$

$$W^2 = J^2 P_0 + \epsilon^{2jk} K_j P_k = 0 + \epsilon^{231} K_3 P_1 + \epsilon^{213} K_1 P_3 = 0$$

$$W^3 = J^3 P_0 + \epsilon^{3jk} K_j P_k = J_3 P_0 + \epsilon^{312} K_1 P_2 + \epsilon^{321} K_2 P_1 = (J^3 + \epsilon^{312} K_1) P_0$$

For $m^2 > 0 \rightarrow P = (m, 0, 0, 0)$

$$W^0 = 0$$

$$W^1 = 0$$

$$W^2 = 0$$

$$W^3 = J^3 m$$

2.
a.

$$\begin{aligned}
W_\sigma P^\sigma &= -\frac{1}{2}\epsilon_{\sigma\mu\nu\lambda}J^{\mu\nu}P^\lambda P^\sigma \\
&= \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}J^{\mu\nu}P^\lambda P^\sigma \\
&= \frac{1}{2}\epsilon_{\mu\nu\sigma\lambda}J^{\mu\nu}P^\sigma P^\lambda \\
&= -\frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}J^{\mu\nu}P^\lambda P^\sigma,
\end{aligned}$$

where in the second line we permuted σ three times across; in the third line we changed the dummy summation indices; in the fourth line we again permute λ and σ and exchange P^λ and P^σ . Hence, the second line is equal to minus the fourth line, which can only hold if they vanish.

b.

We use the identity

$$[A, BC] = [A, B]C + B[A, C],$$

where A , B and C are operators. Hence, we have

$$\begin{aligned}
[J_{\rho\sigma}, W_\mu W^\mu] &= \eta^{\mu\nu} [J_{\rho\sigma}, W_\nu W_\mu] \\
&= \eta^{\mu\nu} ([J_{\rho\sigma}, W_\nu] W_\mu + W_\nu [J_{\rho\sigma}, W_\mu]) \\
&= \eta^{\mu\nu} (i(\eta_{\sigma\nu} W_\rho W_\mu - \eta_{\rho\nu} W_\sigma W_\mu + \eta_{\sigma\mu} W_\nu W_\rho - \eta_{\rho\mu} W_\nu W_\sigma)) \\
&= i\eta^{\mu\nu} \eta_{\sigma\nu} W_\rho W_\mu - i\eta^{\mu\nu} \eta_{\rho\nu} W_\sigma W_\mu + i\eta^{\mu\nu} \eta_{\sigma\mu} W_\nu W_\rho - i\eta^{\mu\nu} \eta_{\rho\mu} W_\nu W_\sigma \\
&= i\delta^\mu_\sigma W_\rho W_\mu - iW_\rho W_\mu + iW_\sigma W_\rho - iW_\sigma W_\rho = 0,
\end{aligned}$$

where we used that

$$\eta^{\mu\nu} \eta_{\sigma\nu} = \delta^\mu_\sigma, \quad \text{etc}$$

That $W_\mu W^\mu$ commutes with P_λ is obvious because $[P_\nu, W_\rho] = 0$.

3. In polar coordinates

$$\begin{aligned}
x &= r \cos \phi \\
y &= r \sin \phi
\end{aligned}$$

radial speed is therefore $\dot{r}$ and tangential speed is $r\dot{\phi}$. So the kinetic energy $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$ and the Lagrangian is given by

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - V$$

Hence

$$\begin{aligned}
\frac{\partial L}{\partial \dot{r}} &= m\dot{r} \\
\frac{\partial L}{\partial r} &= mr\dot{\phi}^2 - \frac{\partial V}{\partial r}
\end{aligned}$$

so the Euler–Lagrange equation for r is

$$m \frac{d\dot{r}}{dt} - mr\dot{\phi}^2 + \frac{\partial V}{\partial r} = 0$$

Similarly,

$$\begin{aligned} \frac{\partial L}{\partial \dot{\phi}} &= mr^2\dot{\phi} \\ \frac{\partial L}{\partial \phi} &= \frac{\partial V}{\partial \phi} \end{aligned}$$

So the Euler–Lagrange equation for ϕ is

$$m \frac{d}{dt}(r^2\dot{\phi}) + \frac{\partial V}{\partial \phi} = 0$$

If the potential is axisymmetric, $\partial V/\partial \phi = 0$. The last equation then states that the angular momentum $mr^2\dot{\phi}$ is constant. If the motion is circular $\dot{r} = 0 = \ddot{r}$ and the radial equation becomes

$$-m \frac{v^2}{r} = -\frac{\partial V}{\partial r},$$

where $v = r\dot{\phi}$ is the speed. The force $\partial V/\partial r$ is equal to m times the centripetal acceleration $-v^2/r$.

4.

$$\dot{p}_0 = -\frac{\partial H}{\partial q_0}$$

Therefore, p_0 is constant in time if H does not depend on q_0 .

An axisymmetric potential does not depend on ϕ so p_ϕ is a constant of the motion.

In polar coordinates

$$\begin{aligned} x &= r \sin \theta \cos \phi \\ y &= r \sin \theta \sin \phi \\ z &= r \cos \theta \end{aligned}$$

The kinetic energy is given by

$$\frac{1}{2}m \left[\dot{r}^2 + (r\dot{\theta})^2 + (r \sin \theta \dot{\phi})^2 \right]$$

and the potential energy is V

$$L = \frac{1}{2}m \left[\dot{r}^2 + (r\dot{\theta})^2 + (r \sin \theta \dot{\phi})^2 \right] - V$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2 \sin^2 \theta \dot{\phi}$$

$$\begin{aligned}
H &= \sum_i p_i q_i - L = m\dot{r}^2 + mr^2\dot{\theta}^2 + mr^2 \sin^2 \theta \dot{\phi}^2 - \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) + V \\
&= \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) + V \\
&= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2 \sin^2 \theta} + V
\end{aligned}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = -\frac{\partial V}{\partial \phi} = 0$$

$\Rightarrow$ angular momentum about the symmetry axis is conserved.