MATH 328 — May 2009: Solutions

All problems are similar to homework problems or material covered in the lectures.

1.

$$ds^2 = dt^2 - dx^2 - dy^2 \tag{1}$$

(a)

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(b) The line element is invariant under 3 translations (dt, dx and dy), 2 boosts ($dtdx$, $dtdy$) and 1 rotation ($dxdy$). The generators associated with the transformations are: $P_0 = i\partial_t$, $P_1 = -i\partial_x$ and $P_2 = -i\partial_y$, the generators of translations. K_1 and K_2 are the boost generators and J_3 is the generator of rotations in the (x, y) plane.

(c)

$$W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma \,,$$

$$W^0 = -\frac{1}{2}\epsilon^{0ijk} J_{ij} P_k = \frac{1}{2}\epsilon_{0ijk} J_{ij} P_k = \frac{1}{2}\epsilon_{kij} J_{ij} P_k = J_k P_k \,,$$

$$\begin{aligned} W^i &= -\frac{1}{2}\epsilon^{i0jk} J_{0j} P_k - \frac{1}{2}\epsilon^{ij0k} J_{j0} P_k - \frac{1}{2}\epsilon^{ijk0} J_{jk} P_0 = \\ &-\frac{1}{2}\epsilon^{i0jk} J_{0j} P_k - \frac{1}{2}\epsilon^{i0jk} J_{0j} P_k + \frac{1}{2}\epsilon^{0ijk} J_{jk} P_0 = \\ &+\frac{1}{2}\epsilon^{0ijk} J_{0j} P_k + \frac{1}{2}\epsilon^{0ijk} J_{0j} P_k + \frac{1}{2}\epsilon^{0ijk} J_{jk} P_0 = \\ &-\epsilon_{0ijk} J_{0j} P_k - \frac{1}{2}\epsilon_{0ijk} J_{jk} P_0 = -\epsilon_{ijk} K_j P_k - J_i P_0 \,. \end{aligned}$$

Hence

$$\begin{aligned} W_0 &= \vec{J} \cdot \vec{P} \,, \\ W_i &= J_i P_0 + \epsilon_{ijk} K_j P_k \,. \end{aligned}$$

In the case of the one+two dimensional line element Eq. (1), embedded in our usual one+three dimensions,

$$\vec{K} = (K_1, K_2, 0) \quad \text{and} \quad \vec{\mathrm{J}} = (0, 0, \mathrm{J}_3)\,.$$

For $m = 0$ we can take, without loss of generality, that $P^\mu = (p, 0, p, 0)$, $P_\mu = (p, 0, -p, 0)$. Then

$$\begin{aligned} W^0 &= 0\,, \\ W^1 &= -\epsilon_{1jk} K_j P_k = -\epsilon_{123} K_2 P_3 - \epsilon_{132} K_3 P_2 = 0\,, \\ W^2 &= -\epsilon_{2jk} K_j P_k = -\epsilon_{213} K_1 P_3 - \epsilon_{231} K_3 P_1 = 0\,, \\ W^3 &= -J_3 P_0 - \epsilon_{3jk} K_j P_k = -J_3 P_0 - \epsilon_{312} K_1 P_2 - \epsilon_{321} K_2 P_1 = \\ &= -J_3 P_0 - K_1 P_0 = -(J_3 + K_1) P_0 \,. \end{aligned}$$

For $m > 0$ we can evaluate the Pauli-Lubanski vector most conveniently in the rest frame, $P^\mu = (m, 0, 0, 0)$. Then

$$
\begin{aligned}
W_0 &= 0\,, \\
W_1 &= 0\,, \\
W_2 &= 0\,, \\
W_3 &= -J_3 m\,.
\end{aligned}
$$

2. (a) In the lectures we have defined the operators $J_{i\,\pm}$ by

$$
\vec{J}_\pm = \frac{1}{2}(\vec{J} \pm i\vec{K})
$$

and seen that $\vec{J}_+$ and $\vec{J}_-$ are both generating the algebra $SU(2)$ but commute between each other. In this way we have rewritten the Lorentz algebra as the algebra of $SU(2) \times SU(2)$. As $\vec{J}_+^2$ and $\vec{J}_-^2$ are Casimir operators of the two commuting $SU(2)$ groups they are thus also invariants of the Lorentz group. We have

$$
\begin{aligned}
\vec{J}_+^2 &= \frac{1}{4}(\vec{J}^2 - \vec{K}^2 + 2\,i\,\vec{J}\cdot\vec{K})\,, \\
\vec{J}_-^2 &= \frac{1}{4}(\vec{J}^2 - \vec{K}^2 - 2\,i\,\vec{J}\cdot\vec{K})\,,
\end{aligned}
$$

and hence

$$
\begin{aligned}
\vec{J}^2 - \vec{K}^2 &= 2(\vec{J}_+^2 + \vec{J}_-^2)\,, \\
\vec{J}\cdot\vec{K} &= -i(J_+^2 - J_-^2)\,.
\end{aligned}
$$

Therefore $\vec{J}^2 - \vec{K}^2$ and $\vec{J}\cdot\vec{K}$ are Lorentz invariants as well, being the sum and difference of Lorentz invariants.

(b) For the representation (j_1, j_2) of the $SU(2) \times SU(2)$ algebra the number of states is $(2j_1 + 1)(2j_2 + 1)$. The total spin is given by $j = j_1 + j_2$. Therefore the composition $j_1 \otimes j_2$ breaks under $SU(2)_J$ with the following spin states

$$
j_1 + j_2 \oplus j_1 + j_2 - 1 \oplus \cdots \oplus |j_1 - j_2|\,.
$$

3. (i) The Lagrangian of the two-dimensional harmonic oscillator is

$$L \;=\; \frac{1}{2}m(\dot{x}^2+\dot{y}^2)-\frac{1}{2}k(x^2+y^2)\,.$$

(ii) The Euler-Lagrange equations are

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}}-\frac{\partial L}{\partial x} \;&=\; m\ddot{x}+kx \;=\; 0\,,\\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{y}}-\frac{\partial L}{\partial y} \;&=\; m\ddot{y}+ky \;=\; 0\,.\end{aligned}$$

(iii) The Hamiltonian is

$$H \;=\; \frac{1}{2m}(p_x^2+p_y^2)+\frac{1}{2}mw^2(x^2+y^2)\;, \quad \text{with} \quad w=\sqrt{\frac{k}{m}}\,.$$

(iv) In polar coordintes we derive:

$$L \;=\; \frac{1}{2}m(\dot{r}^2+r^2\dot{\phi}^2)-\frac{1}{2}kr^2\,,$$

$$\begin{aligned}p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r}\,,\\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}\,.\end{aligned}$$

Hence we get

$$\begin{aligned}H \;&=\; \sum_i p_i\dot{q}_i - L \;=\; m\dot{r}^2+mr^2\dot{\phi}^2-\frac{1}{2}m(\dot{r}^2+r^2\dot{\phi}^2)+\frac{1}{2}mw^2r^2\\ &=\; \frac{1}{2}m(\dot{r}^2+mr^2\dot{\phi}^2)+\frac{1}{2}mw^2r^2\\ &=\; \frac{{p_r}^2}{2m}+\frac{{p_\phi}^2}{2mr^2}+\frac{1}{2}mw^2r^2\,.\end{aligned}$$

(v) There are two constants of the motion.
Since the Hamiltonian does not depend explicitly on time, the energy is a constant of the motion with $E = H(q_0, p_0)$. Since it does not depend explicitly on ϕ, also p_ϕ is a constant of the motion, corresponding to conservation of the angular momentum w.r.t. the symmetry axis.

4. (a) The generalised momentum

$$\pi(x) \;=\; \frac{\partial \mathcal{L}}{\partial \dot\phi}\,.$$

The Hamiltonian density is defined by

$$\mathcal{H} \;=\; \dot\phi(x)\pi(x) - \mathcal{L}\,,$$

so

$$H \;=\; \int \mathrm{d}^3\mathbf{x}\,\dot\phi(\mathbf{x},t)\pi(\mathbf{x},t) - L \;=\; \int \mathcal{H}\,\mathrm{d}^3\mathbf{x}\,.$$

For the free Klein-Gordon field $\pi = \dot\phi$ (as derived in the lecture) and hence

$$\mathcal{H} \;=\; \tfrac{1}{2}\,\dot\phi^2 + \tfrac{1}{2}\,(\nabla\phi)^2 + \tfrac{1}{2}\,m^2\phi^2\,.$$

(b)

$$\begin{aligned}
[\phi(\mathbf{x},t),H] &= [\phi(\mathbf{x},t), \int \left(\tfrac{1}{2}\pi(\mathbf{x}',t)^2 + \tfrac{1}{2}(\nabla'\phi(\mathbf{x}',t))^2 + \tfrac{1}{2}m^2\phi(\mathbf{x}',t)^2\right)\,\mathrm{d}^3\mathbf{x}'] \\
&= \tfrac{1}{2}\int \mathrm{d}^3\mathbf{x}'\,\left([\phi(\mathbf{x}),\pi(\mathbf{x}')^2] + [\phi(\mathbf{x}),(\nabla'\phi(\mathbf{x}'))^2] + m^2[\phi(\mathbf{x}),\phi(\mathbf{x}')^2]\right)\,.
\end{aligned}$$

Now

$$[\phi(\mathbf{x}),\phi(\mathbf{x}')] \;=\; 0 \quad\Rightarrow\quad [\phi(\mathbf{x}),\nabla'\phi(\mathbf{x}')] \;=\; 0\,,$$

and hence

$$\begin{aligned}
i\hbar\dot\phi(\mathbf{x}) &= [\phi(\mathbf{x}),H] \;=\; \tfrac{1}{2}\int \left(\pi(\mathbf{x}')[\phi(\mathbf{x}),\pi(\mathbf{x}')] + [\phi(\mathbf{x}),\pi(\mathbf{x}')]\pi(\mathbf{x}')\right)\,\mathrm{d}^3\mathbf{x}' \\
&= i\hbar\int \pi(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')\,\mathrm{d}^3\mathbf{x}' \;=\; i\hbar\pi(\mathbf{x})\,.
\end{aligned}$$

Next calculate

$$\begin{aligned}
i\hbar\dot\pi(\mathbf{x}) &= [\pi(\mathbf{x},t),H] \\
&= -\,[\int \mathrm{d}^3\mathbf{x}'\,\left(\tfrac{1}{2}\pi(\mathbf{x}')^2 + \tfrac{1}{2}(\nabla'\phi(\mathbf{x}'))^2 + \tfrac{1}{2}m^2\phi(\mathbf{x}')^2\right),\pi(\mathbf{x})] \\
&= -\,\tfrac{1}{2}\int \mathrm{d}^3\mathbf{x}'\,\left(\nabla'\phi(\mathbf{x}')[\nabla'\phi(\mathbf{x}'),\pi(\mathbf{x})] + [\nabla'\phi(\mathbf{x}'),\pi(\mathbf{x})]\nabla'\phi(\mathbf{x}')\right) \\
&\qquad -\,\tfrac{1}{2}m^2\int \mathrm{d}^3\mathbf{x}'\,\left(\phi(\mathbf{x}')[\phi(\mathbf{x}'),\pi(\mathbf{x})] + [\phi(\mathbf{x}'),\pi(\mathbf{x})]\phi(\mathbf{x}')\right) \\
&= -\,i\hbar\int \mathrm{d}^3\mathbf{x}'\,\left(\nabla'\phi(\mathbf{x}').\nabla'\delta(\mathbf{x}-\mathbf{x}') + m^2\phi(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')\right) \\
&= i\hbar\int \mathrm{d}^3\mathbf{x}'\,\left((\nabla')^2\phi(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}') - m^2\phi(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')\right) \\
&= i\hbar\left(\nabla^2\phi(\mathbf{x}) - m^2\phi(\mathbf{x})\right)\,.
\end{aligned}$$

Together with the previous step we immediately get the Klein-Gordon equation

$$\ddot\phi(\mathbf{x}) \;=\; \nabla^2\phi(\mathbf{x}) - m^2\phi(\mathbf{x})\,.$$

5.

$$\gamma^\mu \partial_\mu \psi + im\psi \;=\; 0\,, \qquad (\partial_\mu \psi^\dagger)\gamma^{\mu\dagger} - im\psi^\dagger \;=\; 0$$

$$\Rightarrow \qquad (\partial_\mu \psi^\dagger)\gamma^0\gamma^\mu\gamma^0 - im\psi^\dagger \;=\; 0 \qquad \text{and} \qquad (\partial_\mu \bar{\psi})\gamma^\mu - im\bar{\psi} \;=\; 0\,.$$

(a) With this

$$\partial_\mu(\bar{\psi}\gamma^\mu\psi) \;=\; (\partial_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu(\partial_\mu\psi) \;=\; (im\bar{\psi})\psi + \bar{\psi}(-im\psi) \;=\; 0\,.$$

(b) Similarly, and with $\{\gamma^5, \gamma^\mu\} = 0$ we get

$$\partial_\mu(\bar{\psi}\gamma^\mu\gamma^5\psi) = (\partial_\mu\bar{\psi})\gamma^\mu\gamma^5\psi + \bar{\psi}\gamma^\mu\gamma^5(\partial_\mu\psi) = (im\bar{\psi})\gamma^5\psi - \bar{\psi}\gamma^5(\gamma^\mu\partial_\mu\psi) = 2im\bar{\psi}\gamma^5\psi\,.$$

(c) From the Dirac equation for the spinors $\bar{u}_f$ and u_i we have

$$0 \;=\; \bar{u}_f(\not{p}_f - m)\gamma^\mu u_i \;=\; \bar{u}_f\gamma^\mu(\not{p}_i - m)u_i$$

$$\begin{aligned} \Rightarrow \qquad 2m\bar{u}_f\gamma^\mu u_i \;&=\; \bar{u}_f(\not{p}_f\gamma^\mu + \gamma^\mu\not{p}_i)u_i \\ \text{and} \qquad \not{p}_f\gamma^\mu + \gamma^\mu\not{p}_i \;&=\; \gamma^\nu\gamma^\mu p_{f_\nu} + \gamma^\mu\gamma^\nu p_{i_\nu}\,. \end{aligned}$$

Now

$$\begin{aligned} \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \;&=\; 2g^{\mu\nu}\,, \\ \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu \;&=\; -2i\sigma^{\mu\nu} \end{aligned}$$

$$\Rightarrow \qquad \gamma^\mu\gamma^\nu \;=\; g^{\mu\nu} - i\sigma^{\mu\nu} \qquad \text{and} \qquad \gamma^\nu\gamma^\mu \;=\; g^{\mu\nu} + i\sigma^{\mu\nu}\,.$$

So we get

$$\not{p}_f\gamma^\mu + \gamma^\mu\not{p}_i \;=\; g^{\mu\nu}(p_f + p_i)_\nu + i\sigma^{\mu\nu}(p_f - p_i)_\nu \;=\; (p_f + p_i)^\mu + i\sigma^{\mu\nu}(p_f - p_i)_\nu\,,$$

and finally have derived the Gordon decomposition

$$\bar{u}_f\gamma^\mu u_i \;=\; \frac{1}{2m}\,\bar{u}_f\left[(p_f + p_i)^\mu + i\sigma^{\mu\nu}(p_f - p_i)_\nu\right]u_i\,.$$

6. Unitary equivalence and the '$SU(2)$ miracle':

(a) *First step:*

$$i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ,$$

so $i\sigma_2$ is unitary.

Second step: Prove by explicit matrix multiplication that $\sigma_2\,\sigma_a^\star\,\sigma_2 = -\sigma_a$ $(i = 1, 2, 3)$. For $i = 1$ we have e.g.

$$\sigma_2\,\sigma_1^\star\,\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -\sigma_1 .$$

Third step: Choose $W = i\sigma_2$, and with $\sigma_2^\star\sigma_2^\star = \mathbb{1}$ and the result of step two we write:

$$\begin{aligned} W^\dagger U^\star W &= -\,i\sigma_2^\dagger U^\star i\sigma_2 = \sigma_2 \exp\left(-\tfrac{i}{2}\theta_a\sigma_a^\star\right)\sigma_2 \\ &= \sigma_2 \left(\mathbb{1} - \frac{i}{2}\theta_a\sigma_a^\star - \frac{1}{4}\frac{1}{2!}\theta_a\theta_a - \frac{i}{8}\frac{1}{3!}\theta_a\theta_a\theta_b\sigma_b^\star - \ldots\right)\sigma_2 \\ &= \left(\mathbb{1} + \frac{i}{2}\theta_a\sigma_a - \frac{1}{4}\frac{1}{2!}\theta_a\theta_a + \frac{i}{8}\frac{1}{3!}\theta_a\theta_a\theta_b\sigma_b - \ldots\right) = U . \end{aligned}$$

Taking the complex conjugate of

$$W^\dagger U^\star W = U$$

we now also have

$$W^{\dagger\star} U W^\star = U^\star$$

and as $W = i\sigma_2 = W^\star$ we arrive at the desired relation

$$U^\star = W^\dagger\,U\,W .$$

(b) In the Standard Model, fermion and gauge boson masses are obtained in a gauge invariant way through electroweak symmetry breaking which is mediated by a Higgs potential. Because of the unitary equivalence between the fundamental and the complex conjugate representations of $SU(2)$, gauge invariant mass terms for both up- and down quarks (which are grouped together in $SU(2)$ doublets) can be constructed from only one complex Higgs doublet. In other words, it is due to this special property of $SU(2)$ that the Higgs sector in the Standard Model is the minimal one resulting in only one physical Higgs boson.

7.

$$L \;=\; \frac{1}{2}(\partial_\mu \phi_i)^2 - \frac{1}{2}\mu^2\phi_i^2 - \frac{1}{4}\lambda(\phi_i^2)^2$$

with $\mu^2 < 0$ and $\lambda > 0$.

(a) The first term in the Lagrangian contains the kinetic terms of the three fields which lead to the propagators in the Feynman rules. The second and third term are the quadratic and quartic terms of the scalar potential. The coefficient λ is chosen positive as otherwise there would be no stable vacuum. Only with $\mu^2 < 0$ we get a non-trivial vacuum which allows for spontaneous symmetry breaking. As shown in (b) this leads to mass terms (quadratic in the field) and interaction terms which lead to cubic and quartic vertices in the Feynman rules.

(b)

$$L \;=\; \frac{1}{2}\left((\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2 + (\partial_\mu\phi_3)^2\right) - \frac{1}{2}\mu^2(\phi_1^2+\phi_3^2+\phi_3^2) - \frac{1}{4}\lambda(\phi_1^2+\phi_3^2+\phi_3^2)^2\,.$$

The potential is

$$V \;=\; \frac{1}{2}\mu^2(\phi_1^2+\phi_3^2+\phi_3^2) + \frac{1}{4}\lambda(\phi_1^2+\phi_3^2+\phi_3^2)^2\,.$$

For spontaneous symmetry breaking we need a non-trivial minimum of the potential:

$$\frac{\partial V}{\partial \phi_i} \;=\; \left(\mu^2 + \lambda(\phi_1^2+\phi_3^2+\phi_3^2)\right)\phi_i \;=\; 0\,,$$

so we require

$$\mu^2 + \lambda(\phi_1^2+\phi_3^2+\phi_3^2) \;=\; 0\,.$$

We choose the non-vanishing vacuum expectation value to be

$$\langle\phi_1\rangle \;=\; \sqrt{-\frac{\mu^2}{\lambda}} \;=\; v$$

and expand ϕ_1 around the new vacuum after spontaneous symmetry breaking (keeping ϕ_2 and ϕ_3):

$$\phi_1(x) \;=\; v + h(x)\,.$$

Inserting this into the Lagrangian we get

$$\frac{1}{2}\left((\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2 + (\partial_\mu\phi_3)^2\right) - \frac{1}{2}\mu^2((v+h(x))^2+\phi_2^2+\phi_3^2) - \frac{1}{4}\lambda((v+h(x))^2+\phi_2^2+\phi_3^2)^2$$

and with $-\mu^2 = v^2\lambda$ we arrive at

$$\begin{aligned} L \;=\; & \frac{1}{2}((\partial_\mu h)^2 + (\partial_\mu\phi_2)^2 + (\partial_\mu\phi_3)^2) + \\ & \frac{v^2\lambda}{2}(h^2(x) + v^2 + 2vh(x) + \phi_2^2 + \phi_3^2) - \frac{\lambda}{4}(6v^2h^2(x) + \ldots) \;=\; \\ & \frac{1}{2}\left((\partial_\mu h)^2 + (\partial_\mu\phi_2)^2 + (\partial_\mu\phi_3)^2\right) - \frac{1}{2}\lambda v^2 h^2(x) + \\ & \qquad\qquad \text{cubic and quartic interaction terms + constant}\,. \end{aligned}$$

Hence the Lagrangian describes one massive scalar field and two massless 'Goldstone bosons'.