MATH431 Modern Particle Physics Solutions 7

4.

The unitary groups are discussed in the notes from page 100–112. These are simple extensions of the $SU(2)$ group and you should think of them as such. A unitary $SU(n)$ group has $n-1$ diagonal generators

4a. For $SU(4)$ there are 3 diagonal generators.

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_{15} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

where I chose a convenient normalisation for the third diagonal generator.
4b. $D = 15$. Three diagonal generators of part $(4a)$ plus:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_9 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_{10} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\lambda_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad , \quad \lambda_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

4c.

For SU(4) the fundamental is a 4 representation
$4 = \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$

Just as for SU(2) the fundamental is a 2 representation
$2 = \{(1,0),(0,1)\}$
A maximal subgroup of $SU(4)$ is $SU(3)\times U(1)$.
The 4 representation of $SU(4)$ decomposes under $SU(3)\times U(1)$ as

$$4 = (3, 1/3) + (1, -1)$$

You obtain the charges under the $U(1)$ by acting with the third diagonal generator in (4a) on the 4 eigenvectors in the 4 representation. You see that three of them will return eigenvalue 1/3 and one will return eigenvalue -1. Three first three states transform non–trivially when acting with the matrices in the adjoint with non–zero entries in the upper 3x3 block, whereas the last eigenstate $(0,0,0,1)$ transforms trivially. Hence, the first three eigenstates transform as a triplet of $SU(3)$, whereas the last eigenstate is a singlet. Hence the above decomposition.

4d.

see separate figure in VITAL.

4e. To find the product you take the product of the $4\times\bar{4}$ with the decomposition found in part (4c), where $\bar{4} = (\bar{3}, -1/3) + (1, +1)$. Then

$$\begin{aligned} 4\times\bar{4} &= \{(3,1/3)+(1,-1)\}\times\{(\bar{3},-1/3)+(1,+1)\} = \\ 15+1 &= \{(3,+4/3)+(8,0)+(1,0)+(\bar{3},-4/3)\}+(1,0) \end{aligned}$$

where under $SU(3)$ $3\times\bar{3} = 8+1$, as worked out in the lecture notes, and the $U(1)$ charges just add up. The components are combined into representations of $SU(4)$ by demanding the trace over the $U(1)$ add up to zero.

4f. Done in a similar way to give

$$\begin{aligned} 4\times 4 &= \{(3,1/3)+(1,-1)\}\times\{(3,1/3)+(1,-1)\} = \\ 6+10 &= \{(6,2/3)+(3,-2/3)+(1,-2)\}+\{(\bar{3},2/3)+(3,-2/3)\} \end{aligned}$$

For HW sheet 8 problem 2 the analysis is similar. Again this is a simple extension of the $SU(2)$, $SU(3)$ and $SU(4)$ cases.

2a.

There are 4 diagonal generators in $SU(5)$. A particular choice is:

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_{15} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_{24} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix}.$$

This basis corresponds to the decomposition $SU(5) \to SU(4) \times U(1)$. Another basis

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad ,$$

$$\lambda_{15} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \quad ,$$

$$\lambda_{24} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix} \quad ,$$

which corresponds to the decomposition $SU(5) \to SU(3) \times SU(2) \times U(1)$

2b. The dimension of the group is $D = 24$. Again a simple extension of $SU(2)$. In general the dimension of an $SU(n)$ group has $n^2 - 1$ generators.

2c. The fundamental representation is a 5 representation. To find the $U(1)$ charges act with the last matrix in part $(2a)$ on the 5 states in the 5 representation.

$$5 = (3, 1, 1/3) + (1, 2, -1/2)$$

under $SU(3) \times SU(2) \times U(1)$, where I multiplied the $U(1)$ generator in $(2a)$ by $\frac{1}{6}$.

2d. The $\bar{5}$ representation is obtained by flipping the charges of the 5 representation,

$$\bar{5} = (\bar{3}, 1, -1/3) + (1, 2, 1/2).$$

To take the product and find the $SU(5)$ representations use the decomposition under $SU(3) \times SU(2) \times U(1)$ and group them into traceless combinations. In general, in $SU(n)$ the product of the fundamental times the anti–fundamental is:

$$n \times \bar{n} = [n^2 - 1] + 1$$

hence for $SU(5)$.

$$\begin{aligned} 5 \times \bar{5} = \{(3, 1, 1/3) + (1, 2, -1/2)\} \times \{(\bar{3}, 1, -1/3) + (1, 2, 1/2)\} = \\ 24 + 1 = \{(8, 1, 0) + (1, 3, 0) + (1, 1, 0) + (3, 2, 5/6) + (\bar{3}, 2, -5/6)\} + (1, 0) \end{aligned}$$

We did not cover the weight lattice of $SO(10)$ in the lecture. It was only discussed in problem set 8. The aim being to show how the Standard Model states fit into the representation. The answer in the solution set is quite detailed. We can discuss it over skype.