MATH431 - Modern Particle Physics
Set Work: Sheet 6; Due:

1. The Dirac wave function for the ground state of the hydrogen atom has the following for (in the standard Dirac matrix representation):

$$\psi(r,\theta,\phi) = R(r)\begin{pmatrix} 1 \\ 0 \\ ia\cos\theta \\ iae^{i\phi}\sin\theta \end{pmatrix},$$

where $a \approx \alpha/2$.

(a)Investigate whether ψ is an eigenstate of L_z.

(b)Find the expectation value of L_z and comment on the result.

(c)Show that ψ is an eigenstate of J_z and find its eigenvalue.

[Don't forget to normalize for one electron]

2. (a) Consider the Dirac equation in four dimensions

$$(i\gamma^\mu\partial_\mu - m)\,\psi(x) = 0$$

where ψ is a Dirac spinor. Show that each of the four components of the Dirac spinor satisfies the Klein-Gordon equation.

(b) Derive the conservation equation $\partial_\mu J^\mu_V = 0$ for the four vector current density $J^\mu_V = \bar{\psi}\gamma^\mu\psi$, using the covariant form of the Dirac equation and the relation $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$.

(c) Show that the axial 4-vector current density $J^\mu_A = \bar{\psi}\gamma^\mu\gamma^5\psi$ is not conserved but instead satisfies the covariant equation

$$\partial_\mu J^\mu_A = 2im\bar{\psi}\gamma^5\psi\ .$$

3. Derive the *Gordon decomposition* of the Dirac transition current:

$$\bar{\psi}_f\gamma^\mu\psi_i = \frac{1}{2m}\bar{\psi}_f[(p_f+p_i)^\mu + i\sigma^{\mu\nu}(p_f-p_i)_\nu]\psi_i\ ,$$

where $\sigma^{\mu\nu} = \frac{1}{2}i(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$. [Hint: Use the Dirac equations $\bar{\psi}_f(\not{p}_f - m) = (\not{p}_i - m)\psi_i = 0$.]

4. Consider an electron in a positive constant magnetic field along the z-axis.

(a) write down the vector potential.

(b) Write down the Dirac equation in terms of the two spinor components $\psi = (\phi, \chi)$

(c) Assuming a solution of the form

$$\psi = (\phi(\vec{x}), \chi(\vec{x}))e^{-iEt}$$

solve the Dirac equation in the presence of the constant magnetic field and find the energy eigenvalues.

4. Defining $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, show that

$$\begin{aligned} \gamma^{5\dagger} =& \gamma^5 \\ \{\gamma^5, \gamma^\mu\} =& 0. \end{aligned}$$

5. By inserting $(\gamma^\mu)^2 = 1$ for some $\mu = 0, 1, 2, 3$, write each of $\gamma^0\gamma^1\gamma^2$ and $\gamma^0\gamma^1\gamma^3$ as a product $\gamma^5\gamma^\nu$ for some $\nu = 0, 1, 2, 3$.

6. Show that

$$\mathrm{tr}[\gamma_\mu\gamma_\nu] = 4\eta_{\mu\nu}.$$

Now show that

$$\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma = 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + 2\eta_{\mu\sigma}\gamma_\nu\gamma_\rho - \gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu.$$

Hence show that

$$\mathrm{tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] = 4[\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}]$$

7. Show that the number operator, defined by

$$N \ := \ \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{p}}{2p^0}\, a^\dagger(\mathbf{p})a(\mathbf{p})\,,$$

satisfies

$$[N, a^\dagger(\mathbf{p})] \ = \ a^\dagger(\mathbf{p})\,,$$

and hence

$$N\,|\mathbf{p}_1 \ldots \mathbf{p}_n\rangle \ = \ n\,|\mathbf{p}_1 \ldots \mathbf{p}_n\rangle\,.$$