MATH431 — Introduction to Modern Particle Theory: FEEDBACK/SOLUTIONS 2

1. With $\rho = |\psi|^2 = \psi^*\psi$, we have

$$\frac{\partial \rho}{\partial t} = \frac{\partial \psi^*}{\partial t}\psi + \psi^*\frac{\partial \psi}{\partial t}\,.$$

The Schrödinger equation is given by

$$i\hbar\frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x})\right)\psi\,. \tag{1}$$

Assuming that $V(\mathbf{x})$ is real, its complex conjugate is

$$-i\hbar\frac{\partial \psi^*}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x})\right)\psi^*\,. \tag{2}$$

Now we multiply Eq. (1) by ψ^* from the left and Eq. (2) by ψ from the right, then subtract the resulting equations to arrive at

$$i\hbar\left(\psi^*\frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t}\psi\right) = -\frac{\hbar^2}{2m}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) = -\frac{\hbar^2}{2m}\nabla.\left[\psi^*\nabla\psi - \psi\nabla\psi^*\right]\,.$$

Hence

$$\frac{\partial \rho}{\partial t} = \frac{i\hbar}{2m}\nabla.\left[\psi^*\nabla\psi - \psi\nabla\psi^*\right]\,,$$

i.e.

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\mathbf{j} = 0$$

with

$$\mathbf{j} = -\frac{i\hbar}{2m}\left[\psi^*\nabla\psi - \psi\nabla\psi^*\right]\,. \qquad \text{qed.}$$

2. (a) The generalised momentum

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot\phi}\,.$$

The Hamiltonian density is defined by

$$\mathcal{H} = \dot\phi(x)\pi(x) - \mathcal{L}\,,$$

so

$$H = \int \mathrm{d}^3\mathbf{x}\,\dot\phi(\mathbf{x},t)\pi(\mathbf{x},t) - L = \int \mathcal{H}\,\mathrm{d}^3\mathbf{x}\,.$$

For the free Klein-Gordon field $\pi = \dot\phi$ (as derived in the lecture) and hence

$$\mathcal{H} = \tfrac{1}{2}\,\dot\phi^2 + \tfrac{1}{2}\,(\nabla\phi)^2 + \tfrac{1}{2}\,m^2\phi^2\,.$$

(b)

$$
\begin{aligned}
[\phi(\mathbf{x},t),H] &= [\phi(\mathbf{x},t),\int\left(\tfrac{1}{2}\pi(\mathbf{x}',t)^2+\tfrac{1}{2}(\nabla'\phi(\mathbf{x}',t))^2+\tfrac{1}{2}m^2\phi(\mathbf{x}',t)^2\right)\,\mathrm{d}^3\mathbf{x}'] \\
&= \tfrac{1}{2}\int \mathrm{d}^3\mathbf{x}'\left([\phi(\mathbf{x}),\pi(\mathbf{x}')^2]+[\phi(\mathbf{x}),(\nabla'\phi(\mathbf{x}'))^2]+m^2[\phi(\mathbf{x}),\phi(\mathbf{x}')^2]\right)\,.
\end{aligned}
$$

Now

$$
[\phi(\mathbf{x}),\phi(\mathbf{x}')] \;=\; 0 \quad\Rightarrow\quad [\phi(\mathbf{x}),\nabla'\phi(\mathbf{x}')] \;=\; 0\,,
$$

and hence

$$
\begin{aligned}
i\hbar\dot{\phi}(\mathbf{x}) &= [\phi(\mathbf{x}),H] \;=\; \tfrac{1}{2}\int\left(\pi(\mathbf{x}')[\phi(\mathbf{x}),\pi(\mathbf{x}')]+[\phi(\mathbf{x}),\pi(\mathbf{x}')]\pi(\mathbf{x}')\right)\,\mathrm{d}^3\mathbf{x}' \\
&= i\hbar\int\pi(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')\,\mathrm{d}^3\mathbf{x}' \;=\; i\hbar\pi(\mathbf{x})\,.
\end{aligned}
$$

Next calculate

$$
\begin{aligned}
i\hbar\dot{\pi}(\mathbf{x}) &= [\pi(\mathbf{x},t),H] \\
&= -\,[\int\mathrm{d}^3\mathbf{x}'\left(\tfrac{1}{2}\pi(\mathbf{x}')^2+\tfrac{1}{2}(\nabla'\phi(\mathbf{x}'))^2+\tfrac{1}{2}m^2\phi(\mathbf{x}')^2\right),\pi(\mathbf{x})] \\
&= -\,\tfrac{1}{2}\int\mathrm{d}^3\mathbf{x}'\left(\nabla'\phi(\mathbf{x}')[\nabla'\phi(\mathbf{x}'),\pi(\mathbf{x})]+[\nabla'\phi(\mathbf{x}'),\pi(\mathbf{x})]\nabla'\phi(\mathbf{x}')\right) \\
&\quad -\tfrac{1}{2}m^2\int\mathrm{d}^3\mathbf{x}'\left(\phi(\mathbf{x}')[\phi(\mathbf{x}'),\pi(\mathbf{x})]+[\phi(\mathbf{x}'),\pi(\mathbf{x})]\phi(\mathbf{x}')\right) \\
&= -\,i\hbar\int\mathrm{d}^3\mathbf{x}'\left(\nabla'\phi(\mathbf{x}').\nabla'\delta(\mathbf{x}-\mathbf{x}')+m^2\phi(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')\right) \\
&= i\hbar\int\mathrm{d}^3\mathbf{x}'\left((\nabla')^2\phi(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')-m^2\phi(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')\right) \\
&= i\hbar\left(\nabla^2\phi(\mathbf{x})-m^2\phi(\mathbf{x})\right)\,.
\end{aligned}
$$

Together with the previous step we immediately get the Klein-Gordon equation

$$
\ddot{\phi}(\mathbf{x}) \;=\; \nabla^2\phi(\mathbf{x})-m^2\phi(\mathbf{x})\,.
$$

(c) We write

$$
(\partial\phi)^2 \;=\; \eta_{\rho\sigma}(\partial^\rho\phi)(\partial^\sigma\phi)\,,
$$

then using

$$
\frac{\partial(\partial^\rho\phi)}{\partial(\partial^\mu\phi)} \;=\; \delta^\rho{}_\mu
$$

we have

$$
\frac{\partial}{\partial(\partial^\mu\phi)}(\partial\phi)^2 \;=\; \eta_{\rho\sigma}(\delta^\rho{}_\mu\partial^\sigma\phi+\partial^\rho\phi\delta^\sigma{}_\mu)=2\partial_\mu\phi\,.
$$

With this the equation of motion becomes

$$
\partial^\mu\left(\partial_\mu\phi\right) \;=\; -m^2\phi-\frac{1}{2}\lambda_3\phi^2-\frac{1}{3!}\lambda_4\phi^3\,,
$$

or

$$
\partial^2\phi+m^2\phi+\frac{1}{2}\lambda_3\phi^2+\frac{1}{3!}\lambda_4\phi^3 \;=\; 0\,.
$$

3. (a) Two-particle states are defined by

$$|\mathbf{p}_1, \mathbf{p}_2\rangle \;=\; a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)|0\rangle\,,$$

with $|\mathbf{p}_1, \mathbf{p}_2\rangle = |\mathbf{p}_2, \mathbf{p}_1\rangle$ as $[a^\dagger(\mathbf{p}_1), a^\dagger(\mathbf{p}_2)] = 0$. Hence

$$\begin{aligned}
\langle \mathbf{p}'_1, \mathbf{p}'_2|\mathbf{p}_1, \mathbf{p}_2\rangle &= \langle 0|a(\mathbf{p}'_1)a(\mathbf{p}'_2)a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)|0\rangle \\
&= \langle 0|a(\mathbf{p}'_1)\{a^\dagger(\mathbf{p}_1)a(\mathbf{p}'_2) + (2\pi)^3 2p_1^0\delta(\mathbf{p}_1 - \mathbf{p}'_2)\}a^\dagger(\mathbf{p}_2)|0\rangle \\
&= \langle 0|a(\mathbf{p}'_1)a^\dagger(\mathbf{p}_1)\{a^\dagger(\mathbf{p}_2)a(\mathbf{p}'_2) + (2\pi)^3 2p_1^0\delta(\mathbf{p}_2 - \mathbf{p}'_2)\}|0\rangle \\
&\quad + (2\pi)^3 2p_1^0\delta(\mathbf{p}_1 - \mathbf{p}'_2)\langle 0|a(\mathbf{p}'_1)a^\dagger(\mathbf{p}_2)|0\rangle \\
&= (2\pi)^3 2p_2^0\delta(\mathbf{p}_2 - \mathbf{p}'_2)\langle 0|\{a^\dagger(\mathbf{p}_1)a(\mathbf{p}'_1) + (2\pi)^3 2p_1^0\delta(\mathbf{p}_1 - \mathbf{p}'_1)\}|0\rangle \\
&\quad + (2\pi)^3 2p_1^0\delta(\mathbf{p}_1 - \mathbf{p}'_2)\langle 0|\{a^\dagger(\mathbf{p}_2)a(\mathbf{p}'_1) + (2\pi)^3 2p_2^0\delta(\mathbf{p}_2 - \mathbf{p}'_1)\}|0\rangle \\
&= (2\pi)^6(2p_1^0)(2p_2^0)\{\delta(\mathbf{p}_1 - \mathbf{p}'_1)\delta(\mathbf{p}_2 - \mathbf{p}'_2) + \delta(\mathbf{p}_1 - \mathbf{p}'_2)\delta(\mathbf{p}_2 - \mathbf{p}'_1)\}\,.
\end{aligned}$$

(b) The number operator is

$$N \;:=\; \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{p}'}{2p'^0}\, a^\dagger(\mathbf{p}')a(\mathbf{p}')\,.$$

$$\begin{aligned}
\text{With this}\quad [N, a^\dagger(\mathbf{p})] &= \left[\frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{p}'}{2p'^0}\, a^\dagger(\mathbf{p}')a(\mathbf{p}'), a^\dagger(\mathbf{p})\right] \\
&= \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{p}'}{2p'^0}\, [a^\dagger(\mathbf{p}')a(\mathbf{p}'), a^\dagger(\mathbf{p})] \\
&= \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{p}'}{2p'^0}\, \{a^\dagger(\mathbf{p}')[a(\mathbf{p}'), a^\dagger(\mathbf{p})] + [a^\dagger(\mathbf{p}'), a^\dagger(\mathbf{p})]a(\mathbf{p}')\} \\
&= \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{p}'}{2p'^0}\, a^\dagger(\mathbf{p}')2p'^0(2\pi)^3\delta(\mathbf{p} - \mathbf{p}') \\
&= a^\dagger(\mathbf{p})\,.
\end{aligned}$$

So we have

$$\begin{aligned}
Na^\dagger(\mathbf{p}) - a^\dagger(\mathbf{p})N &= a^\dagger(\mathbf{p}) \\
\Rightarrow Na^\dagger(\mathbf{p}) &= a^\dagger(\mathbf{p})(N+1)\,, \\
\text{and}\quad N|\mathbf{p}_1 \ldots \mathbf{p}_n\rangle &= Na^\dagger(\mathbf{p}_1)\ldots a^\dagger(\mathbf{p}_n)|0\rangle \\
&= a^\dagger(\mathbf{p}_1)(N+1)a^\dagger(\mathbf{p}_2)\ldots a^\dagger(\mathbf{p}_n)|0\rangle \\
&= a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)(N+2)\ldots a^\dagger(\mathbf{p}_n)|0\rangle \\
&= \;\ldots\; = \; a^\dagger(\mathbf{p}_1)\ldots a^\dagger(\mathbf{p}_n)(N+n)|0\rangle \\
&= n\, a^\dagger(\mathbf{p}_1)\ldots a^\dagger(\mathbf{p}_n)|0\rangle \quad (\text{as } a(\mathbf{p})|0\rangle = 0 \;\text{ and so } N|0\rangle = 0) \\
&= n|\mathbf{p}_1 \ldots \mathbf{p}_n\rangle\,.
\end{aligned}$$