UNIVERSITY OF LIVERPOOL

MATH 328 — May 2009

Examiner: Dr. T. Teubner, Extension 43791.

Time allowed: Two and a half hours

In this paper bold-face quantities like $\mathbf{x}$ and quantities with a vector arrow like $\vec{K}$ represent three-dimensional vectors.

Full marks can be obtained for complete answers to FIVE questions. Only the best FIVE answers will be counted.

1. Consider the Poincaré group in 1+2 dimensions, with the infinitesimal line element

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 .$$

(a) Write down in matrix form the metric for this line element and its inverse. What are the transformations under which this line element is invariant? Give the generators associated with each transformation. [6 marks]

(b) The Pauli-Lubanski vector in four dimensions is given by

$$W^\mu = -\frac{1}{2}\,\epsilon^{\mu\nu\sigma\rho}\,J_{\nu\sigma}\,P_\rho\,,$$

where $K_i = J_{i0}$ and $J_i = \frac{1}{2}\epsilon_{ijk}\,J_{jk}$ are the generators of boosts and rotations, respectively, and P_ρ is the momentum four-vector. Give W^μ in terms of K_i, J_i, P_0, P_i. Write down the four components of the Pauli-Lubanski vector in the case where the line element $\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2$ is viewed as embedded in four space-time dimensions, for both massless and massive particle states. [14 marks]

2. (a) Let $\vec{J}$ and $\vec{K}$ be the generators of rotations and boosts, respectively. Show that

$$\vec{J}^2 - \vec{K}^2 \qquad \text{and} \qquad \vec{J}\cdot\vec{K}$$

are Lorentz invariants, i.e. that they commute with all the generators of the Lorentz group. [10 marks]

(b) Assume a representation (j_1, j_2) of $SU(2)\times SU(2)$. How many states are in this representation? How does this representation decompose into irreducible representations of $SU(2)_J$, where J is the total angular momentum? [10 marks]

3. The potential function of a two-dimensional harmonic oscillator is

$$V(x,y) = \frac{1}{2}\,k\,(x^2+y^2)\,.$$

(i) Write down the Lagrangian of this system. [3 marks]

(ii) Write down the Euler-Lagrange equations of motion. [3 marks]

(iii) Write down the Hamiltonian. [5 marks]

(iv) Write down the Lagrangian and Hamiltonian in polar coordinates (r,ϕ) with $(x = r\cos\phi\,,\ y = r\sin\phi)$. [5 marks]

(v) How many constants of the motion are there? What are they? [4 marks]

4. (a) The Lagrange density for a free real scalar field is given by

$$\mathcal{L}_0 \;=\; \tfrac{1}{2}\,(\partial\phi)^2 - \tfrac{1}{2}\,m^2\phi^2\,.$$

Show that the Hamiltonian H_0 is given by

$$H_0 \;=\; \tfrac{1}{2}\int \left(\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2\right) \mathrm{d}^3x\,.$$

[5 marks]

(b) The canonical commutation relations are given by

$$\begin{aligned} [\phi(\mathbf{x},t),\pi(\mathbf{x}',t)] &= i\hbar\delta(\mathbf{x}-\mathbf{x}')\,,\\ [\phi(\mathbf{x},t),\phi(\mathbf{x}',t)] &= 0\,,\\ [\pi(\mathbf{x},t),\pi(\mathbf{x}',t)] &= 0\,. \end{aligned}$$

Show that with these commutation relations the (Heisenberg) equations of motion for the time evolution of the operators ϕ and π,

$$i\hbar\dot{\phi} \;=\; [\phi, H_0] \qquad \text{and} \qquad i\hbar\dot{\pi} \;=\; [\pi, H_0]\,,$$

imply the Klein-Gordon equation for the field ϕ. [15 marks]

5. (a) Given the four-vector current density $J_V^\mu = \bar{\psi}\gamma^\mu\psi$, derive the current conservation, $\partial_\mu J_V^\mu = 0$, by using the covariant form of the Dirac equation and the relation $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$. [6 marks]

(b) Show that the axial four-vector current density $J_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$ is not conserved but instead satisfies the covariant equation

$$\partial_\mu J_A^\mu \;=\; 2im\,\bar{\psi}\gamma^5\psi\,.$$

[6 marks]

(c) Derive the *Gordon decomposition* of the Dirac transition current,

$$\bar{\psi}_f\gamma^\mu\psi_i \;=\; \frac{1}{2m}\,\bar{\psi}_f\left[(p_f+p_i)^\mu + i\sigma^{\mu\nu}(p_f-p_i)_\nu\right]\psi_i\,,$$

where $\sigma^{\mu\nu} = \frac{1}{2}i(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$. [8 marks]

6. (a) Prove that for the group $SU(2)$ the fundamental representation (the representation by complex 2×2 matrices) is equivalent to the complex conjugate representation. This means that there exists a unitary 2×2 matrix W such that $U^\star = W^\dagger U W$ (unitary equivalence).

Hints: The complex conjugate representation can be written as

$$U^\star \ = \ \exp(-\frac{i}{2}\,\theta_a\,\sigma_a^\star)\,,$$

with the three Pauli-matrices in the usual representation,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

First show that $i\sigma_2$ is unitary and that $\sigma_2\,\sigma_a^\star\,\sigma_2 = -\sigma_a$. [16 marks]

(b) Explain the consequences this has for the construction of mass terms in the Standard Model. [4 marks]

7. Let the Lagrangian for three interacting real scalar fields ϕ_1, ϕ_2, ϕ_3 be given by

$$L \ = \ \frac{1}{2}(\partial_\mu\phi_i)^2 - \frac{1}{2}\mu^2\phi_i^2 - \frac{1}{4}\lambda(\phi_i^2)^2\,,$$

with $\mu^2 < 0$ and $\lambda > 0$, and where the summation over i is implied.

(a) Explain the meaning of the different terms and to which elements of the corresponding Feynman rules they lead. Why is μ^2 chosen negative, but λ positive? [5 marks]

(b) Show that after spontaneous symmetry breaking this Lagrangian describes one massive field with mass $\sqrt{-2\mu^2}$ and two massless 'Goldstone bosons'. [15 marks]