MATH 431 — Aug 2019: Solutions

All problems are similar to homework problems or material covered in the lectures.

1.

$$\begin{aligned} & a_0 = a^0,\ a_1 = -a^1,\ a_2 = -a^2,\ a_3 = -a^3 \\ & a'_0 = a'^0 = \gamma(a^0 - \beta a^1) = \gamma(a_0 + \beta a_1) \\ & a'_1 = -a'^1 = -\gamma(-\beta a^0 + a^1) = \gamma(\beta a_0 + a_1) \\ \text{and} \ \ & a'_2 = a_2,\ a'_3 = a_3 \end{aligned} \tag{1}$$

b.

Suppose we have a function $f(x^0, x^1, x^2, x^3)$ which we express as a function of x'^0, x'^1, x'^2, x'^3 by expressing x^μ as a function of x'^μ. The standard chain rule for partial differentiation says that

$$\frac{\partial f(x^\nu(x'^\mu))}{\partial x'^\mu} = \sum_{\nu=0}^{3} \frac{\partial f}{\partial x^\nu}\frac{\partial x^\nu}{\partial x'^\mu} \quad \text{for} \quad \mu = 0, 1, 2, 3$$

Using the summation convention and writing as an operator equation we get

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}$$

We need x as a function of x', the inverse of the Lorentz transformation that gives x' as a function of x. For a boost along the x' axis, the inverse is a boost with the opposite speed, so

$$x^0 = \gamma(x'^0 + \beta x'^1),\ x^1 = \gamma(\beta x'^0 + x'^1),\ x^2 = x'^2 x^3 = x'^3$$

Hence

$$\frac{\partial x^0}{\partial x'^0} = \gamma, \quad \frac{\partial x^0}{\partial x'^1} = \gamma\beta, \quad \frac{\partial x^1}{\partial x'^0} = \gamma\beta, \quad \frac{\partial x^1}{\partial x'^1} = \gamma$$

and

$$\frac{\partial}{\partial x'^0} = \gamma(\frac{\partial}{\partial x^0} + \beta\frac{\partial}{\partial x^1}), \quad \frac{\partial}{\partial x'^1} = \gamma(\beta\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}), \quad \frac{\partial}{\partial x'^2} = \frac{\partial}{\partial x^2}, \quad \frac{\partial}{\partial x'^3} = \frac{\partial}{\partial x^3}$$

which is the same as as (1) with $a_\mu = \frac{\partial}{\partial x^\mu}$

c.

The operator for momentum $\vec{p}$ is $-i\hbar\vec{\nabla}$, *i.e.*

$$p^1 = -i\hbar\frac{\partial}{\partial x_1},\ p^2 = -i\hbar\frac{\partial}{\partial x_2},\ p^3 = -i\hbar\frac{\partial}{\partial x_3} \tag{5}$$

and

$$p_1 = i\hbar\frac{\partial}{\partial x^1},\ p_2 = i\hbar\frac{\partial}{\partial x^2},\ p_3 = i\hbar\frac{\partial}{\partial x^3} \qquad (6)$$

The Schrödinger equation says

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi$$

where H is the energy operator. Since $p^0 = \frac{E}{c}$ and $x^0 = ct$, this can be written as

$$p_0 = i\hbar\frac{\partial}{\partial x^0} \qquad (7)$$

If we write (5) and (7) in terms of p_μ, we remove the sign difference between the 0 component and the others.

$$p_\mu = i\hbar\frac{\partial}{\partial x^\mu},$$

which transforms as a Lorentz covariant vector, as we have shown in (b).

2. (a) In the lecture we have derived

$$\dot{p}_0 = -\frac{\partial H}{\partial q_0}.$$

Therefore, p_0 is constant in time if H does not depend on q_0.

One example for such a system is given in part (b) below, where H does not depend on ϕ. A second example would be a one dimensional harmonic oscillator (in x) embedded in two dimensions such that there is no y dependence and therefore the momentum $p_y = constant$.

(b) In polar coordinates

$$\begin{aligned} x &= r\sin\theta\cos\phi\,, \\ y &= r\sin\theta\sin\phi\,, \\ z &= r\cos\theta\,, \end{aligned}$$

the kinetic energy is given by

$$\frac{1}{2}m\left[\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\dot{\phi})^2\right],$$

and with the potential energy V the Lagrangian is given by

$$L = \frac{1}{2}m\left[\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\dot{\phi})^2\right] - V\,.$$

Now

$$\begin{aligned} p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r}\,, \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}\,, \\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\dot{\phi}\,. \end{aligned}$$

Hence

$$
\begin{aligned}
H = \sum_i p_i \dot{q}_i - L &= m\dot{r}^2 + mr^2\dot{\theta}^2 + mr^2 \sin^2\theta\dot{\phi}^2 - \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) + V \\
&= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) + V \\
&= \frac{{p_r}^2}{2m} + \frac{{p_\theta}^2}{2mr^2} + \frac{{p_\phi}^2}{2mr^2\sin^2\theta} + V\,.
\end{aligned}
$$

With this we get

$$
\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = -\frac{\partial V}{\partial \phi} = 0\,.
$$

An axisymmetric potential does not depend on ϕ, so p_ϕ is a constant of the motion, and the angular momentum about the symmetry axis is conserved.

3a.

$$
\vec{J}_\pm = \frac{1}{2}(\vec{J} \pm i\vec{K})
$$

J_+ and J_- generate the algebra $SU(2)\otimes SU(2)^\dagger$. J_+^2 and J_-^2 are the Casimir operators of $SU(2)$ and $SU(2)^\dagger$, respectively, and are therefore invariants of the Lorentz group.

$$
\begin{aligned}
J_+^2 &= \frac{1}{4}(J^2 - K^2 + 2i\vec{J}\cdot\vec{K}) \\
J_-^2 &= \frac{1}{4}(J^2 - K^2 - 2i\vec{J}\cdot\vec{K})
\end{aligned}
$$

$$
\begin{aligned}
J^2 - K^2 &= 2(J_+^2 + J_-^2) \\
\vec{J}\cdot\vec{K} &= -i(J_+^2 - J_-^2)
\end{aligned}
$$

Therefore $J^2 - K^2$ and $\vec{J}\cdot\vec{K}$ are Lorentz invariants as well, being the sum and difference of Lorentz invariants.

3b. For the representation (j_1, j_2) of the $SU(2)\otimes SU(2)^\dagger$ algebra the number of states is $(2j_1+1)(2j_2+1)$.

The total spin is given by $j_1 + j_2$. Therefore the composition $j_1 \otimes j_2$ breaks under $SU(2)_J$ with the following spin states

$$
j_1 + j_2 \oplus j_1 + j_2 - 1 \oplus \cdots \oplus |j_1 - j_2|
$$

4.

$$
\begin{aligned}
&\gamma^\mu\partial_\mu\psi + im\psi = 0\,, \qquad (\partial_\mu\psi^\dagger)\gamma^{\mu\dagger} - im\psi^\dagger = 0 \\
\Rightarrow \quad &(\partial_\mu\psi^\dagger)\gamma^0\gamma^\mu\gamma^0 - im\psi^\dagger = 0 \quad \text{and} \quad (\partial_\mu\bar{\psi})\gamma^\mu - im\bar{\psi} = 0\,.
\end{aligned}
$$

(a) With this

$$\partial_\mu(\bar\psi\gamma^\mu\psi) \;=\; (\partial_\mu\bar\psi)\gamma^\mu\psi + \bar\psi\gamma^\mu(\partial_\mu\psi) \;=\; (im\bar\psi)\psi + \bar\psi(-im\psi) \;=\; 0\,.$$

(b) Similarly, and with $\{\gamma^5, \gamma^\mu\} = 0$ we get

$$\partial_\mu(\bar\psi\gamma^\mu\gamma^5\psi) = (\partial_\mu\bar\psi)\gamma^\mu\gamma^5\psi + \bar\psi\gamma^\mu\gamma^5(\partial_\mu\psi) = (im\bar\psi)\gamma^5\psi - \bar\psi\gamma^5(\gamma^\mu\partial_\mu\psi) = 2im\bar\psi\gamma^5\psi\,.$$

(c) From the Dirac equation for the spinors $\bar u_f$ and u_i we have

$$0 \;=\; \bar u_f(\not p_f - m)\gamma^\mu u_i \;=\; \bar u_f\gamma^\mu(\not p_i - m)u_i$$

$$\begin{aligned} \Rightarrow \qquad 2m\bar u_f\gamma^\mu u_i &= \bar u_f(\not p_f\gamma^\mu + \gamma^\mu \not p_i)u_i \\ \text{and} \qquad \not p_f\gamma^\mu + \gamma^\mu \not p_i &= \gamma^\nu\gamma^\mu p_{f_\nu} + \gamma^\mu\gamma^\nu p_{i_\nu}\,. \end{aligned}$$

Now

$$\begin{aligned} \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu &= 2g^{\mu\nu}\,, \\ \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu &= -2i\sigma^{\mu\nu} \end{aligned}$$

$$\Rightarrow \qquad \gamma^\mu\gamma^\nu \;=\; g^{\mu\nu} - i\sigma^{\mu\nu} \qquad \text{and} \qquad \gamma^\nu\gamma^\mu \;=\; g^{\mu\nu} + i\sigma^{\mu\nu}\,.$$

So we get

$$\not p_f\gamma^\mu + \gamma^\mu \not p_i \;=\; g^{\mu\nu}(p_f + p_i)_\nu + i\sigma^{\mu\nu}(p_f - p_i)_\nu \;=\; (p_f + p_i)^\mu + i\sigma^{\mu\nu}(p_f - p_i)_\nu\,,$$

and finally have derived the Gordon decomposition

$$\bar u_f\gamma^\mu u_i \;=\; \frac{1}{2m}\,\bar u_f\left[(p_f + p_i)^\mu + i\sigma^{\mu\nu}(p_f - p_i)_\nu\right]u_i\,.$$

5.

To construct a consistent three dimensional theory, we must ensure that the dynamics do not depend on the z–direction. The motion of the particle must be confined in the (x, y) plane. Since the Lorents force is perpendicular to the magnetic field, it follows that the components of the magnetic fields in the (x, y) plane has to vanish. Similarly, the component of the electric field in the z–direction is zero. Hence, we take

$$E_z = B_x = B_y = 0.$$

The remaining components E_x, E_y and B_z can only depend on x and y. Similarly, the velocity and current components in the z–directions are zero, *i.e.* $v_z = j_z = 0$. Maxwell's equations then become

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = \rho \quad \text{from} \qquad \vec\nabla\cdot\vec E = \rho$$

$$\frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y} = -\frac{1}{c}\frac{\partial B_z}{\partial t} \quad \text{from} \quad \vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\begin{aligned} \frac{\partial B_z}{\partial y} &= \frac{j_x}{c} + \frac{1}{c}\frac{\partial E_x}{\partial t} \\ -\frac{\partial B_z}{\partial x} &= \frac{j_y}{c} + \frac{1}{c}\frac{\partial E_y}{\partial t} \end{aligned}$$

$$\text{from} \quad \vec{\nabla} \times \vec{B} = \frac{\vec{j}}{c} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

The remaining Maxwell equation $\vec{\nabla} \cdot \vec{B} = 0$ is trivial because $B_x = B_y = 0$ and $B_z = B_z(x, y)$. The Lorentz force law gives nontrivial equations only for the x and y components:

$$\begin{aligned} \frac{dp_x}{dt} &= q\left(E_x + \frac{v_y}{c} B_x\right), \\ \frac{dp_y}{dt} &= q\left(E_y - \frac{v_x}{c} B_y\right). \end{aligned}$$

5b. In three dimensions we have $A^\mu = \left(\Phi, A^1, A^2\right)$, $A_\mu = (\Phi, -A_1, -A_2)$, and $j^\mu = \left(c\rho, j^1, j^2\right)$. Moreover, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, so

$$\begin{aligned} F_{0i} &= \frac{1}{c}\frac{\partial A_i}{\partial t} + \frac{\partial \Phi}{\partial x^i} \equiv -E_i \\ F_{12} &= \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \equiv B_z \end{aligned}$$

Thus, the field strength tensor takes the form

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y \\ E_x & 0 & B_z \\ E_y & -B_z & 0 \end{pmatrix} \qquad F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y \\ -E_x & 0 & B_z \\ -E_y & -B_z & 0 \end{pmatrix}$$

The above $D = 3$ field strength F can be viewed as the $D = 4$ one with $E_z = B_x = B_y = 0$. The $D = 3$ current j can be viewed as the $D = 4$ current with $j_z = 0$. The three dimensional Maxwell equations are therefore the truncation to $E_z = B_x = B_y = 0$ and $j_z = 0$ of the original four dimensional ones

6a.

+	+	+	+	+
+	+	+	-	-
-	+	+	-	+
+	-	+	-	+
+	+	-	-	+
-	+	+	+	-
+	-	+	+	-
+	+	-	+	-
-	-	+	+	+
-	+	-	+	+
+	-	-	+	+
-	-	+	-	-
+	+	+	-	-
+	+	+	-	-
-	-	-	-	+
-	-	-	+	-

The weight lattice of the spinorial 16 representation of $SO(10)$. Each entry is multiplied by the $\frac{1}{2}$. These are the charges with respect to the five $U(1)$ generators of the Cartan subalgebra. The product of the five charges should be either positive or negative. In the positive case we can have zero, two or four negative charges of the total five. In the negative case there can be one, three or five negative charges. In either case the total number of possibilities is 16. These are the two spinorial representations of $SO(10)$ being the chiral 16 and the anti–chiral $\overline{16}$.

6(b).

Under $SU(5) \times U(1)_X$, using the combinatorial notation introduced in the class

$$16 = \binom{5}{0} + \binom{5}{2} + \binom{5}{4}$$

where the combinatorial factor counts the number of $-$ in a given state. Hence,

$$16 = (1, \frac{5}{2}) + (10, \frac{1}{2}) + (\bar{5}, -\frac{3}{2})$$

where the $U(1)_X$ charges are obtained by taking the trace $Q_X = Q_1 + Q_2 + Q_3 + Q_4 + Q_5$, and the Q_i are the $\pm\frac{1}{2}$ charges with respect to the five generators of the Cartan subalgebra. To find the decomposition under $SO(6) \times SO(4) \equiv SU(4) \times SU(2)_L \times SU(2)_R$ we split the five slots into the first three which correspond to $SO(6)$ and the last two which correspond to $SO(4)$. Using the same combinatorial notation gives

$$16 = \left[\binom{3}{0} + \binom{3}{2}\right]\left[\binom{2}{0} + \binom{2}{2}\right] + \left[\binom{3}{1} + \binom{3}{3}\right]\left[\binom{2}{1}\right]$$

Hence, under $SO(6) \times SO(4) \equiv SU(4) \times SU(2)_L \times SU(2)_R$ it decomposes as:

$$16 = (4, 2, 1) + (\bar{4}, 1, 2)$$

The decomposition under $SU(3) \times U(1)_C \times SU(2) \times U(1)_L$ is

$$16 = \left[\binom{3}{0}\binom{2}{0}\right] + \left[\binom{3}{2}\binom{2}{0}\right] + \left[\binom{3}{0}\binom{2}{2}\right] + \left[\binom{3}{2}\binom{2}{2}\right] + \left[\binom{3}{1}\binom{2}{1}\right] + \left[\binom{3}{3}\binom{2}{1}\right]$$

The charges under the $U(1)$s are given by the sums under $Q_C = Q_1 + Q_2 + Q_3$ and $Q_L = Q_4 + Q_5$ for $U(1)_C$ and $U(1)_L$, respectively. Hence, under $SU(3) \times U(1)_C \times SU(2) \times U(1)_L$ the 16 decomposes as:

$$16 = (1, \frac{3}{2}, 1, +1) + (\bar{3}, -\frac{1}{2}, 1, +1) + (1, \frac{3}{2}, 1, -1) + (\bar{3}, -\frac{1}{2}, 1, -1) + (3, \frac{1}{2}, 2, 0) + (1, -\frac{3}{2}, 2, 0).$$

6(c). From the last line we can read off the Standard Model states. These are in order respectively

$$\mathrm{e}^c_L\ , \mathrm{d}^c_L\ , \mathrm{N}^c_L\ , \mathrm{u}^c_L\ , Q\ , L$$

where the subscript L indicates that these are all left–handed fields and the upperscript c denotes charge conjugation (corresponding to antiparticles). The weak hypercharge is given by

$$U(1)_Y = \frac{1}{3}U_C + \frac{1}{2}U_L$$

and the electric charge by $U(1)_{e.m.} = T_{3_L} + U(1)_Y$ where T_{3_L} is the diagonal generator of the $SU(2)$ subgroup.

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7(a). The $SU(2) \times U(1)_Y$ charges of the electroweak Higgs doublet ϕ.

$$\phi : T = \frac{1}{2}\ ;\ T_3 = \pm\frac{1}{2}\ ;\ Y = 1$$

(7b.) The electric charge is given by:

$$Q_{\mathrm{e.m.}}(\phi_+) = T_3 + \frac{1}{2}Y = \frac{1}{2} + \frac{1}{2} = +1$$
$$Q_{\mathrm{e.m.}}(\phi_0) = T_3 + \frac{1}{2}Y = -\frac{1}{2} + \frac{1}{2} = 0$$

(7c.) The Lagrangian density for the Higgs field include the kinetic and potential terms

$$\left|\left(\partial_\mu - ig\vec{T}\cdot\vec{W}_\mu - ig'\frac{Y}{2}B_\mu\right)\phi\right|^2 - V(\phi)$$

where the potential is given by

$$\mu^2\phi^\dagger\phi + \lambda\left(\phi^\dagger\phi\right)^2$$

(7d.) The relevant term for the gauge boson masses

$$\begin{aligned}
\left|\left(g\vec{T}\cdot\vec{W}_\mu + g'\frac{Y}{2}B_\mu\right)\phi\right|^2 = \left|\left(g\frac{\vec{\tau}}{2}\cdot\vec{W}_\mu + g'\frac{Y}{2}B_\mu\right)\phi\right|^2 &= \\
\frac{1}{4}\left|\left(\frac{g}{2}\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}W^1_\mu + \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}W^2_\mu + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}W^3_\mu\right] + \frac{g'}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}B_\mu\right)\phi\right|^2 &= \\
\frac{1}{4}\left|\left[g\begin{pmatrix}W^3_\mu & W^2_\mu - iW^1_\mu\\ W^2_\mu + iW^1_\mu & -W^3_\mu\end{pmatrix} + g'\begin{pmatrix}B_\mu & 0\\ 0 & B_\mu\end{pmatrix}\right]\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix}\right|^2 &= \\
\frac{1}{8}\left|\begin{pmatrix}gW^3_\mu + g'B_\mu & g(W^2_\mu - iW^1_\mu)\\ g(W^2_\mu + iW^1_\mu) & -gW^3_\mu + g'B_\mu\end{pmatrix}\begin{pmatrix}0\\ v\end{pmatrix}\right|^2 &= \\
\frac{1}{8}v^2\left(g\left(W^2_\mu + iW^1_\mu\right), \left(-gW^3_\mu + g'B_\mu\right)\right)\begin{pmatrix}g\left(W^2_\mu - iW^1_\mu\right)\\ -gW^3_\mu + g'B_\mu\end{pmatrix} &= \\
\frac{1}{4}g^2v^2W^{\mu+}W^{\mu-} + \frac{1}{8}v^2(g^2+g'^2)\frac{\left(-gW^3_\mu + g'B_\mu\right)^2}{\left(\sqrt{g^2+g'^2}\right)^2} &= \\
\left(\frac{1}{2}gv\right)^2 W^{\mu+}W^{\mu-} + \frac{1}{2}v^2\frac{(g^2+g'^2)}{4}\left(\frac{-gW^3_\mu + g'B_\mu}{\sqrt{g^2+g'^2}}\right)^2 &= \\
M^2_{W^\pm} \quad W^{\mu+}W^{\mu-} \quad + \qquad \frac{1}{2}M^2_Z Z^2 &
\end{aligned}$$

reading off from the last two lines we have

$$M_{W^\pm} = \frac{gv}{2} \quad ; \quad M_Z = \frac{1}{2}v\left(g^2+g'^2\right)^{\frac{1}{2}}$$

hence

$$\frac{M_W}{M_Z} = \frac{g}{\left(g^2+g'^2\right)^{\frac{1}{2}}} = \cos\theta_W$$