MATH431 — Introduction to Modern Particle Theory: FEEDBACK/SOLUTIONS 6

1. The Dirac spinor for the hydrogen ground state (with spin up) is given by

$$\psi(r,\theta,\phi) \;=\; R(r)\begin{pmatrix} 1 \\ 0 \\ ia\cos\theta \\ iae^{i\phi}\sin\theta \end{pmatrix}.$$

For the normalisation we calculate:

$$1 \;=\; \int \mathrm{d}^3\mathbf{r}\,\psi^\dagger\psi \;=\; \int 4\pi r^2\mathrm{d}r\,|R|^2(1+a^2)$$

$$\Rightarrow \quad N^2 \;:=\; \int_0^\infty r^2|R|^2\mathrm{d}r \;=\; [4\pi(1+a^2)]^{-1}.$$

(a) ($\hbar = 1$ here and in part (b))

$$L_z \;=\; -i\frac{\partial}{\partial\phi} \quad\Rightarrow\quad L_z\psi \;=\; R\begin{pmatrix} 0 \\ 0 \\ 0 \\ iae^{i\phi}\sin\theta \end{pmatrix} \not\propto \psi,$$

so ψ is NOT an eigenstate of L_z.

(b)

$$\langle L_z\rangle \;=\; \int \mathrm{d}^3\mathbf{r}\,\psi^\dagger L_z\psi \;=\; 2\pi\int r^2\mathrm{d}r\,\mathrm{d}\cos\theta\,|R|^2a^2\sin^2\theta.$$

Now

$$\int_{-1}^{1}\mathrm{d}\cos\theta(1-\cos^2\theta) \;=\; 2-\frac{2}{3} \;=\; \frac{4}{3} \quad\Rightarrow\quad \langle L_z\rangle \;=\; \frac{8\pi}{3}a^2\,\frac{1}{4\pi(1+a^2)}$$

$$\Rightarrow \quad \langle L_z\rangle \;=\; \frac{2a^2}{3(1+a^2)}.$$

In the H-atom, $v/c \sim \alpha \;\Rightarrow\; \langle L_z\rangle = O(v^2/c^2)$ is a relativistic effect which is due to the spin-orbit interaction. In the non-relativistic limit $a \to 0$, and the four-component Dirac spinors for the spin-up and spin-down states reduce to (decoupled) solutions of the Schroedinger equation multiplied by the two-component Pauli-spinors.

(c)

$$
\begin{aligned}
S_z &= \frac{1}{2}\hbar\,\Sigma_z \;=\; \frac{1}{2}\hbar \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
\Rightarrow \quad S_z\psi &= \frac{1}{2}\hbar\, R \begin{pmatrix} 1 \\ 0 \\ ia\cos\theta \\ -iae^{i\phi}\sin\theta \end{pmatrix}\,,
\end{aligned}
$$

$$
\text{hence} \quad J_z\psi \;=\; (L_z + S_z)\psi \;=\; \frac{1}{2}\hbar\, R \begin{pmatrix} 1 \\ 0 \\ ia\cos\theta \\ iae^{i\phi}\sin\theta \end{pmatrix} \;=\; \frac{1}{2}\hbar\,\psi \quad \Rightarrow \quad J_z \;=\; +\frac{\hbar}{2}\,.
$$

2. The Dirac equation reads $(i\gamma^\mu\partial_\mu - m)\psi = 0$, hence

$$\gamma^\mu\partial_\mu\psi + im\psi = 0 \quad \Rightarrow \quad (\partial_\mu\psi^\dagger)\gamma^{\mu\dagger} - im\psi^\dagger = 0\,.$$

In addition we need to remember that $\bar\psi := \psi^\dagger\gamma^0$ and $\gamma^{\mu\,\dagger} = \gamma^0\gamma^\mu\gamma^0$. Then

$$(\partial_\mu\psi^\dagger)\gamma^0\gamma^\mu\gamma^0 - im\psi^\dagger \;=\; 0$$

and, multiplying with γ^0 from the right,

$$(\partial_\mu\bar\psi)\gamma^\mu - im\bar\psi \;=\; 0\,.$$

Using this we get

$$\partial_\mu(\bar\psi\gamma^\mu\psi) \;=\; (\partial_\mu\bar\psi)\gamma^\mu\psi + \bar\psi\gamma^\mu(\partial_\mu\psi) \;=\; (im\bar\psi)\psi + \bar\psi(-im\psi) \;=\; 0\,.$$

Similarly, and with $\{\gamma^5, \gamma^\mu\} = 0$, we get

$$\partial_\mu(\bar\psi\gamma^\mu\gamma^5\psi) = (\partial_\mu\bar\psi)\gamma^\mu\gamma^5\psi + \bar\psi\gamma^\mu\gamma^5(\partial_\mu\psi) = (im\bar\psi)\gamma^5\psi - \bar\psi\gamma^5(\gamma^\mu\partial_\mu\psi) = 2im\bar\psi\gamma^5\psi\,.$$

Hence the axial-vector current is not conserved for $m \neq 0$.

3. From the Dirac equation for the spinors $\bar u_f = \bar u(p_f)$ and $u_i = u(p_i)$ we have

$$\begin{aligned} 0 \;=\bar u_f(\not p_f - m)\gamma^\mu u_i \;&=\; \bar u_f\gamma^\mu(\not p_i - m)u_i \\ \Rightarrow \qquad 2m\bar u_f\gamma^\mu u_i \;&=\; \bar u_f(\not p_f\gamma^\mu + \gamma^\mu\not p_i)u_i\,, \qquad\qquad (\star) \end{aligned}$$

where

$$\not p_f\gamma^\mu + \gamma^\mu\not p_i \;=\; \gamma^\nu\gamma^\mu p_{f\nu} + \gamma^\mu\gamma^\nu p_{i\nu}\,.$$

Now

$$\begin{aligned} \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \;&=\; 2g^{\mu\nu}\,, \\ \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu \;&=\; -2i\sigma^{\mu\nu} \end{aligned}$$

$$\Rightarrow \qquad \gamma^\mu\gamma^\nu \;=\; g^{\mu\nu} - i\sigma^{\mu\nu} \qquad \text{and} \qquad \gamma^\nu\gamma^\mu \;=\; g^{\mu\nu} + i\sigma^{\mu\nu}\,.$$

So we get

$$\not p_f\gamma^\mu + \gamma^\mu\not p_i \;=\; g^{\mu\nu}(p_f + p_i)_\nu + i\sigma^{\mu\nu}(p_f - p_i)_\nu \;=\; (p_f + p_i)^\mu + i\sigma^{\mu\nu}(p_f - p_i)_\nu\,,$$

and finally have derived from $(\star)$ the *Gordon decomposition*

$$\bar u_f\gamma^\mu u_i \;=\; \frac{1}{2m}\,\bar u_f\left[(p_f + p_i)^\mu + i\sigma^{\mu\nu}(p_f - p_i)_\nu\right]u_i\,.$$

Note: In the non-relativistic limit, the Gordon decomposition separates the electron's interactions with the electromagnetic field A_μ into a part steming from its charge, $-e$, and into a part due to its magnetic moment, $-e/(2m)$.