Properties of Lorentz transformations

Lorentz transformations are transformations that preserve the scalar product and the length of Lorentz four vectors, just like the rotations that preserve the length of three vectors in three dimensional spatial space. The only difference is that in three dimensions we are in Euclidean space with its Euclidean metric, whereas Lorentz transformations operate in Minkowski spacetime with its Minkowski metric. The length of four vectors in Minkowski space is given by

$$\eta_{\mu\nu} X^{\mu} X^{\nu}$$

which is invariant under Lorentz transformations, *i.e.* there no change in its size and shape. The invariance implies the existence of a symmetry, which is generated by a group, the Lorentz group.

Assume $X^\mu \to X'^\mu$ under some Lorentz transformation, *i.e.*

$$X^\mu \to X'^\mu = \Lambda^\mu{}_\nu X^\nu$$

Hence

$$\eta_{\mu\nu} X^\mu X^\nu \to \eta_{\mu\nu} X'^\mu X'^\nu = \eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta X^\alpha X^\beta = \eta_{\alpha\beta} X^\alpha X^\beta$$

$$\Rightarrow \eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta} \tag{1}$$

or in matrix notation

$$\Lambda^T \eta \Lambda = \eta$$

where η is the 4x4 Minkowski metric and Λ is the 4x4 matrix of Lorentz transformations. The identity in eq. (1) defines the Lorentz transformations.

$$\Rightarrow (\mathrm{Det}\Lambda)^2 = 1 \Rightarrow \mathrm{Det}\Lambda = \pm 1$$

The physical transformations correspond to those that can be continuously connected to the identity, for which $\mathrm{Det}\Lambda = +1$.

- $\mathrm{Det}\Lambda = +1 \rightarrow$ Proper Lorentz tranformations
- $\mathrm{Det}\Lambda = -1 \rightarrow$ Imroper Lorentz tranformations

We can look at the 00 component of the identity $\eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}$

$$\begin{aligned} & \eta_{\mu\nu}\Lambda^{\mu}{}_{0}\Lambda^{\nu}{}_{0} = +1 \\ \Rightarrow \quad & (\Lambda^{0}{}_{0})^2 - \sum_i (\Lambda^{i}{}_{0})^2 = +1 \\ \Rightarrow \quad & (\Lambda^{0}{}_{0})^2 = 1 + \sum_i (\Lambda^{i}{}_{0})^2 \geq 1 \end{aligned}$$

We have that as,

$$\begin{aligned} (\Lambda^{0}{}_{0})^2 \geq 1 \Rightarrow \qquad & \Lambda^{0}{}_{0} \geq +1 \quad \text{orthochronos LT} \\ \text{or} \quad & \Lambda^{0}{}_{0} \leq -1 \quad \text{non orthochronos LT} \end{aligned}$$

An example of a nonorthochronos Lorentz transformation is given by **reflections**:

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The Lorentz transformations that are continuously connected to the identity are proper and orthochronos *i.e.*

- 1. $\mathrm{Det}\Lambda = 1 \leftrightarrow \quad \text{proper}$
- 2. $\quad \Lambda^0{}_0 \geq 1 \leftrightarrow \quad \text{orthochronos}$

Examples:

- $$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- **rotations**:

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \vec{R} \end{pmatrix}$$

where $\vec{R}$ are 3x3 rotation matrices in the three spatial dimensions. We have that $\mathrm{Det}\Lambda = \mathrm{Det}R = \pm 1$ and $\mathrm{Det}R = +1$ for proper rotations.

- **boosts**: For a boost along the x–axis

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

we have $\mathrm{Det}\Lambda = \cosh^2\eta - \sinh^2\eta = 1$ and $\Lambda^0{}_0 = \cosh\eta \geq 1$.

- **time inversion**:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

$\mathrm{Det}\Lambda = -1$ and $\Lambda^0{}_0 = -1$. An improper non–orthochronos Lorentz transformation.

- **full inversion**:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\mathrm{Det}\Lambda = +1$ and $\Lambda^0{}_0 = -1$. A proper non–orthochronos LT.

All Lorentz Transformations (LT) are generated by the above transformations. The proper orthochronos LT correspond to rotations and boosts. These are the physical LT that can be continuously connected to the identity LT, *i.e.* we can write them in infinitesimal form. The proper orthochronos LT form a group. We can parametrise these transformations in terms of six parameters, similar to the θ parameter in 2D rotation

$$6 \text{ parameters} = 3 \text{ angles} + 3 \text{ boosts}$$

Write an infinitesimal proper orthochronos Lorentz tranformation in the form

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu} \tag{3}$$

$\delta^{\mu}{}_{\nu} \quad \rightarrow \quad$ identity transformation; $\qquad \omega^{\mu}{}_{\nu} \quad \rightarrow \quad$ a 4x4 infinitesimal matrix

$$\delta^{\mu}{}_{\nu} = +1 \text{ for } \mu = \nu \quad ; \quad \delta^{\mu}{}_{\nu} = \ 0 \text{ for } \mu \neq \nu$$

Next, we expand eq. (1) to first order in the infinitesimal parameters in $\omega^{\mu}{}_{\nu}$.

$$\eta_{\mu\nu}(\delta^{\mu}{}_{\alpha} + \omega^{\mu}{}_{\alpha})(\delta^{\nu}{}_{\beta} + \omega^{\nu}{}_{\beta}) = \eta_{\alpha\beta}$$

open brackets

$$\eta_{\mu\nu}\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta} + \eta_{\mu\nu}\omega^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta} + \eta_{\mu\nu}\delta^{\mu}{}_{\alpha}\omega^{\nu}{}_{\beta} + \eta_{\mu\nu}\omega^{\mu}{}_{\alpha}\omega^{\nu}{}_{\beta} = \eta_{\alpha\beta}$$

$$\Rightarrow \qquad \eta_{\alpha\beta} + \omega_{\beta\alpha} + \omega_{\alpha\beta} + O(\omega^2) = \eta_{\alpha\beta}$$

$$\Rightarrow \qquad \omega_{\beta\alpha} + \omega_{\alpha\beta} = 0$$

$\rightarrow \qquad$ The tensor of infinitesimal transformations is antisymmetric.

rotations: rotation group in two dimensions

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{for small } \theta \tag{4}$$

The number of degrees of freedom in a 4x4 antisymmetric matrix?

for general n: $\frac{n^2-n}{2} = \frac{n(n-1)}{2}$

$$\text{for } n = 4 \rightarrow \frac{4\cdot 3}{2} = 6 \rightarrow 3 \text{ rotation angles} + 3 \text{ boosts}$$

$$\Rightarrow \omega_{\mu\nu} = \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & b_3 & -b_2 \\ -a_2 & -b_3 & 0 & b_1 \\ -a_3 & b_2 & -b_1 & 0 \end{pmatrix} \tag{5}$$