All problems are similar to homework problems

Solution to Problem 1

a.

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1}$$

b.

$$\begin{aligned} x &\to x + \epsilon A(x,y) \\ y &\to y + \epsilon B(x,y) \end{aligned}$$

$$\begin{aligned} dx &\to dx + \epsilon(\tfrac{\partial A}{\partial x}dx + \tfrac{\partial A}{\partial y}dy) \\ dy &\to dy + \epsilon(\tfrac{\partial B}{\partial x}dx + \tfrac{\partial B}{\partial y}dy) \end{aligned}$$

$$ds^2 \to [(1+\epsilon\frac{\partial A}{\partial x})dx + \epsilon\frac{\partial A}{\partial y}dy]^2 + [(1+\epsilon\frac{\partial B}{\partial y})dy + \epsilon\frac{\partial B}{\partial x}dx]^2$$

we require invariance of ds^2. Expanding to first order in ϵ we impose that the coefficients of the addtional terms vanish. These yield the constraints on the functions A and B.

$$\begin{aligned} dx^2 \quad &: \quad \tfrac{\partial A}{\partial x} = 0 \Rightarrow A = A(y) \\ dy^2 \quad &: \quad \tfrac{\partial B}{\partial y} = 0 \Rightarrow B = B(x) \\ dxdy \quad &: \quad \tfrac{\partial A}{\partial y} + \tfrac{\partial B}{\partial x} = 0 \Rightarrow \tfrac{dA}{dy} = -\tfrac{dB}{dx} = \text{constant} = c \end{aligned}$$

$$\begin{aligned} \Rightarrow A(y) &= cy + a \\ B(x) &= -cx + b \end{aligned}$$

we obtained three constants of integration a, b and c. These correspond to a shift in x a, a shift in y b, and a rotation c.

Solution to Problem 2

1a The electromagnetic field strength tensor,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

$$\begin{aligned}
L_{e.m.} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\
&= -\frac{1}{4}(A_{\mu,\nu} - A_{\nu,\mu})(A^{\mu,\nu} - A_{\nu,\mu}) \\
&= -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(A_{\beta,\alpha} - A_{\alpha,\beta})(A_{\mu,\nu} - A_{\nu,\mu})
\end{aligned}$$

The Euler–Lagrange equations of motion are obtained from the Lagrangian

$$\frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial A_{\mu,\nu}}\right) - \frac{\partial L}{\partial A_\mu} = 0$$

$$\frac{\partial L}{\partial A_\mu} = 0 \Rightarrow \frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial A_{\mu,\nu}}\right) = 0$$

$$\begin{aligned}
\frac{\partial L}{\partial A_{p,q}} &= -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(\delta^p_\alpha \delta^q_\beta - \delta^p_\beta \delta^q_\alpha)(A_{\mu,\nu} - A_{\nu,\mu}) + (A_{\alpha,\beta} - A_{\beta,\alpha})(\delta^p_\mu \delta^q_\nu - \delta^p_\nu \delta^q_\mu) \\
&= -\frac{1}{4}(\eta^{p\mu}\eta^{q\nu} - \eta^{q\mu}\eta^{p\nu})(A_{\mu,\nu} + A_{\nu,\mu}) + (\eta^{\alpha p}\eta^{\beta q} - \eta^{\alpha q}\eta^{\beta p})(A_{\beta,\alpha} - A_{\alpha,\beta}) \\
&= -4\frac{1}{4}(A^{q,p} - A^{p,q}) \\
&= -F^{pq}
\end{aligned}$$

$$\frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial A_{\mu,\nu}}\right) = -\frac{1}{4}\frac{\partial}{\partial x^\nu} F^{\mu\nu} = 0$$

which are Maxwell's equation in the absence of sources.

$$A_\mu \to \tilde{A}_\mu = A_\mu + \partial_\mu \Lambda$$

$$\begin{aligned}
\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu &= \partial_\mu(A_\nu - \partial_\nu \Lambda) - \partial_\nu(A_\mu - \partial_\mu \Lambda) \\
&= \partial_\mu A_\nu - \partial_\nu A_\mu - \partial_\mu \partial_\nu \Lambda + \partial_\nu \partial_\mu \Lambda = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}
\end{aligned}$$

Therefore $L_{e.m.}$ is also invariant under the transformation.
From the Euler–Lagrange eq. of motion the second term gives

$$\frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial A_{\mu,\nu}}\right) - \frac{\partial L}{\partial A_\mu} = 0$$

$$\frac{\partial L}{\partial A_\mu} = j^\mu \Rightarrow \frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial A_{\mu,\nu}}\right) = \partial_\nu F^{\mu\nu} = j^\mu$$

$$\partial_\mu \partial_\nu F^{\mu\nu} = 0 \to \partial_\mu j^\mu = 0$$

1b

In the Lorentz gauge we impose $\partial_\mu A^\mu = 0$. The derivative of the mass term $\frac{1}{2}m^2 A_\mu A^\mu$ with respect to A^ν gives $m^2 A^\nu$. Hence

$$(\partial_\mu \partial^\mu A^\nu + m^2 A^\nu) = j^\nu$$

.