Math431 Particle Theory Class Test 15.4.21

1a. $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dx^2 + dy^2$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1b. Want to find functions

$$A(x,y) = \xi^1(x,y)$$
$$B(x,y) = \xi^2(x,y)$$

Such that ds^2 remains invariant under the transformations:

$$x \to x + \varepsilon A(x,y)$$
$$y \to y + \varepsilon B(x,y)$$

Keeping terms to 1st order in ε

$$dx \to \left(1 + \varepsilon \frac{\partial A}{\partial x}\right) dx + \varepsilon \frac{\partial A}{\partial y} dy$$

$$dy \to \left(1 + \varepsilon \frac{\partial B}{\partial y}\right) dy + \varepsilon \frac{\partial B}{\partial x} dx$$

$$dx^2 \to \left(1 + 2\varepsilon \frac{\partial A}{\partial x}\right) dx^2 + 2\varepsilon \frac{\partial A}{\partial y} dx dy$$

$$dy^2 \to \left(1 + 2\varepsilon \frac{\partial B}{\partial y}\right) dy^2 + 2\varepsilon \frac{\partial B}{\partial x} dx dy$$

Demanding ds^2 invariant so

$$dx^2 + dy^2 \to dx^2 + dy^2 + \underbrace{0}_{\text{vanishing terms.}}$$

$$dx^2 : \frac{\partial A}{\partial x} = 0 \Rightarrow A = A(y) = f'(y) = \frac{df}{dy}$$

$$dy^2 : \frac{\partial B}{\partial y} = 0 \Rightarrow B = B(x) = g'(x) = \frac{dg}{dx}$$

$dxdy$: $\frac{\partial A}{\partial y} + \frac{\partial B}{\partial x} = 0 \Rightarrow \frac{\partial A}{\partial y} = -\frac{\partial B}{\partial x}$

Only holds if $\frac{\partial A}{\partial y} = \text{constant} = -\frac{\partial B}{\partial x}$

Since $A = A(y)$; $\frac{\partial A}{\partial y} = \frac{dA}{dy} \left(= f''(y) \right)$

$\frac{dA}{dy} = c$ likewise $\frac{dB}{dx} = -c$

$A = cy + d$ and $B = -cx + e$

where d, e constants of integration.

We are left with three parameters, c, d and e which we denote as α, β and γ respectively;

$$A = \alpha y + \beta$$

$$B = \gamma - \alpha x$$

Lorentz boosts.

2a. $L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^{\mu} A_{\mu}$ $\quad F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$
$\quad F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$

$$L = -\frac{1}{4}\left(\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}\right)\left(\partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}\right) - j^{\mu} A_{\mu}$$

$$L = -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}\left(A_{\beta,\alpha} - A_{\alpha,\beta}\right)\left(A_{\mu,\nu} - A_{\nu,\mu}\right) - j^{\mu} A_{\mu}$$

Euler-Lagrange equations:

$$\frac{\partial}{\partial x^{\nu}}\left(\frac{\partial L}{\partial A_{\mu,\nu}}\right) - \frac{\partial L}{\partial A_{\mu}} = 0$$

$$\frac{\partial L}{\partial A_{\mu}} = -j^{\mu} \Rightarrow \frac{\partial}{\partial x^{\nu}}\left(\frac{\partial L}{\partial A_{\mu,\nu}}\right) = j^{\mu}$$

$$\frac{\partial L}{\partial A_{\rho,q}} = -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}\Big(\left(\delta^{\rho}_{\alpha}\delta^{q}_{\beta} - \delta^{\rho}_{\beta}\delta^{q}_{\alpha}\right)\left(A_{\mu,\nu} - A_{\nu,\mu}\right)$$
$$+ \left(A_{\alpha,\beta} - A_{\beta,\alpha}\right)\left(\delta^{\rho}_{\mu}\delta^{q}_{\nu} - \delta^{\rho}_{\nu}\delta^{q}_{\mu}\right)\Big)$$

$$= -\frac{1}{4}\Big(\left(\eta^{\rho\mu}\eta^{q\nu} - \eta^{q\mu}\eta^{\rho\nu}\right)\left(A_{\mu,\nu} + A_{\nu,\mu}\right)$$
$$+ \left(\eta^{\alpha\rho}\eta^{\beta q} - \eta^{\alpha q}\eta^{\beta\rho}\right)\left(A_{\beta,\alpha} - A_{\alpha,\beta}\right)\Big)$$

$$= -4 \times \frac{1}{4}\left(A^{q,\rho} - A^{\rho,q}\right)$$

$$= -F^{\rho q}$$

$$\Rightarrow \frac{\partial}{\partial x^{\nu}}\left(\frac{\partial L}{\partial A_{\mu,\nu}}\right) = \frac{\partial}{\partial x^{\nu}}\left(-F^{\mu\nu}\right) = j^{\mu}$$

$$\Rightarrow \frac{\partial}{\partial x^{\mu}}\left(-F^{\nu\mu}\right) = j^{\nu}$$

$$\Rightarrow \partial_{\mu} F^{\mu\nu} = j^{\nu}$$

$$A_\mu \to \tilde{A}_\mu = A_\mu + \partial_\mu \Lambda$$

$$F_{\mu\nu} \to \tilde{F}_{\mu\nu}$$

$$\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu = \partial_\mu (A_\nu - \partial_\nu \Lambda) - \partial_\nu (A_\mu - \partial_\mu \Lambda)$$

$$= \partial_\mu A_\nu - \partial_\mu \partial_\nu \Lambda - \partial_\nu A_\mu + \partial_\nu \partial_\mu \Lambda$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$= F_{\mu\nu}$$

Therefore L is also invariant under transformation and j^ν is conserved. Since $\tilde{j}^\nu = \partial_\mu \tilde{F}^{\mu\nu}$.

2b. $L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^{\mu} A_{\mu} + \frac{1}{2} m^2 A_{\mu} A^{\mu}$

$$= -\frac{1}{4} \eta^{\alpha\mu} \eta^{\beta\nu} \left(A_{\beta,\alpha} - A_{\alpha,\beta}\right)\left(A_{\mu,\nu} - A_{\nu,\mu}\right)$$

$$- j^{\mu} A_{\mu} + \frac{1}{2} m^2 A_{\mu} \eta^{\mu\rho} A_{\rho}$$

$$\cancel{\frac{\partial L}{\partial A_{\mu}} = -j^{\mu} + \frac{1}{2} m^2 \eta^{\mu\rho} A_{\rho} + \frac{1}{2} m^2 \eta^{\mu\rho} A_{\mu}}$$

$$\cancel{= j^{\mu} + \frac{1}{2} m^2 \left(A^{\mu} + A^{\rho}\right)}$$

$$L = -\frac{1}{4} \eta^{\alpha\mu} \eta^{\beta\nu} \left(A_{\beta,\alpha} - A_{\alpha,\beta}\right)\left(A_{\mu,\nu} - A_{\nu,\mu}\right)$$

$$+ A_{\mu}\left(\frac{1}{2} m^2 A^{\mu} - j^{\mu}\right)$$

$$\cancel{\partial_\mu}\ \partial_\nu(-F^{\mu\nu}) = \cancel{\partial}\ j^\mu - \frac{1}{2}m^2A^\mu$$

$$\partial_\mu F^{\mu\nu} = j^\nu - \frac{1}{2}m^2A^\nu$$

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = j^\nu - \frac{1}{2}m^2A^\nu$$

$$\partial_\mu\partial^\mu A^\nu - \partial_\mu\partial^\nu A^\mu = j^\nu - \frac{1}{2}m^2A^\nu$$

$$\frac{\partial L}{\partial A_\mu} = -j^\mu + \frac{1}{2}m^2A^\mu + \frac{1}{2}m^2$$

$$\frac{\partial}{\partial x^\nu}(-F^{\mu\nu}) - \frac{\partial}{\partial A_\mu}\left(-j^\mu A_\mu + \frac{1}{2}m^2A_\mu A^\mu\right) = 0$$