MATH 431: Introduction to Modern Particle Theory, Summer 2011

EXAMINER: Dr. T. Teubner, EXTENSION 43791.

TIME ALLOWED: Two and a half hours

In this paper bold-face quantities like $\mathbf{p}$ and quantities with a vector arrow like $\vec{x}$ represent three-dimensional vectors.

Full marks can be obtained for complete answers to FIVE questions. Only the best FIVE answers will be counted.

1. Consider the Poincaré group in 1+2 dimensions, with the infinitesimal line element

$$\mathrm{d}s^2 \;=\; \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2\,.$$

(a) Write down in matrix form the metric for this line element and its inverse. What are the transformations under which this line element is invariant? Give the generator associated with the transformation for (at least) three of the transformations.

[6 marks]

(b) The Pauli-Lubanski vector in four dimensions is given by ($\epsilon^{0123} = +1$)

$$W^\mu \;=\; -\frac{1}{2}\,\epsilon^{\mu\nu\sigma\rho}\,J_{\nu\sigma}\,P_\rho\,,$$

where $K_i = J_{i0}$ and $J_i = \frac{1}{2}\epsilon_{ijk}\,J_{jk}$ are the generators of boosts and rotations, respectively, and P_ρ is the momentum four-vector. Give W^μ in terms of K_i, J_i, P_0, P_i. Write down the four components of the Pauli-Lubanski vector in the case where the line element $\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2$ is viewed as embedded in four space-time dimensions, for both massless and massive particle states.

[14 marks]

2. (a) The Hamiltonian H_0 for a free real scalar field is given by

$$H_0 \;=\; \tfrac{1}{2}\int \left(\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2\right) \mathrm{d}^3x\,.$$

Show that the (Heisenberg) equation of motion for the time evolution of the generalised momentum operator π,

$$i\hbar\dot{\pi} \;=\; [\pi, H_0]\,,$$

together with the canonical commutation relations, given by

$$\begin{aligned}
[\phi(\mathbf{x},t), \pi(\mathbf{x}',t)] &= i\hbar\delta(\mathbf{x}-\mathbf{x}')\,,\\
[\phi(\mathbf{x},t), \phi(\mathbf{x}',t)] &= [\pi(\mathbf{x},t), \pi(\mathbf{x}',t)] \;=\; 0\,,
\end{aligned}$$

imply the Klein-Gordon equation for the field ϕ. (You may use without derivation that for the free scalar field $\pi = \dot{\phi}\,$.)

[10 marks]

(b) Assume a five-dimensional space-time $(t, \mathbf{x}, y)$, where $x = (t, \mathbf{x})$ are the usual four-dimensional space-time coordinates and y is the coordinate of an additional compact extra dimension, $-R/2 \le y \le R/2$.

Consider the free Klein-Gordon equation in this space-time,

$$(\Box_5 + m^2)\phi \;=\; 0\,, \tag{1}$$

where $\Box_5 \equiv \Box - \partial^2/\partial y^2$, with $\Box = \partial_0^2 - \nabla^2$ the usual d'Alembert operator. The general solution of equation (1) is obtained by a Fourier expansion and by imposing appropriate boundary conditions.

Show that

$$\phi(x, y) \;=\; \sum_{n=1}^{\infty} \phi_n(x)\,\mathrm{cs}\left(\frac{n\pi y}{R}\right)\,,$$

where $\mathrm{cs}(n\pi y/R) = \cos(n\pi y/R)$ if n is odd and $\mathrm{cs}(n\pi y/R) = \sin(n\pi y/R)$ for even n, is a solution of equation (1), provided that the Fourier coefficients $\phi_n(x)$ are solutions of a four-dimensional Klein-Gordon equation. Determine the mass spectrum as seen by a four-dimensional observer, and show that for $m = 0$ the masses are equally spaced. What is this infinite set of massive particles called?

[10 marks]

3. (a) Prove the following relations:

(i) $\mathrm{tr}[\gamma_\mu\gamma_\nu] \;=\; 4\eta_{\mu\nu}\,,$

(ii) $\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma \;=\; 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + 2\eta_{\mu\sigma}\gamma_\nu\gamma_\rho - \gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu\,,$

(iii) $\mathrm{tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] \;=\; 4[\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}]\,.$

[9 marks]

(b) Given the four-vector current density $J_V^\mu = \bar{\psi}\gamma^\mu\psi$, derive the current conservation, $\partial_\mu J_V^\mu = 0$, by using the covariant form of the Dirac equation and the relation $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0\,.$ [4 marks]

(c) Using the definition $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, show that

(i) $\{\gamma^5, \gamma^\mu\} = 0\,,$

(ii) a (charge-lowering) weak current of the form

$$\bar{u}_e\gamma^\mu\frac{1}{2}\left(1-\gamma^5\right)u_\nu$$

involves only left-handed electrons (or right-handed positrons).

[7 marks]

4. Weak decays of leptons:

(i) Explain why the decay $\mu^- \to e^-\gamma$ is not allowed in the Standard Model.

[3 marks]

(ii) Draw the (lowest order) Feynman diagram for the muon decay in the Standard Model,

$$\mu^- \to \nu_\mu\, e^-\, \bar{\nu}_e\,.$$

Name and explain the different elements of the Feynman diagram and give their corresponding algebraic expressions. (You can leave out the labels for the momenta and spins of the external particles.) [7 marks]

(iii) Given that the weak coupling g is of the order of the electromagnetic coupling e ($g\sin\theta_W = e$ with the sine of the Weinberg angle $\sin\theta_W \approx 0.5$), why are the weak decays suppressed compared to typical electromagnetic or strong decays? [4 marks]

(iv) In analogy to the μ decay, discuss the possible decays of the τ lepton.

[6 marks]

5. (i) Prove that for the group $SU(2)$ the fundamental representation (the representation by complex 2×2 matrices) is equivalent to the complex conjugate representation. This means that there exists a unitary 2×2 matrix W such that $U^\star = W^\dagger U W$ (unitary equivalence).

Hints: The complex conjugate representation can be written as

$$U^\star \;=\; \exp(-\frac{i}{2}\,\theta_a\,\sigma_a^\star)\,,$$

with the three Pauli-matrices in the usual representation,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

First show that $i\sigma_2$ is unitary and that $\sigma_2\,\sigma_a^\star\,\sigma_2 = -\sigma_a$. [15 marks]

(ii) Explain which consequences this has for the construction of mass terms in the Standard Model. [5 marks]

6. Let the Lagrangian for three interacting real scalar fields ϕ_1, ϕ_2, ϕ_3 be given by

$$L \;=\; \frac{1}{2}\,(\partial_\mu \phi_i)^2 - \frac{1}{2}\,\mu^2 \phi_i^2 - \frac{1}{4}\,\lambda(\phi_i^2)^2\,,$$

with $\mu^2 < 0$ and $\lambda > 0$, and where the summation over i is implied.

(i) Explain the meaning of the different terms and to which elements of the corresponding Feynman rules they lead. Why is μ^2 chosen negative, but λ positive? [6 marks]

(ii) Show that after spontaneous symmetry breaking this Lagrangian describes one massive field with mass $\sqrt{-2\mu^2}$ and two massless 'Goldstone bosons'. [14 marks]

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7. In perturbative QCD the scale (energy) dependence of the 'running' coupling $\alpha_s = g_s^2/(4\pi)$ is described by the differential equation

$$\frac{\mathrm{d}\alpha_s}{\mathrm{d}\ln E} = -b_0\alpha_s^2 - b_1\alpha_s^3 + \mathcal{O}(\alpha_s^4)\,,$$

with b_0, $b_1 > 0$ the first two coefficients of the QCD beta-function.

(i) Neglect the term $\sim \alpha_s^3$ and verify that the solution is given by

$$\alpha_s(E) = \frac{\alpha_s(\mu)}{1 + b_0\alpha_s(\mu)\ln(E/\mu)}\,,$$

where the initial condition is fixed by the coupling $\alpha_s(\mu)$ at a reference scale μ. [4 marks]

(ii) Show that by defining a mass scale

$$\Lambda_{\mathrm{QCD}} = \mu \exp\left[-\frac{1}{b_0\alpha_s(\mu)}\right]$$

the solution reads

$$\alpha_s(E) = \frac{1}{b_0\ln(E/\Lambda_{\mathrm{QCD}})}\,.$$

Experimentally, $\Lambda_{\mathrm{QCD}} \simeq 200$ MeV. Sketch the behaviour of $\alpha_s(E)$ and discuss the physical consequences in the two cases $E \to \Lambda_{\mathrm{QCD}}$ and at asymptotically large energies. [9 marks]

(iii) Derive a solution similar to the one in part (ii), but taking into account the term $-b_1\alpha_s^3$ in the differential equation and chosing a suitable redefinition of Λ_{QCD} (formula for Λ_{QCD} at next order not required). [7 marks]