MATH 431: Introduction to Modern Particle Theory

EXAMINER: Dr. T. Teubner, EXTENSION 43791.

TIME ALLOWED: Two and a half hours

In this paper bold-face quantities like **p** and quantities with a vector arrow like $\vec{x}$ represent three-dimensional vectors.

Full marks can be obtained for complete answers to FIVE questions. Only the best FIVE answers will be counted.

1.

(a) Show that if the Hamiltonian is independent of a generalised coordinate q_0, then the conjugate momentum p_0 is a constant ot the motion. Such coordinates are called *cyclic coordinates*. Briefly describe two examples of physical systems which have a cyclic coordinate. [6 marks]

(b) Show that in three-dimensional spherical polar coordinates the Hamiltonian of a particle of mass m moving in a potential $V(\vec{x})$ is

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + V(\vec{x})\,.$$

Now show that $p_\phi = \mathit{constant}$ if $\partial V/\partial\phi \equiv 0$ and give a physical interpretation of this result. [14 marks]

2. Suppose that we live on a two dimensional surface with a line element on it given by

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\,.$$

(A convenient notation is $\theta^\mu \equiv (\theta, \phi)\,,\ \mu = 1, 2.$)

(a) Write the corresponding metric $g_{\mu\nu}$ in explicit matrix form. Give $g^{\mu\nu}$ in matrix form. [5 marks]

(b) Find the set of infinitesimal transformations of the form

$$\theta^\mu \;\rightarrow\; \theta^\mu + \epsilon\,\zeta^\mu(\theta, \phi)$$

for which the line element $\mathrm{d}s^2$ is invariant.

How many parameters occur in the most general transformation of this kind? [10 marks]

(c) What is the geometric meaning of the transformations in this example? What is the algebra which is satisfied by the operators generating the transformations? [5 marks]

3.

(a) Two-particle states are defined by $|\mathbf{p}_1, \mathbf{p}_2\rangle \; = \; a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)|0\rangle$. Show that

$$\langle \mathbf{p}_1', \mathbf{p}_2' | \mathbf{p}_1, \mathbf{p}_2\rangle \; = \; (2\pi)^6 (2p_1^0)(2p_2^0)\left[\delta(\mathbf{p}_1 - \mathbf{p}_1')\delta(\mathbf{p}_2 - \mathbf{p}_2') + \delta(\mathbf{p}_1 - \mathbf{p}_2')\delta(\mathbf{p}_2 - \mathbf{p}_1')\right] .$$

[6 marks]

(b) Show that the number operator, defined by

$$N \; := \; \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{p}}{2p^0}\, a^\dagger(\mathbf{p})a(\mathbf{p})\,,$$

satisfies

$$[N, a^\dagger(\mathbf{p})] \; = \; a^\dagger(\mathbf{p})\,,$$

and hence

$$N\,|\mathbf{p}_1 \ldots \mathbf{p}_n\rangle \; = \; n\,|\mathbf{p}_1 \ldots \mathbf{p}_n\rangle\,.$$

[14 marks]

4. Let A_μ be the electromagnetic vector potential. The electromagnetic field strength tensor is then defined as

$$F_{\mu\nu} \; = \; \partial_\mu A_\nu - \partial_\nu A_\mu\,.$$

(a) Show that Maxwell's equations (in the absence of sources) in four-vector notation can be derived from the electromagnetic Lagrangian given by

$$L_{\text{e.m.}} \; = \; -\frac{1}{4}\,F_{\alpha\beta}F^{\alpha\beta}\,.$$

[10 marks]

(b) Show that $L_{\text{e.m.}}$ is invariant under the transformation $A_\mu \to A_\mu - \partial_\mu\Lambda$, where Λ is a scalar function. [4 marks]

(c) Show that by imposing a local $U(1)$ symmetry a mass term for the photon is forbidden. [6 marks]

5. The Dirac wave function for the ground state of the hydrogen atom has the form (spin-up state, standard Dirac matrix representation)

$$\psi_\uparrow(r,\theta,\phi) \;=\; R(r)\begin{pmatrix} 1 \\ 0 \\ ia\cos\theta \\ iae^{i\phi}\sin\theta \end{pmatrix},$$

where $a = (1-\sqrt{1-\alpha^2})/\alpha \approx \alpha/2$ (with $\alpha \approx 1/137$ the QED coupling) and R is a function of the radial variable r only.

(a) Investigate whether $\psi_\uparrow$ is an eigenstate of L_z. [4 marks]

(b) Calculate the expectation value of L_z and discuss the result. What would happen in the non-relativistic limit? [8 marks]

(c) Show that $\psi_\uparrow$ is an eigenstate of J_z and find its eigenvalue. [8 marks]

Hint: Don't forget to normalise the state $\psi_\uparrow$.

6. The Lagrangian density for an interacting complex scalar field

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

is

$$\mathcal{L} = (\partial_\mu\phi)^*(\partial^\mu\phi) - \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2\,.$$

(a) What are the transformations under which $\mathcal{L}$ is invariant? [2 marks]

(b) Explain what would happen if $\lambda < 0$. [1 marks]

(c) Briefly discuss the case where $\mu^2 > 0$ and $\lambda > 0$. [2 marks]

(d) Now consider $\mathcal{L}$ with $\mu^2 < 0$ and $\lambda > 0$. Show that it describes a massive field of mass $\sqrt{-2\mu^2}$ and one massless Goldstone boson. [15 marks]

7. In perturbative QCD the scale (energy) dependence of the 'running' coupling $\alpha_s = g_s^2/(4\pi)$ is described by the differential equation

$$\frac{\mathrm{d}\alpha_s}{\mathrm{d}\ln E} = -b_0\alpha_s^2 - b_1\alpha_s^3 + \mathcal{O}(\alpha_s^4)\,,$$

with b_0, $b_1 > 0$ the first two coefficients of the QCD beta-function.

(a) Neglect the term $\sim \alpha_s^3$ and verify that the solution is given by

$$\alpha_s(E) = \frac{\alpha_s(\mu)}{1 + b_0\alpha_s(\mu)\ln(E/\mu)}\,,$$

where the initial condition is fixed by the coupling $\alpha_s(\mu)$ at a reference scale μ. [3 marks]

(b) Show that by defining a mass scale

$$\Lambda_{\mathrm{QCD}} = \mu \exp\left[-\frac{1}{b_0\alpha_s(\mu)}\right]$$

the solution reads

$$\alpha_s(E) = \frac{1}{b_0\ln(E/\Lambda_{\mathrm{QCD}})}\,.$$

Experimentally, $\Lambda_{\mathrm{QCD}} \simeq 200$ MeV. Sketch the behaviour of $\alpha_s(E)$ and briefly discuss what happens at $E \to \Lambda_{\mathrm{QCD}}$ and at asymptotically large energies. [8 marks]

(c) Derive a solution similar to the one in part (b), but taking into account the term $-b_1\alpha_s^3$ in the differential equation and chosing a suitable redefinition of Λ_{QCD} (formula for Λ_{QCD} at next order not required). [9 marks]