MATH 431 — May 2010: Solutions

All problems are similar to homework problems or material covered in the lectures.

1. (a) In the lecture we have derived

$$\dot{p}_0 \;=\; -\frac{\partial H}{\partial q_0}\,.$$

Therefore, p_0 is constant in time if H does not depend on q_0.

One example for such a system is given in part (b) below, where H does not depend on ϕ. A second example would be a one dimensional harmonic oscillator (in x) embedded in two dimensions such that there is no y dependence and therefore the momentum $p_y = \mathit{constant}$.

(b) In polar coordinates

$$\begin{aligned} x &= r\sin\theta\cos\phi\,, \\ y &= r\sin\theta\sin\phi\,, \\ z &= r\cos\theta\,, \end{aligned}$$

the kinetic energy is given by **Bookwork**

$$\frac{1}{2}m\left[\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\dot{\phi})^2\right]\,,$$

and with the potential energy V the Lagrangian is given by

$$L \;=\; \frac{1}{2}m\left[\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\dot{\phi})^2\right] - V\,.$$

Now

$$\begin{aligned} p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r}\,, \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}\,, \\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\dot{\phi}\,. \end{aligned}$$

Hence

$$\begin{aligned} H = \sum_i p_i\dot{q}_i - L &= m\dot{r}^2 + mr^2\dot{\theta}^2 + mr^2\sin^2\theta\dot{\phi}^2 - \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) + V \\ &= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) + V \\ &= \frac{{p_r}^2}{2m} + \frac{{p_\theta}^2}{2mr^2} + \frac{{p_\phi}^2}{2mr^2\sin^2\theta} + V\,. \end{aligned}$$

With this we get

$$\dot{p}_\phi \;=\; -\frac{\partial H}{\partial \phi} \;=\; -\frac{\partial V}{\partial \phi} \;=\; 0\,.$$

An axisymmetric potential does not depend on ϕ, so p_ϕ is a constant of the motion, and the angular momentum about the symmetry axis is conserved.

2. We consider the line element on a two dimensional surface

$$\mathrm{d}s^2 \;=\; \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\,.$$

(a) The corresponding metric is

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}, \quad \text{and its inverse} \qquad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix}.$$

(b) Now we want to determine the infinitesimal transformations

$$\theta^\mu \;\rightarrow\; \theta^\mu + \epsilon\,\zeta^\mu(\theta,\phi)\,.$$

For this we want to find the two functions

$$A(\theta,\phi) \;=\; \zeta^1(\theta,\phi) \quad \text{and} \qquad B(\theta,\phi) \;=\; \zeta^2(\theta,\phi)\,,$$

such that $\mathrm{d}s^2$ remains invariant under the transformations

$$\begin{aligned} \theta \;&\rightarrow\; \theta + \epsilon\,A(\theta,\phi)\,, \\ \phi \;&\rightarrow\; \phi + \epsilon\,B(\theta,\phi)\,, \\ \text{with which} \qquad \sin\theta \;&\rightarrow\; \sin(\theta+\epsilon A) \;\approx\; \sin\theta + \epsilon\,A\,\cos\theta \\ \text{and} \qquad \sin^2\theta \;&\rightarrow\; \sin^2\theta + 2\epsilon A\sin\theta\cos\theta + \mathcal{O}(\epsilon^2)\,. \end{aligned}$$

Keeping terms to first order in ϵ we get the transformations

$$\begin{aligned} \mathrm{d}\theta \;&\rightarrow\; (1+\epsilon\,\frac{\partial A}{\partial\theta})\mathrm{d}\theta + \epsilon\,\frac{\partial A}{\partial\phi}\mathrm{d}\phi\,, \\ \mathrm{d}\phi \;&\rightarrow\; \epsilon\,\frac{\partial B}{\partial\theta}\mathrm{d}\theta + (1+\epsilon\,\frac{\partial B}{\partial\phi})\mathrm{d}\phi\,; \\ \mathrm{d}\theta^2 \;&\rightarrow\; (1+2\epsilon\,\frac{\partial A}{\partial\theta})\mathrm{d}\theta^2 + 2\epsilon\,\frac{\partial A}{\partial\phi}\mathrm{d}\theta\mathrm{d}\phi\,, \\ \mathrm{d}\phi^2 \;&\rightarrow\; (1+2\epsilon\,\frac{\partial B}{\partial\phi})\mathrm{d}\phi^2 + 2\epsilon\,\frac{\partial B}{\partial\theta}\mathrm{d}\theta\mathrm{d}\phi\,, \\ \sin^2\theta\,\mathrm{d}\phi^2 \;&\rightarrow\; \sin^2\theta\,(1+2\epsilon\,\frac{\partial B}{\partial\phi})\,\mathrm{d}\phi^2 + \sin^2\theta\,2\epsilon\,\frac{\partial B}{\partial\theta}\,\mathrm{d}\theta\,\mathrm{d}\phi + 2\epsilon\,A\sin\theta\,\cos\theta\,\mathrm{d}\phi^2\,. \end{aligned}$$

We now demand that $\mathrm{d}s^2$ remains invariant:

$$\mathrm{d}\theta^2 + \sin\theta \mathrm{d}\,\phi^2 \;\to\; \mathrm{d}\theta^2 + \sin\theta\,\mathrm{d}\phi^2 + \underbrace{\cdots\cdots\cdots\cdots\cdots}_{\text{terms must vanish}}.$$

Thus we obtain the following constraints:

$$\begin{aligned}
\mathrm{d}\theta^2 : \frac{\partial A}{\partial \theta} &= 0 \Rightarrow A = A(\phi) = f'(\phi) = \frac{\mathrm{d}f}{\mathrm{d}\phi}\,, \\
\mathrm{d}\phi^2 : \sin^2\theta \frac{\partial B}{\partial \phi} + A \sin\theta\cos\theta = 0 &\Rightarrow \frac{\partial B}{\partial \phi} = -\frac{\mathrm{d}f}{\mathrm{d}\phi}\frac{\cos\theta}{\sin\theta} \Rightarrow B = -f(\phi)\frac{\cos\theta}{\sin\theta} + g(\theta)\,, \\
\mathrm{d}\theta\mathrm{d}\phi : \frac{\partial A}{\partial \phi} + \sin^2\theta\frac{\partial B}{\partial \theta} = 0 &\Rightarrow f'' + \sin^2\theta\left(\frac{f(\phi)}{\sin^2\theta} + g'(\theta)\right) = 0 \\
&\Rightarrow f''(\phi) + f(\phi) = -\sin^2\theta\, g'(\theta) = \mathit{constant} = c\,.
\end{aligned}$$

Solving the two differential equations for f and g we get

$$\begin{aligned}
f'' + f = c \;&\Rightarrow\; f = a\sin\phi + b\cos\phi + c\,, \\
\sin^2\theta\,\frac{\mathrm{d}g}{\mathrm{d}\theta} = -c \;&\Rightarrow\; g = c\frac{\cos\theta}{\sin\theta} + d\,.
\end{aligned}$$

For A and B we thus arrive at

$$\begin{aligned}
A &= f'(\phi) = a\cos\phi - b\sin\phi\,, \\
B &= -\frac{\cos\theta}{\sin\theta}(a\sin\phi + b\cos\phi + c) + c\frac{\cos\theta}{\sin\theta} + d = -\frac{\cos\theta}{\sin\theta}(a\sin\phi + b\cos\phi) + d\,.
\end{aligned}$$

We are left with the three parameters a, b and d, which in the following we denote as α, β and γ, respectively:

$$\begin{aligned}
A &= \alpha\cos\phi + \beta\sin\phi\,, \\
B &= -\frac{\cos\theta}{\sin\theta}(\alpha\sin\phi - \beta\cos\phi) + \gamma\,.
\end{aligned}$$

Hence we are left with three degrees of freedom needed to parametrise the transformation.

(c) The metric $\mathrm{d}s^2$ is the metric on the surface of a sphere. The transformations are generated by the rotations in (θ, ϕ) of a vector fixed at the origin. The generators associated with the three degrees of freedom are the operators for the three components of the angular momentum, J_i. They obey the $SU(2)$ algebra,

$$[J_i, J_j] = i\,\epsilon_{ijk}J_k\,.$$

3. (a) Two-particle states are defined by

$$|\mathbf{p}_1, \mathbf{p}_2\rangle \;=\; a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)|0\rangle\,,$$

with $|\mathbf{p}_1, \mathbf{p}_2\rangle = |\mathbf{p}_2, \mathbf{p}_1\rangle$ as $[a^\dagger(\mathbf{p}_1), a^\dagger(\mathbf{p}_2)] = 0$.

We also know that **Bookwork**

$$[a(\mathbf{p}_1), a^\dagger(\mathbf{p}_2)] = (2\pi)^3 2p_1^0 \delta(\mathbf{p}_1 - \mathbf{p}_2)\,.$$

Hence

$$\begin{aligned}
\langle \mathbf{p}_1', \mathbf{p}_2' | \mathbf{p}_1, \mathbf{p}_2\rangle &= \langle 0|a(\mathbf{p}_1')a(\mathbf{p}_2')a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)|0\rangle \\
&= \langle 0|a(\mathbf{p}_1')\{a^\dagger(\mathbf{p}_1)a(\mathbf{p}_2') + (2\pi)^3 2p_1^0\delta(\mathbf{p}_1 - \mathbf{p}_2')\}a^\dagger(\mathbf{p}_2)|0\rangle \\
&= \langle 0|a(\mathbf{p}_1')a^\dagger(\mathbf{p}_1)\{a^\dagger(\mathbf{p}_2)a(\mathbf{p}_2') + (2\pi)^3 2p_1^0\delta(\mathbf{p}_2 - \mathbf{p}_2')\}|0\rangle \\
&\quad + (2\pi)^3 2p_1^0\delta(\mathbf{p}_1 - \mathbf{p}_2')\langle 0|a(\mathbf{p}_1')a^\dagger(\mathbf{p}_2)|0\rangle \\
&= (2\pi)^3 2p_2^0\delta(\mathbf{p}_2 - \mathbf{p}_2')\langle 0|\{a^\dagger(\mathbf{p}_1)a(\mathbf{p}_1') + (2\pi)^3 2p_1^0\delta(\mathbf{p}_1 - \mathbf{p}_1')\}|0\rangle \\
&\quad + (2\pi)^3 2p_1^0\delta(\mathbf{p}_1 - \mathbf{p}_2')\langle 0|\{a^\dagger(\mathbf{p}_2)a(\mathbf{p}_1') + (2\pi)^3 2p_2^0\delta(\mathbf{p}_2 - \mathbf{p}_1')\}|0\rangle \\
&= (2\pi)^6(2p_1^0)(2p_2^0)\{\delta(\mathbf{p}_1 - \mathbf{p}_1')\delta(\mathbf{p}_2 - \mathbf{p}_2') + \delta(\mathbf{p}_1 - \mathbf{p}_2')\delta(\mathbf{p}_2 - \mathbf{p}_1')\}\,.
\end{aligned}$$

(b) The number operator is

$$N \;:=\; \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{p}'}{2p'^0}\, a^\dagger(\mathbf{p}')a(\mathbf{p}')\,.$$

With this

$$\begin{aligned}
[N, a^\dagger(\mathbf{p})] &= \left[\frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{p}'}{2p'^0}\, a^\dagger(\mathbf{p}')a(\mathbf{p}'), a^\dagger(\mathbf{p})\right] \\
&= \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{p}'}{2p'^0}\, [a^\dagger(\mathbf{p}')a(\mathbf{p}'), a^\dagger(\mathbf{p})] \\
&= \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{p}'}{2p'^0}\, \left\{a^\dagger(\mathbf{p}')[a(\mathbf{p}'), a^\dagger(\mathbf{p})] + [a^\dagger(\mathbf{p}'), a^\dagger(\mathbf{p})]a(\mathbf{p}')\right\} \\
&= \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{p}'}{2p'^0}\, a^\dagger(\mathbf{p}')2p'^0(2\pi)^3\delta(\mathbf{p} - \mathbf{p}') \\
&= a^\dagger(\mathbf{p})\,.
\end{aligned}$$

So we have

$$\begin{aligned}
Na^\dagger(\mathbf{p}) - a^\dagger(\mathbf{p})N &= a^\dagger(\mathbf{p}) \\
\Rightarrow Na^\dagger(\mathbf{p}) &= a^\dagger(\mathbf{p})(N+1)\,, \\
\text{and}\quad N|\mathbf{p}_1 \ldots \mathbf{p}_n\rangle &= Na^\dagger(\mathbf{p}_1)\ldots a^\dagger(\mathbf{p}_n)|0\rangle \\
&= a^\dagger(\mathbf{p}_1)(N+1)a^\dagger(\mathbf{p}_2)\ldots a^\dagger(\mathbf{p}_n)|0\rangle \\
&= a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)(N+2)\ldots a^\dagger(\mathbf{p}_n)|0\rangle \\
&= \;\ldots\; = \; a^\dagger(\mathbf{p}_1)\ldots a^\dagger(\mathbf{p}_n)(N+n)|0\rangle \\
&= n\, a^\dagger(\mathbf{p}_1)\ldots a^\dagger(\mathbf{p}_n)|0\rangle \quad (\text{as } a(\mathbf{p})|0\rangle = 0 \text{ and so } N|0\rangle = 0) \\
&= n|\mathbf{p}_1 \ldots \mathbf{p}_n\rangle\,.
\end{aligned}$$

4. (a) The electromagnetic field strength tensor is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,,$$

and the electromagnetic Lagrangian is

$$\begin{aligned} L_{\text{e.m.}} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\ &= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\ &= -\frac{1}{4}\eta^{\mu\alpha}\eta^{\nu\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial_\mu A_\nu - \partial_\nu A_\mu)\,. \end{aligned}$$

The Euler-Lagrange equations of motion for this Lagrangian are obtained from

$$\frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial(\partial_\nu A_\mu)}\right) - \frac{\partial L}{\partial A_\mu} = 0\,.$$

Now

$$\frac{\partial L}{\partial A_\mu} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial(\partial_\nu A_\mu)}\right) = 0\,.$$

Hence calculate

$$\begin{aligned} \frac{\partial L}{\partial(\partial_\delta A_\epsilon)} &= -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}\left[(\delta^\delta_\alpha\delta^\epsilon_\beta - \delta^\delta_\beta\delta^\epsilon_\alpha)(\partial_\mu A_\nu - \partial_\nu A_\mu)\right. \\ &\qquad\qquad \left. + (\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\delta^\delta_\mu\delta^\epsilon_\nu - \delta^\delta_\nu\delta^\epsilon_\mu)\right] \\ &= -\frac{1}{4}\left[(\eta^{\delta\mu}\eta^{\epsilon\nu} - \eta^{\epsilon\mu}\eta^{\delta\nu})(\partial_\mu A_\nu - \partial_\nu A_\mu)\right. \\ &\qquad \left. + (\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\eta^{\alpha\delta}\eta^{\beta\epsilon} - \eta^{\alpha\epsilon}\eta^{\beta\delta})\right] \\ &= -\frac{1}{4}\,4\left[\partial^\delta A^\epsilon - \partial^\epsilon A^\delta\right] \\ &= -F^{\delta\epsilon}\,. \end{aligned}$$

So

$$\frac{\partial}{\partial x^\delta}\left(\frac{\partial L}{\partial(\partial_\delta A_\epsilon)}\right) = -\frac{\partial}{\partial x^\delta}F^{\delta\epsilon} = 0\,,$$

which are the Maxwell equations in the absence of sources.

(b)

$$A_\mu \rightarrow \tilde{A}_\mu = A_\mu + \partial_\mu\Lambda\,.$$

Now

$$\begin{aligned} \tilde{F}_{\mu\nu} = \partial_\mu\tilde{A}_\nu - \partial_\nu\tilde{A}_\mu &= \partial_\mu(A_\nu + \partial_\nu\Lambda) - \partial_\nu(A_\mu + \partial_\mu\Lambda) \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\mu\partial_\nu\Lambda - \partial_\nu\partial_\mu\Lambda \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu} \end{aligned}$$

as the derivatives of the scalar function Λ commute. With $F_{\mu\nu}$ also $L_{\mathrm{e.m.}}$ is invariant under the transformation.

(c) The Lagrangian for a massive photon without sources is given by

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + m^2 A_\mu A^\mu \,.$$

As before the first term is invariant under the transformation considered above. However, the second term transforms as

$$m^2 A_\mu A^\mu \;\rightarrow\; m^2 \tilde{A}_\mu \tilde{A}^\mu \;=\; m^2 A_\mu A^\mu + m^2(\partial_\mu\Lambda\partial^\mu\Lambda + \partial_\mu\Lambda A^\mu + A_\mu\partial^\mu\Lambda) \,.$$

The extra terms do not vanish, so the mass term would not be invariant under the transformation, i.e. *break the invariance.* Insisting on the invariance therefore forbids the mass term, i.e. $m^2 = 0$ and the photon has to remain massless.

5. The Dirac spinor for the hydrogen ground state (with spin up) is given by

$$\psi(r,\theta,\phi) \;=\; R(r)\begin{pmatrix} 1 \\ 0 \\ ia\cos\theta \\ iae^{i\phi}\sin\theta \end{pmatrix} .$$

For the normalisation we calculate:

$$1 \;=\; \int \mathrm{d}^3\mathbf{r}\,\psi^\dagger\psi \;=\; \int 4\pi r^2 \mathrm{d}r\,|R|^2(1+a^2)$$

$$\Rightarrow \quad N^2 \;:=\; \int_0^\infty r^2|R|^2\mathrm{d}r \;=\; [4\pi(1+a^2)]^{-1} \,.$$

(a) ($\hbar = 1$ here and in part (b))

$$L_z \;=\; -i\frac{\partial}{\partial\phi} \quad\Rightarrow\quad L_z\psi \;=\; R\begin{pmatrix} 0 \\ 0 \\ 0 \\ iae^{i\phi}\sin\theta \end{pmatrix} \not\propto \psi \,,$$

so ψ is NOT an eigenstate of L_z.

(b)

$$\langle L_z\rangle \;=\; \int \mathrm{d}^3\mathbf{r}\,\psi^\dagger L_z\psi \;=\; 2\pi\int r^2\mathrm{d}r\,\mathrm{d}\cos\theta\,|R|^2 a^2\sin^2\theta \,.$$

Now

$$\int_{-1}^{1}\mathrm{d}\cos\theta(1-\cos^2\theta) \;=\; 2-\frac{2}{3} \;=\; \frac{4}{3} \quad\Rightarrow\quad \langle L_z\rangle \;=\; \frac{8\pi}{3}a^2\,\frac{1}{4\pi(1+a^2)}$$

$$\Rightarrow \quad \langle L_z \rangle \;=\; \frac{2a^2}{3(1+a^2)}\,.$$

In the H-atom, $v/c \sim \alpha \;\Rightarrow\; \langle L_z \rangle = O(v^2/c^2)$ is a relativistic effect which is due to the spin-orbit interaction. In the non-relativistic limit $a \longrightarrow 0$, and the four-component Dirac spinors for the spin-up and spin-down states reduce to (decoupled) solutions of the Schroedinger equation multiplied by the two-component Pauli-spinors.

(c) **Bookwork**

$$S_z \;=\; \frac{1}{2}\hbar\,\Sigma_z \;=\; \frac{1}{2}\hbar \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\Rightarrow \quad S_z\psi \;=\; \frac{1}{2}\hbar\, R \begin{pmatrix} 1 \\ 0 \\ ia\cos\theta \\ -iae^{i\phi}\sin\theta \end{pmatrix},$$

$$\text{hence} \quad J_z\psi \;=\; (L_z+S_z)\psi \;=\; \frac{1}{2}\hbar\, R \begin{pmatrix} 1 \\ 0 \\ ia\cos\theta \\ iae^{i\phi}\sin\theta \end{pmatrix} \;=\; \frac{1}{2}\hbar\,\psi \quad \Rightarrow \quad J_z \;=\; +\frac{\hbar}{2}\,.$$

6. (a) The Lagrangian density is invariant under global transformations of the phase of the field ϕ,

$$\phi \longrightarrow e^{i\alpha}\phi\,.$$

(b) If the coefficient of the highest power, i.e. λ, is negative, then there is no stable vacuum as the system could gain infinite potential energy by choosing large field values. Therefore the case $\lambda < 0$ is not physical. In contrast, the Lagrangian with $\mu^2 > 0$ and $\lambda > 0$ descibes a charged, interacting scalar field with mass $m_\phi = \mu$.

(c) The choice $\mu^2 < 0$ and $\lambda > 0$ leads to spontaneous symmetry breaking (SSB). To see this, we first write the Lagrangian in terms of two real components of ϕ, ϕ_1 and ϕ_2:

$$\mathcal{L} = \frac{1}{2}\left((\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2\right) - \frac{1}{2}\mu^2(\phi_1^2+\phi_2^2) - \frac{1}{4}\lambda(\phi_1^2+\phi_2^2)^2\,.$$

The potential is then given by

$$V(\phi_1,\phi_2) = \frac{1}{2}\mu^2(\phi_1^2+\phi_2^2) + \frac{1}{4}\lambda(\phi_1^2+\phi_2^2)^2\,,$$

with the minima determined by

$$\frac{\partial V}{\partial \phi_i} = \mu^2\phi_i + \lambda(\phi_1^2+\phi_2^2)\phi_i = \phi_i\left(\mu^2 + \lambda(\phi_1^2+\phi_2^2)\right) = 0$$

$$\Rightarrow \mu^2 + \lambda(\phi_1^2 + \phi_2^2) = 0\,.$$

We can choose

$$\langle\phi_1\rangle = \sqrt{\frac{-\mu^2}{\lambda}} =: v\,, \qquad \langle\phi_2\rangle = 0$$

as our new vacuum after SSB. Expanding the field around the new vacuum we can write

$$\phi(x) = v + \eta(x) + i\zeta(x)$$

and finally obtain for the Lagrangian $\mathcal{L}'$ after SSB

$$\mathcal{L}' = \frac{1}{2}\left((\partial_\mu\eta)^2 + (\partial_\mu\zeta)^2\right) + \mu^2\eta^2 + \text{constant} + (\text{cubic \& quartic terms in } \eta \,\&\, \zeta)\,.$$

The term $\mu^2\eta^2$ is a mass term for the field η with $m_\eta = \sqrt{-2\mu^2}$. There is no corresponding mass term for the field ζ which is a massless scalar field and hence a Goldstone Boson.

7. (a) Setting $u = 1/\alpha_s$ the differential equation becomes

$$\frac{\mathrm{d}u}{\mathrm{d}\ln E} = b_0 + \frac{b_1}{u}\,.$$

Truncating at lowest order, the solution is given by $u(E) = u(\mu) + b_0 \ln(E/\mu)$, with a free integration constant $u(\mu)$. After sustituting back $u \to 1/\alpha_s$ this is the proposed solution, with the 'initial condition' $\alpha_s(\mu)$.

(b) The proposed definition is equivalent to the substitution

$$\mu = \Lambda_{\mathrm{QCD}} \exp\left[\frac{1}{b_0\alpha_s(\mu)}\right]$$

with which we obtain the form

$$\alpha_s(E) = \frac{1}{b_0 \ln(E/\Lambda_{\mathrm{QCD}})}\,.$$

At energies $E \to \Lambda_{\mathrm{QCD}}$ the couplings develops a pole, the so-called Landau-pole. For large values of the coupling the perturbative series breaks down, and the Landau pole indicates the non-perturbative region of QCD. At large energies, the coupling becomes small. This so-called 'asymptotic freedom' of QCD as an $SU(3)$ gauge field theory allows us to perform perturbative calculations on the parton level which in turn reliably describe QCD processes at large momentum transfer. The sketch should show a monotonically decreasing positive function, indicating the Landau pole at $E = \Lambda_{\mathrm{QCD}}$ and a small value of α_s at large energies.

(c) The solution valid at the next order of perturbation theory is obtained by inserting our lowest-order solution into the differential equation including the next order term $\sim b_1$, leading to

$$\frac{\mathrm{d}u}{\mathrm{d}\ln E} = b_0 + \frac{b_1}{b_0 \ln(E/\Lambda_{\mathrm{QCD}})} \, .$$

The solution of this differential equation is given by

$$u(E) = \frac{1}{\alpha_s(E)} = b_0 \ln(E/\Lambda_{\mathrm{QCD}}) + (b_1/b_0) \ln\ln(E/\Lambda_{\mathrm{QCD}}) \, ,$$

with a suitable redefinition of Λ_{QCD} to next order so that the integration constant vanishes (formula not requested).