MATH431 - Modern Particle Physics
Set Work: Sheet 5;

1. (i) The Lagrangian density for a real scalar field is given by

$$\mathcal{L}_0 = \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}m^2\phi^2$$

Show that the Hamiltonian H_0 is given by

$$H_0 = \tfrac{1}{2}\int [\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2] d^3x.$$

(ii) Show that if we take the usual canonical commutation relations,

$$\begin{aligned}
[\phi(\mathbf{x},t), \pi(\mathbf{x}',t)] =& i\hbar\delta(\mathbf{x}-\mathbf{x}'),\\
[\phi(\mathbf{x},t), \phi(\mathbf{x}',t)] =& 0,\\
[\pi(\mathbf{x},t), \pi(\mathbf{x}',t)] =& 0,
\end{aligned}$$

the equations of motion are obtained from

$$i\hbar\dot{\phi} = [\phi, H_0] \quad \text{and} \quad i\hbar\dot{\pi} = [\pi, H_0] \ .$$

(iii) Consider the Lagrangian density

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I,$$

with

$$\mathcal{L}_I = -\frac{\lambda_3}{3!}\phi^3 - \frac{\lambda_4}{4!}\phi^4 \ .$$

Derive the equation of motion from

$$\partial^\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\right] - \frac{\partial\mathcal{L}}{\partial\phi} = 0.$$

2. Two-particle states are defined by

$$|\mathbf{p}_1, \mathbf{p}_2 >= a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_1)|0> .$$

Show that

$$\begin{aligned}
&< \mathbf{p}_1', \mathbf{p}_2'|\mathbf{p}_1, \mathbf{p}_2 >\\
=&(2\pi)^6(2p_1^0)(2p_2^0)\{\delta(\mathbf{p}_1-\mathbf{p}_1')\delta(\mathbf{p}_2-\mathbf{p}_2') + \delta(\mathbf{p}_1-\mathbf{p}_2')\delta(\mathbf{p}_2-\mathbf{p}_1')\}.
\end{aligned}$$

3. (a) Let A_μ be the electromagnetic vector potential. The electromagnetic field strength tensor is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Show that Maxwell's equation in four vector notation can be derived from the electromagnetic Lagrangian given by

$$L_{e.m.} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}.$$

Show that $L_{e.m.}$ is invariant under the transformation $A_\mu \to A_\mu - \partial_\mu \Lambda$, where Λ is a scalar function.

(b) Show that imposing a local $U(1)$ symmetry forbids the photon from attaining a mass.