MATH431 — Introduction to Modern Particle Theory: FEEDBACK/SOLUTIONS 7

1. See copy of separate handwritten sheet.

2.

$$UU^\dagger \;=\; 1\!\!\mathrm{I} \quad\Rightarrow\quad U \;=\; \mathrm{e}^{iH}\,; \quad U^\dagger \;=\; \mathrm{e}^{-iH^\dagger} \;=\; \mathrm{e}^{-iH}\,,$$

hence we must have $H = H^\dagger$ hermitian.

3. Unitary equivalence and the '$SU(2)$ miracle':

(a) *First step:*

$$i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,,$$

so $i\sigma_2$ is unitary.

Second step: Prove by explicit matrix multiplication that $\sigma_2\,\sigma_a^\star\,\sigma_2 = -\sigma_a$ $(i = 1, 2, 3)$. For $i = 1$ we have e.g.

$$\sigma_2\,\sigma_1^\star\,\sigma_2 \;=\; \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \;=\; \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \;=\; -\sigma_1\,.$$

Third step: Choose $W = i\sigma_2$, and with $\sigma_2^\star\sigma_2^\star = 1\!\!\mathrm{I} = \sigma_2\sigma_2$ and the result of step two we write:

$$\begin{aligned}
W^\dagger U^\star W \;&=\; -\,i\sigma_2^\dagger U^\star i\sigma_2 \;=\; \sigma_2 \exp\left(-\tfrac{i}{2}\theta_a\sigma_a^\star\right)\sigma_2 \\
&= \sigma_2\left(1\!\!\mathrm{I} - \frac{i}{2}\theta_a\sigma_a^\star - \frac{1}{4}\frac{1}{2!}\theta_a\theta_a - \frac{i}{8}\frac{1}{3!}\theta_a\theta_a\theta_b\sigma_b^\star - \ldots\right)\sigma_2 \\
&= \left(1\!\!\mathrm{I} + \frac{i}{2}\theta_a\sigma_a - \frac{1}{4}\frac{1}{2!}\theta_a\theta_a + \frac{i}{8}\frac{1}{3!}\theta_a\theta_a\theta_b\sigma_b - \ldots\right) \;=\; U\,.
\end{aligned}$$

Taking the complex conjugate of

$$W^\dagger U^\star W \;=\; U$$

we now also have

$$W^{\dagger\star} U W^\star \;=\; U^\star$$

and as $W = i\sigma_2 = W^\star$ we arrive at the desired relation

$$U^\star \;=\; W^\dagger\, U\, W\,.$$

(b) *Will be discussed later.*

In the Standard Model, fermion and gauge boson masses are obtained in a gauge invariant way through electroweak symmetry breaking which is mediated by a Higgs potential. Because of the unitary equivalence between the fundamental and the complex conjugate representations of $SU(2)$, gauge invariant mass terms for both up- and down quarks (which are grouped together in $SU(2)$ doublets) can be constructed from only one complex Higgs doublet. In other words, it is due to this special property of $SU(2)$ that the Higgs sector in the Standard Model is the minimal one resulting in only one physical Higgs boson.