from the previous lecture ...

an infinitesimal proper orthochronos LT takes the form

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}$$

expanding the defining identity of LT $\quad \Rightarrow$

$$\omega_{\mu\nu} = -\omega_{\nu\mu}$$

$\rightarrow$ The tensor of infinitesimal transformations is antisymmetric.

rotations: rotation group in two dimensions

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{for small } \theta \tag{1}$$

The number of degrees of freedom in a 4x4 antisymmetric matrix?

for general n: $\frac{n^2-n}{2} = \frac{n(n-1)}{2}$

$$\text{for } n = 4 \rightarrow \frac{4\cdot 3}{2} = 6 \rightarrow 3 \text{ rotation angles} + 3 \text{ boosts}$$

$$\Rightarrow \omega_{\mu\nu} = \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & b_3 & -b_2 \\ -a_2 & -b_3 & 0 & b_1 \\ -a_3 & b_2 & -b_1 & 0 \end{pmatrix} \tag{2}$$

Algebraic properties of the Lorentz group

Associate an operator $U(\Lambda)$ with the Lorentz transformation Λ.

For the special case $\Lambda = \delta \;\rightarrow\; U(\delta) = I$, the identity operator.

Find an operator associated with $\Lambda = \delta + \omega$. To order $O(\omega)$

$$U(\delta + \omega) = I + \frac{1}{2} i J_{\mu\nu} \omega^{\mu\nu} + \cdots$$

where $J_{\mu\nu}$ are a set of operators and the $\omega^{\mu\nu}$ are the infinitesimal parameters.

to see the relevant structure we can look at the example of rotations

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & \alpha & -\beta \\ -\alpha & 1 & \gamma \\ \beta & -\gamma & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

we can write the matrix in the form

$$A = I + \alpha \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

the 3x3 matrices are the operators, analogs of the operators $J^{\mu\nu}$ and $\alpha, \beta, \gamma \in \omega^{\mu\nu}$. The operators should be be hermitian $\rightarrow$ multiply by i,

$$A = I + i\alpha \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + i\beta \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} + i\gamma \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$\Rightarrow$ the operators are hemitian.

To extract the algebraic properties of the Lorentz group, we will first discuss the problem in general and then specialise to the generators of the Lorentz group. Our observations here will then be valid in different cases as well.
In quantum mechanics, which underlies quantum field theory, and hence modern particle physics, operators are unitary,

$$U_{QM}^{-1} = U_{QM}^{\dagger} \quad \Rightarrow \quad U_{QM}^{\dagger} U_{QM} = I$$

The reason that operators have to be unitary is that for a pure state probability should be preserved. Under a unitary transformation we have

$$|\psi\rangle \to |\psi'\rangle = U|\psi\rangle$$

$$P = \langle\psi'||\psi'\rangle = \langle\psi|U^{\dagger}U|\psi\rangle = \langle\psi|\psi\rangle.$$

Hence, the probability is preserved if U is unitary. A convenient way to write a unitary opetor is using exponentiation

$$
\begin{aligned}
& U = \mathrm{e}^{iO} \to U^\dagger = \mathrm{e}^{-iO^\dagger} \\
\Rightarrow \qquad & U^\dagger U = \mathrm{e}^{-iO^\dagger}\mathrm{e}^{iO} = I
\end{aligned}
$$

We see that provided that the operator O is hermitian, exponentiation is a good way to represent unitary operators. A property of exponents is

$$
\mathrm{e}^a \mathrm{e}^b = \mathrm{e}^{a+b}
$$

provided that a and b commute *i.e.* $ab = ba$. What happens, however, if a and b do not commute? Write

$$
\mathrm{e}^A \mathrm{e}^B \stackrel{?}{=} \mathrm{e}^{A+B} \tag{3}
$$

and A and B are two operators that do not commute *i.e.* $[A, B] \neq 0$. To test the identity in (3) we can expand the two sides and compare at equal orders in the expansion

$$(I + A + \frac{A^2}{2!})(I + B + \frac{B^2}{2!}) \stackrel{?}{=} (I + A + B + \frac{(A+B)^2}{2!})$$
$$I + A + B + AB + \frac{A^2}{2!} + \frac{B^2}{2!} \stackrel{?}{=} I + A + B + \frac{A^2}{2!} + \frac{AB}{2!} + \frac{BA}{2!} + \frac{B^2}{2!}$$

We see that the two sides are not equal! To remedy the inequality we can fix the left hand side

$$\begin{aligned} & I + A + B + \frac{A^2}{2!} + \frac{AB}{2!} + \frac{BA}{2!} + \frac{B^2}{2!} + \frac{AB}{2!} - \frac{BA}{2!} \\ = \quad & I + (A+B) + \frac{(A+B)^2}{2!} + \frac{[A,B]}{2!} \end{aligned}$$

Hence, to second order we derived the identity

$$\mathrm{e}^A \mathrm{e}^B = \mathrm{e}^{A+B+\frac{1}{2}[A,B]} \tag{4}$$

Now, suppose we have the exponential representation of a unitary group

$$\mathrm{e}^{i\alpha_a X_a},$$

where X_a are hermitian generators and form a vector space; α_a are infinisetimal numbers; and summation over repeated indecis is implied. In general, as we saw,

$$\mathrm{e}^{i\alpha_a X_A}\mathrm{e}^{i\beta_b X_b} \neq \mathrm{e}^{i(\alpha_a X_a+\beta_b X_b)}$$

but as the elements

$$\mathrm{e}^{i\gamma_a X_a}$$

form a group, we must have

$$\mathrm{e}^{i\alpha_a X_A}\mathrm{e}^{i\beta_b X_b} = \mathrm{e}^{i\delta_a X_a}$$

for some δ_a, where summation over repeated indices is implied.

Digression: properties of a group

Groups will crop up throughout the lectures. It is useful to recall some of their properties. Assume group G under some product $\otimes$:

1. if $g_1,\ g_2 \in G$ then $g_1 \otimes g_2 = g_3 \in G$
2. there exist an identity element $e \in G$ and $eg = ge = g$ for all $g \in G$.
3. for each $g \in G$ there exist an inverse $g^{-1} \in G$ and $gg^{-1} = e$

Examples:

[1] $\quad G = \text{integers}; \quad \otimes = +$

if $n, m \in G \ \Rightarrow \ n + m \in G;\ n + 0 = 0 + n = n;\ n + (-n) = 0 \in G$

$\Rightarrow$ the group criteria are satisfied but G = integers is not a group under $\otimes = \times$ as there is no inverse

i.e. if $n,\ m \in G \Rightarrow n \times 1 = 1 \times n = n$

but $n \times \dfrac{1}{n} = 1$ and $\dfrac{1}{n} \notin G$

Hence, the group criteria are not satisfied.

[2] $G =$ rational numbers form a group under; $\otimes = \times$

if $X, Y \in G \Rightarrow X \times Y = Z \in G$;

$X \times 1 = 1 \times X = X; \rightarrow$ 1 is the identity

$$X = \frac{n}{m} \Rightarrow X^{-} = \frac{m}{n} \Rightarrow X \times X^{-1} = \frac{m}{n}\frac{n}{m} = 1 \rightarrow X^{-1} \in G$$

$\Rightarrow$ the group criteria are satisfied