Mia Riley.

Particle Physics - Class test.

1a) $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dx^2 + dy^2$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = g_{\mu\nu}$$

b) Find the set of indep. transformations of the form
$x \to x + \epsilon A(x,y)$
$y \to y + \epsilon B(x,y)$ that leave ds^2 invariant.
State what each transformation represents in space time.

$$dx \to dx + \epsilon \frac{\partial A}{\partial x} dx + \epsilon \frac{\partial A}{\partial y} dy = \left(1 + \epsilon \frac{\partial A}{\partial x}\right) dx + \epsilon \frac{\partial A}{\partial y} dy$$

$$dy \to dy + \epsilon \frac{\partial B}{\partial y} dy + \epsilon \frac{\partial B}{\partial x} dx = \epsilon \frac{\partial B}{\partial x} dx + \left(1 + \epsilon \frac{\partial B}{\partial y}\right) dy$$

$$dx^2 \to \left[\left(1 + \epsilon \frac{\partial A}{\partial x}\right) dx + \epsilon \frac{\partial A}{\partial y} dy\right]^2$$

$$= dx^2 + 2\epsilon \frac{\partial A}{\partial x} dx^2 + 2\epsilon \frac{\partial A}{\partial y} dx dy$$

(neglecting terms of order ϵ^2)

$$dy^2 \to dy^2 + 2\epsilon \frac{\partial B}{\partial y} dy^2 + 2\epsilon \frac{\partial B}{\partial x} dy dx$$

Demand that $ds^2 = dx^2 + dy^2$ remains invariant.

$$dx^2 + dy^2 \to dx^2 + dy^2 + 2\epsilon \frac{\partial A}{\partial x} dx^2 + 2\epsilon \frac{\partial B}{\partial y} dy^2$$

$$+ 2\epsilon \left(\frac{\partial A}{\partial y} + \frac{\partial B}{\partial x}\right) dx dy.$$

we get $\frac{\partial A}{\partial x} = 0 \Rightarrow A(y)$

$\frac{\partial B}{\partial y} = 0 \Rightarrow B(x)$

$\frac{\partial A}{\partial y} + \frac{\partial B}{\partial x} = 0 \Rightarrow A' + B' = 0$

$\Rightarrow A' = -B' = \text{constant}, c.$

Solve the 2 different differential equations:

① $A' = c \Rightarrow A(y) = cy + d$

② $-B' = c \Rightarrow -B(x) = cx + e \Rightarrow B(x) = -cx - e$

We are left with 3 parameters, c, d, e, so we are left with 3 degrees of freedom.

$A = cy + d.$

$B = -cx - e.$

The three d.o.f are represented by the 3 constants of the motion: c, d, e.

If we choose $d = e = 0$ then $x \to x + \epsilon c y$
$y \to y - \epsilon c x$

we have a boost (rotation in 2D)

If we choose $c = d = 0$ then $x \to x$
$y \to y - \epsilon e$

if we choose $c = e = 0$ then $x \to x + \epsilon d$
$y \to y$

these are both translations in space, if x, y are spatial coordinates.

If x were to represent ~~a~~ time, then choosing $c = e = 0$ represents a translation in time, since $x \to x + \epsilon d$.

2a) $L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu$

$$\frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial A_{\mu,\nu}}\right) - \frac{\partial L}{\partial A_\mu} = 0$$

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu$$

$$= -\frac{1}{4}(A_{\mu,\nu} - A_{\nu,\mu})(A^{\mu,\nu} - A^{\nu,\mu}) - j^\mu A_\mu$$

$$= -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(A_{\beta,\alpha} - A_{\alpha,\beta})(A_{\mu,\nu} - A_{\nu,\mu}) - j^\mu A_\mu$$

$$\frac{\partial L}{\partial A_\mu} = -j^\mu$$

$$\frac{\partial L}{\partial A_{\mu,\nu}} = -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}\big[(\delta^\mu{}_\alpha \delta^\nu{}_\beta - \delta^\mu{}_\beta \delta^\nu{}_\alpha)(A_{\mu,\nu} - A_{\nu,\mu})$$
$$+ (A_{\alpha,\beta} - A_{\beta,\alpha})(\delta^\mu{}_\mu \delta^\nu{}_\nu - \delta^\mu{}_\nu \delta^\nu{}_\mu)\big]$$

$$= -\frac{1}{4}\big[(\eta^{\mu\mu}\eta^{\nu\nu} - \eta^{\nu\mu}\eta^{\mu\nu})(A_{\mu,\nu} + A_{\nu,\mu})$$
$$+ (\eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\alpha\nu}\eta^{\beta\mu})(A_{\beta,\alpha} - A_{\alpha,\beta})\big]$$

$$= -4\,\frac{1}{4}(A^{\nu,\mu} - A^{\mu,\nu}) = F^{\mu\nu}$$

$$\Rightarrow \frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial A_{\mu,\nu}}\right) - \frac{\partial L}{\partial A_\mu} = \frac{\partial}{\partial x^\nu} F^{\mu\nu} + j^\mu = \partial_\nu F^{\mu\nu} \cancel{+}\, j^\mu = 0$$

$$\Rightarrow \partial_\nu F^{\mu\nu} + j^\mu = 0 \Rightarrow -\partial_\nu F^{\mu\nu} = j^\mu$$
$$\Rightarrow \partial_\nu F^{\nu\mu} = j^\mu$$

switching indices $\Rightarrow$ $\partial_\mu F^{\mu\nu} = j^\nu$ as required.

Show j^ν is conserved.

$\partial_\nu j^\nu = \partial_\nu \partial_\mu F^{\mu\nu} = -\partial_\nu \partial_\mu F^{\nu\mu} = -\partial_\mu \partial_\nu F^{\nu\mu} = -\partial_\nu \partial_\mu F^{\mu\nu}$

by using $F^{\mu\nu} = -F^{\nu\mu}$, using that partial derivatives commute + then relabelling indices.

$\Rightarrow \partial_\nu \partial_\mu F^{\mu\nu} = 0 \Rightarrow \partial_\nu j^\nu = 0$ ie: j^ν is conserved.

b) $L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^{\mu} A_{\mu} + \frac{1}{2} m^2 A_{\mu} A^{\mu}$

$\frac{1}{2} m^2 A_{\mu} A^{\mu} = \frac{1}{2} m^2 \eta_{\mu\mu} A^{\mu} A^{\mu} = \frac{1}{2} m^2 \eta_{\mu\mu} (A^{\mu})^2$

$\frac{\partial L}{\partial A_{\mu}} = -j^{\mu} + \frac{2}{2} m^2 A^{\mu} = -j^{\mu} + m^2 A^{\mu}$

$\frac{\partial L}{\partial A_{\mu,\nu}} = F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$

$\Rightarrow \quad \partial_{\nu}(\partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}) - (-j^{\mu} + m^2 A^{\mu}) = 0$

$\partial_{\nu} \partial^{\mu} A^{\nu} - \partial_{\nu} \partial^{\nu} A^{\mu} + j^{\mu} - m^2 A^{\mu} = 0$

$\Rightarrow \quad j^{\mu} = \partial_{\nu} \partial^{\nu} A^{\mu} + m^2 A^{\mu}$

$= (\partial_{\nu} \partial^{\nu} + m^2) A^{\mu}$

$\Rightarrow \quad j^{\nu} = (\partial_{\mu} \partial^{\mu} + m^2) A^{\mu}$