MATH431 — Introduction to Modern Particle Theory: FEEDBACK/SOLUTIONS 5

1. (a) The electromagnetic field strength tensor is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu ,$$

and the electromagnetic Lagrangian is

$$\begin{aligned} L_{\text{e.m.}} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\ &= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\ &= -\frac{1}{4}\eta^{\mu\alpha}\eta^{\nu\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial_\mu A_\nu - \partial_\nu A_\mu) . \end{aligned}$$

The Euler-Lagrange equations of motion for this Lagrangian are obtained from

$$\frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial(\partial_\nu A_\mu)}\right) - \frac{\partial L}{\partial A_\mu} = 0 .$$

Now

$$\frac{\partial L}{\partial A_\mu} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial(\partial_\nu A_\mu)}\right) = 0 .$$

Hence calculate

$$\begin{aligned} \frac{\partial L}{\partial(\partial_\delta A_\epsilon)} &= -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}\left[(\delta^\delta_\alpha \delta^\epsilon_\beta - \delta^\delta_\beta \delta^\epsilon_\alpha)(\partial_\mu A_\nu - \partial_\nu A_\mu)\right. \\ &\qquad\qquad \left. + (\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\delta^\delta_\mu \delta^\epsilon_\nu - \delta^\delta_\nu \delta^\epsilon_\mu)\right] \\ &= -\frac{1}{4}\left[(\eta^{\delta\mu}\eta^{\epsilon\nu} - \eta^{\epsilon\mu}\eta^{\delta\nu})(\partial_\mu A_\nu - \partial_\nu A_\mu)\right. \\ &\qquad \left. + (\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\eta^{\alpha\delta}\eta^{\beta\epsilon} - \eta^{\alpha\epsilon}\eta^{\beta\delta})\right] \\ &= -\frac{1}{4}\, 4\left[\partial^\delta A^\epsilon - \partial^\epsilon A^\delta\right] \\ &= -F^{\delta\epsilon} . \end{aligned}$$

So

$$\frac{\partial}{\partial x^\delta}\left(\frac{\partial L}{\partial(\partial_\delta A_\epsilon)}\right) = -\frac{\partial}{\partial x^\delta} F^{\delta\epsilon} = 0 ,$$

which are the Maxwell equations in the absence of sources.

(b)

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x) .$$

Now

$$
\begin{aligned}
F'_{\mu\nu} \;=\; \partial_\mu A'_\nu - \partial_\nu A'_\mu \;&=\; \partial_\mu(A_\nu + \partial_\nu \Lambda) - \partial_\nu(A_\mu + \partial_\mu \Lambda) \\
&=\; \partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\mu \partial_\nu \Lambda - \partial_\nu \partial_\mu \Lambda \\
&=\; \partial_\mu A_\nu - \partial_\nu A_\mu \;=\; F_{\mu\nu}
\end{aligned}
$$

as the derivatives of the scalar function Λ commute. With $F_{\mu\nu}$ also $L_{\mathrm{e.m.}}$ is invariant under the transformation.

(c) First note that the *gauge* transformation in (b) is exactly the $U(1)$ phase transformation which we have used to derive (scalar) electrodynamics; gauge invariance required the introduction of a gauge field which we called a_μ, with transformation $a_\mu(x) \to a'_\mu(x) = a_\mu(x) + i\partial_\mu \alpha(x)$ (here the scalar function $\Lambda(x)$ is the space-time dependent phase $\alpha(x)$).
The Lagrangian for a massive photon without sources is given by

$$
-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + m^2 A_\mu A^\mu \,.
$$

As before the first term is invariant under the transformation considered above. However, the second term transforms as

$$
m^2 A_\mu A^\mu \;\to\; m^2 A'_\mu A'^\mu \;=\; m^2 A_\mu A^\mu + m^2(\partial_\mu \Lambda \partial^\mu \Lambda + \partial_\mu \Lambda A^\mu + A_\mu \partial^\mu \Lambda)\,.
$$

The extra terms do not vanish, so the mass term would not be invariant under the transformation, i.e. *break the invariance.* Insisting on the invariance therefore forbids the mass term, i.e. $m^2 = 0$ and the photon has to remain massless.

(d) With the source-term $-j^\mu A_\mu$ in the Lagrangian we get, in addition to the terms in the source-free case, the term

$$
-\frac{\partial L}{\partial A_\mu} \;=\; j^\mu \,.
$$

This leads, with the additional minus sign obtained from the first term of the Euler-Lagrange equation, to a term j^μ on the right hand side, and hence to the covariant form of Maxwell's equation including sources as given in the lecture.
Now $\partial_\mu j^\mu = \partial_\mu \partial_\nu F^{\nu\mu} = 0$ directly, as the order of differentiation can be interchanged ('*symmetric* $\times$ *anti-symmetric* $= 0$').

2. (a)

$$\begin{aligned}
\gamma^{5\dagger} &= (i\gamma^0\gamma^1\gamma^2\gamma^3)^\dagger \\
&= -\,i\gamma^{3\dagger}\gamma^{2\dagger}\gamma^{1\dagger}\gamma^{0\dagger} \\
&= -\,i(\gamma^0\gamma^3\gamma^0)(\gamma^0\gamma^2\gamma^0)(\gamma^0\gamma^1\gamma^0)\gamma^0 \\
&= -\,i\gamma^0\gamma^3\gamma^2\gamma^1 = i\gamma^0\gamma^2\gamma^3\gamma^1 \;=\; -i\gamma^0\gamma^2\gamma^1\gamma^3 = i\gamma^0\gamma^1\gamma^2\gamma^3 \;=\; \gamma^5\,.
\end{aligned}$$

For the second part, it is best to work out the anti-commutator for each μ in turn:

$$\begin{aligned}
\gamma^5\gamma^0 &= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 \;=\; -i\gamma^0\gamma^1\gamma^2\gamma^0\gamma^3 \;=\; i\gamma^0\gamma^1\gamma^0\gamma^2\gamma^3 \\
&= -\,i\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 \;=\; -\gamma^0\gamma^5 \;\Rightarrow\; \{\gamma^5,\gamma^0\} \;=\; 0\,, \\
\gamma^5\gamma^1 &= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1 \;=\; -i\gamma^0\gamma^1\gamma^2\gamma^1\gamma^3 \;=\; i\gamma^0\gamma^1\gamma^1\gamma^2\gamma^3 \\
&= -\,i\gamma^1\gamma^0\gamma^1\gamma^2\gamma^3 \;=\; -\gamma^1\gamma^5 \;\Rightarrow\; \{\gamma^5,\gamma^1\} \;=\; 0\,.
\end{aligned}$$

It is clear that the other two calculations will be similar.

(b)

$$\begin{aligned}
\gamma^0\gamma^1\gamma^2 &= -\,\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3 \;=\; i\gamma^5\gamma^3\,, \\
\gamma^0\gamma^1\gamma^3 &= -\,\gamma^0\gamma^1\gamma^2\gamma^2\gamma^3 \;=\; \gamma^0\gamma^1\gamma^2\gamma^3\gamma^2 \;=\; -i\gamma^5\gamma^2\,.
\end{aligned}$$

(c)

$$\{\gamma_\mu,\gamma_\nu\} \;=\; 2\eta_{\mu\nu}\mathbb{I}_4 \quad\Rightarrow\quad \gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu \;=\; 2\eta_{\mu\nu}\mathbb{I}_4\,,$$

where $\mathbb{I}_4$ is the 4-dimensional identity matrix, usually not written explicitly. Taking the trace, and using $\mathrm{Tr}(AB)=\mathrm{Tr}(BA)$, $\mathrm{Tr}\,\mathbb{I}_4=4$, we get

$$\mathrm{Tr}[\gamma_\mu\gamma_\nu] \;=\; 4\eta_{\mu\nu}\,.$$

Now

$$\begin{aligned}
\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma &= -\,\gamma_\nu\gamma_\mu\gamma_\rho\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma \\
&= \gamma_\nu\gamma_\rho\gamma_\mu\gamma_\sigma - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma \\
&= -\,\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu + 2\eta_{\mu\sigma}\gamma_\nu\gamma_\rho - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma \\
\Rightarrow\quad \gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma + \gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu &= 2\eta_{\mu\sigma}\gamma_\nu\gamma_\rho - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma\,.
\end{aligned}$$

Taking the trace and using

$$\mathrm{Tr}[\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu] \;=\; \mathrm{Tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma]\,,$$

together with $\mathrm{Tr}[\gamma_\mu\gamma_\nu]=4\eta_{\mu\nu}$, we find

$$\mathrm{Tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] \;=\; 4[\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}]\,.$$