

## MATH328 Modern Particle Physics Solutions 4

1. The Lagrangian of the given two-dimensional potential is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{4}k(x^2 + y^2)^2.$$

- (ii) The Euler-Lagrange equations are

$$\begin{aligned}\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= m\ddot{x} + k(x^2 + y^2)x = 0, \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} &= m\ddot{y} + k(x^2 + y^2)y = 0.\end{aligned}$$

- (iii) The Hamiltonian is

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{4}mw^2(x^2 + y^2)^2 \quad \text{with} \quad w = \sqrt{\frac{k}{m}}.$$

- (iv) In polar coordinates we derive:

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{1}{4}kr^4,$$

$$\begin{aligned}p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}.\end{aligned}$$

Hence we get

$$\begin{aligned}H &= \sum_i p_i \dot{q}_i - L = m\dot{r}^2 + mr^2\dot{\phi}^2 - \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{4}mw^2r^4 \\ &= \frac{1}{2}m(\dot{r}^2 + mr^2\dot{\phi}^2) + \frac{1}{4}mw^2r^4 \\ &= \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + \frac{1}{4}mw^2r^4.\end{aligned}$$

- (v) There are two constants of the motion.

Since the Hamiltonian does not depend explicitly on time, the energy is a constant of the motion with  $E = H(q_0, p_0)$ . Since it does not depend explicitly on  $\phi$ , also  $p_\phi$  is a constant of the motion, corresponding to conservation of the angular momentum w.r.t. the symmetry axis.

**2.** Considering the Lagrangian

$$L(q(t), \dot{q}(t), t)$$

and the variation

$$\begin{aligned} q(t) &\rightarrow q(t) + \delta(q(t)) = q(t) + \epsilon h(q(t), t) \\ \dot{q}(t) &\rightarrow \dot{q}(t) + \frac{d}{dt}(\delta(q(t))) \end{aligned}$$

Variation of the action gives rise to the Euler-Lagrange equation of motion

$$S(t) = \int dt L(q(t), \dot{q}(t), t) \Rightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0$$

since  $L$  is invariant under the variation

$$\begin{aligned} L(q + \delta q, \dot{q} + \frac{d}{dt}\delta q, t) &= L + \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}\delta q + \dots = L \\ \Rightarrow \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}\delta q &= 0 \end{aligned}$$

Now take

$$\begin{aligned} \epsilon \frac{dQ}{dt} &= \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}}\delta q \right) = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \delta q + \frac{\partial L}{\partial \dot{q}} \frac{d\delta q}{dt} = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}\delta q = 0 \\ \Rightarrow \frac{dQ}{dt} &= 0, \end{aligned}$$

where we have used the Euler–Lagrange equation to go from the second to third equality. Therefore, we have that

$$Q(t) = \frac{\partial L}{\partial \dot{q}} h(q(t), t)$$

is conserved.

**3.** With  $\rho = |\psi|^2 = \psi^* \psi$ , we have

$$\frac{\partial \rho}{\partial t} = \frac{\partial \psi^*}{\partial t} \psi + \psi^* \frac{\partial \psi}{\partial t}.$$

Now from the Schrödinger equation,

$$i\hbar \frac{\partial \psi}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) \right) \psi, \quad (1)$$

and so, taking the complex conjugate

$$-i\hbar \frac{\partial \psi^*}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) \right) \psi^* \quad (2)$$

(assuming that  $V(\mathbf{x})$  is real.) Multiplying (1) by  $\psi^*$  and (2) by  $\psi$  and subtracting, we have

$$i\hbar \left( \psi^* \frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t} \psi \right) = -\frac{\hbar^2}{2m} (\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*) = -\frac{\hbar^2}{2m} \nabla \cdot [\psi^* \nabla \psi - \psi \nabla \psi^*],$$

and so

$$\frac{\partial \rho}{\partial t} = \frac{i\hbar}{2m} \nabla \cdot [\psi^* \nabla \psi - \psi \nabla \psi^*],$$

i.e.

$$\frac{\partial \rho}{\partial t} + \text{div} \mathbf{j} = 0,$$

$$\text{where } \mathbf{j} = -\frac{i\hbar}{2m} [\psi^* \nabla \psi - \psi \nabla \psi^*]$$

#### 4. Inserting the given solution

$$\phi(x, y) = \sum_{n=1}^{\infty} \phi_n(x) \text{cs} \left( \frac{n\pi y}{R} \right)$$

into the five-dimensional Klein-Gordon equation and carrying out  $\partial^2/\partial y^2$ , one obtains a four-dimensional Klein-Gordon equation for each of the Fourier coefficients  $\phi_n(x)$ :

$$\left( \partial_0^2 - \nabla^2 + m^2 + \left( \frac{n\pi}{R} \right)^2 \right) \phi_n(x) = 0.$$

Therefore the given  $\phi(x, y)$  is a solution of the five-dimensional Klein-Gordon equation if the  $\phi_n(x)$  are solutions of the four-dimensional equations. The masses  $m_n$  of the fields  $\phi_n$  are given by

$$m_n^2 = m^2 + \left( \frac{n\pi}{R} \right)^2.$$

If the five-dimensional mass  $m$  is zero, we get the equally spaced mass spectrum

$$m_n = \frac{n\pi}{R}.$$

The infinite set of particles are called Kaluza-Klein tower.