

MATH 431 CLASS TEST

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$$1) ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dx^2 + dy^2$$

$$a) g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

we

$$b) x \rightarrow x + \epsilon A(x, y)$$

$$y \rightarrow y + \epsilon B(x, y)$$

$$dx \rightarrow dx + \epsilon \left(\frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy \right)$$

$$dy \rightarrow dy + \epsilon \left(\frac{\partial B}{\partial x} dx + \frac{\partial B}{\partial y} dy \right)$$

~~$\Rightarrow ds^2 \rightarrow$~~

$$dx^2 \rightarrow dx^2 + 2\epsilon \left(\frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy \right) dx + O(\epsilon^2)$$

$$dy^2 \rightarrow dy^2 + 2\epsilon \left(\frac{\partial B}{\partial x} dx + \frac{\partial B}{\partial y} dy \right) dy + O(\epsilon^2)$$

~~$\Rightarrow ds^2 \rightarrow$~~

$$\Rightarrow ds^2 \rightarrow dx^2 + dy^2 + 2\epsilon \left(\frac{\partial A}{\partial x} dx^2 + \frac{\partial B}{\partial y} dy^2 \right)$$

$$\Rightarrow ds^2 \rightarrow dx^2 + dy^2 + 2\epsilon \left(\frac{\partial A}{\partial x} dx^2 + \frac{\partial B}{\partial y} dy^2 + \left(\frac{\partial A}{\partial y} + \frac{\partial B}{\partial x} \right) dx dy \right) + O(\epsilon^2)$$

$$\text{for } dx^2: \frac{\partial A}{\partial x} = 0 \Rightarrow A = A(y)$$

$$\text{for } dy^2: \frac{\partial B}{\partial y} = 0 \Rightarrow B = B(x)$$

$$\text{for } dx dy: \frac{\partial A}{\partial y} + \frac{\partial B}{\partial x} = 0$$

$$\Rightarrow \frac{\partial A}{\partial y} = -\frac{\partial B}{\partial x} = C$$

$$\Rightarrow A = Cy + a$$

$$B = -Cx + b$$

$$B(x) = -Cx + b$$

$$\text{for } C = b = 0 \text{ and } a \neq 0$$

$$x \rightarrow x + \epsilon a$$

$$y \rightarrow y$$

which corresponds to a translation ~~along the x-axis~~
along the x-axis
(in space)

$$\text{for } C = a = 0 \text{ and } b \neq 0$$

$$x \rightarrow x$$

$$y \rightarrow y + \epsilon b$$

which corresponds to a translation ~~along the y-axis~~
along the y-axis
(in space)

$$\text{for } a = b = 0 \text{ and } C \neq 0$$

$$x \rightarrow x + \epsilon Cy$$

$$y \rightarrow y - \epsilon Cx$$

which corresponds to a translation in the x-y plane
(in space)

Rotation

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$$2) L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu$$

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$$L = -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu) (\partial^\mu A^\nu - \partial^\nu A^\mu) - j^\mu A_\mu$$

$$= -\frac{1}{4} \eta^{\alpha\mu} \eta^{\beta\nu} (\partial_\alpha A_\beta - \partial_\beta A_\alpha) (\partial_\mu A_\nu - \partial_\nu A_\mu) - j^\mu A_\mu$$

~~$$\frac{\partial L}{\partial x^\mu} = \frac{\partial L}{\partial (\partial_\mu A_\nu)} \frac{\partial (\partial_\mu A_\nu)}{\partial x^\mu}$$~~

~~$$\frac{\partial L}{\partial A_\mu}$$~~
$$\frac{\partial}{\partial x^\mu} \left(\frac{\partial L}{\partial (\partial_\mu A_\nu)} \right) - \frac{\partial L}{\partial A_\nu} = 0$$

$$\frac{\partial L}{\partial A_\nu} = -j^\nu$$

~~$$\frac{\partial L}{\partial (\partial_\mu A_\nu)}$$~~

$$\frac{\partial L}{\partial (\partial_\alpha A_\beta)} = -\frac{1}{4} \eta^{\alpha\mu} \eta^{\beta\nu} \left((\delta_\alpha^\mu \delta_\beta^\nu - \delta_\beta^\mu \delta_\alpha^\nu) (\partial_\nu A_\mu - \partial_\mu A_\nu) + (\partial_\beta A_\alpha - \partial_\alpha A_\beta) (\delta_\mu^\nu \delta_\nu^\mu - \delta_\nu^\mu \delta_\mu^\nu) \right)$$

$$= -\frac{1}{4} \left((\eta^{\alpha\mu} \eta^{\beta\nu} - \eta^{\beta\mu} \eta^{\alpha\nu}) (\partial_\nu A_\mu + \partial_\mu A_\nu) + (\eta^{\alpha\mu} \eta^{\beta\nu} - \eta^{\alpha\nu} \eta^{\beta\mu}) (\partial_\alpha A_\beta - \partial_\beta A_\alpha) \right)$$

$$= -(\partial^\mu A^\alpha - \partial^\alpha A^\mu) = -F^{\mu\alpha}$$

$$\Rightarrow \frac{\partial L}{\partial (\partial_\mu A_\nu)} = F^{\nu\mu} \Rightarrow \frac{\partial}{\partial x^\mu} \left(\frac{\partial L}{\partial (\partial_\mu A_\nu)} \right) = \partial_\mu F^{\nu\mu}$$

$$\Rightarrow \partial_\mu F^{\nu\mu} = -\partial_\mu F^{\mu\nu}$$

$$-\partial_\mu F^{\mu\nu} + j^\nu = 0 \Rightarrow \partial_\mu F^{\mu\nu} = j^\nu$$

$$\partial_\nu j^\nu = 0$$

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to show that j^ν is conserved it can be shown that $F^{\mu\nu}$ is invariant under a transformation

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

consider transformation $A^\mu \rightarrow \tilde{A}^\mu = A^\mu + \partial^\mu \Lambda$

$$\begin{aligned} \Rightarrow \tilde{F}^{\mu\nu} &= \partial^\mu \tilde{A}^\nu - \partial^\nu \tilde{A}^\mu = \partial^\mu (A^\nu + \partial^\nu \Lambda) - \partial^\nu (A^\mu + \partial^\mu \Lambda) \\ &= \partial^\mu A^\nu - \partial^\nu A^\mu - \partial^\mu \partial^\nu \Lambda + \partial^\nu \partial^\mu \Lambda = \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu} \end{aligned}$$

$\Rightarrow \partial_\mu F^{\mu\nu}$ is invariant under transformation, and j^ν is conserved

$$b) L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu + \frac{1}{2} m^2 A_\mu A^\mu$$

$$\frac{\partial L}{\partial A_\nu} = -j^\nu + \frac{1}{2} m^2 A^\nu$$

$$\Rightarrow \frac{\partial L}{\partial x^\mu} \left(\frac{\partial L}{\partial (\partial_\mu A_\nu)} \right) = \partial_\mu F^{\mu\nu} = -\partial_\mu F^{\mu\nu}$$

$$\Rightarrow \partial_\mu F^{\mu\nu} + \frac{1}{2} m^2 A^\nu = j^\nu$$

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu$$

$$\partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu + \frac{1}{2} m^2 A^\nu = j^\nu \quad 4$$

Lorentz gauge $\partial_\mu A^\mu = 0$