All problems are similar to homework problems

Solution to Problem 1

$$\psi(r,\theta,\phi) = R(r)\left(10ia\cos\theta iae^{i\phi}\sin\theta\right)$$

Normalize

$$\int d^3\mathbf{r}\psi^\dagger\psi = 1 = \int 4\pi r^2 dr(|R|^2(1+a^2))$$

$$\Rightarrow \int_0^\infty r^2|R|^2 dr = [4\pi(1+a^2)]^{-1}$$

(a.)

$$L_z = -i\frac{\partial}{\partial\phi} \Rightarrow L_z\psi = R\begin{pmatrix}0\\0\\0\\iae^{i\phi}\sin\theta\end{pmatrix} \not\propto \psi$$

so ψ is not an eigenstate of L_z.

(b.)

$$\langle L_z\rangle = \int d^3\mathbf{r}\psi^\dagger L_z\psi = \int 2\pi r^2 d\cos\theta|R|^2 a^2\sin^2\theta$$

$$\int_{-1}^{1} d\cos\theta(1-\cos^2\theta) = 2-\frac{2}{3} = \frac{4}{3} \Rightarrow \langle L_z\rangle = \frac{8\pi}{3}a^2\cdot\frac{1}{4\pi(1+a^2)}$$

$$\Rightarrow \langle L_z\rangle = \frac{2a^2}{3(1+a^2)}$$

In H-atom , $v/c \sim \alpha \Rightarrow \langle L_z\rangle = O(v^2/c^2)$. This is a relativistic effect - spin–orbit interaction.

(c.)

$$\begin{aligned} S_z &= \frac{1}{2}\hbar\begin{pmatrix}+1&0&0&0\\0&-1&0&0\\0&0&+1&0\\0&0&0&-1\end{pmatrix} \\ \Rightarrow S_z\psi &= \frac{1}{2}\hbar R\begin{pmatrix}1\\0\\ia\cos\theta\\-iae^{i\phi}\sin\theta\end{pmatrix} \\ (L_z+S_z)\psi &= R\begin{pmatrix}1\\0\\ia\cos\theta\\iae^{i\phi}\sin\theta\end{pmatrix} = \frac{1}{2}\psi \Rightarrow J_z = +\frac{1}{2} \qquad (1)\end{aligned}$$

2. 2a. 3 diagonal generators.

$$\lambda_3 = \begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} \quad , \quad \lambda_8 = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-2&0\\0&0&0&0\end{pmatrix} \quad , \quad \lambda_{15} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-3\end{pmatrix}$$

2b. $D = 15$. Three diagonal generators of part $(2a)$ plus:

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\lambda_6 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\lambda_9 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_{10} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \\
\lambda_{11} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \\
\lambda_{13} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad , \quad \lambda_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}
\end{aligned} \tag{2}$$

2c.

$$4 = (3, 1/3) + (1, -1)$$

under the maximal subgroup $SU(3) \times U(1)$

2d.

$SU(4)$ decomposes as $SU(3) \times U(1)$ under the Standard Model, where $SU(3)$ corresponds to the gauge symmetry of the strong interactions and the $U(1)$ can be indentified as baryon minus lepton number. The fundamental representation decomposes as a triplet and a singlet of $SU(3)$ with the triplet corresponding to a quark, with baryon number $1/3$, and the singlet to a lepton, with lepton number -1.

2e.

$$\begin{aligned} 4 \times \bar{4} \qquad &= \{(3, 1/3) + (1, -1)\} \times \{(\bar{3}, -1/3) + (1, +1)\} = \\ 15 + 1 &= \{(3, +4/3) + (8, 0) + (1, 0) + (\bar{3}, -4/3)\} + (1, 0) \end{aligned} \tag{3}$$

2f.

$$\begin{aligned} 4 \times 4 \qquad &= \{(3, 1/3) + (1, -1)\} \times \{(3, 1/3) + (1, -1)\} = \\ 6 + 10 &= \{(6, 2/3) + (3, -2/3) + (1, -2)\} + \{(\bar{3}, 2/3) + (3, -2/3)\} \end{aligned} \tag{4}$$

3a. The $SU(2) \times U(1)_Y$ charges of the electroweak Higgs doublet ϕ.

$$\phi : T = \frac{1}{2} \ ; \ T_3 = \pm\frac{1}{2} \ ; \ Y = 1$$

(3b.)
The electric charge is given by:

$$\begin{aligned} Q_{\text{e.m.}}(\phi_+) &= T_3 + \tfrac{1}{2}Y = \tfrac{1}{2} + \tfrac{1}{2} = +1 \\ Q_{\text{e.m.}}(\phi_0) &= T_3 + \tfrac{1}{2}Y = -\tfrac{1}{2} + \tfrac{1}{2} = 0 \end{aligned} \tag{5}$$

(3c.) The Lagrangian density for the Higgs field include the kinetic and potential terms

$$\left| \left(\partial_\mu - i g \vec{T} \cdot \vec{W}_\mu - i g' \frac{Y}{2} B_\mu \right) \phi \right|^2 - V(\phi)$$

where the potential is given by

$$\mu^2 \phi^\dagger \phi + \lambda \left(\phi^\dagger \phi \right)^2$$

(3d.) The relevant term for the gauge boson masses

$$\begin{aligned}
\left| \left(g \vec{T} \cdot \vec{W}_\mu + g' \frac{Y}{2} B_\mu \right) \phi \right|^2 = \left| \left(g \frac{\vec{\tau}}{2} \cdot \vec{W}_\mu + g' \frac{Y}{2} B_\mu \right) \phi \right|^2 &= \\
\frac{1}{4} \left| \left(\frac{g}{2} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} W_\mu^1 + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} W_\mu^2 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} W_\mu^3 \right] + \frac{g'}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} B_\mu \right) \phi \right|^2 &= \\
\frac{1}{4} \left| \left[g \begin{pmatrix} W_\mu^3 & W_\mu^2 - i W_\mu^1 \\ W_\mu^2 + i W_\mu^1 & -W_\mu^3 \end{pmatrix} + g' \begin{pmatrix} B_\mu & 0 \\ 0 & B_\mu \end{pmatrix} \right] \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \right|^2 &= \\
\frac{1}{8} \left| \begin{pmatrix} g W_\mu^3 + g' B_\mu & g(W_\mu^2 - i W_\mu^1) \\ g(W_\mu^2 + i W_\mu^1) & -g W_\mu^3 + g' B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \right|^2 &= \\
\frac{1}{8} v^2 \left(g \left(W_\mu^2 + i W_\mu^1 \right), \left(-g W_\mu^3 + g' B_\mu \right) \right) \begin{pmatrix} g \left(W_\mu^2 - i W_\mu^1 \right) \\ -g W_\mu^3 + g' B_\mu \end{pmatrix} &= \\
\frac{1}{4} g^2 v^2 W^{\mu +} W^{\mu -} + \frac{1}{8} v^2 (g^2 + g'^2) \frac{\left(-g W_\mu^3 + g' B_\mu \right)^2}{\left(\sqrt{g^2 + g'^2} \right)^2} &= \\
\left(\frac{1}{2} g v \right)^2 W^{\mu +} W^{\mu -} + \frac{1}{2} v^2 \frac{(g^2 + g'^2)}{4} \left(\frac{-g W_\mu^3 + g' B_\mu}{\sqrt{g^2 + g'^2}} \right)^2 &= \\
M_{W^\pm}^2 \quad W^{\mu +} W^{\mu -} \quad + \qquad \frac{1}{2} M_Z^2 Z^2 &
\end{aligned} \tag{6}$$

reading off from the last two lines we have

$$M_{W^\pm} = \frac{g v}{2} \quad ; \quad M_Z = \frac{1}{2} v \left(g^2 + g'^2 \right)^{\frac{1}{2}}$$

hence

$$\frac{M_W}{M_Z} = \frac{g}{\left(g^2 + g'^2 \right)^{\frac{1}{2}}} = \cos \theta_W$$