PAPER CODE NO.
MATH 431

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Introduction to Modern Particle Physics

TIME ALLOWED: Two and half hours

INSTRUCTIONS TO CANDIDATES: In this paper bold-face quantities like **x** represent three-dimensional vectors.

Full marks can be obtained for complete answers to FIVE questions. Only the best FIVE answers will be counted.

1. (a) Give the Lorentz transformations for the components a_μ of a vector under a boost along the x^1 axis.

[7 marks]

(b) Show explicitly that the object $\frac{\partial}{\partial x^\mu}$ transforms under a boost along the x^1 axis as the a_μ vector considered in (a) does.

[7 marks]

(c) Show that, in quantum mechanics, the expression for the energy and momentum in terms of derivatives can be written compactly as $p_\mu = i\hbar \frac{\partial}{\partial x^\mu}$.

[6 marks]

2. (a) Show that if the Hamiltonian is independent of a generalised coordinate q_0, then the conjugate momentum p_0 is a constant of the motion. Such coordinates are called *cyclic coordinates*. Briefly describe two examples of physical systems which have a cyclic coordinate.

[6 marks]

(b) Show that in three-dimensional spherical polar coordinates the Hamiltonian of a particle of mass m moving in a potential $V(\vec{x})$ is

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) + V(\vec{x}) \, .$$

Now show that $p_\phi = \textit{constant}$ if $\partial V/\partial\phi \equiv 0$ and give a physical interpretation of this result.

[14 marks]

3. (a) Let $\vec{J}$ and $\vec{K}$ be the generators of rotations and boosts, respectively. Show that

$$\vec{J}^2 - \vec{K}^2 \quad \text{and} \quad \vec{J}\cdot\vec{K}$$

are Lorentz invariants (*i.e.* that they commute with all the generators of the Lorentz group). [10 marks]

(b) Assume a representation (j_1, j_2) of $SU(2)\times SU(2)^\dagger$. How many states are there in this representation? How does this representation decompose in irreducible representations of $SU(2)_J$, where J is the total angular momentum? [10 marks]

4. (a) Given the four-vector current density $J_V^\mu = \bar{\psi}\gamma^\mu\psi$, derive the current conservation equation, $\partial_\mu J_V^\mu = 0$, by using the covariant form of the Dirac equation and the relation $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$.

[6 marks]

(b) Show that the axial four-vector current density $J_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$ is not conserved but instead satisfies the covariant equation

$$\partial_\mu J_A^\mu \;=\; 2im\,\bar{\psi}\gamma^5\psi\,.$$

[6 marks]

(c) Derive the *Gordon decomposition* of the Dirac transition current,

$$\bar{\psi}_f\gamma^\mu\psi_i \;=\; \frac{1}{2m}\,\bar{\psi}_f\left[(p_f+p_i)^\mu + i\sigma^{\mu\nu}(p_f-p_i)_\nu\right]\psi_i\,,$$

where $\sigma^{\mu\nu} = \frac{1}{2}i(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$.

[8 marks]

5. Consider the electromagnetic field in three spacetime dimensions.

(a) Find the reduced Maxwell equations in three dimensions by starting with Maxwell's equations and the force law in four dimensions, using the ansatz $E_z = B_x = B_y = 0$, and assuming that no field can depend on the z direction.

[10 marks]

(b) Write down the field strength tensor in three dimensions in terms of the three dimensional scalar and vector potentials. Write down the field strength tensor in three dimensions in terms of the three dimensional electric and magnetic fields.

[10 marks]

6. Consider the $SO(10)$ Grand Unified Theory

(a) Write down the weight lattice of the spinorial 16 representation of $SO(10)$.

[5 marks]

(b) Show how the spinorial 16 representation of $SO(10)$ decomposes under the $SU(5) \times U(1)$, $SO(6) \times SO(4)$ and $SU(3) \times SU(2) \times U(1)^2$ subgroups.

[10 marks]

(c) Identify how the Standard Model states fit into the spinorial 16 of $SO(10)$.

[5 marks]

7. Consider the $SU(2)_W \times U(1)_Y$ Weinberg–Salam model of electroweak symmetry breaking.

(a) The Higgs bosons of the model reside in a complex doublet representation of $SU(2)_W$. Write down the weak and hypercharge charges of the components of the Higgs doublet.

[2 marks]

(b) Calculate the electric charges of the components in part (a).

[2 marks]

(c) Write down the Lagrangian for the Higgs field, including the kinetic and potential term.

[4 marks]

(d) Assuming a VEV of the form

$$\phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

use the Lagrangian to show that

$$\frac{M_W}{M_Z} = \cos\theta_W$$

where M_W and M_Z are the masses of the charged weak vector bosons and the neutral electroweak vector boson respectively.

[12 marks]