

$$a) \quad ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dx^2 + dy^2$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad g^{\mu\nu} = (g_{\mu\nu})^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$b) \quad x \rightarrow x + \epsilon A(x, y)$$

$$y \rightarrow y + \epsilon B(x, y)$$

$$dx \rightarrow dx + \epsilon \left(\frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy \right)$$

$$dy \rightarrow dy + \epsilon \left(\frac{\partial B}{\partial x} dx + \frac{\partial B}{\partial y} dy \right)$$

$$dx^2 \rightarrow \left(dx + \epsilon \left(\frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy \right) \right)^2$$

$$dx^2 \rightarrow \left[\left(1 + \epsilon \frac{\partial A}{\partial x} \right) dx + \epsilon \frac{\partial A}{\partial y} dy \right]^2$$

$$dy^2 \rightarrow \left[\left(1 + \epsilon \frac{\partial B}{\partial y} \right) dy + \epsilon \frac{\partial B}{\partial x} dx \right]^2$$

$$\begin{aligned}
 ds^2 &\rightarrow \left[\left(1 + \epsilon \frac{\partial A}{\partial x}\right) dx + \epsilon \frac{\partial A}{\partial y} dy \right]^2 + \left[\left(1 + \epsilon \frac{\partial B}{\partial y}\right) dy + \epsilon \frac{\partial B}{\partial x} dx \right]^2 \\
 &= \left[dx + \epsilon \frac{\partial A}{\partial x} dx + \epsilon \frac{\partial A}{\partial y} dy \right]^2 + \left[dy + \epsilon \frac{\partial B}{\partial y} dy + \epsilon \frac{\partial B}{\partial x} dx \right]^2 \\
 &= \left[dx^2 + 2\epsilon \frac{\partial A}{\partial x} dx^2 + 2\epsilon \frac{\partial A}{\partial y} dx dy + O(\epsilon)^2 \right] + \left[dy^2 + 2\epsilon \frac{\partial B}{\partial y} dy^2 + 2\epsilon \frac{\partial B}{\partial x} dx dy + O(\epsilon)^2 \right] \\
 &= \left(1 + 2\epsilon \frac{\partial A}{\partial x}\right) dx^2 + \left(1 + 2\epsilon \frac{\partial B}{\partial y}\right) dy^2 + 2\epsilon \left(\frac{\partial A}{\partial y} + \frac{\partial B}{\partial x}\right) dx dy + O(\epsilon)^2
 \end{aligned}$$

Since ϵ is an infinitesimal constant, terms including ϵ^2 can be approximated to 0.

$$\therefore dx^2 + dy^2 \equiv \left(1 + 2\epsilon \frac{\partial A}{\partial x}\right) dx^2 + \left(1 + 2\epsilon \frac{\partial B}{\partial y}\right) dy^2 + 2\epsilon \left(\frac{\partial A}{\partial y} + \frac{\partial B}{\partial x}\right) dx dy$$

To keep ds^2 invariant under transformations the additional terms must be equal to 0.

$$\text{Equate } dx^2: \quad x = x + 2\epsilon \frac{\partial A}{\partial x} \Rightarrow 2\epsilon \frac{\partial A}{\partial x} = 0 \Rightarrow \frac{\partial A}{\partial x} = 0 \Rightarrow A \neq A(x) \quad \therefore A = A(y)$$

$$dy^2: \quad y = y + 2\epsilon \frac{\partial B}{\partial y} \Rightarrow 2\epsilon \frac{\partial B}{\partial y} = 0 \Rightarrow \frac{\partial B}{\partial y} = 0 \Rightarrow B \neq B(y) \quad \therefore B = B(x)$$

$$dx dy: \quad 0 = 2\epsilon \left(\frac{\partial A}{\partial y} + \frac{\partial B}{\partial x}\right) \Rightarrow \frac{\partial A(x)}{\partial y} = -\frac{\partial B(y)}{\partial x} = \text{constant} = c$$

$$\Rightarrow \frac{\partial A}{\partial y} = c \Rightarrow \underline{\cancel{A(y)} A(y) = cy + a}$$

$a, b \& c$ are constants

$$-\frac{\partial B}{\partial x} = c \Rightarrow \underline{B(x) = b - cx}$$

$$\therefore \begin{aligned} x &\rightarrow x + \epsilon(y + ea) \\ y &\rightarrow y + \epsilon(b - cx) \end{aligned}$$

$$\text{Set } a=b=0, c \neq 0$$

$$\Rightarrow \begin{aligned} x &\rightarrow x + \epsilon c y \\ y &\rightarrow y - \epsilon c x \end{aligned}$$

$\therefore c$ corresponds to a boost

$$\text{Set } a=c=0, b \neq 0$$

$$\Rightarrow \begin{aligned} x &\rightarrow x \\ y &\rightarrow y + \epsilon b \end{aligned}$$

$\therefore b$ is a shift along the y ~~axis~~ coordinate.

$$\text{Set } a=b=c=0, a \neq 0$$

$$\Rightarrow \begin{aligned} x &\rightarrow x + \epsilon a \\ y &\rightarrow y \end{aligned}$$

$\therefore a$ is a shift along the x coordinate.

$$2a) \quad L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu$$

~~For $F^{\mu\nu}$~~

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$\therefore L = -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu) (\partial^\mu A^\nu - \partial^\nu A^\mu) - j^\mu A_\mu$$

$$= -\frac{1}{4} (\partial^\mu A^\nu - \partial^\nu A^\mu) (\partial_\mu A_\nu - \partial_\nu A_\mu) - j^\mu A_\mu$$

$$= -\frac{1}{4} \eta^{\alpha\mu} \eta^{\beta\nu} (\partial_\alpha A_\beta - \partial_\beta A_\alpha) (\partial_\mu A_\nu - \partial_\nu A_\mu) - j^\mu A_\mu$$

Euler-Lagrange eqn: $\frac{\partial}{\partial x^\nu} \left(\frac{\partial L}{\partial (\partial_\nu A_\mu)} \right) - \frac{\partial L}{\partial A_\mu} = 0$

$$\frac{\partial L}{\partial (\partial_\nu A_\mu)} = -\frac{1}{4} \left[\eta^{\alpha\mu} \eta^{\beta\nu} \left(\delta_\alpha^\rho \delta_\beta^q - \delta_\beta^\rho \delta_\alpha^q \right) (\partial_\mu A_\nu - \partial_\nu A_\mu) + (\partial_\alpha A_\beta - \partial_\beta A_\alpha) (\delta_\mu^\rho \delta_\nu^q - \delta_\nu^\rho \delta_\mu^q) \right]$$

$$= -\frac{1}{4} \left[(\eta^{\rho\mu} \eta^{q\nu} - \eta^{q\mu} \eta^{\rho\nu}) (\partial_\mu A_\nu - \partial_\nu A_\mu) + (\eta^{\alpha q} \eta^{\beta\rho} - \eta^{\alpha\rho} \eta^{\beta q}) (\partial_\alpha A_\beta - \partial_\beta A_\alpha) \right]$$

$$= -\frac{1}{4} [4(\partial^\rho A^q - \partial^q A^\rho)]$$

$$= -F^{\rho q}$$

$$\therefore \frac{\partial L}{\partial (\partial_\nu A_\mu)} = -F^{\mu\nu}$$

$$\frac{\partial L}{\partial A_\mu} = \frac{\partial}{\partial A_\mu} (-j^\mu A_\mu) = -j^\mu$$

$$\therefore \frac{\partial}{\partial x^\nu} \left(\frac{\partial L}{\partial (\partial_\nu A_\mu)} \right) - \frac{\partial L}{\partial A_\mu} = 0$$

$$\Rightarrow -\partial_\nu F^{\mu\nu} + j^\mu = 0$$

$$\therefore \partial_\nu F^{\mu\nu} = j^\mu$$

~~is conserved~~

For j^μ to be conserved. $\partial_\mu j^\mu = 0$

$$\therefore \partial_\mu j^\mu = \partial_\mu (\partial_\nu F^{\mu\nu}) = \partial_\mu \partial_\nu [\partial^\mu A^\nu - \partial^\nu A^\mu]$$

$$= \underbrace{\partial_\nu \partial_\mu \partial^\mu A^\nu}_{\partial^2} - \underbrace{\partial_\mu \partial_\nu \partial^\nu A^\mu}_{\partial^2}$$

$$= \partial^2 [\partial_\nu A^\nu - \partial_\mu A^\mu] \quad \text{or } \partial^2 [0]$$

$$= \cancel{\partial_\mu \partial_\nu [\partial^\mu A^\nu - \partial^\nu A^\mu]}$$

Re label indices

$$= \partial^2 [\partial_\mu A^\mu - \partial_\nu A^\nu]$$

$$= \partial_\nu \partial_\mu [\partial^\nu A^\mu - \partial^\mu A^\nu] = \partial_\nu \partial_\mu F^{\nu\mu} = -\partial_\nu \partial_\mu F^{\mu\nu}$$

For this to be possible $\partial_\nu \partial_\mu F^{\mu\nu} = 0$

$$\therefore \partial_\mu j^\mu = 0$$

And j^μ is conserved

b)
$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu + \frac{1}{2} m^2 A_\mu A^\mu$$

$$+ \frac{1}{2} m^2 A_\mu A^\mu =$$

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$$\frac{1}{2} m^2 \eta^{\mu\nu} A_\mu A_\nu$$

$$\therefore \frac{\partial L}{\partial A_\mu} = -j^\mu + \frac{1}{2} m^2 \left(A_\mu \frac{\partial A_\nu}{\partial A_\mu} + \frac{\partial A_\mu}{\partial A_\mu} A_\nu \right) \eta^{\mu\nu} = -j^\mu + \frac{1}{2} m^2 (A_\mu \delta_\mu^\nu + A_\nu) \eta^{\mu\nu}$$

$$= -j^\mu + 2 \frac{1}{2} m^2 A_\nu \eta^{\mu\nu}$$

$$= -j^\mu + m^2 A^\mu$$

$$\therefore \frac{\partial}{\partial x^\nu} \left(\frac{\partial L}{\partial (\partial_\nu A_\mu)} \right) - \frac{\partial L}{\partial A_\mu} = 0 \Rightarrow \partial_\nu F^{\mu\nu} - (-j^\mu + m^2 A^\mu) = 0$$

$$\Rightarrow -\partial_\nu F^{\mu\nu} - (-j^\mu + m^2 A^\mu) = 0$$

$$\Rightarrow \partial_\mu F^{\mu\nu} + m^2 A^\mu = j^\mu$$

$$\Rightarrow \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) + m^2 A^\mu = j^\mu$$