from the previous lecture ...

we require:

$$U(\Lambda_2, a_2)U(\Lambda_1, a_1) = U(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2) \tag{1}$$

where $U(\lambda, a)$ is given by

$$U(\Lambda, a) = 1 + i\vec{\alpha}\cdot\vec{J} - i\vec{\beta}\cdot\vec{K} + ia^{\mu}P_{\mu} \tag{2}$$

Here, α_i, β_i and a^{μ} are infinitesimal parameters. Hence, (2) is an expansion of $U(\Lambda, a)$ to first order in the infinitesimal parameters. Inserting (2) into (1) and keeping terms to second order in the inifinitesimal parameters, we derive the commutation relations. Alternatively, we can use the differential form of the operators P^{μ} and $L^{\mu\nu}$ that we presented in the previous lecture to find the commutation relations between the operators. We saw that the linear momentum generators commute among themselved whereas $L^{\mu\nu}$ satisfy

$$[L_{\mu\nu}, L_{\rho\sigma}] = i\eta_{\nu\rho}L_{\mu\sigma} - i\eta_{\mu\rho}L_{\nu\sigma} - i\eta_{\nu\sigma}L_{\mu\rho} + i\eta_{\mu\rho}L_{\nu\sigma} \tag{3}$$

which are the commutation relations of the $SO(1,3)$ Lie algebra. The most general representation of the generators of the $SO(1,3)$ algebra that obeys eq. (3) is given by

$$J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}$$

where $S_{\mu\nu}$ obeys eq. (3) and commutes with $L_{\mu\nu}$.

$$[J_{\mu\nu}, P_{\rho}] = -i\eta_{\mu\rho}P_{\nu} + i\eta_{\nu\rho}P_{\mu} \tag{4}$$

In terms of J_i and K_i the commutation relations become

$$[J_i, P_j] = i\epsilon_{ijk}P_k$$

$$[K_i, P_j] = iH\delta_{ij} \qquad \text{where} \quad \boxed{H = P_0}$$

$$[J_i, H] = 0 \quad , \quad [P_i, H] = 0 \quad , \quad [K_i, H] = iP_i$$

Pauli–Lubanski vector

A more elegant form of the commutation relations is obtained by defining the Pauli–Lubanski vector

$$W_\sigma = -\frac{1}{2}\epsilon_{\sigma\mu\nu\lambda} J^{\mu\nu} P^\lambda \tag{5}$$

We note the appearance of the antisymmetric tensor in four dimensions $\epsilon_{\sigma\mu\nu\lambda}$, which generalises the antisymmetric tensor in three dimensions ϵ_{ijk}. Before proceeding to examine the Pauli–Lubasnki vectort, we digress to discuss the generalisation of the antisymmetric tensor in any number of dimensions.

antisymmetric tensor in 3D: $\qquad \epsilon_{ijk} \qquad i,j,k = 1,2,3$

$$\begin{array}{ll} \epsilon_{123} = +1 & \epsilon_{132} = -1 \\ \epsilon_{231} = +1 & \epsilon_{213} = -1 \\ \epsilon_{312} = +1 & \epsilon_{321} = -1 \end{array}$$

i.e. $\epsilon_{123} = \epsilon_{\text{even permutations}} = \epsilon_{\text{odd permutations}}$

For a 3×3 matrix $\qquad A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$

$$
\begin{aligned}
\mathrm{Det} A &= \sum_{i,j,k} \epsilon_{ijk} a_i b_j c_k \\
&= \epsilon_{123} a_1 b_2 c_3 + \epsilon_{132} a_1 b_3 c_2 + \epsilon_{213} a_2 b_1 c_3 \\
&\quad + \epsilon_{231} a_2 b_3 c_1 + \epsilon_{312} a_3 b_1 c_2 + \epsilon_{321} a_3 b_2 c_1 \\
&= a_1(b_2 c_3 - b_3 c_2) - a_2(b_1 c_3 - b_3 c_1) + a_3(b_1 c_2 - b_2 c_1)
\end{aligned}
$$

<u>useful identities</u> : $\epsilon_{ijk}\epsilon_{ilm} = (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})$

$\epsilon_{ijl}\epsilon_{ijk} = 2\delta_{lk}$

facilitates vector calculus calculations in $3D$.

Generalises to nD $\qquad \epsilon_{\mu\nu\rho\,\cdots\,\sigma}\ :\qquad \epsilon_{123\cdots n} = +1 = \epsilon_{e.p.} = -\epsilon_{o.p.}$

For a $n \times n$ matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

$$\mathrm{Det} A = \sum_{i,j,k,\cdots} \epsilon_{ijk\,\cdots}\, a_{1i} a_{2j} \cdots a_{n\rho}$$

in $4D$ $\qquad \epsilon_{\mu\rho\sigma\tau} \qquad \mu, \rho, \sigma, \tau = 0, 1, 2, 3$

Returning to the Pauli–Lubanski vector eq. (5) we have

$$
\begin{aligned}
W_\sigma P^\sigma &= -\frac{1}{2}\epsilon_{\sigma\mu\nu\lambda} J^{\mu\nu} P^\lambda P^\sigma = +\frac{1}{2}\epsilon_{\mu\nu\lambda\sigma} J^{\mu\nu} P^\lambda P^\sigma = 0 \\
W_0 &= -\frac{1}{2}\epsilon_{0ijk} J^{ij} P^k = -J_k P^k = -\vec{J}\cdot\vec{P} \\
W_i &= -\frac{1}{2}\epsilon_{ijk0} J^{jk} P^0 - \frac{1}{2}\epsilon_{ij0k} J^{j0} P^k - \frac{1}{2}\epsilon_{i0jk} J^{0j} P^k \\
&= \frac{1}{2}\epsilon_{0ijk} J^{jk} P^0 - \frac{1}{2}\epsilon_{0ijk} J^{j0} P^k - \frac{1}{2}\epsilon_{0ijk} J^{j0} P^k \\
&= P_0 J_i + \epsilon_{ijk} P_j K_k
\end{aligned}
$$

$$
\vec{W} = P_0 \vec{J} + \vec{P} \times \vec{K}
$$

For $\vec{P} = 0 \quad , \quad P_0 = m \quad \Rightarrow \quad \vec{W} = +m\vec{J} = +m\vec{S}$

The commutation relations become:

$$
\begin{aligned}
&[J_{\mu\nu}, W_\rho] = i(\eta_{\nu\rho} W_\mu - \eta_{\mu\rho} W_\nu) \\
&[W_\mu, P_\nu] = 0 \\
&[W_\mu, W_\nu] = i\epsilon_{\mu\nu\rho\sigma} W^\rho P^\sigma
\end{aligned}
$$

Casimir invariants of the Poincare group

A Casimir operator is one that commutes with all the generators of the group.

Eigenvalues of the Casimir operators are labels of 1–particle states.

The Casimir operator C commutes with the Hamiltonian

$$H = P_0 \Rightarrow [C, H] = 0$$
$$P_0 = i\hbar \frac{\partial}{\partial t} \leftrightarrow \text{translation in time}$$

Hence, the eigenvalues of the Casimir operator are constants in time $\rightarrow$ constants of the motion.

single particle states: $\psi(x) = |\vec{P}, S\rangle$

$\vec{P}$ represents the momentum vector

S stands for other quantum numbers

$P_\mu P^\mu$ is the first Casimir operator of the Poincare group

$$P_\mu P^\mu |\vec{P}, S\rangle = m_0^2 |\vec{P}, S\rangle \quad \rightarrow \quad m_0 \text{ rest mass of the particle}$$

The rest mass of the particle is a Poincare invariant.

The second Casimir operator of the Poincare group is given by $W_\mu W^\mu$.

We saw that W_μ is a Lorentz four vector. Hence, $W_\mu W^\mu$ is a Lorentz scalar, and commutes with $J_{\mu\nu}$, the generators of the Lorentz group.

From the explicit form of W_μ it also follows that

$$[W_\mu, P_\nu] = 0$$

by using the asymmetry of $\epsilon_{\mu\nu\rho\sigma}$ and the symmetry of $P_\nu = i\partial_\nu$.

it follows that

$$[W_\mu W^\mu, P^\nu] = 0$$

Since, $W_\mu W^\mu$ is a Lorentz invariant, we can compute it in a convenient frame. If $m \neq 0$ it is convenient to choose the rest frame of the particle. In this frame

$$P^\mu = (m, 0, 0, 0).$$

$$W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho 0} J_{\nu\rho} P_0 + \cdots = -\frac{m}{2}\epsilon^{\mu\nu\rho 0} J_{\nu\rho} \cdots = \frac{m}{2}\epsilon^{0\mu\nu\rho} J_{\nu\rho}$$

$$\begin{aligned} \Rightarrow \quad W^0 &= 0 \\ W^i &= \frac{m}{2}\epsilon^{0ijk} J_{jk} = \frac{m}{2}\epsilon^{ijk} J_{jk} = mJ^i \end{aligned}$$

Therefore, on a one particle state with mass m and spin j we have $-W_\mu W^\mu = m^2 J_i J^i = m^2 \vec{J}^2$

$$-W_\mu W^\mu |\vec{P}, S\rangle = m^2 j(j+1)|\vec{P}, S\rangle \qquad (m \neq 0)$$