

$$\textcircled{1} \quad ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dx^2 + dy^2$$

$$\frac{37}{40}$$

$$a) \quad g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$b) \quad x \rightarrow x + \epsilon A(x, y)$$

$$y \rightarrow y + \epsilon B(x, y)$$

$$dx \rightarrow dx + \epsilon \left(\frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy \right) = \left(1 + \epsilon \frac{\partial A}{\partial x} \right) dx + \epsilon \frac{\partial A}{\partial y} dy$$

$$dy \rightarrow dy + \epsilon \left(\frac{\partial B}{\partial x} dx + \frac{\partial B}{\partial y} dy \right) = \left(1 + \epsilon \frac{\partial B}{\partial y} \right) dy + \epsilon \frac{\partial B}{\partial x} dx$$

$$dx^2 \Rightarrow dx^2 + 2\epsilon \frac{\partial A}{\partial x} \frac{\partial A}{\partial y} dx dy + \left(1 + 2\epsilon \frac{\partial A}{\partial x} \right) dx^2 + 2\epsilon \frac{\partial A}{\partial y} dx dy$$

$$dy^2 \Rightarrow \left(1 + 2\epsilon \frac{\partial B}{\partial y} \right) dy^2 + 2\epsilon \frac{\partial B}{\partial x} dx dy$$

$$ds^2 = dx^2 + dy^2 = \left(1 + 2\epsilon \frac{\partial A}{\partial x} \right) dx^2 + 2\epsilon \frac{\partial A}{\partial y} dx dy$$

$$+ \left(1 + 2\epsilon \frac{\partial B}{\partial y} \right) dy^2 + 2\epsilon \frac{\partial B}{\partial x} dx dy$$

\therefore to be invariant,

$$(1 + 2e \frac{\partial A}{\partial x}) = 1$$

$$(1 + 2e \frac{\partial B}{\partial y}) = 1$$

$$2e \frac{\partial A}{\partial y} + 2e \frac{\partial B}{\partial x} = 0$$

$$\Rightarrow \frac{\partial A}{\partial x} = 0$$

$$\therefore A \neq A(x)$$

$$A = A(y)$$

$$\frac{\partial B}{\partial y} = 0$$

$$\therefore B \neq B(y)$$

$$B = B(x)$$

$$\frac{\partial A}{\partial y} = -\frac{\partial B}{\partial x} = \text{Constant} = C$$

$$\therefore A(y) = Cy + a$$

$$B(x) = -Cx + b$$

~~It is a shift in the y-direction.~~
~~It is a shift in the x-direction.~~

'a' is a shift in space

'b' is also a shift in space

'c' is a ~~shift~~ rotation.

$$(2) a) F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu$$

$$= -\frac{1}{4} (A_{\mu\nu} - A_{\nu\mu}) (A^{\mu\nu} - A^{\nu\mu}) - j^\mu A_\mu$$

$$= -\frac{1}{4} \eta^{\alpha\mu} \eta^{\beta\nu} (A_{\beta\alpha} - A_{\alpha\beta}) (A_{\mu\nu} - A_{\nu\mu}) - j^\mu A_\mu$$

The Euler-Lagrange equations of motion,

$$\frac{\partial}{\partial x^\nu} \left(\frac{\partial \mathcal{L}}{\partial A_{\mu\nu}} \right) - \frac{\partial \mathcal{L}}{\partial A_\mu} = 0$$

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = -j^\mu$$

$$\frac{\partial}{\partial x^\nu} \left(\frac{\partial \mathcal{L}}{\partial A_{\mu\nu}} \right) \Rightarrow \frac{\partial \mathcal{L}}{\partial A_{\rho q}} = -\frac{1}{4} \eta^{\alpha\mu} \eta^{\beta\nu} (S_\alpha^\rho S_\beta^\nu - S_\beta^\rho S_\alpha^\nu) (A_{\mu\nu} - A_{\nu\mu})$$

$$+ (A_{\alpha\beta} - A_{\beta\alpha}) (S_\mu^\rho S_\nu^\nu - S_\nu^\rho S_\mu^\nu)$$

$$= -\frac{1}{4} (\eta^{\rho\mu} \eta^{\nu\nu} - \eta^{\nu\mu} \eta^{\rho\nu}) (A_{\mu\nu} + A_{\nu\mu}) + (\eta^{\rho\mu} \eta^{\beta\nu} - \eta^{\nu\mu} \eta^{\rho\beta}) (A_{\beta\alpha} - A_{\alpha\beta})$$

$$= -\frac{1}{4} (A^{\alpha\rho} - A^{\rho\alpha}) = -F^{\rho\alpha}$$

$$\therefore \frac{\partial \mathcal{L}}{\partial x^\nu} = -j^\nu \Rightarrow \partial_\mu F^{\mu\nu} = j^{\mu\nu}$$

$$\partial_\mu j^\mu = \partial_\mu \partial_\nu F^{\mu\nu} = 0 \quad \therefore \text{Current is conserved.}$$

② b) New Lagrangian,

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu + \frac{1}{2} m^2 A_\mu A^\mu$$

$$\begin{aligned} \text{New } \frac{\partial L}{\partial A_\mu} &= -j^\mu + \frac{1}{2} m^2 \eta^{\mu\alpha} A_\alpha + \frac{1}{2} m^2 \eta^{\mu\alpha} \partial_\mu (A_\alpha A_\alpha) \\ &= -j^\mu + \frac{1}{2} m^2 \eta^{\mu\alpha} [S_\mu A_\alpha + S_\mu A_\alpha] \\ &= -j^\mu + m^2 A^\mu \end{aligned}$$

$$\text{as } \frac{\partial}{\partial x^\nu} \left(\frac{\partial L}{\partial A_{\mu\nu}} \right) = \cancel{\partial_\mu F^\mu{}_\nu} - \partial_\mu F^{\mu\nu}$$

$$\therefore -\partial_\mu F^{\mu\nu} + j^\nu - m^2 A^\nu = 0$$

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu - j^\nu = 0$$

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = j^\nu$$

$$\Rightarrow (\partial_\mu \partial^\mu + m^2) A^\nu = j^\nu$$

$\partial_\mu F^\mu{}_\nu = 0 \leftarrow \text{ Lorenz gauge.}$