from the previous lecture ...

Pauli–Lubanski vector

$$W_\sigma = -\frac{1}{2}\epsilon_{\sigma\mu\nu\lambda} J^{\mu\nu} P^\lambda$$

The commutation relations become:

$$[J_{\mu\nu}, W_\rho] = i(\eta_{\nu\rho} W_\mu - \eta_{\mu\rho} W_\nu)$$
$$[W_\mu, P_\nu] = 0$$
$$[W_\mu, W_\nu] = i\epsilon_{\mu\nu\rho\sigma} W^\rho P^\sigma$$

Casimir invariants of the Poincare group : $P_\mu P^\mu = m^2$ and $W_\mu W^\mu$

If $m \neq 0$ take $P^\mu = (m, 0, 0, 0) \quad \Rightarrow \quad -W_\mu W^\mu = m^2 J_i J^i = m^2 \vec{J}^2$

$$-W_\mu W^\mu |\vec{P}, S\rangle = m^2 j(j+1)|\vec{P}, S\rangle \qquad (m \neq 0)$$

If $m = 0$ there is no rest frame and $|\vec{v}| = c$, *i.e.* the particle is travelling at the speed of light. We choose a frame with

$$P^\mu = (w, 0, 0, w).$$

$$\begin{aligned}
\Rightarrow W^0 &= -\frac{1}{2}\epsilon^{0ijk} J_{ij} P_k = wJ^3 = W^3 \\
W^1 &= -\frac{1}{2}\epsilon^{10jk} J_{0j} P_k - \frac{1}{2}\epsilon^{1j0k} J_{j0} P_k - \frac{1}{2}\epsilon^{1jk0} J_{jk} P_0 \\
&= w(J^1 - K^2) \\
\text{Similarly,} \qquad W^2 &= w(J^2 + K^1)
\end{aligned}$$

$$\text{Therefore,} \qquad -W_\mu W^\mu = w^2\left[(K^2 - J^1)^2 + (K^1 + J^2)^2\right] \qquad (m = 0)$$

$\lim m \to 0$ is nontrivial. Study separately the massive and massless cases.

- Massive representations

$$
\begin{aligned}
P_\mu P^\mu &= m^2 \\
W_\mu W^\mu &= -m^2 j(j+1)
\end{aligned}
$$

Take $m > 0$ massive states are labeled by their mass m and spin j.

For massive states we can choose the frame $P^\mu = (m, 0, 0, 0)$.

$\Rightarrow$ invariant under spatial rotations

The "Little group" of LT is the set of LT that leaves P^μ invariant.

$$P^\mu = \Lambda^\mu{}_\alpha P^\alpha$$

From the form of P^μ in the rest frame

$\Rightarrow$ "little group" for massive representations $\rightarrow$ spatial rotations

$$\Rightarrow \quad \text{the little group for massive representations is } SU(2)$$

massive representations of mass m are labeled by their spin $j = 0,\ \frac{1}{2},\ 1,\ \cdots$

states within each representations are labelled by
$j_z = -j, -j+1, \cdots, j-1, j$

$$\Rightarrow \quad \text{massive particles of spin } j \text{ have } \underline{2j+1} \text{ degrees of freedom}$$

- Massless representations $\quad P_\mu P^\mu = P^2 = m^2 = 0.$
  Two choices satisfy this condition: $P^\mu = (w, 0, 0, w)$ or $P^\mu = (0, 0, 0, 0)$.
  The second case is unphysical. It is unchanged under LT.
  In the case of massless states there is no rest frame.
  We want to find the little group in the case of massless states, *i.e.* the set of Lorentz transformations $\Lambda^\mu{}_\nu$ that leaves the momentum 4–vector invariant. Our problem is to solve the equation

$$\Lambda^\mu{}_\nu(p) P^\nu = P^\mu$$

where $P^\nu = (w, 0, 0, w)$.

First, we note that rotations in the $x, \; y$ plane leaves this p^ν invariant

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ o & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

This is an $SO(2)$ subgroup of $SU(2)$ generated by J^3, the generator of rotations in the $x - y$ plane. To find the most general transformation that leaves $P^\mu = (w, 0, 0, w)$ invariant, it is sufficient to look at the infinitesimal Lorentz transformations,

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

We look for the most general matrix $\omega^{\mu\nu}$ that satisfies

$$\omega^{\mu\nu} = -\omega^{\nu\mu}$$

and

$$\Lambda^{\mu\nu} P_\nu = (\delta^{\mu\nu} + \omega^{\mu\nu}) P_\nu = \delta^{\mu\nu} P_\nu + \omega^{\mu\nu} P_\nu = P^\mu$$

Hence, $\qquad \omega^{\mu\nu} P_\nu = 0$

For $P^\nu = (w, 0, 0, w) \ \rightarrow \ P_\nu = (w, 0, 0, -w)$

$$\Rightarrow \quad \begin{pmatrix} 0 & w^{01} & w^{02} & w^{03} \\ -w^{01} & 0 & w^{12} & w^{13} \\ -w^{02} & -w^{12} & 0 & w^{23} \\ -w^{03} & -w^{13} & -w^{23} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = 0$$

This gives the constraints

$$\begin{aligned} w^{03} &= 0 \\ w^{01} + w^{13} &= 0 \\ w^{02} + w^{23} &= 0 \end{aligned}$$

Denoting $w^{01} = \alpha;\ w^{02} = \beta;\ w^{12} = \theta$, the most general transformation that leaves P^μ invariant is given by

$$\Lambda = \mathrm{e}^{-i(\alpha A + \beta B + \theta C)}$$

where (with a lower second index)

$$A^\mu{}_\nu = i \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad B^\mu{}_\nu = i \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{2}$$

$$C^\mu{}_\nu = i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The matrix C corrspond to rotations in the $x - y$ plane, whereas the A and B matrices are given by combinations of boost and rotation generators.

$$\begin{aligned} C^{\mu}{}_{\nu} &= \left(J^3\right)^{\mu}{}_{\nu} \\ A^{\mu}{}_{\nu} &= \left(K^1 + J^2\right)^{\mu}{}_{\nu} \\ B^{\mu}{}_{\nu} &= \left(K^2 - J^1\right)^{\mu}{}_{\nu} \end{aligned}$$

We showed that for massless states the Poincare invariant $-W_{\mu}W^{\mu}$ is given by

$$-W_{\mu}W^{\mu} = w^2\left[(K^2 - J^1)^2 + (K^1 + J^2)^2\right]$$

Hence, we obtained

$$-W_{\mu}W^{\mu} = w^2\left[A^2 + B^2\right]$$

Using the expressions for the matrices A, B and C that we found or the commutation relations of the Lorentz algebra we find that the J^3, A and B generators close an algebra

$$\left[J^3, A\right] = iB \quad ; \quad \left[J^3, B\right] = -iA \quad ; \quad [A, B] = 0$$

This is the same as the algebra generated by

$$P^x, \; P^y, \text{ and } L^z = (xP^y - yP^x) \; ,$$

i.e. translations and rotations in the Euclidean $x - y$ plane with A and B playing the role of translation operators. The algebra is denoted as $ISO(2)$. It is a non–compact algebra, *i.e.* it is infinite dimensional due to the continuous eigenvalues of the momentum operators P^x and P^y. Since A and B commute their eigenvalues are continuous and non–compact. We find that for massless particles there exists a continuous degree of freedom that is not realised physically.

We demand therefore that the operators A and B annihilate the physical massless states

$$\begin{aligned}\hat{A}|\vec{P}, a, b\rangle &= a|\vec{P}, a, b\rangle \\ \hat{B}|\vec{P}, a, b\rangle &= b|\vec{P}, a, b\rangle\end{aligned}$$

with $a = b = 0$ for physical massless states. Therefore,

$$-W_\mu W^\mu = 0$$

for physical massless states. This agrees well with

$$\lim_{m\to 0} W_\mu W^\mu = \lim_{m\to 0} -m^2(j(j+1)) = 0$$

that we found in the massive case. For massless states with $a, \; b = 0$ the little group is $SO(2)$ or $U(1)$. The generator of rotations in the $x - y$ plane is J^3. Hence, the representations are labeled by the eigenvalue h of J^3, which is the angular momentum in the direction of propagation.