

MATH431.

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φ1

(a) $ds^2 = g_{uv} dx^u dx^v = dx^2 + dy^2$

$$g_{uv} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad |$$

inverse = $(g_{uv})^{-1} = g^{uv} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad |$

(b) $x \rightarrow x + \epsilon A(x, y)$

$$y \rightarrow y + \epsilon B(x, y)$$

$$dx \rightarrow dx + \epsilon \left(\frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy \right)$$

$$dy \rightarrow dy + \epsilon \left(\frac{\partial B}{\partial y} dy + \frac{\partial B}{\partial x} dx \right)$$

$$ds^2 \rightarrow \left[\left(1 + \epsilon \frac{\partial A}{\partial x}\right) dx + \epsilon \frac{\partial A}{\partial y} dy \right]^2 + \left[\left(1 + \epsilon \frac{\partial B}{\partial x}\right) dx + \epsilon \frac{\partial B}{\partial y} dy \right]^2$$

we require invariance of ds^2 .

Expanding to first order in ϵ .

$$dx^2 \rightarrow \left(1 + 2\epsilon \frac{\partial A}{\partial x}\right) dx^2 + 2\epsilon \left(\frac{\partial A}{\partial y}\right) dx dy$$

$$dy^2 \rightarrow \left(1 + 2\epsilon \frac{\partial B}{\partial y}\right) dy^2 + 2\epsilon \left(\frac{\partial B}{\partial x}\right) dx dy$$

we then find the following

$$ds^2 \rightarrow \left[\underset{\textcircled{1}}{\left(1 + \epsilon \frac{\partial A}{\partial x}\right) dx + \epsilon \frac{\partial A}{\partial y} dy} \right]^2 + \left[\underset{\textcircled{2}}{\left(1 + \epsilon \frac{\partial B}{\partial x}\right) dx + \epsilon \frac{\partial B}{\partial y} dy} \right]^2$$

(to check).

$$\textcircled{1} \left[\left(1 + \epsilon \frac{\partial A}{\partial x}\right) dx + \epsilon \frac{\partial A}{\partial y} dy \right] \left[\left(1 + \epsilon \frac{\partial A}{\partial x}\right) dx + \epsilon \frac{\partial A}{\partial y} dy \right]$$

$$= \left(1 + \epsilon \frac{\partial A}{\partial x}\right)^2 dx^2 + 2 \left(1 + \epsilon \frac{\partial A}{\partial x}\right) dx \epsilon \frac{\partial A}{\partial y} dy + \epsilon^2 \left(\frac{\partial A}{\partial y}\right)^2 dy^2$$

$$= (dx^2) \left(1 + 2\epsilon \frac{\partial A}{\partial x} + \epsilon^2 \frac{\partial^2 A}{\partial x^2}\right) + 2\epsilon \frac{\partial A}{\partial y} dx dy + \dots \mathcal{O}(\epsilon^2)$$

$\overline{\quad}$ 2nd order term.

$$= (dx)^2 + 2\epsilon \frac{\partial A}{\partial x} dx^2 + 2\epsilon \frac{\partial A}{\partial y} dx dy$$

$$\textcircled{2} \left[\left(1 + \epsilon \frac{\partial B}{\partial y} \right) dx + \epsilon \frac{\partial B}{\partial x} dy \right] \left[\left(1 + \epsilon \frac{\partial B}{\partial y} \right) dy + \epsilon \frac{\partial A}{\partial y} dx \right]$$

$$\Rightarrow \left(1 + \epsilon \frac{\partial B}{\partial y} \right)^2 dx^2 + 2 \left(1 + \epsilon \frac{\partial B}{\partial y} \right) dx \epsilon \frac{\partial A}{\partial y} + \epsilon^2 \frac{\partial B^2}{\partial y} dy^2$$

$$\Rightarrow (dy)^2 + 2\epsilon \frac{\partial B}{\partial y} dy^2 + 2\epsilon \frac{\partial B}{\partial x} dx dy$$

$$ds^2 \rightarrow dx^2 + dy^2 + 2\epsilon \left(\frac{\partial A}{\partial x} dx^2 + \frac{\partial B}{\partial y} dy^2 + \left(\frac{\partial A}{\partial y} + \frac{\partial B}{\partial x} \right) dx dy \right) + O(\epsilon^2)$$

terms of additional coefficients vanish.

$$dx^2: \frac{\partial A}{\partial x} = 0 \Rightarrow A = A(y)$$

$$dy^2: \frac{\partial B}{\partial y} = 0 \Rightarrow B = B(x)$$

$$dx dy: \left(\frac{\partial A}{\partial y} + \frac{\partial B}{\partial x} \right) = 0 \Rightarrow \frac{\partial A}{\partial y} = -\frac{\partial B}{\partial x} = \text{constant} = c.$$

$$\textcircled{3} A(y) = cy + a, \quad B(x) = cx + b$$

We obtain 3 degrees of freedom represented by the constants a, b, c .

- $a = b = 0$; $y \rightarrow y + a$ - $y \rightarrow y + b$
 $b \neq 0$ (translation in space)

- We end up from the line element with 2 translations in space and 1 rotation.
translation in x and y and rotation in xy plane.

$$\frac{18}{20}$$

○

Q2

(a) $L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu$

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$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} \eta^{\alpha\mu} \eta^{\beta\nu} (A_{\beta,\alpha} - A_{\alpha,\beta}) (A_{\mu,\nu} - A_{\nu,\mu})$$

$$= -\frac{1}{4} (A_{\mu,\nu} - A_{\nu,\mu}) (A^{\mu\nu} - A^{\nu\mu})$$

○

From Euler Lagrange eqn.

$$\frac{\partial}{\partial x^\nu} \left(\frac{\partial L}{\partial A_{\mu,\nu}} \right) - \frac{\partial L}{\partial A_\mu} = 0 \quad 2$$

$$\frac{\partial L}{\partial A_{p,q}} = -\frac{1}{4} \eta^{\alpha\mu} \eta^{\beta\nu} (\delta_\alpha^p \delta_\beta^q - \delta_\beta^p \delta_\alpha^q) (A_{\mu,\nu} - A_{\nu,\mu})$$

$$+ (A_{\alpha,\beta} - A_{\beta,\alpha}) (\delta_\mu^p \delta_\nu^q - \delta_\nu^p \delta_\mu^q)$$

○

$$= -\frac{1}{4} (\eta^{\rho\mu} \eta^{q\nu} - \eta^{q\mu} \eta^{\rho\nu}) (A_{\mu,\nu} + A_{\nu,\mu})$$

$$+ (\eta^{\alpha\rho} \eta^{\beta q} - \eta^{\alpha q} \eta^{\beta\rho}) (A_{\beta,\alpha} - A_{\alpha,\beta})$$

$$= -4 \times \frac{1}{4} (A^{p,q} - A^{q,p}) = -F^{pq} \quad 4$$

$$\frac{\partial}{\partial x^\nu} \left(\frac{\partial L}{\partial A_{\mu,\nu}} \right) - \left(\frac{\partial L}{\partial A_\mu} \right) = \frac{1}{4} \frac{\partial}{\partial x^\nu} (-F^{pq}) = -j^\nu$$

○

j^μ comes from $\left(\frac{\partial L}{\partial A_\mu} \right)$ term. $\therefore \frac{\partial}{\partial x^\nu} = \partial_\mu \Rightarrow \partial_\mu F^{\mu\nu} = j^\nu$ 3

hence show ~~$j^\mu = 0$~~ .

$$\partial_\mu \partial_\nu F^{\mu\nu} = \partial_\mu j^\mu = 0$$

given a symmetry transformation, we can show that the four current is conserved.

i.e show $\partial_\mu j^\mu = 0$.

$$\partial_\mu j^\mu = \nabla \cdot \vec{J} = 0$$

(6) $\frac{1}{2} m^2 A_\mu A^\mu$

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu + \frac{1}{2} m^2 A_\mu A^\mu$$

$$\left(\frac{\partial \mathcal{L}}{\partial A_{\mu,\nu}} \right) = \frac{\partial}{\partial x^\nu} \left(\frac{\partial \mathcal{L}}{\partial A_{\mu,\nu}} \right)$$

$$\frac{\partial}{\partial x^\nu} \left(\frac{\partial \mathcal{L}}{\partial A_{\mu,\nu}} \right) = (A^{\mu,\nu} - A^{\nu,\mu}) + \frac{\partial}{\partial x^\nu} \left(\frac{1}{2} m^2 A_\mu A^\mu \right)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\textcircled{1} = -(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{\partial}{\partial x^\nu} \left(\frac{1}{2} m^2 A_\mu A^\mu \right)$$

$$\Rightarrow \frac{1}{2} m^2 \frac{\partial}{\partial x^\nu} (A_\mu A^\mu)$$

...

$$\Rightarrow (\partial_\mu A^\mu + \partial^\mu A_\mu + m^2 A^\mu)$$

$$\Rightarrow \partial_\mu A^\mu A_\mu + \partial_\mu A^\mu A_\mu + \partial_\mu A_\mu A^\mu$$

(cancels with term in ①)