

MATH431 Modern Particle Physics Solutions 6

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$$\psi(r, \theta, \phi) = R(r) \begin{pmatrix} 1 \\ 0 \\ ia \cos \theta \\ ia e^{i\phi} \sin \theta \end{pmatrix}$$

Normalize

$$\begin{aligned} \int d^3\mathbf{r} \psi^\dagger \psi &= 1 = \int 4\pi r^2 dr (|R|^2 (1 + a^2)) \\ \Rightarrow \int_0^\infty r^2 |R|^2 dr &= [4\pi(1 + a^2)]^{-1} \end{aligned}$$

(a.)

$$L_z = -i \frac{\partial}{\partial \phi} \Rightarrow L_z \psi = R \begin{pmatrix} 0 \\ 0 \\ 0 \\ ia e^{i\phi} \sin \theta \end{pmatrix} \not\propto \psi$$

so ψ is not an eigenstate of L_z .

(b.)

$$\begin{aligned} \langle L_z \rangle &= \int d^3\mathbf{r} \psi^\dagger L_z \psi = \int 2\pi r^2 d \cos \theta |R|^2 a^2 \sin^2 \theta \\ \int_{-1}^1 d \cos \theta (1 - \cos^2 \theta) &= 2 - \frac{2}{3} = \frac{4}{3} \Rightarrow \langle L_z \rangle = \frac{8\pi}{3} a^2 \cdot \frac{1}{4\pi(1 + a^2)} \\ \Rightarrow \langle L_z \rangle &= \frac{2a^2}{3(1 + a^2)} \end{aligned}$$

In H-atom, $v/c \sim \alpha \Rightarrow \langle L_z \rangle = O(v^2/c^2)$. This is a relativistic effect - spin-orbit interaction.

(c.)

$$\begin{aligned} S_z &= \frac{1}{2} \hbar \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\ \Rightarrow S_z \psi &= \frac{1}{2} \hbar R \begin{pmatrix} 1 \\ 0 \\ ia \cos \theta \\ -ia e^{i\phi} \sin \theta \end{pmatrix} \\ (L_z + S_z) \psi &= R \begin{pmatrix} 1 \\ 0 \\ ia \cos \theta \\ ia e^{i\phi} \sin \theta \end{pmatrix} = \frac{1}{2} \psi \Rightarrow J_z = +\frac{1}{2} \end{aligned}$$

2.

$$\gamma^\mu \partial_\mu \psi + im\psi = 0 \quad , \quad (\partial\psi^\dagger)\gamma^{\mu\dagger} - im\psi^\dagger = 0$$

$$\Rightarrow (\partial\psi^\dagger)\gamma^0\gamma^\mu\gamma^0 - im\psi^\dagger = 0$$

$$\Rightarrow (\partial_\mu\bar{\psi})\gamma^\mu - im\bar{\psi} = 0$$

- $\partial_\mu(\bar{\psi}\gamma^\mu\psi) = (\partial_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu(\partial_\mu\psi)$
 $= (im\bar{\psi})\psi + \bar{\psi}(-im\psi) = 0$

- $\partial_\mu(\bar{\psi}\gamma^\mu\gamma^5\psi) = (\partial_\mu\bar{\psi})\gamma^\mu\gamma^5\psi + \bar{\psi}\gamma^\mu\gamma^5(\partial_\mu\psi)$
 $= (im\bar{\psi})\gamma^5\psi - \bar{\psi}\gamma^5(\gamma^\mu\partial_\mu\psi) = 2im\bar{\psi}\gamma^5\psi$

3.

$$\bar{u}_f(\not{p}_f - m)\gamma^\mu u_i = \bar{u}_f\gamma^\mu(\not{p}_i - m)u_i = 0 \quad (\text{Dirac eq.})$$

$$\Rightarrow 2m\bar{u}_f\gamma^\mu u_i = \bar{u}_f(\not{p}_f\gamma^\mu + \gamma^\mu\not{p}_i)u_i$$

$$\not{p}_f\gamma^\mu + \gamma^\mu\not{p}_i = \gamma^\nu\gamma^\mu p_{f\nu} + \gamma^\mu\gamma^\nu p_{i\nu}$$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

$$\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu = -2i\sigma^{\mu\nu}$$

Hence

$$\gamma^\mu\gamma^\nu = g^{\mu\nu} - i\sigma^{\mu\nu}; \gamma^\nu\gamma^\mu = g^{\mu\nu} + i\sigma^{\mu\nu}$$

$$\Rightarrow \not{p}_f\gamma^\mu + \gamma^\mu\not{p}_i = g^{\mu\nu}(p_f + p_i)_\nu + i\sigma^{\mu\nu}(p_f - p_i)_\nu = (p_f + p_i)^\mu + i\sigma^{\mu\nu}(p_f - p_i)_\nu$$

$$\Rightarrow \bar{u}_f\gamma^\mu U_i = \frac{1}{2m}\bar{u}_f[(p_f + p_i)^\mu + i\sigma^{\mu\nu}(p_f - p_i)_\nu]u_i$$

4. We consider an electron in a constant magnetic field $\vec{B} = (0, 0, B)$ with $B > 0$.

(a.)

The vector potential

$$A^\mu = (0, 0, Bx, 0)$$

(b.)

$$(i\partial_0 - m)\phi = \vec{\sigma} \cdot (\vec{p} - e\vec{A})\chi$$

$$(i\partial_0 + m)\chi = \vec{\sigma} \cdot (\vec{p} - e\vec{A})\phi$$

where, as usual, $\vec{p} = -i\nabla$.

(c.) Assuming a solution of the form

$$\phi(x) = \phi(\vec{x})e^{-iEt}, \chi(x) = \chi(\vec{x})e^{-iEt}$$

Inserting into the equations from (b.) these equations become

$$\begin{aligned}(E - m)\phi(\vec{x}) &= \vec{\sigma} \cdot (\vec{p} - e\vec{A})\chi(\vec{x}) \\ (E + m)\chi(\vec{x}) &= \vec{\sigma} \cdot (\vec{p} - e\vec{A})\phi(\vec{x})\end{aligned}$$

Substituting $\chi(\vec{x})$ from the second equation into the first and repeating the steps that we too in class when deriving the gyromagnetic factor from the Dirac equation, we get

$$\begin{aligned}(E^2 - m^2)\phi(\vec{x}) &= [(\vec{p} - e\vec{A})^2 - e\vec{\sigma} \cdot \vec{B}]\phi(\vec{x}) \\ &= [\vec{p}^2 + e^2 B^2 x^2 - 2ep_y Bx - e\sigma_z B]\phi(\vec{x})\end{aligned}$$

Since p_x, p_y commute with x , we can search for solutions of the form

$$\phi(\vec{x}) = e^{i(p_y y + p_z z)} f(x)$$

where p_y and p_z are c -numbers and $f(x)$, as $\phi(\vec{x})$, is a two component spinor. The equation for $f(x)$ becomes

$$\left[-\frac{d^2}{dx^2} + (p_y - eBx)^2 - eB\sigma_z\right]f(x) = (E^2 - m^2 - p_z^2)f(x)$$

$f(x)$ can be taken to be an eigenfunction of σ_z with eigenvalues $\sigma = \pm 1$, $\sigma_z f = \sigma f$. Then

$$\left[-\frac{d^2}{dx^2} + \frac{1}{2}(2e^2 B^2)(x - \frac{p_y}{eB})^2\right]f(x) = (E^2 - m^2 - p_z^2 + eB\sigma)f(x)$$

This is formally identical to the Schrödinger equation of an harmonic oscillator with frequency $2|e|B$. The energy levels are therefore given by

$$E^2 - m^2 - p_z^2 + eB\sigma = (n + \frac{1}{2})2|e|B$$

or

$$E = [m^2 + p_z^2 + (2n + 1 + \sigma)|e|B]^{\frac{1}{2}}$$

Observe that there is a continuous degeneracy in p_x and p_y , as well as a discrete degeneracy

$$E(n, p_z, \sigma = +1) = E(n + 1, p_z, \sigma = -1).$$

In the nonrelativistic limit $p_z \ll m$, $(2n + 1)|e|B \ll m^2$ the nonrelativistic limit therefore gives

$$E(n, p_z, \sigma) \simeq m + \frac{p_z^2}{2m} + \left(n + \frac{1 + \sigma}{2}\right)\omega_B$$

with $\omega_B = |e|B/m$. These are the Landau levels of nonrelativistic quantum mechanics.