from the previous lecture ...

Action principle for fixed values of $q(t_i) = q_{in}$, $q(t_f) = q_{out}$, then the classical trajectory which satisfies these boundary conditions is an extremum of the action

$$\delta \int_{t_{in}}^{t_{out}} dt \; L(q_i, \dot{q}_i, t) = 0$$

$$\Rightarrow \qquad \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

Classical field theory

the action is given by

$$S = \int d^4x \; \mathcal{L}(\phi, \partial^\mu \phi) \quad \leftarrow \quad \text{relativistic notation}$$

The equations of motions for the field ϕ are obtained by requiring that the variation of the action vanishes and demanding that $\delta\phi = 0$ at t_1 and t_2.

$$\delta \int \mathcal{L}(\phi, \partial^\mu \phi) d^4x = 0$$

we note that there is no explicit dependence of $\mathcal{L}$ on x^μ.
Since this holds for any $\delta\phi$ we have that

$$\frac{\partial \mathcal{L}(\phi, \partial^\mu \phi)}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}(\phi, \partial^\mu \phi)}{\partial(\partial_\mu \phi)} \right) = 0$$

These are the Euler–Lagrange equations of motion for the field ϕ.

Example: in the case of the harmonic chain

$$\mathcal{L} = \frac{1}{2}\mu\dot{y}^2 - \frac{1}{2}\tau {y'}^2$$

$$\frac{\partial\mathcal{L}}{\partial\dot{y}} = \mu\dot{y} \quad ; \quad \frac{\partial\mathcal{L}}{\partial y'} = \tau y' \quad ; \quad \frac{\partial\mathcal{L}}{\partial y} = 0$$

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu y)} - \frac{\partial\mathcal{L}}{\partial y} = \mu\frac{\partial^2 y}{\partial t^2} - \tau\frac{\partial^2 y}{\partial x^2} = 0$$

which is the familiar wave equation.

If $\quad \mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\eta^{\alpha\beta}(\partial_\alpha\phi)(\partial_\beta\phi) - \frac{1}{2}m^2\phi^2$

$$
\begin{aligned}
\frac{\partial\mathcal{L}(\phi,\partial^\mu\phi)}{\partial(\partial_\mu\phi)} &= \frac{1}{2}\eta^{\alpha\beta}\frac{\partial(\partial_\alpha\phi)}{\partial(\partial_\mu\phi)}(\partial_\beta\phi) + \frac{1}{2}\eta^{\alpha\beta}(\partial_\alpha\phi)\frac{\partial(\partial_\beta\phi)}{\partial(\partial_\mu\phi)} \\
&= \frac{1}{2}\eta^{\alpha\beta}\delta_{\alpha\mu}(\partial_\beta\phi) + \frac{1}{2}\eta^{\alpha\beta}\delta_{\beta\mu}(\partial_\alpha\phi) \\
&= \frac{1}{2}\eta^{\mu\beta}(\partial_\beta\phi) + \frac{1}{2}\eta^{\mu\alpha}(\partial_\alpha\phi) = \frac{1}{2}2\eta^{\mu\nu}(\partial_\nu\phi)
\end{aligned}
$$

$$
\begin{aligned}
\partial_\mu\left(\frac{\partial\mathcal{L}(\phi,\partial^\mu\phi)}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}(\phi,\partial^\mu\phi)}{\partial\phi} &= \partial_\mu(\eta^{\mu\nu}\partial_\nu\phi) + m^2\phi \\
&= (\partial_\mu\partial^\mu + m^2)\phi(\vec{x},t) = 0
\end{aligned}
$$

This is the Klein–Gordon equation of a reletivistic free scalar field that we will encounter in more detail later on.

The Hamiltonian for a field can be written has

$$H = \int h(\phi, \pi) dxdydz$$

where the generalised momentum density is given by

$$\pi(\vec{x}, t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

where the generalised Hamiltonian density is defined by

$$h = \dot{\phi}(\vec{x}, t)\pi(\vec{x}, t) - \mathcal{L}$$

and

$$H = \int d^3x(\dot{\phi}(\vec{x}, t)\pi(\vec{x}, t) - \mathcal{L}) = \int d^3x \; h(\phi, \pi)$$

For the Klein–Gordon field,

$$\mathcal{L} = \frac{1}{2}(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2)$$

We used here $\partial^\mu = (\partial_t, \vec{\nabla})$, $\partial_\mu = (\partial_t, -\vec{\nabla})$ $\rightarrow$ $\Box = \partial_\mu \partial^\mu = (\partial_t^2 - \vec{\nabla}^2)$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$$

$$h = \dot{\phi}^2 - \left(\frac{1}{2}(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2)\right) = \frac{1}{2}\left(\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2\right)$$

the total energy $\int h d^3x$ should be conserved if $H \neq H(t)$, and be the zero component of some four vaector P^μ. To construct it we introduce the energy momentum tensor

$$T^{\mu\nu} = \partial^\mu \phi \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} - \mathcal{L}\eta^{\mu\nu}$$

$$\Rightarrow P^\mu = \int T^{\mu 0} d^3x$$

$$\text{and } H = P^0 = \int T^{00} d^3x$$

$T^{\mu\nu}$ is conserved as it satisfies the continuity equation

$$
\begin{aligned}
& \partial_\nu T^{\mu\nu} = \\
= \ & \partial_\nu \left(\partial^\mu \phi \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} - \mathcal{L}\eta^{\mu\nu} \right) \\
= \ & \partial^\mu \phi \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \right) + \partial_\nu \partial^\mu \phi \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} - \eta^{\mu\nu} \left(\frac{\partial \mathcal{L}}{\partial \phi} \partial_\nu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi)} \partial_\nu \partial_\lambda \phi \right) \\
= \ & \left(\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} \right) \partial^\mu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\nu \partial^\mu \phi - \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi)} \partial^\mu \partial_\lambda \phi = 0
\end{aligned}
$$

The conservation of $T^{\mu\nu}$ implies that

$$P^\mu = \int T^{\mu 0} d^3 x$$

transforms as a 4–vector and is time independent.

P^μ is the energy–momentum 4–vector.

The Klein–Gordon equation

In non–relativistic quantum mechanics

$$\vec{P} \to -i\hbar\vec{\nabla} \quad , \quad E \to i\hbar\frac{\partial}{\partial t} \text{ quantum operators}$$

The Hamiltonian for an energy conserving system is

$$H = \frac{\vec{P}}{2m} + V(\vec{q}) = E$$

leads to the Shrödinger equation by substitution

$$\left(-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{q})\right)\Psi(\vec{q},t) = i\hbar\frac{\partial}{\partial t}\Psi(\vec{q},t)$$

In special relativity the four vector P^μ is given by

$$P^\mu = (\frac{E}{c}, \vec{P})$$

we have $\partial^\mu = (\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla})$; $\partial_\mu = (\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla})$ Therefore we can identify $P^\mu = i\hbar\partial^\mu$.

In special relativity $E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$, $\vec{P} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}$ Therefore,

$P_\mu P^\mu = \frac{E^2}{c^2} - \vec{P}^2 = m^2c^2$. Following the example of non–relativistic quantum mechanics we use $P^\mu \to i\hbar\partial^\mu$ and obtain the wave equation

$$-\hbar^2\partial_\mu\partial^\mu\phi = m^2c^2\phi$$

or

$$\left(\partial^2 + \frac{m^2c^2}{\hbar^2}\right)\phi(\vec{X}, t) = 0 \qquad \leftarrow \qquad \text{the Klein–Gordon equation}$$

Its interpretation as a single particle is problematic. The equation describes a scalar field but not in a single state but a multi–state, *i.e.* a quantised field.