MATH 431 — Exam Summer 2011: Solutions

All problems are similar to homework problems or material covered in the lectures. Marking as indicated on the question sheet.

1. *Poincaré group in 1+2 dimensions; Pauli-Lubanski vector*

$$\mathrm{d}s^2 \;=\; \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 \tag{1}$$

(a)

$$g_{\mu\nu} \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\,, \qquad g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\,.$$

The line element is invariant under 3 translations ($\mathrm{d}t$, $\mathrm{d}x$ and $\mathrm{d}y$), 2 boosts ($\mathrm{d}t\mathrm{d}x$, $\mathrm{d}t\mathrm{d}y$) and 1 rotation ($\mathrm{d}x\mathrm{d}y$). The generators associated with the transformations of the three translations are: $P_0 = i\partial_t$, $P_1 = -i\partial_x$ and $P_2 = -i\partial_y$. (The generators of the boosts, K_1 and K_2, and the generator of rotations in the (x, y) plane, J_3, are the 1+2 dimensional versions of the usual 1+3 dimensional generators which were discussed in the lectures.) **Bookwork**

(b)

$$W^\mu \;=\; -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}J_{\nu\rho}P_\sigma\,,$$

$$\begin{aligned}
W^0 \;&=\; -\frac{1}{2}\epsilon^{0ijk}J_{ij}P_k \;=\; \frac{1}{2}\epsilon_{0ijk}J_{ij}P_k \;=\; \frac{1}{2}\epsilon_{kij}J_{ij}P_k \;=\; J_kP_k\,,\\
W^i \;&=\; -\frac{1}{2}\epsilon^{i0jk}J_{0j}P_k - \frac{1}{2}\epsilon^{ij0k}J_{j0}P_k - \frac{1}{2}\epsilon^{ijk0}J_{jk}P_0 \;=\\
&\quad -\frac{1}{2}\epsilon^{i0jk}J_{0j}P_k - \frac{1}{2}\epsilon^{i0jk}J_{0j}P_k + \frac{1}{2}\epsilon^{0ijk}J_{jk}P_0 \;=\\
&\quad +\frac{1}{2}\epsilon^{0ijk}J_{0j}P_k + \frac{1}{2}\epsilon^{0ijk}J_{0j}P_k + \frac{1}{2}\epsilon^{0ijk}J_{jk}P_0 \;=\\
&\quad -\epsilon_{0ijk}J_{0j}P_k - \frac{1}{2}\epsilon_{0ijk}J_{jk}P_0 \;=\; \epsilon_{ijk}K_jP_k - J_iP_0\,.
\end{aligned}$$

In the case of the 1+2 dimensional line element Eq. (1), embedded in our usual 1+3 dimensions,

$$\vec{K} \;=\; (K_1, K_2, 0) \quad \text{and} \quad \vec{\mathrm{J}} \;=\; (0, 0, \mathrm{J}_3)\,.$$

For $m = 0$ we can choose, without loss of generality, $P^\mu = (p, 0, p, 0)$, and hence $P_\mu = (p, 0, -p, 0)$. Then

$$\begin{aligned}
W^0 \;&=\; 0\,,\\
W^1 \;&=\; \epsilon_{1jk}K_jP_k \;=\; \epsilon_{123}K_2P_3 + \epsilon_{132}K_3P_2 \;=\; 0\,,\\
W^2 \;&=\; \epsilon_{2jk}K_jP_k \;=\; \epsilon_{213}K_1P_3 + \epsilon_{231}K_3P_1 \;=\; 0\,,\\
W^3 \;&=\; -J_3P_0 + \epsilon_{3jk}K_jP_k \;=\; -J_3P_0 + \epsilon_{312}K_1P_2 + \epsilon_{321}K_2P_1 \;=\\
&=\; -J_3P_0 - K_1P_0 \;=\; -(J_3 + K_1)P_0\,.
\end{aligned}$$

For $m > 0$ we can evaluate the Pauli-Lubanski vector most conveniently in the rest frame, hence $P^\mu = (m, 0, 0, 0)$. Then

$$\begin{aligned}
W^0 &= 0\,,\\
W^1 &= 0\,,\\
W^2 &= 0\,,\\
W^3 &= -J_3 m\,.
\end{aligned}$$

2. (a) The Hamiltonian H_0 for a free real scalar field is

$$H_0 = \tfrac{1}{2}\int \left(\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2\right) \mathrm{d}^3x\,.$$

From the Heisenberg equation of motion for the generalised momentum operator π and the canonical commutation relations we calculate:

$$\begin{aligned}
i\hbar\dot{\pi}(\mathbf{x}) &= [\pi(\mathbf{x},t), H]\\
&= -[\int \mathrm{d}^3\mathbf{x}' \left(\tfrac{1}{2}\pi(\mathbf{x}')^2 + \tfrac{1}{2}(\nabla'\phi(\mathbf{x}'))^2 + \tfrac{1}{2}m^2\phi(\mathbf{x}')^2\right), \pi(\mathbf{x})]\\
&= -\tfrac{1}{2}\int \mathrm{d}^3\mathbf{x}' \left((\nabla'\phi(\mathbf{x}'))\,[\nabla'\phi(\mathbf{x}'), \pi(\mathbf{x})] + [\nabla'\phi(\mathbf{x}'), \pi(\mathbf{x})]\,(\nabla'\phi(\mathbf{x}'))\right)\\
&\quad -\tfrac{1}{2}m^2\int \mathrm{d}^3\mathbf{x}' \left(\phi(\mathbf{x}')[\phi(\mathbf{x}'), \pi(\mathbf{x})] + [\phi(\mathbf{x}'), \pi(\mathbf{x})]\phi(\mathbf{x}')\right)\\
&= -i\hbar\int \mathrm{d}^3\mathbf{x}' \left((\nabla'\phi(\mathbf{x}')) \,.\, (\nabla'\delta(\mathbf{x}-\mathbf{x}')) + m^2\phi(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')\right)\\
&= i\hbar\int \mathrm{d}^3\mathbf{x}' \left(\left((\nabla')^2\phi(\mathbf{x}')\right)\delta(\mathbf{x}-\mathbf{x}') - m^2\phi(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')\right)\\
&= i\hbar\left(\nabla^2\phi(\mathbf{x}) - m^2\phi(\mathbf{x})\right)\,.
\end{aligned}$$

We now use (as derived in the lecture) that the generalised momentum $\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}$ for the free Klein-Gordon field is $\pi = \dot{\phi}$, and hence immediately get the Klein-Gordon equation

$$\ddot{\phi}(\mathbf{x}) = \nabla^2\phi(\mathbf{x}) - m^2\phi(\mathbf{x})\,,$$

or, written with the usual d'Alembert operator ($\Box \equiv \partial^\mu\partial_\mu$):

$$\left(\Box + m^2\right)\phi = 0\,.$$

(b) Inserting the given solution

$$\phi(x,y) = \sum_{n=1}^{\infty}\phi_n(x)\,\mathrm{cs}\left(\frac{n\pi y}{R}\right)$$

into the five-dimensional Klein-Gordon equation and carrying out $\partial^2/\partial y^2$, one obtains a four-dimensional Klein-Gordon equation for each of the Fourier coefficients $\phi_n(x)$:

$$\left(\partial_0^2 - \nabla^2 + m^2 + \left(\frac{n\pi}{R}\right)^2\right)\phi_n(x) = 0\,.$$

Therefore the given $\phi(x,y)$ is a solution of the five-dimensional Klein-Gordon equation *if* the $\phi_n(x)$ are solutions of the four-dimensional equations. The masses m_n of the fields ϕ_n are given by

$$m_n^2 = m^2 + \left(\frac{n\pi}{R}\right)^2\,.$$

If the five-dimensional mass m is zero, we get the equally spaced mass spectrum

$$m_n = \frac{n\pi}{R}\,.$$

The infinite set of particles are called Kaluza-Klein tower.

Bookwork

3. (a) (i)

$$\{\gamma_\mu, \gamma_\nu\} \;=\; 2\eta_{\mu\nu}\mathbb{I}_4 \quad\Rightarrow\quad \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu \;=\; 2\eta_{\mu\nu}\mathbb{I}_4\,,$$

where $\mathbb{I}_4$ is the 4-dimensional identity matrix. Taking the trace, and using $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, $\mathrm{tr}\mathbb{I}_4 = 4$, we get

$$\mathrm{tr}[\gamma_\mu\gamma_\nu] \;=\; 4\eta_{\mu\nu}\,.$$

(ii)

$$\begin{aligned}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma &= -\,\gamma_\nu\gamma_\mu\gamma_\rho\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma \\ &= \gamma_\nu\gamma_\rho\gamma_\mu\gamma_\sigma - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma \\ &= -\,\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu + 2\eta_{\mu\sigma}\gamma_\nu\gamma_\rho - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma\end{aligned}$$

(iii) From (ii) we have

$$\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma + \gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu \;=\; 2\eta_{\mu\sigma}\gamma_\nu\gamma_\rho - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma\,.$$

Taking the trace and using

$$\mathrm{tr}[\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu] \;=\; \mathrm{tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma]\,,$$

together with $\mathrm{tr}[\gamma_\mu\gamma_\nu] = 4\eta_{\mu\nu}$ from part (i), we find

$$\mathrm{tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] \;=\; 4[\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}]\,.$$

(b) The Dirac equation reads $(i\gamma^\mu\partial_\mu - m)\psi = 0$, hence

$$\gamma^\mu\partial_\mu\psi + im\psi = 0 \quad\Rightarrow\quad (\partial_\mu\psi^\dagger)\gamma^{\mu\dagger} - im\psi^\dagger = 0\,.$$

In addition we need to remember that $\bar{\psi} := \psi^\dagger \gamma^0$ (and $\gamma^{\mu\,\dagger} = \gamma^0 \gamma^\mu \gamma^0$ as given in the question). Then

$$(\partial_\mu \psi^\dagger)\gamma^0\gamma^\mu\gamma^0 - im\psi^\dagger \;=\; 0$$

and, multiplying with γ^0 from the right,

$$(\partial_\mu \bar{\psi})\gamma^\mu - im\bar{\psi} \;=\; 0\,.$$

Using this we get

$$\partial_\mu(\bar{\psi}\gamma^\mu\psi) \;=\; (\partial_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu(\partial_\mu\psi) \;=\; (im\bar{\psi})\psi + \bar{\psi}(-im\psi) \;=\; 0\,.$$

(c) (i) It is best to work out the anti-commutator for each specific μ. For $\mu = 0$ we have:

$$\begin{aligned}
\gamma^5\gamma^0 &= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 \;=\; -i\gamma^0\gamma^1\gamma^2\gamma^0\gamma^3 \;=\; i\gamma^0\gamma^1\gamma^0\gamma^2\gamma^3 \\
&= -\,i\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 \;=\; -\gamma^0\gamma^5 \;\Rightarrow\; \{\gamma^5,\gamma^0\} \;=\; 0\,, \\
\gamma^5\gamma^1 &= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1 \;=\; -i\gamma^0\gamma^1\gamma^2\gamma^1\gamma^3 \;=\; i\gamma^0\gamma^1\gamma^1\gamma^2\gamma^3 \\
&= -\,i\gamma^1\gamma^0\gamma^1\gamma^2\gamma^3 \;=\; -\gamma^1\gamma^5 \;\Rightarrow\; \{\gamma^5,\gamma^1\} \;=\; 0\,.
\end{aligned}$$

The other two cases, with γ^2 and γ^3, follow the same steps, with the same number of sign changes when commuting γ^2 or γ^3 from right to left.

(ii) With (i) we can write

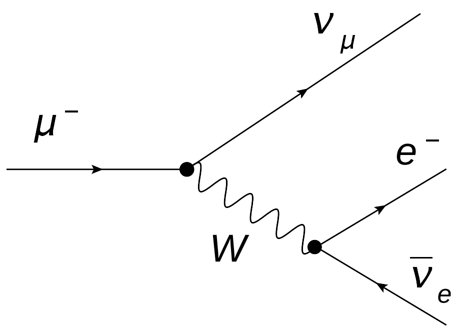
$$\bar{u}_e\gamma^\mu\tfrac{1}{2}\left(1-\gamma^5\right)u_\nu \;=\; \bar{u}_e\tfrac{1}{2}\left(1+\gamma^5\right)\gamma^\mu u_\nu\,,$$

which contains the left-handed electron field $\bar{u}_e^L$, as

$$\bar{u}_e^L \;=\; u_e^{L\,\dagger}\gamma^0 \;=\; u_e^\dagger\tfrac{1}{2}\left(1-\gamma^5\right)\gamma^0 \;=\; \bar{u}_e\tfrac{1}{2}\left(1+\gamma^5\right)\,.$$

4. (i) The decay $\mu \to e\gamma$ is allowed by kinematics and w.r.t. charge conservation, but is forbidden in the Standard Model as it would violate the separate conservation of the lepton numbers N_μ and N_e. Empirically, $N_{e,\mu,\tau}$ are conserved and the decay $\mu \to e\gamma$ is not observed. **Bookwork**

(ii) Feynman diagram for $\mu^- \to \nu_\mu \, e^- \, \bar{\nu}_e$ in the Standard Model to lowest order:



The solid 'external' lines stand for the incoming and outgoing spin-1/2 particles, the muon, electron, muon-neutrino and electron anti-neutrino. The wavy internal line is the propagator of the W^- boson, the carrier of the weak interaction. It mediates the transition from the initial state μ^- to the final state ν_μ and creates the e^- and $\bar{\nu}_e$ in the final state. The solid dots denote the vertices of this weak interaction.
The algebraic expressions for the different elements are:

– incoming μ^-: spinor u

– outgoing ν_μ: spinor $\bar{u}$

– outgoing e^-: spinor $\bar{u}$

– outgoing $\bar{\nu}_e$: spinor v

– vertices: $-i\frac{g}{\sqrt{2}}\gamma^\alpha \frac{1}{2}(1-\gamma^5)$ and $-i\frac{g}{\sqrt{2}}\gamma^\beta \frac{1}{2}(1-\gamma^5)$, with weak coupling constant g

– propagator: $i\eta_{\alpha\beta}/(q^2 - m_W^2)$, where q is the four-momentum transfer and m_W is the mass of the W boson **Bookwork**

(iii) The momentum transfer squared, q^2, is of the order of (but limited by) m_μ^2, which is very small compared to m_W^2. Thefore the propagator is well approximated by $-i\eta_{\alpha\beta}/m_W^2$. This is a very strong suppression factor, which would not be present if the W would be massless. **Bookwork**

(iv) The τ lepton is much heavier than the μ or the e, therefore it can decay into both:

$$\tau^- \to \nu_\tau \, e^- \, \bar{\nu}_e \ , \quad \tau^- \to \nu_\tau \, \mu^- \, \bar{\nu}_\mu \, .$$

The decays are very similar, with only small differences due to the different masses of the final state particles. In addition, as the τ is also heavier than light hadrons, it can decay in many hadronic final states like pions (the ν_τ must always be there due to N_τ conservation). In these cases the W intially couples to a quark pair which then hadronises. One example is the decay

$$\tau^- \to \nu_\tau \, \pi^- \, \pi^0 \, .$$

Bookwork

5. (i) *First step:*

$$i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,,$$

so $i\sigma_2$ is unitary.

Second step: Prove by explicit matrix multiplication that $\sigma_2\,\sigma_a^\star\,\sigma_2 = -\sigma_a$ $(i = 1, 2, 3)$. For $i = 1$ we have e.g.

$$\sigma_2\,\sigma_1^\star\,\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -\sigma_1\,.$$

Third step: Choose $W = i\sigma_2$, and with $\sigma_2^\star\sigma_2^\star = 1\!\!\mathrm{I} = \sigma_2\sigma_2$ and the result of step two we write:

$$\begin{aligned} W^\dagger U^\star W &= -\,i\sigma_2^\dagger U^\star i\sigma_2 \;=\; \sigma_2 \exp\left(-\tfrac{i}{2}\theta_a\sigma_a^\star\right)\sigma_2 \\ &= \sigma_2\left(1\!\!\mathrm{I} - \frac{i}{2}\theta_a\sigma_a^\star - \frac{1}{4}\frac{1}{2!}\theta_a\theta_a - \frac{i}{8}\frac{1}{3!}\theta_a\theta_a\theta_b\sigma_b^\star - \ldots\right)\sigma_2 \\ &= \left(1\!\!\mathrm{I} + \frac{i}{2}\theta_a\sigma_a - \frac{1}{4}\frac{1}{2!}\theta_a\theta_a + \frac{i}{8}\frac{1}{3!}\theta_a\theta_a\theta_b\sigma_b - \ldots\right) \;=\; U\,. \end{aligned}$$

Taking the complex conjugate of

$$W^\dagger U^\star W = U$$

we now also have

$$W^{\dagger\star} U W^\star = U^\star$$

and as $W = i\sigma_2 = W^\star$ we arrive at the desired relation

$$U^\star = W^\dagger\, U\, W\,.$$

Bookwork

(ii) In the Standard Model, fermion and gauge boson masses are obtained in a gauge invariant way through electroweak symmetry breaking which is mediated by a Higgs potential. Because of the unitary equivalence between the fundamental and the complex conjugate representations of $SU(2)$, gauge invariant mass terms for both up- and down quarks (which are grouped together in $SU(2)$ doublets) can be constructed from only one complex Higgs doublet. In other words, it is due to this special property of $SU(2)$ that the Higgs sector in the Standard Model is the minimal one resulting in only one physical Higgs boson.

Bookwork

6. (i)

$$L = \frac{1}{2}(\partial_\mu \phi_i)^2 - \frac{1}{2}\mu^2 \phi_i^2 - \frac{1}{4}\lambda(\phi_i^2)^2$$

with $\mu^2 < 0$ and $\lambda > 0$. The first term in the Lagrangian contains the kinetic terms of the three fields which lead to the propagators in the Feynman rules. The second and third term are the quadratic and quartic terms of the scalar potential. The coefficient λ is chosen positive as otherwise there would be no stable vacuum. Only with $\mu^2 < 0$ we get a non-trivial vacuum which allows for spontaneous symmetry breaking. This leads to mass terms (quadratic in the field) and interaction terms which lead to cubic and quartic vertices in the Feynman rules. **Bookwork**

(ii)

$$L = \frac{1}{2}\left((\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2 + (\partial_\mu\phi_3)^2\right) - \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2 + \phi_3^2) - \frac{1}{4}\lambda(\phi_1^2 + \phi_2^2 + \phi_3^2)^2 .$$

The potential is

$$V = \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2 + \phi_3^2) + \frac{1}{4}\lambda(\phi_1^2 + \phi_2^2 + \phi_3^2)^2 .$$

For spontaneous symmetry breaking we need a non-trivial minimum of the potential:

$$\frac{\partial V}{\partial \phi_i} = \left(\mu^2 + \lambda(\phi_1^2 + \phi_2^2 + \phi_3^2)\right)\phi_i = 0 ,$$

so we require

$$\mu^2 + \lambda(\phi_1^2 + \phi_2^2 + \phi_3^2) = 0 .$$

We choose the non-vanishing vacuum expectation value to be

$$\langle \phi_1 \rangle = \sqrt{-\frac{\mu^2}{\lambda}} = v ,$$

so the vacuum (after spontaneous symmetry breaking) is $\Phi_0 = (v, 0, 0)$. We now expand ϕ_1 around the new vacuum (keeping ϕ_2 and ϕ_3):

$$\phi_1(x) = v + h(x) .$$

Inserting this into the Lagrangian we get

$$\frac{1}{2}\left((\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2 + (\partial_\mu\phi_3)^2\right) - \frac{1}{2}\mu^2((v+h(x))^2 + \phi_2^2 + \phi_3^2) - \frac{1}{4}\lambda((v+h(x))^2 + \phi_2^2 + \phi_3^2)^2$$

and with $-\mu^2 = v^2\lambda$ we arrive at

$$\begin{aligned} L = {} & \frac{1}{2}((\partial_\mu h)^2 + (\partial_\mu\phi_2)^2 + (\partial_\mu\phi_3)^2) + \\ & \frac{v^2\lambda}{2}(h^2(x) + v^2 + 2vh(x) + \phi_2^2 + \phi_3^2) - \frac{\lambda}{4}(6v^2h^2(x) + \ldots) = \\ & \frac{1}{2}\left((\partial_\mu h)^2 + (\partial_\mu\phi_2)^2 + (\partial_\mu\phi_3)^2\right) - \lambda v^2 h^2(x) + \\ & \qquad \text{cubic and quartic interaction terms + constant} . \end{aligned}$$

Note that the terms quadratic in ϕ_2 and ϕ_3 have cancelled, so there are no mass terms for these fields. Hence the Lagrangian, after SSB, describes two massless 'Goldstone bosons' and one massive scalar Higgs field h with mass $\sqrt{-2\mu^2}$.

7. (i) Setting $u = 1/\alpha_s$ the differential equation becomes

$$\frac{\mathrm{d}u}{\mathrm{d}\ln E} = b_0 + \frac{b_1}{u}\,.$$

Truncating at lowest order, the solution is given by $u(E) = u(\mu) + b_0 \ln(E/\mu)$, with a free integration constant $u(\mu)$. After sustituting back $u \to 1/\alpha_s$ this is the proposed solution, with the 'initial condition' $\alpha_s(\mu)$.

(ii) The proposed definition is equivalent to the substitution

$$\mu = \Lambda_{\mathrm{QCD}} \exp\left[\frac{1}{b_0\alpha_s(\mu)}\right]$$

with which we obtain the form

$$\alpha_s(E) = \frac{1}{b_0 \ln(E/\Lambda_{\mathrm{QCD}})}\,.$$

At energies $E \to \Lambda_{\mathrm{QCD}}$ the coupling develops a pole, the so-called Landau-pole. For large values of the coupling the perturbative series breaks down, and the Landau pole indicates the non-perturbative region of QCD. This is the region where quarks and gluons are confined into colourless hadrons (baryons and mesons), which are the observed degrees of freedom at low energy scales.
At large energies, the coupling becomes small. This so-called 'asymptotic freedom' of QCD as an $SU(3)$ gauge field theory allows us to perform perturbative calculations on the parton level which in turn reliably describe QCD processes at large momentum transfer, like e.g. jet-production.
The sketch should show a monotonically decreasing positive function, indicating the Landau pole at $E = \Lambda_{\mathrm{QCD}}$ and a small value of α_s at large energies.

(iii) The solution valid at the next order of perturbation theory is obtained by inserting our lowest-order solution into the differential equation including the next order term $\sim b_1$, leading to

$$\frac{\mathrm{d}u}{\mathrm{d}\ln E} = b_0 + \frac{b_1}{b_0 \ln(E/\Lambda_{\mathrm{QCD}})}\,.$$

The solution of this differential equation is given by

$$u(E) = \frac{1}{\alpha_s(E)} = b_0 \ln(E/\Lambda_{\mathrm{QCD}}) + (b_1/b_0) \ln\ln(E/\Lambda_{\mathrm{QCD}})\,,$$

with a suitable redefinition of Λ_{QCD} to next order so that the integration constant vanishes (formula not requested).