

MATH431 Modern Particle Physics Solutions 5

1. Two-particle states are defined by

$$\begin{aligned}
 |\mathbf{p}_1, \mathbf{p}_2\rangle &= a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)|0\rangle \\
 |\mathbf{p}_1, \mathbf{p}_2\rangle &= |\mathbf{p}_2, \mathbf{p}_1\rangle \quad \text{as} \quad [a(\mathbf{p}_1), a(\mathbf{p}_2)] = 0 \\
 \langle \mathbf{p}'_1, \mathbf{p}'_2 | \mathbf{p}_1, \mathbf{p}_2 \rangle &= \langle 0 | a(\mathbf{p}'_1)a(\mathbf{p}'_2)a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2) | 0 \rangle \\
 &= \langle 0 | a(\mathbf{p}'_1) \{ a^\dagger(\mathbf{p}_1)a(\mathbf{p}'_2) + (2\pi)^3 2p_1^0 \delta(\mathbf{p}_1 - \mathbf{p}'_2) \} a^\dagger(\mathbf{p}_2) | 0 \rangle \\
 &= \langle 0 | a(\mathbf{p}'_1)a^\dagger(\mathbf{p}_1) \{ a^\dagger(\mathbf{p}_2)a(\mathbf{p}'_2) + (2\pi)^3 2p_1^0 \delta(\mathbf{p}_2 - \mathbf{p}'_2) \} | 0 \rangle \\
 &\quad + (2\pi)^3 2p_1^0 \delta(\mathbf{p}_1 - \mathbf{p}'_2) \langle 0 | a(\mathbf{p}'_1)a^\dagger(\mathbf{p}_2) | 0 \rangle \\
 &= (2\pi)^3 2p_2^0 \delta(\mathbf{p}_2 - \mathbf{p}'_2) \langle 0 | \{ a^\dagger(\mathbf{p}_1)a(\mathbf{p}'_1) + (2\pi)^3 2p_1^0 \delta(\mathbf{p}_1 - \mathbf{p}'_1) \} | 0 \rangle \\
 &\quad + (2\pi)^3 2p_1^0 \delta(\mathbf{p}_1 - \mathbf{p}'_2) \langle 0 | \{ a^\dagger(\mathbf{p}_2)a(\mathbf{p}'_1) + (2\pi)^3 2p_2^0 \delta(\mathbf{p}_2 - \mathbf{p}'_1) \} | 0 \rangle \\
 &= (2\pi)^6 (2p_1^0)(2p_2^0) \{ \delta(\mathbf{p}_1 - \mathbf{p}'_1)\delta(\mathbf{p}_2 - \mathbf{p}'_2) + \delta(\mathbf{p}_1 - \mathbf{p}'_2)\delta(\mathbf{p}_2 - \mathbf{p}'_1) \}.
 \end{aligned}$$

2(a).

The electromagnetic field strength tensor,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

$$\begin{aligned}
 L_{e.m.} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\
 &= -\frac{1}{4}(A_{\mu,\nu} - A_{\nu,\mu})(A^{\mu,\nu} - A_{\nu,\mu}) \\
 &= -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(A_{\beta,\alpha} - A_{\alpha,\beta})(A_{\mu,\nu} - A_{\nu,\mu})
 \end{aligned}$$

The Euler–Lagrange equations of motion are obtained from the Lagrangian

$$\frac{\partial}{\partial x^\nu} \left(\frac{\partial L}{\partial A_{\mu,\nu}} \right) - \frac{\partial L}{\partial A_\mu} = 0$$

$$\frac{\partial L}{\partial A_\mu} = 0 \Rightarrow \frac{\partial}{\partial x^\nu} \left(\frac{\partial L}{\partial A_{\mu,\nu}} \right) = 0$$

$$\begin{aligned}
 \frac{\partial L}{\partial A_{p,q}} &= -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(\delta_\alpha^p\delta_\beta^q - \delta_\beta^p\delta_\alpha^q)(A_{\mu,\nu} - A_{\nu,\mu}) + (A_{\alpha,\beta} - A_{\beta,\alpha})(\delta_\mu^p\delta_\nu^q - \delta_\nu^p\delta_\mu^q) \\
 &= -\frac{1}{4}(\eta^{p\mu}\eta^{q\nu} - \eta^{q\mu}\eta^{p\nu})(A_{\mu,\nu} + A_{\nu,\mu}) + (\eta^{\alpha p}\eta^{\beta q} - \eta^{\alpha q}\eta^{\beta p})(A_{\beta,\alpha} - A_{\alpha,\beta}) \\
 &= -4\frac{1}{4}(A^{q,p} - A^{p,q}) \\
 &= -F^{pq}
 \end{aligned}$$

$$\frac{\partial}{\partial x^\nu} \left(\frac{\partial L}{\partial A_{\mu,\nu}} \right) = -\frac{1}{4} \frac{\partial}{\partial x^\nu} F^{\mu\nu} = 0$$

which are Maxwell's equation in the absence of sources.

$$A_\mu \rightarrow \tilde{A}_\mu = A_\mu + \partial_\mu \Lambda$$

$$\begin{aligned} \tilde{F}_{\mu\nu} &= \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu = \partial_\mu (A_\nu - \partial_\nu \Lambda) - \partial_\nu (A_\mu - \partial_\mu \Lambda) \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - \partial_\mu \partial_\nu \Lambda + \partial_\nu \partial_\mu \Lambda = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu} \end{aligned}$$

Therefore $L_{e.m.}$ is also invariant under the transformation.

(b) the Lagrangian for a massive photon without sources is given by

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + m^2 A_\mu A^\mu$$

The first term is invariant under transformation. The second term transforms to

$$\rightarrow m^2 \tilde{A}_\mu \tilde{A}^\mu = m^2 A_\mu A^\mu + m^2 (\partial_\mu \Lambda \partial^\mu \Lambda - \partial_\mu \Lambda \partial^\mu \Lambda - A_\mu \partial^\mu \Lambda)$$

therefore the mass term does not vanish under the transformation. Requiring invariance imposes $m^2 = 0$.

3.

$$\begin{aligned} \gamma^{5\dagger} &= (i\gamma^0 \gamma^1 \gamma^2 \gamma^3)^\dagger \\ &= -i\gamma^{3\dagger} \gamma^{2\dagger} \gamma^{1\dagger} \gamma^{0\dagger} \\ &= -i(\gamma^0 \gamma^3 \gamma^0)(\gamma^0 \gamma^2 \gamma^0)(\gamma^0 \gamma^1 \gamma^0) \gamma^0 \\ &= -i\gamma^0 \gamma^3 \gamma^2 \gamma^1 = i\gamma^0 \gamma^2 \gamma^3 \gamma^1 = -i\gamma^0 \gamma^2 \gamma^1 \gamma^3 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \gamma^5. \end{aligned}$$

For the second part, it's best to do for each μ in turn:

$$\begin{aligned} \gamma^5 \gamma^0 &= i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^0 = -i\gamma^0 \gamma^1 \gamma^2 \gamma^0 \gamma^3 = i\gamma^0 \gamma^1 \gamma^0 \gamma^2 \gamma^3 \\ &= -i\gamma^0 \gamma^0 \gamma^1 \gamma^2 \gamma^3 = -\gamma^0 \gamma^5 \Rightarrow \{\gamma^5, \gamma^0\} = 0, \\ \gamma^5 \gamma^1 &= i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^1 = -i\gamma^0 \gamma^1 \gamma^2 \gamma^1 \gamma^3 = i\gamma^0 \gamma^1 \gamma^1 \gamma^2 \gamma^3 \\ &= -i\gamma^1 \gamma^0 \gamma^1 \gamma^2 \gamma^3 = -\gamma^1 \gamma^5 \Rightarrow \{\gamma^5, \gamma^1\} = 0. \end{aligned}$$

It is clear that the other two calculations will be similar.

4. N.B. Of course the question was wrong—should have said $(\gamma^0)^2 = 1$, $(\gamma^i)^2 = -1$, $i = 1, 2, 3$.

$$\begin{aligned} \gamma^0 \gamma^1 \gamma^2 &= -\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^3 = i\gamma^5 \gamma^3 \\ \gamma^0 \gamma^1 \gamma^3 &= -\gamma^0 \gamma^1 \gamma^2 \gamma^2 \gamma^3 = \gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^2 = -i\gamma^5 \gamma^2. \end{aligned}$$

5.

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}1_4 \Rightarrow \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}1_4,$$

where 1_4 is the 4-dimensional identity matrix, usually not written explicitly. Taking the trace, and using $\text{tr}(AB) = \text{tr}(BA)$, $\text{tr}1_4 = 4$, we get

$$\text{tr}[\gamma_\mu\gamma_\nu] = 4\eta_{\mu\nu}.$$

Now

$$\begin{aligned} \gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma &= -\gamma_\nu\gamma_\mu\gamma_\rho\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma \\ &= \gamma_\nu\gamma_\rho\gamma_\mu\gamma_\sigma - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma \\ &= -\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu + 2\eta_{\mu\sigma}\gamma_\nu\gamma_\rho - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma \\ \Rightarrow \gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma + \gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu &= 2\eta_{\mu\sigma}\gamma_\nu\gamma_\rho - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma. \end{aligned}$$

Taking the trace and using

$$\text{tr}[\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu] = \text{tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma],$$

together with $\text{tr}[\gamma_\mu\gamma_\nu] = 4\eta_{\mu\nu}$, we find

$$\text{tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] = 4[\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}]$$