from the previous lecture ...

The Klein–Gordon equation

In non–relativistic quantum mechanics

$$\vec{P} \to -i\hbar\vec{\nabla} \quad , \quad E \to i\hbar\frac{\partial}{\partial t} \quad \text{quantum operators}$$

The Hamiltonian for an energy conserving system is

$$H = \frac{\vec{P}^2}{2m} + V(\vec{q}) = E$$

leads to the Shrödinger equation by substitution

$$\left(-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{q})\right)\Psi(\vec{q},t) = i\hbar\frac{\partial}{\partial t}\Psi(\vec{q},t)$$

In special relativity the four vector P^μ is given by

$$P^\mu = (\frac{E}{c}, \vec{P})$$

we have $\partial^\mu = (\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla})$; $\partial_\mu = (\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla})$

In special relativity $E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$, $\vec{P} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}$

$\Rightarrow \quad P_\mu P^\mu = \frac{E^2}{c^2} - \vec{P}^2 = m^2 c^2.$

Set $P^\mu \to i\hbar\partial^\mu$ and obtain the wave equation

$$-\hbar^2 \partial_\mu \partial^\mu \phi = m^2 c^2 \phi$$

or $\left(\partial^2 + \frac{m^2c^2}{\hbar^2}\right)\phi(\vec{X}, t) = 0 \quad \leftarrow$ the Klein–Gordon equation

Its interpretation as a single particle is problematic. The equation describes a scalar field but not in a single state but a multi–state, *i.e.* a quantised field.

To find solutions of the Klein–Gordon equation we put it in a box and impose that the wave–function vanishes on the boundaries. We then take the volume to infinity. We assume that the particle is free *i.e.* $V(\vec{X}) = 0$. The solution in the box is a plane wave solution of the form

$$\phi(\vec{x}, t) \sim \mathrm{e}^{ik_\mu x^\mu}$$

$$\begin{aligned} x^\mu &= (t, \vec{x}) \\ k^\mu &= (w, \vec{k}) \qquad , \qquad k_\mu = (w, -\vec{k}) \quad \leftarrow \text{constant} \\ \phi(\vec{x}, t) &\sim \mathrm{e}^{i(wt - \vec{k}\cdot\vec{x})} \end{aligned}$$

We want the solution to describe a free particle of mass m. We substitute this solution into the Klein–Gordon equation

$$\Rightarrow \quad k_\mu k^\mu = w^2 - |\vec{k}|^2 = m^2$$

The particle is confined in a box $\overset{\text{assume}}{\Rightarrow}$ $\phi(x,y,z,t) = T(t)X(x)Y(y)(Z(z)$

<u>substituting in the KG equation</u> $\rightarrow \; -\frac{\ddot{T}}{T} + \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} \; = \; m^2 \; \rightarrow$ const

<u>all constants</u> $\rightarrow \quad w^2 \; - \; k_x^2 \; - \; k_y^2 \; - \; k_z^2$

To satisfy the boundary conditions we impose:

$$k_x = \tfrac{n_1\pi}{L} \quad , \quad k_y = \tfrac{n_2\pi}{L} \quad , \quad k_z = \tfrac{n_3\pi}{L}$$ where

$n_1, \; n_2, \; n_3$ are integers

and obtain $w^2 = m^2 + \frac{\pi^2}{L^2}(n_1^2 + n_2^2 + n_3^2) \quad \leftarrow$ dispersion relations

The solutions of the KG equations are momentum eigenstates

we have: $$\phi = A\mathrm{e}^{-ik\cdot x} = A\mathrm{e}^{-ik_\mu x^\mu} \Rightarrow \partial^\mu \phi = \frac{\partial \phi}{\partial x_\mu} = -ik^\mu A\mathrm{e}^{-ik\cdot x} = -ik^\mu \phi$$

$$\partial^2\phi = -k_\mu k^\mu \phi = -k^2\phi$$

$$\Rightarrow \quad \left(\partial^2 + \frac{m^2c^2}{\hbar^2}\right)\phi = 0 \quad \Rightarrow \quad k^2 = \frac{m^2c^2}{\hbar^2} = \frac{1}{\lambda^2}$$

$\lambda = \frac{\hbar}{mc}$ has dimension of length $\rightarrow$ Compton wave length of particle of mass m

we have: $$P^\mu \phi = i\hbar\partial^\mu \phi = \hbar k^\mu \phi$$

ϕ describes a momentum eigenstate with eigenvale $\hbar k^\mu$

The condition $k^2 = \frac{m^2c^2}{\hbar^2}$ is the same as $P^2 = m^2c^2 \ \rightarrow$ mass shell condition we have: $P^2 = \frac{E^2}{c^2} - \vec{p}^2 = m^2c^2 \quad \Rightarrow \quad \frac{E}{c} = \pm\sqrt{m^2c^2 + \vec{p}^2}$

$\Rightarrow$ some solutions of the KGE correspond to negative energy states

$\rightarrow$ interpretation as a single particle state is problematic. Such interpretation is in conflict with the probability interpretation of the wave function.

Quantum mechanics $\rightarrow$ non–relativistically $|\psi(x)|^2 = \rho(x)$.

$\rho(x) \rightarrow$ probability density $\rightarrow \int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$

The probability density in quantum mechanics obeys a continuity equation.

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

$$\text{where} \qquad \vec{J} = -\frac{i\hbar}{2m}(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*)$$

Provided that the potential $V(\vec{r})$ is real, $\rho(x)$ is conserved.

We want to construct similar quantities for the KG equation. Furthermore, we want the continuity equation to be covariant *i.e.* to hold in all inertial frames,

$$\rightarrow \qquad \partial_\mu j^\mu = 0 \quad \leftarrow \quad \text{covariant}$$

find a 4–vector $j^\mu = (j^0, \vec{J})$ which obeys $\partial_\mu j^\mu = 0$.

Start with the KGE

1. $\phi^* \cdot / \quad (\partial^2 + m^2)\phi = 0$
2. $\phi \cdot / \quad (\partial^2 + m^2)\phi^* = 0$

$$1 \; - \; 2 \qquad \rightarrow \qquad \phi^* \partial^2 \phi - \phi \partial^2 \phi^* = 0$$

$$\begin{aligned} \eta_{\alpha\beta}(\phi^* \partial^\alpha \partial^\beta \phi - \phi \partial^\alpha \partial^\beta \phi^*) &= \eta_{\alpha\beta} \partial^\alpha (\phi^* \partial^\beta \phi - \phi \partial^\beta \phi^*) \\ &= \partial_\beta (\phi^* \partial^\beta \phi - \phi \partial^\beta \phi^*) = 0 \end{aligned}$$

$$\Rightarrow \quad j^\mu \sim (\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) \quad \Rightarrow \quad \partial_\mu j^\mu = 0$$

In Schrödinger case $\rho = \psi^* \psi$.

$$\text{Here,} \qquad \rho = \frac{i\hbar}{2mc^2}\left(\phi^* \frac{\partial}{\partial t}\phi - \phi \frac{\partial}{\partial t}\phi^*\right)$$

$$\phi \sim \mathrm{e}^{\pm i(Et - \vec{p}\cdot\vec{x})} = \mathrm{e}^{\pm iEt}\mathrm{e}^{\mp \vec{p}\cdot\vec{x}}$$

The case with $+E$ in the exponent corresponds to the antiparticle case. For $\phi \sim \mathrm{e}^{+iEt}(\ )$, with $E \approx mc^2$ we have

$$\rho = \frac{i\hbar}{2mc^2}\left(\frac{imc^2}{\hbar}\phi^*\phi + \frac{imc^2}{\hbar}\phi^*\phi\right) = -\phi^*\phi < 0$$

We get negative probability associated with the antiparticle and positive probability asscociated with the particle. This does not make sense as probability density $\rightarrow$ multiply by charge e.

$$\Rightarrow \quad \text{Charge density} \;\; \rho = \frac{i\hbar e}{2mc^2}\left(\frac{imc^2}{\hbar}\phi^*\phi + \frac{imc^2}{\hbar}\phi^*\phi\right)$$

$$\text{Charge current density} \;\; \rho = \frac{e\hbar}{imc}\left(\phi^*\vec{\nabla}\phi + \phi\vec{\nabla}\phi^*\right)$$

The interpretation of the solution of the KGE makes sense as charge density, not as probability density.

$\rightarrow$ it makes sense as a quantum field $\rightarrow$ creating–annihilating particles