MATH431 Modern Particle Physics Solutions 5

1.

The generalised momentum
$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}.$$
The Hamiltonian density is defined by
$$\mathcal{H} = \dot{\phi}(x)\pi(x) - \mathcal{L},$$
then
$$H = \int d^3\mathbf{x}\dot{\phi}(\mathbf{x},t)\pi(\mathbf{x},t) - L = \int \mathcal{H}d^3\mathbf{x}.$$
For the KG field $\pi = \dot{\phi}$ and
$$\mathcal{H} = \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m^2\phi^2.$$

$$\begin{aligned}
[\phi(\mathbf{x},t), H] =& [\phi(\mathbf{x},t), \int [\tfrac{1}{2}\pi(\mathbf{x}',t)^2 + \tfrac{1}{2}(\nabla\phi(\mathbf{x}',t)^2 + \tfrac{1}{2}m^2\phi(\mathbf{x}',t)^2]d^3x']\\
=& \tfrac{1}{2}\int d^3\mathbf{x}' \left\{[\phi(\mathbf{x}), \pi(\mathbf{x}')^2] + [\phi(\mathbf{x}), (\nabla'\phi(\mathbf{x}'))^2] + m^2[\phi(\mathbf{x}), \phi(\mathbf{x}')^2]\right\}.
\end{aligned}$$
$$[\phi(\mathbf{x}), \phi(\mathbf{x}')] = 0 \Rightarrow [\phi(\mathbf{x}), \nabla'\phi(\mathbf{x}')] = 0.$$
$$\begin{aligned}
\text{So}\quad [\phi(\mathbf{x}), H] =& \tfrac{1}{2}\int \{\pi(\mathbf{x}')[\phi(\mathbf{x}), \pi(\mathbf{x}')] + [\phi(\mathbf{x}), \phi(\mathbf{x}')]\pi(\mathbf{x}')\}\\
=& i\hbar\int \pi(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')d^3\mathbf{x}'\\
=& i\hbar\pi(\mathbf{x}) = i\hbar\dot{\phi}(\mathbf{x}) \quad \text{OK.}
\end{aligned}$$

$$\begin{aligned}
[\pi(\mathbf{x},t), H] =& -\left[\int d^3\mathbf{x}' \left\{\tfrac{1}{2}\pi(\mathbf{x}')^2 + \tfrac{1}{2}(\nabla\phi(\mathbf{x}')^2 + \tfrac{1}{2}m^2\phi(\mathbf{x}')^2\right\}, \pi(\mathbf{x})\right]\\
=& -\tfrac{1}{2}\int d^3\mathbf{x}' \{\nabla'\phi(\mathbf{x}')[\nabla'\phi(\mathbf{x}'), \pi(\mathbf{x})] + [\nabla'\phi(\mathbf{x}'), \pi(\mathbf{x})]\nabla'\phi(\mathbf{x}')\}\\
&- \tfrac{1}{2}m^2\int d^3\mathbf{x}' \{\phi(\mathbf{x}')[\phi(\mathbf{x}'), \pi(\mathbf{x})] + [\phi(\mathbf{x}'), \pi(\mathbf{x})]\phi(\mathbf{x}')\}\\
=& -i\hbar\int d^3\mathbf{x}' \left\{\nabla'\phi(\mathbf{x}').\nabla'\delta(\mathbf{x}-\mathbf{x}') + m^2\phi(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')\right\}\\
=& i\hbar\int d^3\mathbf{x}' \left\{(\nabla')^2\phi(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}') - m^2\phi(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')\right\}\\
=& i\hbar\left\{\nabla^2\phi(\mathbf{x}) - m^2\phi(\mathbf{x})\right\}\\
=& i\hbar\ddot{\phi}(\mathbf{x}) \quad \text{using K-G equation}\\
=& i\hbar\dot{\pi}(\mathbf{x})
\end{aligned}$$

We write

$$(\partial\phi)^2 = \eta_{\rho\sigma}\partial^\rho\phi\partial^\sigma\phi,$$

then using

$$\frac{\partial(\partial^\rho)\phi}{\partial(\partial^\mu\phi)} = \delta^\rho{}_\mu,$$

we have

$$\frac{\partial}{\partial(\partial^\mu\phi)}(\partial\phi)^2 = \eta_{\rho\sigma}(\delta^\rho{}_\mu\partial^\sigma\phi + \partial^\rho\phi\delta^\sigma{}_\mu) = 2\partial_\mu\phi.$$

So equation of motion becomes

$$\partial^\mu(\partial_\mu\phi) = -m^2\phi - \frac{1}{2}\lambda_3\phi^2 - \frac{1}{3!}\lambda_4\phi^3,$$

or

$$\partial^2\phi + m^2\phi + \frac{1}{2}\lambda_3\phi^2 + \frac{1}{3!}\lambda_4\phi^3 = 0.$$

2. Two-particle states are defined by

$$\begin{aligned}
|\mathbf{p}_1, \mathbf{p}_2 >=& a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_1)|0>\\
|\mathbf{p}_1, \mathbf{p}_2 >=& |\mathbf{p}_2, \mathbf{p}_1 > \quad \text{as} \quad [a(\mathbf{p}_1), a(\mathbf{p}_2)] = 0\\
< \mathbf{p}_1', \mathbf{p}_2'|\mathbf{p}_1, \mathbf{p}_2 >=& < 0|a(\mathbf{p}_1')a(\mathbf{p}_2')a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)|0>\\
&= < 0|a(\mathbf{p}_1')\{a^\dagger(\mathbf{p}_1)a(\mathbf{p}_2') + (2\pi)^3 2p_1^0\delta(\mathbf{p}_1 - \mathbf{p}_2')\}a^\dagger(\mathbf{p}_2)|0>\\
&= < 0|a(\mathbf{p}_1')a^\dagger(\mathbf{p}_1)\{a^\dagger(\mathbf{p}_2)a(\mathbf{p}_2') + (2\pi)^3 2p_1^0\delta(\mathbf{p}_2 - \mathbf{p}_2')\}|0>\\
&\quad + (2\pi)^3 2p_1^0\delta(\mathbf{p}_1 - \mathbf{p}_2') < 0|a(\mathbf{p}_1')a^\dagger(\mathbf{p}_2)|0>\\
&=(2\pi)^3 2p_2^0\delta(\mathbf{p}_2 - \mathbf{p}_2') < 0|\{a^\dagger(\mathbf{p}_1)a(\mathbf{p}_1') + (2\pi)^3 2p_1^0\delta(\mathbf{p}_1 - \mathbf{p}_1')\}|0>\\
&+(2\pi)^3 2p_1^0\delta(\mathbf{p}_1 - \mathbf{p}_2') < 0|\{a^\dagger(\mathbf{p}_2)a(\mathbf{p}_1') + (2\pi)^3 2p_2^0\delta(\mathbf{p}_2 - \mathbf{p}_1')\}|0>\\
&=(2\pi)^6(2p_1^0)(2p_2^0)\{\delta(\mathbf{p}_1 - \mathbf{p}_1')\delta(\mathbf{p}_2 - \mathbf{p}_2') + \delta(\mathbf{p}_1 - \mathbf{p}_2')\delta(\mathbf{p}_2 - \mathbf{p}_1')\}.
\end{aligned}$$

3(a).

The electromagnetic field strength tensor,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

$$\begin{aligned}
L_{e.m.} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\\
&= -\frac{1}{4}(A_{\mu,\nu} - A_{\nu,\mu})(A^{\mu,\nu} - A_{\nu,\mu})\\
&= -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(A_{\beta,\alpha} - A_{\alpha,\beta})(A_{\mu,\nu} - A_{\nu,\mu})
\end{aligned}$$

The Euler–Lagrange equations of motion are obtained from the Lagrangian

$$\frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial A_{\mu,\nu}}\right) - \frac{\partial L}{\partial A_\mu} = 0$$

$$\frac{\partial L}{\partial A_\mu} = 0 \Rightarrow \frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial A_{\mu,\nu}}\right) = 0$$

$$\begin{aligned}
\frac{\partial L}{\partial A_{p,q}} &= -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(\delta^p_\alpha\delta^q_\beta - \delta^p_\beta\delta^q_\alpha)(A_{\mu,\nu} - A_{\nu,\mu}) + (A_{\alpha,\beta} - A_{\beta,\alpha})(\delta^p_\mu\delta^q_\nu - \delta^p_\nu\delta^q_\mu) \\
&= -\frac{1}{4}(\eta^{p\mu}\eta^{q\nu} - \eta^{q\mu}\eta^{p\nu})(A_{\mu,\nu} + A_{\nu,\mu}) + (\eta^{\alpha p}\eta^{\beta q} - \eta^{\alpha q}\eta^{\beta p})(A_{\beta,\alpha} - A_{\alpha,\beta}) \\
&= -4\frac{1}{4}(A^{q,p} - A^{p,q}) \\
&= -F^{pq}
\end{aligned}$$

$$\frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial A_{\mu,\nu}}\right) = -\frac{1}{4}\frac{\partial}{\partial x^\nu}F^{\mu\nu} = 0$$

which are Maxwell's equation in the absence of sources.

$$A_\mu \to \tilde{A}_\mu = A_\mu + \partial_\mu\Lambda$$

$$\begin{aligned}
\tilde{F}_{\mu\nu} = \partial_\mu\tilde{A}_\nu - \partial_\nu\tilde{A}_\mu &= \partial_\mu(A_\nu - \partial_\nu\Lambda) - \partial_\nu(A_\mu - \partial_\mu\Lambda) \\
&= \partial_\mu A_\nu - \partial_\nu A_\mu - \partial_\mu\partial_\nu\Lambda + \partial_\nu\partial_\mu\Lambda = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}
\end{aligned}$$

Therefore $L_{e.m.}$ is also invariant under the transformation.

(b) the Lagrangian for a massive photon without sources is given by

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + m^2A_\mu A^\mu$$

The first term is invariant under transformation. The second term transforms to

$$\to m^2\tilde{A}_\mu\tilde{A}^\mu = m^2A_\mu A^\mu + m^2(\partial_\mu\Lambda\partial^\mu\Lambda - \partial_m u\Lambda A^\mu - A_\mu\partial^\mu\Lambda)$$

therefore the mass term does not vanish under the transformation. Requiring invariance imposes $m^2 = 0$.