from the previous lecture ...

Lagrangian & Hamiltonian Mechanics

In modern particle physics calculations are done in the framework of quantum field theories. A bridge between the "old" Newtonian mechanics and the "modern" particle physics is provided by the classical Lagrangian & Hamiltonian formulations of classical mechanics.

The Lagrangian $L = L(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n, t)$

The Hamiltonian $H = H(q_1, \cdots, q_n, p_1, \cdots, q_n, t)$

Configuration space : $(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n),$

Phase space : $(q_1, \cdots, q_n, p_1, \cdots, p_n),$

where $$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

The tranformation is made by a Legendre transformation

$$H = \sum_{k=1}^{n} p_k \dot{q}_k - L(q_1, \cdots, q_{n-1}, \dot{q}_1, \cdots, \dot{q}_n, t) = H(q_k, p_k),$$

when $L \neq L(t) \quad \Rightarrow \quad H \neq H(t) \quad \Rightarrow$ H is a constant of the motion.

H is identified with the conserved energy $H = E = \mathrm{constant}$.

The action principle

The Euler–Lagrange equations of motion can be derived from an action principle. Given the Lagrangian

$$L(q_i, \dot{q}_i) = \sum_i \frac{1}{2} m_i \dot{q}^i - V(q_i)$$

define

$$S = \int dtL(q_i, \dot{q}_i).$$

Action principle for fixed values of $q(t_i) = q_{in}$, $q(t_f) = q_{out}$, then the classical trajectory which satisfies these boundary conditions is an extremum of the action

$$\delta \int_{t_{in}}^{t_{out}} dtL(q_i, \dot{q}_i) = 0$$

$$
\begin{aligned}
\delta S &= \int_{t_{in}}^{t_{out}} dt\ \delta L(q_i, \dot{q}_i = \int_{t_{in}}^{t_{out}} dt \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i \right) \\
&= \int_{t_{in}}^{t_{out}} dt \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \frac{dq_i}{dt} \right) \\
&= \int_{t_{in}}^{t_{out}} dt \left(\frac{\partial L}{\partial q_i}\delta q_i - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \right) + \left. \frac{\partial L}{\partial \dot{q}_i}\delta q_i \right|_{t_{in}}^{t_{out}} = 0 \\
&\Rightarrow \int_{t_{in}}^{t_{out}} dt \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) \right) \delta q_i = 0 \qquad (1)
\end{aligned}
$$

This must hold for any variation δq_i. Hence we must have

$$
\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0 \qquad (2)
$$

Classical field theory

So far we discussed systems with discrete and finite number of particles

Classical field $\Psi(\vec{r}, t)$: function of $\vec{r}$, t, specifying the value of the field at a spacetime point.

A field representation can simplify a many body mechanical problem

<u>Example:</u>

$$\frac{\partial^2 y(x,t)}{\partial t^2} - \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial x^2} = 0$$

where $y(x,t)$ represents the motion of molecules along a chain. With a large number of molecules in the chain, say of the order of 10^{23} we can write the deviation from some equilibrium position as a function of the variables x and t.

Continuous dynamics in the Lagrangian formalism

The Lagrangian for longitudenal motion of a N–particle linear elastic chain.

$$L = \sum_{i=1}^{N} \frac{1}{2} m \dot{y}_i^2 - \sum_{i} \frac{1}{2} K (y_{i+1} - y_i)^2$$

where

$$\begin{aligned} m_i = m \quad &\rightarrow \quad \text{mass of the } i^{th} \text{ particle} \\ y_i \quad &\rightarrow \quad \text{longitudenal displacement from equilibrium} \\ K_i = K \quad &\rightarrow \quad \text{elastic constant} \\ a \quad &\rightarrow \quad \text{equilibrium position} \\ Na \quad &\rightarrow \quad \text{total length of the chain} \end{aligned}$$

$$L = \frac{1}{2}\sum_{i=1}^{N} a\left[\frac{m}{a}\dot{y}_i^2 - Ka\left(\frac{y_{i+1}-y_i}{a}\right)^2\right] = \sum_i aL_i$$

$$\lim_{a\to 0} = \sum_{i=1}^{N} aL_i = \int dx\mathcal{L}(x) = L$$

x is a continuous index replacing the discrete index i.

$$\begin{aligned} \lim_{a\to 0}\frac{m}{a} \quad &\to \quad \frac{dm}{dx} = \mu \to \text{linear mass density} \\ \lim_{\substack{a\to 0\\ N\to\infty}} Ka \quad &\to \quad \tau = \text{elastic tenion; Young's modulus} \end{aligned}$$

$$L = \lim_{a\to 0}\frac{1}{2}\sum a\left\{\frac{m}{a}\dot{y}_i^2 - Ka\left(\frac{y_{i+1}-y_i}{a}\right)^2\right\} \to \frac{1}{2}\int dx\left(\mu\dot{y}^2 - \tau {y'}^2\right)$$

$$\text{Lagrangian density} \quad \mathcal{L}(x) = \frac{1}{2}\left\{\mu \dot{y}^2 - \tau {y'}^2\right\} \quad \text{per unit length}$$

$\mathcal{L}$ is a function of the field velocity $\dot{y}$, the field coordinate $y(x,t)$ and the field gradient ${y'}^2$.

x is an index. A point in the field. It replaced the i variable in the discrete case.

The generalisation to three dimensions

$$L = \int \mathcal{L}(\phi, \dot{\phi}, \nabla\phi) dxdydz \quad \text{where} \quad \phi = \phi(x, y, z, t)$$

the action is given by

$$\begin{aligned} S = \int_{t_1}^{t_2} dt \;\; L[\phi] \;\; &= \;\; \int_{t_1}^{t_2} dtdxdydz \; \mathcal{L}(\phi, \dot{\phi}, \nabla\phi) \\ &= \;\; \int d^4x \; \mathcal{L}(\phi, \partial^\mu \phi) \quad \leftarrow \quad \text{relativistic notation} \end{aligned}$$

The equations of motions for the field ϕ are obtained by requiring that the variation of the action vanishes and demanding that $\delta\phi = 0$ at t_1 and t_2.

$$\delta \int \mathcal{L}(\phi, \partial^\mu \phi) d^4 x =$$
$$\int \left(\frac{\partial \mathcal{L}(\phi, \partial^\mu \phi)}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}(\phi, \partial^\mu \phi)}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right) d^4 x$$

we note that there is no explicit dependence of $\mathcal{L}$ on x^μ. With $\delta(\partial^\mu \phi) = \partial^\mu(\delta\phi)$ we get

$$\begin{aligned}
&= \int_\Omega \left(\frac{\partial \mathcal{L}(\phi, \partial^\mu \phi)}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}(\phi, \partial^\mu \phi)}{\partial(\partial_\mu \phi)} \partial_\mu \delta\phi \right) d^4 x \\
&= \int_\Omega \delta\phi \left(\frac{\partial \mathcal{L}(\phi, \partial^\mu \phi)}{\partial \phi} \right) d^4 x + \int_\Omega \partial_\mu \left(\frac{\partial \mathcal{L}(\phi, \partial^\mu \phi)}{\partial(\partial_\mu \phi)} \delta\phi \right) d^4 x \\
&\qquad - \int_\Omega \delta\phi \partial_\mu \left(\frac{\partial \mathcal{L}(\phi, \partial^\mu \phi)}{\partial(\partial_\mu \phi)} \right) d^4 x
\end{aligned}$$

where we used integration by parts $\int VdU = VU - \int UdV$.
the second term is a boundary term that vanishes by the divergence theorem (Gauss law).

$$= \int \delta\phi \left(\frac{\partial \mathcal{L}(\phi, \partial^\mu \phi)}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}(\phi, \partial^\mu \phi)}{\partial(\partial_\mu \phi)} \right) \right) d^4x = 0$$

Since this holds for any $\delta\phi$ we have that

$$\frac{\partial \mathcal{L}(\phi, \partial^\mu \phi)}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}(\phi, \partial^\mu \phi)}{\partial(\partial_\mu \phi)} \right) = 0$$

These are the Euler–Lagrange equations of motion for the field ϕ.