MATH 431 Midterm

$\frac{35}{40}$

1a) $$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = dx^2 + dy^2$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad g^{\mu\nu} = (g_{\mu\nu})^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

b) $$x \to x + \epsilon A(x,y)$$

$$y \to y + \epsilon B(x,y)$$

$$dx \to dx + \epsilon\left(\frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy\right)$$

$$dy \to dy + \epsilon\left(\frac{\partial B}{\partial x} dx + \frac{\partial B}{\partial y} dy\right)$$

$$dx^2 \to \left[\left(1 + \epsilon\frac{\partial A}{\partial x}\right)dx + \epsilon\frac{\partial A}{\partial y} dy\right]^2$$

$$dy^2 \to \left[\left(1 + \epsilon\frac{\partial B}{\partial y}\right)dy + \epsilon\frac{\partial B}{\partial x} dx\right]^2$$

$$ds^2 \to \left[\left(1+\epsilon\frac{\partial A}{\partial x}\right)dx + \epsilon\frac{\partial A}{\partial y}dy\right]^2 + \left[\left(1+\epsilon\frac{\partial B}{\partial y}\right)dy + \epsilon\frac{\partial B}{\partial x}dx\right]^2$$

$$= \left[dx + \epsilon\frac{\partial A}{\partial x}dx + \epsilon\frac{\partial A}{\partial y}dy\right]^2 + \left[dy + \epsilon\frac{\partial B}{\partial y}dy + \epsilon\frac{\partial B}{\partial x}dx\right]^2$$

$$= \left[dx^2 + 2\epsilon\frac{\partial A}{\partial x}dx^2 + 2\epsilon\frac{\partial A}{\partial y}dx\,dy + O(\epsilon)^2\right] + \cancel{\left[dy^2 + 2\epsilon\frac{\partial B}{\partial y}dy^2 + 2\epsilon\frac{\partial A}{\partial x}dx\,dy\right.}$$

$$+ \left[dy^2 + 2\epsilon\frac{\partial B}{\partial y}dy^2 + 2\epsilon\frac{\partial B}{\partial x}dx\,dy + O(\epsilon)^2\right]$$

$$= \left(1+2\epsilon\frac{\partial A}{\partial x}\right)dx^2 + \left(1+2\epsilon\frac{\partial B}{\partial y}\right)dy^2 + 2\epsilon\left(\frac{\partial A}{\partial y}+\frac{\partial B}{\partial x}\right)dx\,dy + O(\epsilon)^2$$

Since ϵ is an infinitesimal constant, terms including ϵ^2 can be approximated to 0.

$$\therefore dx^2+dy^2 \equiv \left(1+2\epsilon\frac{\partial A}{\partial x}\right)dx^2 + \left(1+2\epsilon\frac{\partial B}{\partial y}\right)dy^2 + 2\epsilon\left(\frac{\partial A}{\partial y}+\frac{\partial B}{\partial x}\right)dx\,dy$$

To keep ds^2 invariant under transformations the additional terms must be equal to 0.

Equate dx^2: $\cancel{1} = \cancel{1} + 2\epsilon\frac{\partial A}{\partial x} \Rightarrow 2\epsilon\frac{\partial A}{\partial x} = 0 \Rightarrow \frac{\partial A}{\partial x} = 0 \Rightarrow A \neq A(x)$

$\therefore A = A(y)$

dy^2: $\cancel{1} = \cancel{1} + 2\epsilon\frac{\partial B}{\partial y} \Rightarrow 2\epsilon\frac{\partial B}{\partial y} = 0 \Rightarrow \frac{\partial B}{\partial y} = 0 \Rightarrow B \neq B(y)$

$\therefore B = B(x)$

$dx\,dy$: $0 = 2\epsilon\left(\frac{\partial A}{\partial y}+\frac{\partial B}{\partial x}\right) \Rightarrow \frac{\partial A(y)}{\partial y} = -\frac{\partial B(x)}{\partial x} = \text{constant} = C$

$\Rightarrow \frac{\partial A}{\partial y} = c \Rightarrow \underline{A(y) = cy + a}$ a, b & c are constants

$-\frac{\partial B}{\partial x} = c \Rightarrow \underline{B(x) = b - cx}$

$\therefore x \to x + \epsilon c y + \epsilon a$

$y \to y + \epsilon b - \epsilon c x$

Set $a = b = 0, c \neq 0$

$\Rightarrow \begin{array}{l} x \to x + \epsilon c y \\ y \to y - \epsilon c x \end{array}$

$\therefore$ c corresponds to a ~~boost~~ Rotation

$dx^2 + dy^2$

Euclidean space

Set $a = c = 0, b \neq 0$

$\Rightarrow \begin{array}{l} x \to x \\ y \to y + \epsilon b \end{array}$

$\therefore$ b is a shift along the y coordinate.

Set $b = c = 0, a \neq 0$

$\Rightarrow \begin{array}{l} x \to x + \epsilon a \\ y \to y \end{array}$

$\therefore$ a is a shift along the x coordinate.

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2a) $L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu$

$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

$$\therefore L = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - j^\mu A_\mu$$

$$= -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) - j^\mu A_\mu$$

$$= -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial_\mu A_\nu - \partial_\nu A_\mu) - j^\mu A_\mu$$

Euler-Lagrange eqn: $\frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial(\partial_\nu A_\mu)}\right) - \frac{\partial L}{\partial A_\mu} = 0$

$$\frac{\partial L}{\partial(\partial_q A_\rho)} = -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}\Big[(\delta^\rho_\alpha \delta^q_\beta - \delta^\rho_\beta \delta^q_\alpha)(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$+ (\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\delta^\rho_\nu \delta^q_\mu - \delta^\rho_\mu \delta^q_\nu)\Big]$$

$$= -\frac{1}{4}\Big[(\eta^{\rho\mu}\eta^{q\nu} - \eta^{q\mu}\eta^{\rho\nu})(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$+ (\eta^{\alpha q}\eta^{\beta\rho} - \eta^{\alpha\rho}\eta^{\beta q})(\partial_\alpha A_\beta - \partial_\beta A_\alpha)\Big]$$

$$= -\frac{1}{\cancel{4}}\left[\cancel{4}(\partial^\rho A^q - \partial^q A^\rho)\right]$$

$$= -F^{\rho q}$$

$$\therefore \frac{\partial L}{\partial(\partial_\nu A_\mu)} = -F^{\mu\nu} \qquad \frac{\partial L}{\partial A_\mu} = \frac{\partial}{\partial A_\mu}(-j^\mu A_\mu) = -j^\mu$$

$$\therefore \underbrace{\frac{\partial}{\partial x^\nu}}_{\partial_\nu}\left(\frac{\partial L}{\partial(\partial_\nu A_\mu)}\right) - \frac{\partial L}{\partial A_\mu} = 0$$

$$\Rightarrow -\partial_\nu F^{\mu\nu} + j^\mu = 0$$

$$\therefore \partial_\nu F^{\mu\nu} = j^\mu$$

~~is conserved~~

For j^μ to be conserved. $\partial_\mu j^\mu = 0$

$$\therefore \partial_\mu j^\mu = \partial_\mu(\partial_\nu F^{\mu\nu}) = \partial_\mu \partial_\nu [\partial^\mu A^\nu - \partial^\nu A^\mu]$$

$$= \partial_\nu \underbrace{\partial_\mu \partial^\mu}_{\partial^2} A^\nu - \partial_\mu \underbrace{\partial_\nu \partial^\nu}_{\partial^2} A^\mu$$

$$= \partial^2 [\partial_\nu A^\nu - \partial_\mu A^\mu]$$

Relabel indicies

$$= \partial^2 [\partial_\mu A^\mu - \partial_\nu A^\nu]$$

$$= \partial_\nu \partial_\mu [\partial^\nu A^\mu - \partial^\mu A^\nu] = \partial_\nu \partial_\mu F^{\nu\mu} = -\partial_\nu \partial_\mu F^{\mu\nu}$$

For this to be possible $\partial_\nu \partial_\mu F^{\mu\nu} = 0$

$$\therefore \partial_\mu j^\mu = 0$$

And j^μ is conserved

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b) $L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu + \frac{1}{2} m^2 A_\mu A^\mu$

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$+\frac{1}{2} m^2 A_\mu A^\mu = \frac{1}{2} m^2 \eta^{\mu\nu} A_\mu A_\nu$

$$\therefore \frac{\partial L}{\partial A_\mu} = -j^\mu + \frac{1}{2} m^2 \left(A_\mu \frac{\partial A_\nu}{\partial A_\mu} + \frac{\partial A_\mu}{\partial A_\mu} A_\nu \right) \eta^{\mu\nu} = -j^\mu + \frac{1}{2} m^2 \left(A_\mu \delta^\mu_\nu + A_\nu \right) \eta^{\mu\nu}$$

$$= -j^\mu + 2 \frac{1}{2} m^2 A_\nu \eta^{\mu\nu}$$

$$= -j^\mu + m^2 A^\mu$$

$$\therefore \frac{\partial}{\partial x^\nu} \left(\frac{\partial L}{\partial(\partial_\nu A_\mu)} \right) - \frac{\partial L}{\partial A_\mu} = 0$$

$$\Rightarrow -\partial_\nu F^{\mu\nu} - \left(-j^\mu + m^2 A^\mu \right) = 0$$

Lorentz gauge

$\partial_\mu A^\mu = 0$

$$\Rightarrow \partial_\mu F^{\mu\nu} + m^2 A^\mu = j^\mu$$

$$\Rightarrow \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu \right) + m^2 A^\mu = j^\mu$$

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