

MATH328 Modern Particle Physics Solutions 4

1. With $\rho = |\psi|^2 = \psi^* \psi$, we have

$$\frac{\partial \rho}{\partial t} = \frac{\partial \psi^*}{\partial t} \psi + \psi^* \frac{\partial \psi}{\partial t}.$$

Now from the Schrödinger equation,

$$i\hbar \frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) \right) \psi, \quad (1)$$

and so, taking the complex conjugate

$$-i\hbar \frac{\partial \psi^*}{\partial t} = \left(-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) \right) \psi^* \quad (2)$$

(assuming that $V(\mathbf{x})$ is real.) Multiplying (1) by ψ^* and (2) by ψ and subtracting, we have

$$i\hbar \left(\psi^* \frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t} \psi \right) = -\frac{\hbar^2}{2m} (\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*) = -\frac{\hbar^2}{2m} \nabla \cdot [\psi^* \nabla \psi - \psi \nabla \psi^*],$$

and so

$$\frac{\partial \rho}{\partial t} = \frac{i\hbar}{2m} \nabla \cdot [\psi^* \nabla \psi - \psi \nabla \psi^*],$$

i.e.

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \text{div} \mathbf{j} &= 0, \\ \text{where } \mathbf{j} &= -\frac{i\hbar}{2m} [\psi^* \nabla \psi - \psi \nabla \psi^*] \end{aligned}$$

2.

The generalised momentum

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}.$$

The Hamiltonian density is defined by

$$\mathcal{H} = \dot{\phi}(x) \pi(x) - \mathcal{L},$$

then

$$H = \int d^3 \mathbf{x} \dot{\phi}(\mathbf{x}, t) \pi(\mathbf{x}, t) - L = \int \mathcal{H} d^3 \mathbf{x}.$$

For the KG field $\pi = \dot{\phi}$ and

$$\mathcal{H} = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2.$$

$$\begin{aligned}
[\phi(\mathbf{x}, t), H] &= [\phi(\mathbf{x}, t), \int [\frac{1}{2}\pi(\mathbf{x}', t)^2 + \frac{1}{2}(\nabla\phi(\mathbf{x}', t))^2 + \frac{1}{2}m^2\phi(\mathbf{x}', t)^2] d^3x'] \\
&= \frac{1}{2} \int d^3\mathbf{x}' \{ [\phi(\mathbf{x}), \pi(\mathbf{x}')^2] + [\phi(\mathbf{x}), (\nabla'\phi(\mathbf{x}'))^2] + m^2[\phi(\mathbf{x}), \phi(\mathbf{x}')^2] \}. \\
[\phi(\mathbf{x}), \phi(\mathbf{x}')] &= 0 \Rightarrow [\phi(\mathbf{x}), \nabla'\phi(\mathbf{x}')] = 0.
\end{aligned}$$

$$\begin{aligned}
\text{So } [\phi(\mathbf{x}), H] &= \frac{1}{2} \int \{ \pi(\mathbf{x}') [\phi(\mathbf{x}), \pi(\mathbf{x}')] + [\phi(\mathbf{x}), \phi(\mathbf{x}')] \pi(\mathbf{x}') \} \\
&= i\hbar \int \pi(\mathbf{x}') \delta(\mathbf{x} - \mathbf{x}') d^3\mathbf{x}' \\
&= i\hbar \pi(\mathbf{x}) = i\hbar \dot{\phi}(\mathbf{x}) \quad \text{OK.}
\end{aligned}$$

$$\begin{aligned}
[\pi(\mathbf{x}, t), H] &= - \left[\int d^3\mathbf{x}' \{ \frac{1}{2}\pi(\mathbf{x}')^2 + \frac{1}{2}(\nabla\phi(\mathbf{x}'))^2 + \frac{1}{2}m^2\phi(\mathbf{x}')^2 \}, \pi(\mathbf{x}) \right] \\
&= - \frac{1}{2} \int d^3\mathbf{x}' \{ \nabla'\phi(\mathbf{x}') [\nabla'\phi(\mathbf{x}'), \pi(\mathbf{x})] + [\nabla'\phi(\mathbf{x}'), \pi(\mathbf{x})] \nabla'\phi(\mathbf{x}') \} \\
&\quad - \frac{1}{2}m^2 \int d^3\mathbf{x}' \{ \phi(\mathbf{x}') [\phi(\mathbf{x}'), \pi(\mathbf{x})] + [\phi(\mathbf{x}'), \pi(\mathbf{x})] \phi(\mathbf{x}') \} \\
&= - i\hbar \int d^3\mathbf{x}' \{ \nabla'\phi(\mathbf{x}') \cdot \nabla' \delta(\mathbf{x} - \mathbf{x}') + m^2\phi(\mathbf{x}') \delta(\mathbf{x} - \mathbf{x}') \} \\
&= i\hbar \int d^3\mathbf{x}' \{ (\nabla')^2 \phi(\mathbf{x}') \delta(\mathbf{x} - \mathbf{x}') - m^2\phi(\mathbf{x}') \delta(\mathbf{x} - \mathbf{x}') \} \\
&= i\hbar \{ \nabla^2 \phi(\mathbf{x}) - m^2\phi(\mathbf{x}) \} \\
&= i\hbar \ddot{\phi}(\mathbf{x}) \quad \text{using K-G equation} \\
&= i\hbar \dot{\pi}(\mathbf{x})
\end{aligned}$$

We write

$$(\partial\phi)^2 = \eta_{\rho\sigma} \partial^\rho \phi \partial^\sigma \phi,$$

then using

$$\frac{\partial(\partial^\rho \phi)}{\partial(\partial^\mu \phi)} = \delta^\rho_\mu,$$

we have

$$\frac{\partial}{\partial(\partial^\mu \phi)} (\partial\phi)^2 = \eta_{\rho\sigma} (\delta^\rho_\mu \partial^\sigma \phi + \partial^\rho \phi \delta^\sigma_\mu) = 2\partial_\mu \phi.$$

So equation of motion becomes

$$\partial^\mu (\partial_\mu \phi) = -m^2 \phi - \frac{1}{2} \lambda_3 \phi^2 - \frac{1}{3!} \lambda_4 \phi^3,$$

or

$$\partial^2 \phi + m^2 \phi + \frac{1}{2} \lambda_3 \phi^2 + \frac{1}{3!} \lambda_4 \phi^3 = 0.$$