from the previous lecture ...

How do we describe the SM interactions?

Standard Model $\rightarrow$ $SU(3)_C \times SU(2)_L \times U(1)_Y$ local gauge interactions.

In the modern language of elementary particles, interactions correspond to invariances of the Lagrangian under some local symmetry.

Interaction $\longleftrightarrow$ invariance under a local gauge symmetry

Interaction $\leftrightarrow$ invariance under a local gauge symmetry

Symmetries of the Lagrangian correspond to conserved currents

consider a free scalar field $\quad \mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$

It is invariant under the discrete symmetry $\phi \to -\phi$

consider two free scalar fields with equal mass m

$$\mathcal{L} = \frac{1}{2}\left[\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2\right] - \frac{1}{2}m^2(\phi_1^2 + \phi_2^2) \tag{1}$$

with the transformations

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$$

where α is a global constant.

The Lagrangian in eq. (1) is invariant under the rotations in the $\phi_1,\ \phi_2$ plane.

α is a continuous parameter $\to$ global continuous symmetry, $\alpha \neq \alpha(x)$.

Consider the complex field

$$\begin{aligned}\Phi &= \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \\ \Phi^* &= \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)\end{aligned}$$

The lagrangian in terms of Φ, with $\Phi^*\Phi = \frac{1}{2}(\phi_1^2 + \phi_2^2)$

$$\mathcal{L} = (\partial_\mu \Phi)^*(\partial^\mu \Phi) - m^2 \Phi^* \Phi \tag{2}$$

In terms of Φ the rotation becomes

$$\begin{aligned}\Phi &= \frac{1}{\sqrt{2}}\left[(\cos\alpha\phi_1 + \sin\alpha\phi_2) + i(-\sin\alpha\phi_1 + \cos\alpha\phi_2)\right] \\ \Phi &= \frac{1}{\sqrt{2}}\left[(\cos\alpha - i\sin\alpha)\phi_1 + i(\cos\alpha - i\sin\alpha)\phi_2\right] \\ &= \mathrm{e}^{-i\alpha}\Phi\end{aligned}$$

$$\Rightarrow \quad \Phi \rightarrow e^{-i\alpha}\Phi$$
$$\Phi^* \rightarrow e^{i\alpha}\Phi^*$$

The continuous symmetry in terms of Φ is a symmetry under a phase transformation. The symmetry is a global $U(1)$ symmetry.

$\rightarrow$ arbitrary choice of the phase $\rightarrow$ continuous global symmetry

What happens if $\alpha = \alpha(x)$?

$$\Phi(x) \rightarrow \tilde{\Phi}(x) = e^{-i\alpha(x)}\Phi(x)$$

$$\partial_\mu \tilde{\Phi} = \left(-i\partial_\mu \alpha(x)\Phi(x) + \partial_\mu \Phi(x)\right) e^{-i\alpha(x)}$$

$$\mathcal{L} \to \quad \eta^{\mu\nu} \left[\partial_\mu(\mathrm{e}^{-i\alpha(x)}\Phi)\right]^* \left[\partial_\nu(\mathrm{e}^{-i\alpha(x)}\Phi)\right] + m^2 \left(\mathrm{e}^{-i\alpha(x)}\Phi\right)^* \left(\mathrm{e}^{-i\alpha(x)}\Phi\right)$$

$$\eta^{\mu\nu} \left[\partial_\mu\Phi(x) - i(\partial_\mu\alpha(x))\Phi(x)\right]^* \left[\partial_\nu\Phi(x) - i(\partial_\nu\alpha(x))\Phi(x)\right] + m^2\Phi^*\Phi$$

if $\alpha \neq \alpha(x)$ the derivative term $\partial\alpha(x)$ drops out.

We have to fix the Lagrangian to get a Lagrangian which is invariant under local phase transformaions $\alpha = \alpha(x)$.

We redefine the derivative as $\partial_\mu \;\to\; \partial_\mu + a_\mu(x)$

where $a_\mu(x)$ is a function of x.

Under local phase transformations

$$a_\mu(x) \;\to\; a'_\mu(x) = a_\mu(x) + i\partial_\mu\alpha(x)$$

where $\alpha(x)$ is a scalar function of x.

$$\begin{aligned} \text{Then} \qquad (\partial_\mu + a_\mu(x))\Phi(x) \;&\rightarrow\; (\partial_\mu + a'_\mu(x))(\mathrm{e}^{-i\alpha(x)}\Phi(x)) \\ &= \mathrm{e}^{-i\alpha(x)}(\partial_\mu + a_\mu(x) + i\partial_\mu\alpha(x) - i\partial_\mu\alpha(x))\Phi(x) \\ &= \mathrm{e}^{-i\alpha(x)}(\partial_\mu + a_\mu(x))\Phi(x) \end{aligned}$$

We now get that the Lagrangian is invariant under the local phase transformations.

$$\Phi(x) \;\rightarrow\; \Phi'(x) = \mathrm{e}^{-i\alpha(x)}\Phi(x)$$

<u>requiring local phase invariance</u> $\rightarrow$ introduce $a_\mu(x)$

$\rightarrow$ local gauge field

$\rightarrow$ the electromagnetic field

The electromagnetic field

All interactions in the Standard Model are gauge interactions

$\rightarrow$ invariance under local phase transformations + internal symmetries

electromagnetic interactions $\leftrightarrow$ continuous local symmetry

Maxwell's equations ($\epsilon_0 = \mu_0 = c = 1$)

$$\vec{\nabla} \cdot \vec{E} = \rho_{em} \tag{3}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{4}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{5}$$

$$\vec{\nabla} \times \vec{B} = \vec{J}_{em} + \frac{\partial \vec{E}}{\partial t} \tag{6}$$

Define $\qquad J^{\mu}_{em} = (\rho_{em}, \vec{j}_{em})$

In tems of scalar and vector potential V and $\vec{A}$

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad (B_i = \epsilon_{ijk}\partial_j A_k)$$

so
$$\begin{aligned} (\vec{\nabla} \times \vec{\nabla} \times \vec{A})_i &= \epsilon_{ijk}\partial_j(\epsilon_{klm}\partial_l A_m)_k \\ &= \epsilon_{ijk}\epsilon_{klm}\partial_j\partial_l A_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j\partial_l Am \\ &= \partial_i\partial_j A_j - \partial_j\partial_j A_i = (\vec{\nabla}(\vec{\nabla} \cdot \vec{A}))_i - \vec{\nabla}^2 \vec{A}_i \end{aligned}$$

$$\Rightarrow \qquad \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A} = \vec{j}_{em} - \frac{\partial^2\vec{A}}{\partial t^2} - \frac{\partial}{\partial t}\vec{\nabla}V$$

$$\vec{\nabla}\cdot\vec{E} = -\vec{\nabla}^2 V - \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = \rho_{em}$$

$$\text{or} \qquad \left(\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}^2\vec{A}\right) + \vec{\nabla}\frac{\partial}{\partial t}V + \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) = \vec{j}_{em}$$

$$\left(\frac{\partial^2 V}{\partial t^2} - \vec{\nabla}^2\vec{V}\right) - \frac{\partial}{\partial t}\frac{\partial}{\partial t}V - \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = \rho_{em}$$

we can write this in four vector notation

<u>defining</u> $\quad 4-\text{vector potential} \quad A^\mu = (V, \vec{A}) \;\; ; \;\; A_\mu = (V, -\vec{A})$

$$\begin{aligned} \partial_\nu\partial^\nu A^\mu - \partial^\mu(\partial_\nu A^\nu) &= \partial_\nu(\partial^\nu A^\mu - \partial^\mu A^\nu) \\ &= \partial_\nu F^{\nu\mu} = J^\mu_{em} \end{aligned}$$

where we defined the electromagnetic field strength tensor

$$F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) = -F_{\nu\mu}$$

$$\begin{aligned} F_{0i} &= \partial_0 A_i - \partial_i A_0 = (-\vec{\nabla} V - \partial_t \vec{A})_i = (\vec{E})_i \\ F_{ij} &= \partial_i A_j - \partial_j A_i = -(\vec{\nabla} \times \vec{A})_k = -\epsilon_{ijk} \partial_i A_j = -(\vec{B})_k \end{aligned}$$

$$\underline{\text{hence}} \quad F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$