from the previous lecture ...

Quantisation of the KG field (real scalar field).

$$\underline{\text{The Lagrangian density :}}\quad \mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2$$

$$\text{the conjugate momentum}\quad \pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}$$

equal–time commutation relations

$$\left[\phi(\vec{x},t),\pi(\vec{x}',t)\right] = i\hbar\delta(\vec{x}-\vec{x}')$$
$$\left[\phi(\vec{x},t),\phi(\vec{x}',t)\right] = \left[\pi(\vec{x},t),\pi(\vec{x}',t)\right] = 0$$

In the Heisenberg picture the equations of motion are given by

$$i\hbar\dot{\alpha} = [\alpha, H]\,, \qquad \text{where } \alpha \to \text{operator, } H\text{–Hamiltonian } \alpha \neq \alpha(t)$$

$$\Rightarrow \quad [\phi(\vec{x}, t), H] = i\hbar\dot{\phi}(\vec{x}, t)$$

we obtained the correct equations of motion.

The connection between the quatised field and its particle interpretation is seen by looking at the Fourier transformed field

$$\phi(\vec{x}, t) = \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{2p_o}\left(\mathrm{e}^{-ip\cdot x} f_+(\vec{p}) + \mathrm{e}^{ip\cdot x} f_-(\vec{p})\right)$$

So far the Fourier decomposition that we discussed is classical,

i.e. we didn't yet impose the commutation relations.

If we impose the commutation relations

$$[\pi, \phi] = \delta^3(\vec{x} - \vec{x}') \ ,$$
$$[\phi, \phi] = [\pi, \pi] = 0$$

amplitudes of the Fourier modes become annihilation and creation operators,

$$\begin{aligned} f_-(\vec{p}) &= (f_+(\vec{p}))^\dagger = a^\dagger(\vec{p}) \\ f_+(\vec{p}) &= \qquad\qquad\;\; = a(\vec{p}) \end{aligned}$$

It can then be shown that $a(\vec{p})$ and $a^\dagger(\vec{p})$ satisfy the commutation relations

$$\left[a(\vec{p}), a^\dagger(\vec{p}')\right] = 2p^0\delta^3(\vec{p} - \vec{p}')(2\pi)^3$$
$$\left[a(\vec{p}), a(\vec{p}')\right] = 0 = \left[a^\dagger(\vec{p}), a^\dagger(\vec{p}')\right]$$

i.e. the quantum field $\phi(x)$ creates and annihilates particle states with momentum $\vec{p}$. The vacuum is defined by

$$a(\vec{p})|0\rangle = 0 \qquad \forall\ \vec{p} \qquad \langle 0|0\rangle = 1$$

$$\begin{aligned} 1-\text{particle state} \qquad & |\vec{p}\rangle = a^\dagger(\vec{p})|0\rangle \\ 2-\text{particle state} \qquad & |\vec{p}_1, \vec{p}_2\rangle = a^\dagger(\vec{p}_1)a^\dagger(\vec{p}_2)|0\rangle \end{aligned}$$

and so forth.

So far we discussed the free KG equation.

How do we incorporate interactions?

In Newtonian mechanics we describe interactions by adding a potential.

$$L = \frac{1}{2} \sum_i \left(\frac{dx_i}{dt} \right)^2 - V(\vec{x})$$

$$m \sum_i \ddot{x}_i = -\vec{\nabla} V(\vec{x} = \vec{F}(\vec{x})$$

if we have more than one particle

$$\sum_j m_j \sum_i \ddot{x}_{ji} = \sum_{ij} -\vec{\nabla} V(\vec{x}_i - \vec{x}_j)$$

typically we consider the interactions to be 2–body interactions and we sum over all the interacting particles

For the harmonic oscillator in one dimension

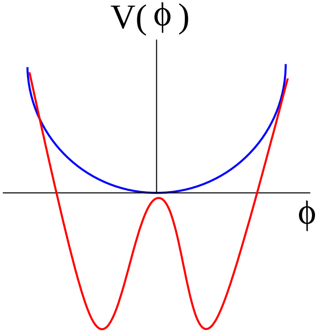
$$\begin{aligned} V(x) &= \frac{1}{2}kx^2 \\ m\ddot{x} &= -\frac{\partial V(x)}{\partial x} = -kx \end{aligned}$$

In the case of the KGE

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2$$

The second term looks like $V(\phi) \sim \frac{1}{2}k\phi^2$.

Classically the particle is in a potential well. If it has initial energy E, it will oscillate about the vacuum $V(\phi) = 0$. The lowest point that the field can be in is the vacuum. So far we are describing a free field. Suppose that we want to describe a field which is not free in a potential



$$V(\phi) = A\phi^4 + B\phi^2 \text{ , with } A > 0, \text{ and } B < 0.$$

In Newtonian mechanics this will correspond to a double well. The point $V(\phi) = 0$ is no longer the vacuum

$$\begin{aligned} \frac{\partial V(\phi)}{\partial \phi} &= 4A\phi^3 + 2B\phi = 0 \\ \text{or} \quad & (2A\phi^2 + B)\phi = 0 \\ \Rightarrow \quad & \phi = 0 \text{ , } \phi = \pm\sqrt{\frac{-B}{2A}} \end{aligned}$$

We can now see how we can use this formalism to describe particles and their interactions.

We developed a diagramatic representation of interactions.

$\rightarrow$ Feynman diagrams

The interactions that we described so far are by using a single scalar field
In nature we are familiar so far with gravity, E&M, weak, and strong interactions.

Force	Gravity	E&M	Weak	Strong
Mediator	Graviton	Photon	$W^{\pm}$, Z-bosons	Gluons
Spin	$+2$	$+1$	$+1$	$+1$
mass	0	0	~ 80, ~ 90GeV	0
gauge symmetry	spacetime diff.	$U(1)$	$SU(2)$	$SU(3)$

How can we describe these interactions?

Standard Model $\rightarrow$ $\quad SU(3)_C \times SU(2)_L \times U(1)_Y \quad$ local gauge interactions.

In the modern language of elementary particles, interactions correspond to invariances of the Lagrangian under some local symmetry.

Interaction $\longleftrightarrow$ invariance under a local gauge symmetry