MATH431 — Introduction to Modern Particle Theory: FEEDBACK/SOLUTIONS 3

1.

$$
\begin{aligned}
L &= \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{pmatrix} \\
\Rightarrow L^T &= L\,. \\
\text{So}\quad L^T\eta L &= \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{pmatrix} \\
&= \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{pmatrix} \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ \frac{\gamma v}{c} & -\gamma \end{pmatrix} \\
&= \begin{pmatrix} \gamma^2 - \left(\frac{\gamma v}{c}\right)^2 & -\gamma\frac{\gamma v}{c} + \gamma\frac{\gamma v}{c} \\ -\gamma\frac{\gamma v}{c} + \gamma\frac{\gamma v}{c} & \left(\frac{\gamma v}{c}\right)^2 - \gamma^2 \end{pmatrix} \\
&= \begin{pmatrix} \gamma^2\left(1 - \frac{v^2}{c^2}\right) & 0 \\ 0 & -\gamma^2\left(1 - \frac{v^2}{c^2}\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \eta\,,
\end{aligned}
$$

since

$$\gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}}\,.$$

2. (i) Taking the inverse of (1), we have

$$(L^T\eta L)^{-1} = \eta^{-1} \Rightarrow L^{-1}\eta^{-1}(L^T)^{-1} = \eta^{-1}\,.$$

Multiplying from the left by L and from the right by L^T, we find

$$
\begin{aligned}
LL^{-1}\eta^{-1}(L^T)^{-1}L^T &= L\eta^{-1}L^T\,, \\
\eta^{-1} &= L\eta^{-1}L^T
\end{aligned}
$$

as required.

(ii) Putting $\mu = \nu = 0$ in (3), we have

$$
\begin{aligned}
&\eta_{\alpha\beta}L^\alpha{}_0 L^\beta{}_0 = 1, \quad \eta^{\alpha\beta}L^0{}_\alpha L^0{}_\beta = 1\,, \\
&\text{i.e.}\quad \left(L^0{}_0\right)^2 - \left(L^1{}_0\right)^2 - \left(L^2{}_0\right)^2 - \left(L^3{}_0\right)^2 = 1, \\
&\left(\bar{L}^0{}_0\right)^2 - \left(\bar{L}^0{}_1\right)^2 - \left(\bar{L}^0{}_2\right)^2 - \left(\bar{L}^0{}_3\right)^2 = 1 \\
&\Rightarrow \left(L^0{}_0\right)^2 - |\mathbf{l}|^2 = 1, \quad \left(\bar{L}^0{}_0\right)^2 - |\bar{\mathbf{l}}|^2 = 1 \\
&\Rightarrow |\mathbf{l}| = \sqrt{\left(L^0{}_0\right)^2 - 1}, \quad |\bar{\mathbf{l}}| = \sqrt{\left(\bar{L}^0{}_0\right)^2 - 1}\,.
\end{aligned}
$$

(iii)

$$
\begin{aligned}
(\bar{L}L)^0{}_0 &= \bar{L}^0{}_\alpha L^\alpha{}_0 \\
&= \bar{L}^0{}_0 L^0{}_0 + \bar{L}^0{}_1 L^1{}_0 + \bar{L}^0{}_2 L^2{}_0 + \bar{L}^0{}_3 L^3{}_0 \\
&= \bar{L}^0{}_0 L^0{}_0 + \bar{\mathbf{l}}.\mathbf{l}
\end{aligned}
$$

(iv)

$$
\begin{aligned}
|\bar{\mathbf{l}}.\mathbf{l}| \le |\mathbf{l}||\bar{\mathbf{l}}| &\Rightarrow -|\mathbf{l}||\bar{\mathbf{l}}| \le \bar{\mathbf{l}}.\mathbf{l} \le |\mathbf{l}||\bar{\mathbf{l}}| \\
\Rightarrow \quad \text{(using (iii))} \quad (\bar{L}L)^0{}_0 - \bar{L}^0{}_0 L^0{}_0 &\ge -|\mathbf{l}||\bar{\mathbf{l}}| \\
\Rightarrow (\bar{L}L)^0{}_0 &\ge \bar{L}^0{}_0 L^0{}_0 - |\mathbf{l}||\bar{\mathbf{l}}| \,, \\
\text{i.e. (using (ii))} \quad (\bar{L}L)^0{}_0 &\ge \bar{L}^0{}_0 L^0{}_0 - \sqrt{\left(L^0{}_0\right)^2 - 1}\,\sqrt{\left(\bar{L}^0{}_0\right)^2 - 1}\,.
\end{aligned}
$$

(v)

$$
\begin{aligned}
(x-y)^2 \ge 0 \Rightarrow x^2 - 2xy + y^2 \ge 0 &\Rightarrow -2xy \ge -x^2 - y^2 \\
\Rightarrow x^2y^2 - 2xy + 1 &\ge x^2y^2 - x^2 - y^2 + 1 \\
\Rightarrow x^2y^2 - 2xy + 1 \ge (x^2-1)(y^2-1) &\Rightarrow (xy-1)^2 \ge (x^2-1)(y^2-1) \\
\Rightarrow \text{either} \quad xy - 1 \ge \sqrt{x^2-1}\sqrt{y^2-1} \quad &\text{or} \quad xy - 1 \le -\sqrt{x^2-1}\sqrt{y^2-1}\,.
\end{aligned}
$$

If $x, y \ge 1$ then $xy - 1$ is positive, and we must have the first inequality, implying

$$
xy - \sqrt{x^2-1}\sqrt{y^2-1} \ge 1\,.
$$

Writing $L^0{}_0 = x$, $\bar{L}^0{}_0 = y$, we have from (iv)

$$
(\bar{L}L)^0{}_0 \ge \bar{L}^0{}_0 L^0{}_0 - \sqrt{\left(L^0{}_0\right)^2 - 1}\,\sqrt{\left(\bar{L}^0{}_0\right)^2 - 1} \ge 1\,.
$$

(vi) Obviously if $\det \bar{L} = \det L = 1$, then

$$
\det(\bar{L}L) = \det \bar{L} \det L = 1\,.
$$

(vii) We have now shown that if $L \in \mathcal{L}_+^\uparrow$, $\bar{L} \in \mathcal{L}_+^\uparrow$, then $\bar{L}L \in \mathcal{L}_+^\uparrow$. It is clear that $1 \in \mathcal{L}_+^\uparrow$.

(viii) We can write (1) as

$$
(\eta^{-1}L^T\eta)L = \eta^{-1}\eta = 1\,,
$$

which shows that $L^{-1} = \eta^{-1}L^T\eta$, i.e. $(L^{-1})^\mu{}_\nu = \eta^{\mu\alpha}(L^T)_\alpha{}^\beta\eta_{\beta\nu} = \eta^{\mu\alpha}L^\beta{}_\alpha\eta_{\beta\nu}$. So $(L^{-1})^0{}_0 = \eta^{00}L^0{}_0\eta_{00} = L^0{}_0$. Moreover,

$$
\det L^{-1} = \det \eta^{-1} \det L^T \det \eta = (-1)\det L(-1) = \det L = 1\,.
$$

So $L^{-1} \in \mathcal{L}_+^\uparrow$. The remaining group property is associativity, $(L_1L_2)L_3 = L_1(L_2L_3)$, which is true for all matrices. So $\mathcal{L}_+^\uparrow$ *is* a group.

3. (a) In the lectures we have defined the operators $J_{i\,\pm}$ by

$$\vec{J}_\pm \;=\; \frac{1}{2}(\vec{J} \pm i\vec{K})$$

and seen that $\vec{J}_+$ and $\vec{J}_-$ are both generating the algebra $SU(2)$ but commute between each other. In this way we have rewritten the Lorentz algebra as the algebra of $SU(2) \times SU(2)$. As $\vec{J}_+^2$ and $\vec{J}_-^2$ are Casimir operators of the two commuting $SU(2)$ groups they are thus also invariants of the Lorentz group. We have

$$\begin{aligned}
\vec{J}_+^2 &= \frac{1}{4}(\vec{J}^2 - \vec{K}^2 + 2\,i\,\vec{J}\cdot\vec{K})\,, \\
\vec{J}_-^2 &= \frac{1}{4}(\vec{J}^2 - \vec{K}^2 - 2\,i\,\vec{J}\cdot\vec{K})\,,
\end{aligned}$$

and hence

$$\begin{aligned}
\vec{J}^2 - \vec{K}^2 &= 2(\vec{J}_+^2 + \vec{J}_-^2)\,, \\
\vec{J}\cdot\vec{K} &= -i(J_+^2 - J_-^2)\,.
\end{aligned}$$

Therefore $\vec{J}^2 - \vec{K}^2$ and $\vec{J}\cdot\vec{K}$ are Lorentz invariants as well, being the sum and difference of Lorentz invariants.

(b) For the representation (j_1, j_2) of the $SU(2) \times SU(2)$ algebra the number of states is $(2j_1+1)(2j_2+1)$. The total spin is given by $j = j_1 + j_2$. Therefore the composition $j_1 \otimes j_2$ breaks under $SU(2)_J$ with the following spin states

$$j_1 + j_2 \oplus j_1 + j_2 - 1 \oplus \cdots \oplus |j_1 - j_2|\,.$$