

MATH431 - Modern Particle Physics
Set Work: Sheet 2

1. Suppose that we live on a two dimensional surface with a line element on it given by

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2$$

(a convenient notation is $\theta^\mu \equiv (\theta, \phi)$ ($\mu = 1, 2$))

- a. Write the metric $g^{\mu\nu}$ in an explicit matrix form. Write $g_{\mu\nu}$ in matrix form.

- b. Find the set of infinitesimal transformations of the form

$$\theta^\mu \rightarrow \theta^\mu + \epsilon \zeta^\mu(\theta, \phi)$$

for which the line element ds^2 is invariant.

- c. Define the operators

$$J = \zeta^\mu \frac{\partial}{\partial x^\mu}$$

(summation over the index μ) show that all three independent operators satisfy

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

- d. What is the geometric meaning of this example

2. Let \vec{J} and \vec{K} be the generators of rotations and boosts, respectively.

- a. Show that

$$J^2 - K^2 \quad \text{and} \quad \vec{J} \cdot \vec{K}$$

are Lorentz invariants (*i.e.* that they commute with all the generators of the Lorentz group).

- b. Assume a representation (j_1, j_2) of $SU(2) \times SU(2)^\dagger$. How many states are there in this representation. How do they decompose under $SU(2)_J$

3. Consider a massive particle moving with velocity $v = \tanh \eta$ along the x -axis. Show that, if E is the energy of the particle and p its momentum along the propagation direction, then

$$\eta = \frac{1}{2} \ln \frac{E + p}{E - p}.$$

Verify that under a second boost in the direction of motion with velocity v' , η' transforms additively.

4. Consider the infinitesimal line element,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - dx^2.$$

- a. Write the metric $g_{\mu\nu}$ and its inverse in an explicit matrix form.
 b. Find the set of independent transformations of the form

$$\begin{aligned}
t &\rightarrow t + \epsilon A(t, x) \\
x &\rightarrow x + \epsilon B(t, x) ,
\end{aligned}$$

where ϵ is an infinitesimal constant and the functions A and B have to be determined by the requirement that ds^2 is invariant. State what each transformation represents in space time.