from the previous lecture ...

The Dirac equation

The Schrödinger equation: $i\hbar\frac{\partial}{\partial t}\psi = \hat{H}\psi$

is linear in $i\hbar\frac{\partial}{\partial t}$ $\rightarrow$ linear in E

since $P_\mu = i\hbar\partial_\mu$ & $P^2 = E^2 - \vec{P}^2 = m^2$, the KGE is quadratic in $\frac{\partial}{\partial t}$.

Dirac wanted to find a relativistically covariant equation which is linear in $i\hbar\frac{\partial}{\partial t}$

i.e. linear in energy $\rightarrow$ linear in the Hamiltonian

$\rightarrow$ linear in the generator of time translations

$\rightarrow$ linear in time + relativistically covariant $\Rightarrow$ linear in $\underline{-i\hbar\vec{\nabla}}$

$-i\hbar\vec{\nabla}$ $\rightarrow$ the quantum generator of spatial translations relativistically $E^2 = \vec{p}^2 c^2 + m^2 c^4 \quad \Rightarrow \quad E = \pm\sqrt{\vec{p}^2 c^2 + m^2 c^4}$

$$i\hbar \frac{1}{c}\frac{\partial}{\partial t}\psi(\vec{r}, t) = \pm\sqrt{\vec{p}^2 + m^2 c^2}\ \psi(\vec{r}, t)$$

we have to get rid of the $\sqrt{}$

$$\text{Write} \qquad \sqrt{\vec{p}^2 + m^2c^2} = \alpha_i p_i + \beta m \tag{1}$$

Diraq $\quad \rightarrow \quad$ find $\alpha_i,\ \beta$ such (1) holds?

$\Rightarrow \quad$ covariant equation linear in $i\hbar\frac{\partial}{\partial t}$ and $-i\hbar\vec{\nabla}$

Take the square of eq. (1)

$$\begin{aligned}
& \vec{p}^2 + m^2c^2 = \sum p_i p_i + m^2c^2 = (\alpha_i p_i + \beta mc)^2 \\
= \quad & (\alpha_i p_i + \beta mc)(\alpha_j p_j + \beta mc) \\
= \quad & \alpha_i \alpha_j p_i p_j + (\alpha_i \beta + \beta \alpha_i) p_i mc + \beta^2 m^2 c^2
\end{aligned}$$

For the equation to hold we must impose the following requirements

1. $\beta^2 = 1$
2. $\alpha_i \beta + \beta \alpha_i = 0 \quad \leftarrow \quad$ no linear term in p_i in the square

   these conditions can hold only if α_i and β are matrices

   with $\alpha_i, \ \beta$ anti–commuting matrices
3. $\alpha_i \alpha_j + \alpha_j \alpha_i = 0 \quad i \neq j$
4. $\alpha_i^2 = 1 \quad i = j$

can we find $\alpha_i, \ \beta$ that satisfy these conditions?

1 2 x 2 matrices $\quad \alpha_i = \sigma_i$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad ; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad ;$$

$$\sigma_i^2 = 1 \qquad ; \qquad \sigma_i \sigma_j + \sigma_j \sigma_i = 0 \ \text{ for } \ i \neq j$$

But we lack a 2x2 β matrix that satisfies 1 & 2.

2 3 x 3 matrices

No solutions $\longleftrightarrow$ solution must be even order.

Proof: Assume an odd order solution

Assume: A β matrix which is diagonal. As in the 2x2 case we can always diagonalise at least one matrix

$$\beta = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & \lambda_n \end{pmatrix} \qquad \beta^2 = 1 \quad \Rightarrow \quad \lambda_i = \pm 1$$

We don't know how many λ_i are positive or negative?

Prove that $\mathrm{Tr}\beta = 0$

In that case: $\#$ of $+1$ eigenvalues $= \#$ of -1 eigenvalues

$\Rightarrow \quad \beta$ must be even

$$
\begin{aligned}
&\alpha_i\beta + \beta\alpha_i = 0 \qquad / \cdot \alpha_i \\
&\alpha_i\beta\alpha_i + \beta\alpha_i^2 = 0 \quad \Rightarrow \quad \alpha_i\beta\alpha_i + \beta = 0 \\
\Rightarrow \qquad &\mathrm{Tr}\beta = -\mathrm{Tr}(\alpha_i\beta\alpha_i) = -\mathrm{Tr}(\alpha_i\alpha_i\beta) = -\mathrm{Tr}\beta \\
\Rightarrow \qquad &\mathrm{tr}\beta = -\mathrm{Tr}\beta \quad \Rightarrow \quad \mathrm{Tr}\beta = 0 \quad \rightarrow \quad \beta_{nxn} \ \text{ with } n\text{–even}
\end{aligned}
$$

3 at order 4 *i.e.* 4x4 matrices $\rightarrow$ there is a solution

$$
\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} I_{2x2} & 0 \\ 0 & -I_{2x2} \end{pmatrix}
$$

where σ_i are 2x2 Pauli matrices

The Dirac equation

$$
i\hbar\frac{1}{c}\frac{\partial}{\partial t}\Psi(\vec{x},t) = (-i\hbar\vec{\alpha}\cdot\vec{\nabla} + \beta mc)\Psi(\vec{x},t) \tag{2}
$$

The wave function $\quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$ is a 4–component vector (not a four spacetime vector)

2 components are spin $\uparrow$ spin $\downarrow$ particles

2 components are spin $\uparrow$ spin $\downarrow$ anti–particles

consequences:

1. The Dirac equation predicts the existence of anti–particles
2. Time and space derivatives are linear

Multiply eq. (2) by β, recalling that $\beta^2 = 1$

$$i\hbar\beta\frac{1}{c}\frac{\partial}{\partial t}\Psi(\vec{x},t) = (-i\hbar\beta\vec{\alpha}\cdot\vec{\nabla} + \beta^2 mc)\Psi(\vec{x},t)$$

or $$i\hbar(\underset{\substack{\downarrow \\ \gamma_0}}{\beta}\frac{1}{c}\frac{\partial}{\partial t} + \underset{\substack{\downarrow \\ +\gamma^i = \beta\alpha_i = -\alpha_i\beta}}{\beta\vec{\alpha}}\cdot\vec{\nabla})\Psi(\vec{x},t) = mc\Psi(\vec{x},t)$$

<u>hence</u> $$i\hbar(\gamma^0\frac{\partial}{\partial ct} + \gamma^i\cdot\partial_i)\Psi(\vec{x},t) = mc\Psi(\vec{x},t)$$

$$\begin{aligned} \text{or} \quad i\hbar(\eta^{\mu\nu}\gamma_\mu\partial_\nu)\Psi &= i\hbar\gamma^\nu\partial_\nu\Psi = i\hbar\not{\partial}\Psi = mc\Psi \\ \gamma^\mu p_\mu\Psi &= mc\Psi \\ \not{p}\Psi &= mc\Psi \quad \rightarrow \quad (\not{p} - mc)\Psi = 0 \quad \leftarrow \quad \text{free Dirac equation} \end{aligned}$$

$$\text{Setting} \quad \hbar = c = 1 \quad \Rightarrow \quad (i\not{\partial} - m)\Psi = 0$$

Lowest order Dirac γ^μ matrices $\rightarrow$ 4x4 $\rightarrow$ massive particles

massless particles $\rightarrow$ no constraint on $\beta \Rightarrow$ 2x2 solution $\rightarrow$ Pauli matrices

The Dirac equation is of the form $i\hbar\frac{\partial}{\partial t}\Psi = H\Psi$

$$H = c\vec{\alpha}\cdot\vec{p} + \beta mc^2$$

$$\underline{\text{hermiticity}} \quad \to \quad H = H^\dagger \quad \Leftrightarrow \quad \alpha_i = \alpha_i^\dagger \quad \beta = \beta^\dagger$$

$$\Rightarrow \quad \gamma^{0\dagger} = \gamma^0 \quad ; \quad \alpha_i = \gamma^0\gamma^i = (\gamma^0\gamma^i)^\dagger = \gamma^{i\dagger}\gamma^{0\dagger} = \gamma^{i\dagger}\gamma^0$$

$$\Rightarrow \quad \gamma^0\gamma^{i\dagger}\gamma^0 = \gamma^i$$

$$\underline{\text{summarise}} \quad \gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu\dagger} \qquad \mu = 0, 1, 2, 3$$

$$\underline{\text{together with}} \qquad \{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$$

are the two properties that define the Dirac γ–matrices

$$\underline{\text{representation}} \qquad \gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \qquad i = 1, 2, 3$$

where σ_i are the Pauli matrices