

Class test

$$1) ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dx^2 + dy^2 \quad (1)$$

$$a) g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad g^{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1}$$

$$b) \left. \begin{aligned} x &\Rightarrow x + \epsilon A(x, y) \\ y &\Rightarrow y + \epsilon B(x, y) \end{aligned} \right\} \text{Transformations}$$

$$dx \Rightarrow dx + \epsilon \left(\frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy \right)$$

$$dy \Rightarrow dy + \epsilon \left(\frac{\partial B}{\partial x} dx + \frac{\partial B}{\partial y} dy \right)$$

$$dx^2 \Rightarrow dx^2 + 2\epsilon \left(\frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy \right) dx + \mathcal{O}(\epsilon^2)$$

$$dy^2 \Rightarrow dy^2 + 2\epsilon \left(\frac{\partial B}{\partial x} dx + \frac{\partial B}{\partial y} dy \right) dy + \mathcal{O}(\epsilon^2)$$

$$(2) \therefore ds^2 = dx^2 + 2\epsilon \left(\frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy \right) dx + dy^2 + 2\epsilon \left(\frac{\partial B}{\partial x} dx + \frac{\partial B}{\partial y} dy \right) dy.$$

For ds^2 to be invariant, it must have the same value before and after undergoing the above transformation.

$$(3) \therefore (1) = (2) \Rightarrow dx^2 + dy^2 = dx^2 + 2\epsilon \left(\frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy \right) dx + dy^2 + 2\epsilon \left(\frac{\partial B}{\partial x} dx + \frac{\partial B}{\partial y} dy \right) dy$$

$$\therefore (3) \Rightarrow dx^2 + dy^2 = dx^2 + 2\epsilon \frac{\partial A}{\partial x} dx^2 + 2\epsilon \frac{\partial A}{\partial y} dy dx + dy^2 + 2\epsilon \frac{\partial B}{\partial x} dx dy + 2\epsilon \frac{\partial B}{\partial y} dy^2$$

Comparing coefficients in (3) gives us the following,

$$dx^2: 1 = 1 + 2\epsilon \frac{\partial A}{\partial x}$$

$$dy^2: 1 = 1 + 2\epsilon \frac{\partial B}{\partial y}$$

0
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$$dydx: 0 = 2\epsilon \frac{\partial A}{\partial y} + 2\epsilon \frac{\partial B}{\partial x}$$

We can then solve these ~~equations~~ equations to obtain a value for our parameters A and B.

$$\therefore dx^2: 2\epsilon \frac{\partial A}{\partial x} = 0 \quad \therefore \frac{\partial A}{\partial x} = 0 \quad \text{as } 2\epsilon \neq 0$$

$$\therefore A = f(y).$$

$$dy^2: \frac{\partial B}{\partial y} = 0 \quad \therefore B = g(x)$$

$$dydx: 2\epsilon \frac{\partial A}{\partial y} = -2\epsilon \frac{\partial B}{\partial x} \quad \therefore \frac{\partial A}{\partial y} = -\frac{\partial B}{\partial x}$$

For $\frac{\partial A}{\partial y} = -\frac{\partial B}{\partial x}$ to be true they both must equal some constant.

$$\therefore \frac{\partial A}{\partial y} = -\frac{\partial B}{\partial x} = c_1,$$

Using the values for A and B found earlier, $A = f(y)$ and $B = g(x)$, we obtain,

$$\frac{\partial A}{\partial y} = f'(y) \quad \& \quad \frac{\partial B}{\partial x} = g'(x)$$

$$\therefore f'(y) = c_1 \quad \therefore f(y) = c_1 y + c_2$$

$$-g'(x) = c_1 \quad \therefore g(x) = c_3 - c_1 x$$

$$\therefore A = c_1 y + c_2, \quad B = c_3 - c_1 x$$

Here c_1 , c_2 and c_3 are parameters where c_1 is a ~~rotation~~ and c_2 & c_3 are shifts in space.

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The transformations then become

$$x \rightarrow x + \epsilon(c_1 y + c_2)$$

$$y \rightarrow y + \epsilon(c_3 - c_1 x)$$

which both represent rotations and have both been translated by a factor of ϵc_2 for x and ϵc_3 for y

2a) $L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu$, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

The Euler Lagrange equation for A_μ is,

$$\frac{\partial}{\partial x^\nu} \left(\frac{\partial L}{\partial A_{\mu,\nu}} \right) - \frac{\partial L}{\partial A_\mu} = 0.$$

We begin by inputting our ^{identity} ~~expression~~ for $F^{\mu\nu}$ into the Lagrangian.

$$\therefore L = -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu) (\partial^\mu A^\nu - \partial^\nu A^\mu) - j^\mu A_\mu$$

which can be re-written as,

$$L = -\frac{1}{4} \eta^{\alpha\mu} \eta^{\beta\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) (\partial_\alpha A_\beta - \partial_\beta A_\alpha) - j^\mu A_\mu$$

by using $\eta^{\alpha\mu} A_\alpha \equiv A^\mu$ etc...

We split our Euler Lagrange equation up into two parts,

$$\underbrace{\frac{\partial}{\partial x^\nu} \left(\frac{\partial L}{\partial A_{\mu,\nu}} \right)}_{\text{first part}}$$

$$\& \underbrace{\frac{\partial L}{\partial A_\mu}}_{\text{second part.}}$$

first part

second part.

We now apply our Lagrangian to the first part of our Euler-Lagrange equation.

We also note that $A_{\mu,\nu} = \partial_\mu A_\nu$

So

$$\therefore \frac{\partial}{\partial x^\nu} \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu)} \right)$$

$$\therefore \frac{\partial}{\partial x^\nu} \left(\frac{\partial (-\frac{1}{4} \eta^{\alpha\mu} \eta^{\beta\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) (\partial_\alpha A_\beta - \partial_\beta A_\alpha))}{\partial (\partial_\rho A_\sigma)} - i j^\mu A_\mu \right)$$

~~Using~~

$$\frac{\partial (\partial_\alpha A_\beta)}{\partial (\partial_\mu A_\nu)} = \delta_\alpha^\mu \delta_\beta^\nu$$

Where we have changed the dummy indices " μ " and " ν " in the denominator.

expanding the brackets in our numerator we get,

$$\frac{\partial}{\partial x^\nu} \left(\frac{\partial (-\frac{1}{4} \eta^{\alpha\mu} \eta^{\beta\nu} (\partial_\mu A_\nu \partial_\alpha A_\beta - \partial_\mu A_\nu \partial_\beta A_\alpha - \partial_\nu A_\mu \partial_\alpha A_\beta + \partial_\nu A_\mu \partial_\beta A_\alpha))}{\partial (\partial_\rho A_\sigma)} - i j^\mu A_\mu \right)$$

(4)

~~Using~~ using $\frac{\partial (\partial_\alpha A_\beta)}{\partial (\partial_\mu A_\nu)} = \delta_\alpha^\mu \delta_\beta^\nu$ we obtain

$$\begin{aligned} (4) \Rightarrow & -\frac{1}{4} \eta^{\alpha\mu} \eta^{\beta\nu} \left(\delta_\rho^\mu \delta_q^\nu \partial_\alpha A_\beta + \delta_\rho^\alpha \delta_q^\beta \partial_\mu A_\nu - \delta_\rho^\mu \delta_q^\nu \partial_\beta A_\alpha - \delta_\rho^\beta \delta_q^\alpha \partial_\mu A_\nu \right. \\ & \left. - \delta_\rho^\nu \delta_q^\mu \partial_\alpha A_\beta - \delta_\rho^\alpha \delta_q^\beta \partial_\nu A_\mu + \delta_\rho^\nu \delta_q^\mu \partial_\beta A_\alpha + \delta_\rho^\beta \delta_q^\alpha \partial_\nu A_\mu \right) \\ \Rightarrow & -\frac{1}{4} \eta^{\alpha\mu} \eta^{\beta\nu} \left((\delta_\rho^\mu \delta_q^\nu - \delta_\rho^\nu \delta_q^\mu) \partial_\alpha A_\beta + (\delta_\rho^\alpha \delta_q^\beta - \delta_\rho^\beta \delta_q^\alpha) \partial_\mu A_\nu \right. \\ & \left. + (\delta_\rho^\nu \delta_q^\mu - \delta_\rho^\mu \delta_q^\nu) \partial_\beta A_\alpha + (\delta_\rho^\beta \delta_q^\alpha - \delta_\rho^\alpha \delta_q^\beta) \partial_\nu A_\mu \right) \end{aligned}$$

which further simplifies to

$$④ \Rightarrow -\frac{1}{4} \eta^{\alpha\mu} \eta^{\beta\nu} \left((\delta_\rho^\mu \delta_\sigma^\nu - \delta_\rho^\nu \delta_\sigma^\mu) (\partial_\alpha A_\beta - \partial_\beta A_\alpha) + (\delta_\rho^\beta \delta_\sigma^\alpha - \delta_\rho^\alpha \delta_\sigma^\beta) (\partial_\mu A_\nu - \partial_\nu A_\mu) \right)$$

Using ~~δ^μ_ρ~~ $\delta^\mu_\rho = \delta^\rho_\mu$ and $\eta^{\alpha\mu} \delta^\rho_\mu = \eta^{\alpha\rho}$,

We obtain

$$④ \Rightarrow -\frac{1}{4} \left[(\eta^{\alpha\rho} \eta^{\beta\sigma} - \eta^{\beta\rho} \eta^{\alpha\sigma}) (\partial_\alpha A_\beta - \partial_\beta A_\alpha) + (-\eta^{\nu\rho} \eta^{\mu\sigma} + \eta^{\mu\rho} \eta^{\nu\sigma}) (\partial_\mu A_\nu - \partial_\nu A_\mu) \right]$$

$$④ \Rightarrow -\frac{1}{4} \left[\eta^{\alpha\rho} \eta^{\beta\sigma} \partial_\alpha A_\beta - \eta^{\beta\rho} \eta^{\alpha\sigma} \partial_\beta A_\alpha + \eta^{\alpha\rho} \eta^{\beta\sigma} \partial_\beta A_\alpha - \eta^{\beta\rho} \eta^{\alpha\sigma} \partial_\alpha A_\beta + \eta^{\nu\rho} \eta^{\mu\sigma} \partial_\mu A_\nu - \eta^{\mu\rho} \eta^{\nu\sigma} \partial_\mu A_\nu + \eta^{\nu\rho} \eta^{\mu\sigma} \partial_\nu A_\mu - \eta^{\mu\rho} \eta^{\nu\sigma} \partial_\nu A_\mu \right]$$

$$④ \Rightarrow -\frac{1}{4} \left[\partial^\rho A^\sigma - \partial^\sigma A^\rho + \partial^\rho A^\sigma - \partial^\sigma A^\rho + \partial^\rho A^\sigma - \partial^\sigma A^\rho + \partial^\rho A^\sigma - \partial^\sigma A^\rho \right]$$

$$\Rightarrow -\frac{1}{4} \left[2 \partial^\rho A^\sigma - 2 \partial^\sigma A^\rho + 2 \partial^\rho A^\sigma - 2 \partial^\sigma A^\rho \right]$$

$$2 - (\partial^\rho A^\sigma - \partial^\sigma A^\rho) = \partial^\sigma A^\rho - \partial^\rho A^\sigma = F^{\sigma\rho}$$

\therefore our first term is $\frac{\partial}{\partial x^\nu} F^{\mu\nu}$

and our second term is trivial, $\frac{\partial \mathcal{L}}{\partial A_\mu} = j^\mu$

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$$\mathcal{L} = \frac{\partial}{\partial x^\nu} F^{\mu\nu} = j^\mu, \quad \partial_\mu j^\mu = 0 \quad \text{as} \quad \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} F^{\mu\nu} = 0$$

$$b) L' = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu + \frac{1}{2} m^2 A_\mu A^\mu$$

$$\therefore \frac{\partial L}{\partial A_\mu} = -j^\mu + \frac{1}{2} m^2 \eta^{\alpha\beta} \partial_\mu (A_\alpha A_\beta)$$

$$\therefore = -j^\mu + \frac{1}{2} m^2 \eta^{\alpha\beta} (\delta^\alpha_\mu A_\alpha + \delta^\beta_\mu A_\beta)$$

$$= -j^\nu + m^2 A^\mu,$$

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applying this to our E-L equation we get

$$-\partial_\mu F^{\mu\nu} = -j^\nu + m^2 A^\mu \quad \text{Lorentz gauge}$$

$$\partial_\mu A^\mu = 0.$$

$$\therefore -\partial_\mu F^{\mu\nu} - m^2 A^\mu = -j^\nu$$

$$\therefore \partial_\mu F^{\mu\nu} + m^2 A^\mu = j^\nu$$

$$\therefore (\partial_\mu \partial^\mu + m^2) A^\mu = j^\nu$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$\partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu + m^2 A^\nu$$

$$= \square A^\nu - \partial_\mu \partial^\nu A^\mu + m^2 A^\nu$$