from the previous lectures ...

The Higgs mechanism

Complex scalar field with continuous $U(1)$ symmetry $\phi(x) \to \mathrm{e}^{i\alpha(x)}\phi(x)$

invariance requires $\partial_\mu \to \partial_\mu - ieA_\mu$ and the gauge field A_μ transforms as

$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha$$

The gauge invariant Lagrangian is

$$\mathcal{L} = (\partial_\mu + ieA_\mu)\phi^*(\partial^\mu - ieA^\mu)\phi - \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

For $\mu^2 < 0$ expand $\phi(x) = \frac{1}{\sqrt{2}}(v+\eta(x)+i\zeta(x)) \cong \frac{1}{\sqrt{2}}(v+\eta(x))\,\mathrm{e}^{i\frac{\zeta(x)}{v}}$

$$\begin{aligned} \Rightarrow \quad \mathcal{L}' \quad = \quad & \frac{1}{2}(\partial_\mu h)^2 - \lambda v^2 h^2 + \frac{1}{2}e^2 v^2 A_\mu A^\mu - \lambda v h^3 - \frac{1}{4}\lambda h^4 \\ & + \frac{1}{2}e^2 A_\mu A^\mu h^2 + v e^2 A_\mu A^\mu h - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \end{aligned}$$

The Goldstone boson disappeared altogether

$\rightarrow$ The Goldstone boson is absorbed as the longitudenal mode of A_μ

$\rightarrow$ only 2 physical fields h and A^μ.

We are ready to see how the Higgs mechanism operates in the Standard Model.

$$\mathcal{L} = (\partial_\mu \Phi)^\dagger (\partial^\mu \Phi) - \mu^2 (\Phi^\dagger \Phi) - \lambda (\Phi^\dagger \Phi)^2$$

$$\text{with} \quad \Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}$$

The Lagrangian is invariant under the gauge transformation

$$\Phi \to \Phi' = e^{i\frac{\alpha_a \tau_a}{2}} \Phi$$

The covariant derivative $\quad D_\mu = \partial_\mu + ig\frac{\tau_a}{2} W_\mu^a \qquad a = 1, 2, 3$

$$\text{under} \quad \Phi(x) \to \Phi'(x) = (1 + i\frac{\vec{\alpha} \cdot \vec{\tau}}{2}) \Phi(x)$$

$$\vec{W}_\mu \to \vec{W}_\mu - \frac{1}{g} \partial_\mu \vec{\alpha} - \vec{\alpha} \times \vec{W}_\mu$$

The gauge invariant Lagrangian is

$$\mathcal{L} = (\partial_\mu \Phi + ig\frac{1}{2}\vec{\tau}\cdot\vec{W}_\mu\Phi)^\dagger(\partial^\mu\Phi + ig\frac{1}{2}\vec{\tau}\cdot\vec{W}^\mu\Phi) - V(\Phi) - \frac{1}{4}\vec{W}_{\mu\nu}\cdot\vec{W}^{\mu\nu}$$

$$\text{where} \qquad \vec{W}_{\mu\nu} = \partial_\mu\vec{W}_\nu - \partial_\nu\vec{W}_\mu - g\vec{W}_\mu\times\vec{W}_\nu$$

Take $\mu^2 < 0 \quad , \quad \lambda > 0$

The potential has a minimum at

$$\Phi^\dagger\Phi = \frac{1}{2}(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2) = -\frac{\mu^2}{2\lambda}$$

Expand about a minimum. Choose

$$\phi_1 = \phi_2 = \phi_4 = 0 \qquad , \qquad \phi_3^2 = -\frac{\mu^2}{\lambda} = v^2$$

$$\text{Expand } \Phi(x) \text{ about the vacuum} \langle \Phi_0 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

$$\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$$

The three additional degrees of freedom are absorbed as the longitudenal components of W_1^μ, W_2^μ , W_3^μ. The mass term

$$\left| \left(g \frac{\vec{\tau}}{2} \cdot \vec{W}_\mu \right) \Phi \right|^2 = \frac{g^2}{8} \left| \begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \right|^2$$

$$= \frac{g^2 v^2}{8} \left[(W_\mu^1)^2 + (W_\mu^2)^2 + (W_\mu^3)^2 \right] \rightarrow \text{3 massive vector bosons}$$

Now consider the Lagrangian of the $SU(2)_W \times U(1)_Y$ of the Standard Model coupled to Φ

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \qquad \text{with} \qquad \begin{aligned} \phi^+ &= \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \\ \phi^0 &= \frac{1}{\sqrt{2}}(\phi_3 + i\phi_4) \end{aligned}$$

The Lagrangian density of the Higgs field is given by:

$$\mathcal{L} = \left| \left(\partial_\mu - ig\vec{T} \cdot \vec{W}_\mu - ig' \frac{Y}{2} B_\mu \right) \Phi \right|^2 - V(\Phi)$$

where $\vec{T} = \frac{\vec{\tau}}{2}$ and

$$V(\Phi) = \mu^2 \Phi^\dagger \Phi + \lambda \left(\Phi^\dagger \Phi \right)^2$$

The relevant term for the gauge boson masses

$$\left|\left(g\frac{\vec{\tau}}{2}\cdot\vec{W}_\mu + g'\frac{1}{2}B_\mu\right)\Phi\right|^2 =$$

$$\left|\left(\frac{g}{2}\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}W_\mu^1 + \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}W_\mu^2 + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}W_\mu^3\right] + \frac{g'}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}B_\mu\right)\Phi\right|^2$$

$$= \frac{1}{4}\left|\left[g\begin{pmatrix}W_\mu^3 & W_\mu^1 - iW_\mu^2\\ W_\mu^1 + iW_\mu^2 & -W_\mu^3\end{pmatrix} + g'\begin{pmatrix}B_\mu & 0\\ 0 & B_\mu\end{pmatrix}\right]\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix}\right|^2$$

$$= \frac{1}{8}\left|\begin{pmatrix}gW_\mu^3 + g'B_\mu & g(W_\mu^1 - iW_\mu^2)\\ g(W_\mu^1 + iW_\mu^2) & -gW_\mu^3 + g'B_\mu\end{pmatrix}\begin{pmatrix}0\\ v\end{pmatrix}\right|^2$$

$$= \frac{1}{8}v^2\left(g\left(W_\mu^1 + iW_\mu^2\right), \left(-gW_\mu^3 + g'B_\mu\right)\right)\begin{pmatrix}g\left(W^{1\mu} - iW^{2\mu}\right)\\ -gW^{3\mu} + g'B^\mu\end{pmatrix}$$

$$= \tfrac{g^2v^2}{8}\left[(W_\mu^1)^2 + (W_\mu^2)^2\right] + \tfrac{v^2}{8}(g'B_\mu - gW_\mu)(g'B^\mu - gW^\mu)$$

$$= \tfrac{1}{4}g^2v^2W^{+\mu}W^{-\mu} + \tfrac{1}{8}v^2(g^2+g'^2)\frac{\left(-gW_\mu^3+g'B_\mu\right)^2}{\left(\sqrt{g^2+g'^2}\right)^2}$$

$$= \left(\tfrac{1}{2}gv\right)^2 W_\mu^+ W^{-\mu} + \tfrac{1}{2}v^2\frac{\left(g^2+g'^2\right)}{4}\left(\frac{-gW_\mu^3+g'B_\mu}{\sqrt{g^2+g'^2}}\right)^2 + 0\left(gW_\mu^3 + g'B_\mu\right)^2$$

$$M_{W^\pm}^2 \quad W_\mu^+ W^{-\mu} \quad + \quad \tfrac{1}{2}M_Z^2 Z_\mu^2 \quad + \quad \tfrac{1}{2}M_A A_\mu^2$$

The first term is the mass term for $W^+,\ W^-$,

$$M_{W^\pm} = \frac{1}{2}gv$$

Recalling that

$$\tan\theta_W = \frac{g'}{g} \qquad ; \qquad \sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}$$

the normalised A_μ and Z_μ field combinations are

$$A_\mu = \frac{g' W^3_\mu + g B_\mu}{\sqrt{g^2+g'^2}} \quad \text{with} \quad M_A = 0$$

$$Z_\mu = \frac{g' W^3_\mu - g B_\mu}{\sqrt{g^2+g'^2}} \quad \text{with} \quad M_Z = \frac{1}{2} v \sqrt{g^2+g'^2}$$

which gives

$$\frac{M_W}{M_Z} = \frac{g}{\sqrt{g^2+g'^2}} = \cos\theta_W$$

which is verified experimentally to high precision