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40Particle Physics - Class test.

1a)  $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dx^2 + dy^2$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = g_{\mu\nu}$$

b) Find the set of indep. transformations of the form

$$x \rightarrow x + \epsilon A(x, y)$$

$$y \rightarrow y + \epsilon B(x, y) \quad \text{that leave } ds^2 \text{ invariant.}$$

State what each transformation represents in space time.

$$dx \rightarrow dx + \epsilon \frac{\partial A}{\partial x} dx + \epsilon \frac{\partial A}{\partial y} dy = (1 + \epsilon \frac{\partial A}{\partial x}) dx + \epsilon \frac{\partial A}{\partial y} dy$$

$$dy \rightarrow dy + \epsilon \frac{\partial B}{\partial y} dy + \epsilon \frac{\partial B}{\partial x} dx = \epsilon \frac{\partial B}{\partial x} dx + (1 + \epsilon \frac{\partial B}{\partial y}) dy$$

$$dx^2 \rightarrow \left[ (1 + \epsilon \frac{\partial A}{\partial x}) dx + \epsilon \frac{\partial A}{\partial y} dy \right]^2$$

$$= dx^2 + 2\epsilon \frac{\partial A}{\partial x} dx^2 + 2\epsilon \frac{\partial A}{\partial y} dx dy$$

(neglecting terms of order  $\epsilon^2$ )

$$dy^2 \rightarrow dy^2 + 2\epsilon \frac{\partial B}{\partial y} dy^2 + 2\epsilon \frac{\partial B}{\partial x} dy dx$$

Demand that  $ds^2 = dx^2 + dy^2$  remains invariant.

$$dx^2 + dy^2 \rightarrow dx^2 + dy^2 + 2\epsilon \frac{\partial A}{\partial x} dx^2 + 2\epsilon \frac{\partial B}{\partial y} dy^2$$

$$+ 2\epsilon \left( \frac{\partial A}{\partial y} + \frac{\partial B}{\partial x} \right) dx dy.$$

we get  $\frac{\partial A}{\partial x} = 0 \Rightarrow A(y)$

$$\frac{\partial B}{\partial y} = 0 \Rightarrow B(x)$$

$$\frac{\partial A}{\partial y} + \frac{\partial B}{\partial x} = 0 \Rightarrow A' + B' = 0$$

$$\Rightarrow A' = -B' = \text{constant}, c.$$

Solve the 2 different differential equations:

$$\textcircled{1} \quad A' = c \Rightarrow A(y) = cy + d$$

$$\textcircled{2} \quad -B' = c \Rightarrow -B(x) = cx + e \Rightarrow B(x) = -cx - e$$

We are left with 3 parameters,  $c, d, e$ , so we are left with 3 degrees of freedom.

$$A = cy + d.$$

$$B = -cx - e.$$

The three d.o.f are represented by the 3 constants of the motion:  $c, d, e$ .

If we choose  $d=e=0$  then  $x \rightarrow x + \epsilon cy$

$$y \rightarrow y - \epsilon cx$$

we have a ~~boost~~ (rotation in 2D)

Rotation not boost

If we choose  $c=d=0$  then  $x \rightarrow x$

$$y \rightarrow y - \epsilon e$$

If we choose  $c=e=0$  then  $x \rightarrow x + \epsilon d$

$$y \rightarrow y$$

these are both translations in space, if  $x, y$  are spatial coordinates.

If  $x$  were to represent ~~a~~ time, then choosing  $c=e=0$  represents a translation in time, since  $x \rightarrow x + \epsilon d$ .

since  $dx^2 + dy^2$

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$$2a) \quad L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu$$

$$\frac{\partial}{\partial x^\nu} \left( \frac{\partial L}{\partial A_{\mu,\nu}} \right) - \frac{\partial L}{\partial A_\mu} = 0$$

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu$$

$$= -\frac{1}{4} (A_{\mu,\nu} - A_{\nu,\mu})(A^{\mu,\nu} - A^{\nu,\mu}) - j^\mu A_\mu$$

$$= -\frac{1}{4} \eta^{\alpha\mu} \eta^{\beta\nu} (A_{\beta,\alpha} - A_{\alpha,\beta})(A_{\mu,\nu} - A_{\nu,\mu}) - j^\mu A_\mu$$

$$\frac{\partial L}{\partial A_\mu} = -j^\mu$$

$$\frac{\partial L}{\partial A_{\mu,\nu}} = -\frac{1}{4} \eta^{\alpha\mu} \eta^{\beta\nu} [(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha)(A_{\mu,\nu} - A_{\nu,\mu}) + (A_{\alpha,\beta} - A_{\beta,\alpha})(\delta^\mu_\mu \delta^\nu_\nu - \delta^\mu_\nu \delta^\nu_\mu)]$$

$$= -\frac{1}{4} [(\eta^{\mu\mu} \eta^{\nu\nu} - \eta^{\nu\mu} \eta^{\mu\nu})(A_{\mu,\nu} - A_{\nu,\mu}) + (\eta^{\alpha\mu} \eta^{\beta\nu} - \eta^{\alpha\nu} \eta^{\beta\mu})(A_{\beta,\alpha} - A_{\alpha,\beta})]$$

$$= -4 \frac{1}{4} (A^{\nu,\mu} - A^{\mu,\nu}) = F^{\mu\nu}$$

$$\Rightarrow \frac{\partial}{\partial x^\nu} \left( \frac{\partial L}{\partial A_{\mu,\nu}} \right) - \frac{\partial L}{\partial A_\mu} = \frac{\partial}{\partial x^\nu} F^{\mu\nu} + j^\mu = \partial_\nu F^{\mu\nu} + j^\mu = 0$$

$$\Rightarrow \partial_\nu F^{\mu\nu} + j^\mu = 0 \Rightarrow -\partial_\nu F^{\mu\nu} = j^\mu$$

$$\Rightarrow \partial_\nu F^{\nu\mu} = j^\mu$$

switching indices  $\Rightarrow \partial_\mu F^{\mu\nu} = j^\nu$  as required.

Show  $j^\nu$  is conserved.

$$\partial_\nu j^\nu = \partial_\nu \partial_\mu F^{\mu\nu} = -\partial_\nu \partial_\mu F^{\nu\mu} = -\partial_\mu \partial_\nu F^{\nu\mu} = -\partial_\nu \partial_\mu F^{\mu\nu}$$

by using  $F_{\mu\nu} = -F_{\nu\mu}$ , using that partial derivatives commute + then relabelling indices.

$$\Rightarrow \partial_\nu \partial_\mu F^{\mu\nu} = 0 \Rightarrow \partial_\nu j^\nu = 0 \quad \text{u: } j^\nu \text{ is conserved.}$$

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$$b) L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_{\mu} A^{\mu} + \frac{1}{2} m^2 A_{\mu} A^{\mu}$$

$$\frac{1}{2} m^2 A_{\mu} A^{\mu} = \frac{1}{2} m^2 \eta^{\mu\mu} A_{\mu} A^{\mu} = \frac{1}{2} m^2 \eta_{\mu\mu} (A^{\mu})^2$$

$$\frac{\partial L}{\partial A^{\mu}} = -j_{\mu} + \frac{2}{2} m^2 A^{\mu} = -j_{\mu} + m^2 A^{\mu}$$

$$\frac{\partial L}{\partial A_{\mu,\nu}} = F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$$

$$\Rightarrow \partial_{\nu} (\partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}) - (-j_{\mu} + m^2 A^{\mu}) = 0$$

$$\partial_{\nu} \partial^{\mu} A^{\nu} - \partial_{\nu} \partial^{\nu} A^{\mu} + j_{\mu} - m^2 A^{\mu} = 0$$

$$\Rightarrow j_{\mu} = \partial_{\nu} \partial^{\nu} A^{\mu} + m^2 A^{\mu}$$

$$= (\partial_{\nu} \partial^{\nu} + m^2) A^{\mu}$$

$$\Rightarrow j^{\nu} = (\partial_{\mu} \partial^{\mu} + m^2) A^{\mu}$$

Lorentz gauge  $\partial_{\mu} A^{\mu} = 0$