**MATH431 - Modern Particle Physics**
**Set Work: Sheet 5;**

**1.** The potential function of a two-dimensional harmonic oscillator is

$$V(x,y) \;=\; \frac{1}{2}\,k\,(x^2+y^2)^2\,.$$

(i) Write down the Lagrangian of this system.

(ii) Write down the Euler-Lagrange equations of motion.

(iii) Write down the Hamiltonian.

(iv) Write down the Lagrangian and Hamiltonian in polar coordinates $(r,\phi)$ with $(x = r\cos\phi\,,\; y = r\sin\phi)\,.$

(v) How many constants of the motion are there?
What are they?

**2.** Two-particle states are defined by

$$|\mathbf{p}_1,\mathbf{p}_2 >= a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_1)|0>\,.$$

Show that

$$\begin{aligned}&< \mathbf{p}_1',\mathbf{p}_2'|\mathbf{p}_1,\mathbf{p}_2 >\\ =&(2\pi)^6(2p_1^0)(2p_2^0)\{\delta(\mathbf{p}_1-\mathbf{p}_1')\delta(\mathbf{p}_2-\mathbf{p}_2')+\delta(\mathbf{p}_1-\mathbf{p}_2')\delta(\mathbf{p}_2-\mathbf{p}_1')\}.\end{aligned}$$

**3.** (a) Let $A_\mu$ be the electromagnetic vector potential. The electromagnetic field strength tensor is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Show that Maxwell's equation in four vector notation can be derived from the electromagnetic Lagrangian given by

$$L_{e.m.} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}.$$

Show that $L_{e.m.}$ is invariant under the transformation $A_\mu \to A_\mu - \partial_\mu\Lambda$, where $\Lambda$ is a scalar function.

(b) Show that imposing a local $U(1)$ symmetry forbids the photon from attaining a mass.

**4.** Defining $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, show that

$$\begin{aligned}\gamma^{5\dagger} =&\gamma^5\\ \{\gamma^5,\gamma^\mu\} =&0.\end{aligned}$$

**5.** By inserting $(\gamma^\mu)^2 = 1$ for some $\mu = 0,1,2,3$, write each of $\gamma^0\gamma^1\gamma^2$ and $\gamma^0\gamma^1\gamma^3$ as a product $\gamma^5\gamma^\nu$ for some $\nu = 0,1,2,3$.

**6.** Show that

$$\mathrm{tr}[\gamma_\mu \gamma_\nu] = 4\eta_{\mu\nu}.$$

Now show that

$$\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma = 2\eta_{\mu\nu} \gamma_\rho \gamma_\sigma - 2\eta_{\mu\rho} \gamma_\nu \gamma_\sigma + 2\eta_{\mu\sigma} \gamma_\nu \gamma_\rho - \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\mu.$$

Hence show that

$$\mathrm{tr}[\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma] = 4[\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}]$$