# from the previous lecture ...

The Klein–Gordon equation

we use $P^\mu \to i\hbar\partial^\mu$ and obtain the wave equation

$$-\hbar^2\partial_\mu\partial^\mu\phi = m^2c^2\phi$$

or

$$\left(\partial^2 + \frac{m^2c^2}{\hbar^2}\right)\phi(\vec{X},t) = 0 \qquad \leftarrow \qquad \text{the Klein–Gordon equation}$$

Its interpretation as a single particle is problematic. The equation describes a scalar field but not in a single state but a multi–state, *i.e.* a quantised field.

$$\Rightarrow \quad \text{Charge density} \quad \rho = \frac{i\hbar e}{2mc^2}\left(\frac{imc^2}{\hbar}\phi^*\phi + \frac{imc^2}{\hbar}\phi^*\phi\right)$$

$$\text{Charge current density} \quad \vec{j} = \frac{e\hbar}{imc}\left(\phi^*\vec{\nabla}\phi + \phi\vec{\nabla}\phi^*\right)$$

$$j^\mu \; = \; (\rho, \vec{j}) \quad \Rightarrow \quad \partial_\mu j^\mu = 0$$

The interpretation of the solution of the KGE makes sense as charge density, not as probability density.

$\rightarrow$ it makes sense as a quantum field $\rightarrow$ creating–annihilating particles

# Quantisation of the KG field (real scalar field).

The Lagrangian density : $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2$

leads to the KG equation : $(\partial^2 + m^2)\phi = 0$

we can write : $\mathcal{L} = \frac{1}{2}\dot{\phi} - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2$

from which we derive $\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}$

In ordinary quantum mechanics we impose the relations:

$$[q_i, p_j] = i\hbar\delta_{ij},\ [q_i, q_j] = 0,\ [p_i, p_j] = 0$$

In classical field theory the coordinates are replaced by the fields

$$q_i \to \phi(x) \qquad , \qquad p_i \to \pi(x)$$

analoguesly, in quantum field theory we impose the equal–time commutation relations

$$\begin{aligned} \left[\phi(\vec{x}, t), \pi(\vec{x}', t)\right] &= i\hbar\delta(\vec{x} - \vec{x}') \\ \left[\phi(\vec{x}, t), \phi(\vec{x}', t)\right] &= \left[\pi(\vec{x}, t), \pi(\vec{x}', t)\right] = 0 \end{aligned}$$

where the Dirac $\delta$–function satisfies the properties

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\delta(x-a)dx &= f(a) \\ \int_{\alpha}^{\beta} f(x)\delta[F(x)]dx &= \frac{f(a)}{|F'(a)|} \quad \text{where } F(a) = 0 \end{aligned}$$

The commutation relations are equal time commutation relations

$\to$ canonical commutation relations

In the Heisenberg picture the equations of motion are given by

$$i\hbar\dot{\alpha} = [\alpha, H] \qquad \text{where } \alpha \to \text{operator, } H\text{–Hamiltonian } \alpha \neq \alpha(t)$$

consider:

$$\begin{aligned}
[\phi(\vec{x}, t), H] &= \left[\phi(\vec{x}, t), \int \left[\frac{1}{2}\pi(\vec{x}', t)^2 + \frac{1}{2}(\vec{\nabla}\phi(x', t))^2 + \frac{1}{2}m^2\phi(x', t)^2\right] d^3x'\right] \\
&= \frac{1}{2}\int d^3x' \left\{\left[\phi(\vec{x}, t), \pi(\vec{x}', t)^2\right] + \left[\phi(\vec{x}, t), (\vec{\nabla}\phi(x', t))^2\right]\right. \\
&\qquad \left. + \, m^2 \left[\phi(\vec{x}, t), \phi(x', t)^2\right]\right\}
\end{aligned}$$

Recalling that $[\phi(\vec{x}, t), \phi(\vec{x}', t)] = 0 \Rightarrow \left[\phi(\vec{x}, t), \vec{\nabla}'\phi(\vec{x}', t)\right] = 0$

$$
\begin{aligned}
\Rightarrow \qquad & [\phi(\vec{x},t),H] \\
& \frac{1}{2}\int d^3x' \left(\left[\phi(\vec{x},t),\pi(\vec{x}',t)^2\right]\right) \\
= \; & \frac{1}{2}\int d^3x' \left(\phi(\vec{x},t)\pi(\vec{x}',t)^2 - \pi(\vec{x}',t)^2\phi(\vec{x},t)\right) \\
= \; & \frac{1}{2}\int d^3x' \left(\phi(\vec{x},t)\pi(\vec{x}',t)\pi(\vec{x}',t) - \pi(\vec{x}',t)\phi(\vec{x},t)\pi(\vec{x}',t)\right. \\
& \qquad \left. + \pi(\vec{x}',t)\phi(\vec{x},t)\pi(\vec{x}',t) - \pi(\vec{x}',t)\pi(\vec{x}',t)\phi(\vec{x},t)\right) \\
= \; & \frac{1}{2}\int d^3x' \left(\left[\phi(\vec{x},t),\pi(\vec{x}',t)\right]\pi(\vec{x}',t) + \pi(\vec{x}',t)\left[\phi(\vec{x},t),\pi(\vec{x}',t)\right]\right) \\
= \; & i\hbar\int d^3x' \left(\pi(\vec{x}',t)\delta^3(\vec{x}-\vec{x}')\right) = i\hbar\pi(\vec{x},t) = i\hbar\dot{\phi}(\vec{x},t)
\end{aligned}
$$

we obtained the correct equations of motion.

The connection between the quatised field and its particle interpretation is seen by looking at the Fourier transformed field

$$
\begin{aligned}
\phi(x) &= \frac{1}{(2\pi)^4}\int d^4p\ \tilde{\phi}(p)\mathrm{e}^{-ip\cdot x} \\
\tilde{\phi}(p) &= \int d^4x\ \phi(x)\mathrm{e}^{ip\cdot x}
\end{aligned}
$$

For $\phi(x)$ to satisfy the KG equation we must have

$$
\begin{aligned}
(\partial^2 + m^2)\phi &= \frac{1}{(2\pi)^4}\int d^4p\ (m^2 - p^2)\tilde{\phi}(p)\mathrm{e}^{-ip\cdot x} d^4p \;=\; 0 \\
& \quad i.e.\ (p^2 - m^2)\tilde{\phi}(p) = 0 \\
& \quad i.e.\ \tilde{\phi}(p) \neq 0 \text{ only when } p^2 = m^2 \\
& \qquad\qquad \tilde{\phi}(p) = (2\pi)\delta(p^2 - m^2)f(p) \\
p^0 &= \pm\sqrt{\vec{p}^2 + m^2}
\end{aligned}
$$

$$\underline{\text{we may set :}} \quad f(\vec{p}) = \theta(p^0) f_+(\vec{p}) + \theta(-p^0) f_-(\vec{p})$$

where

$$\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

<u>from:</u> the properties of the delta function we have

$$\int f(x)\delta(x-a)dx = f(a)$$

$$\text{using } y = \lambda x \ , \ \frac{dy}{\lambda} = dx$$

$$\int f(x)\delta(\lambda x - \lambda a)dx = \int f(\frac{y}{\lambda})\delta(y - \lambda a)\frac{dy}{\lambda} = \frac{f(a)}{\lambda}$$

Further, we evaluate

$$\int_{\alpha}^{\beta} f(x)\delta(F(x))dx =$$

$$= \int_{F(\alpha)}^{F(\beta)} f(F^{-1}(y))\delta(y)\frac{dy}{F'(F^{-1}(y))} = \sum_i \frac{f(a_i)}{F'(a_i)}$$

$$\text{using } y = F(x)\ ,\ dy = F'(x)dx\ ,\ x = F^{-1}(y)\ ,\ dx = \frac{dy}{F'(F^{-1}(y))}$$

where $y = F(a_i) = 0$.

Hence,
$$\begin{aligned}\delta(p^2 - m^2) &= \delta(p_0^2 - (\vec{p}^2 + m^2)) \\ &= \frac{1}{2p_0}\delta(p_0 - (\vec{p}^2 + m^2)^{\frac{1}{2}}) + \frac{1}{2p_0}\delta(p_0 + (\vec{p}^2 + m^2)^{\frac{1}{2}})\end{aligned}$$

$$\phi(\vec{x}, t) = \frac{1}{(2\pi)^3} \int \frac{d^3\vec{p}}{2p_o} \left(\mathrm{e}^{-ip\cdot x} f_+(\vec{p}) + \mathrm{e}^{ip\cdot x} f_-(\vec{p})\right)$$

$$\text{Here } p_0 = \sqrt{\vec{p}^2 + m^2}$$

$$f_+(\vec{p}) = f(+\sqrt{\vec{p}^2}, +\vec{p})$$
$$f_-(\vec{p}) = f(-\sqrt{\vec{p}^2}, -\vec{p})$$

The $f_+$ term corresponds to positive energy states.

The $f_-$ term corresponds to negative energy states.