

Yuhang Tia MATH 431 Class Test

$\frac{7}{40}$

(a) $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dx^2 + dy^2$

$g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\frac{15}{20}$

(b) $x \rightarrow x + \epsilon A(x, y)$
 $y \rightarrow y + \epsilon B(x, y)$

$dx \rightarrow dx + \epsilon \left(\frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy \right)$

~~$dy \rightarrow dy + \epsilon \left(\frac{\partial B}{\partial x} dx + \frac{\partial B}{\partial y} dy \right)$~~

$dy \rightarrow dy + \epsilon \left(\frac{\partial B}{\partial x} dx + \frac{\partial B}{\partial y} dy \right)$

~~$ds^2 \rightarrow \left[\left(1 + \epsilon \frac{\partial A}{\partial x} \right) dx + \epsilon \frac{\partial A}{\partial y} dy \right]^2$~~

~~$ds^2 \rightarrow \left[\left(1 + \epsilon \frac{\partial A}{\partial x} \right) dx + \epsilon \frac{\partial A}{\partial y} dy \right]^2 + \left[\left(1 + \epsilon \frac{\partial B}{\partial x} \right) dx + \epsilon \frac{\partial B}{\partial y} dy \right]^2$~~

\approx

We require the invariance of ds^2 . By comparing the equation above to $ds^2 = dx^2 + dy^2$:

$dx^2 : \frac{\partial A}{\partial x} = 0 \Rightarrow A = A(y)$

$dy^2 : \frac{\partial B}{\partial y} = 0 \Rightarrow B = B(x)$

$$ds^2 \rightarrow \left[\left(1 + \epsilon \frac{\partial A}{\partial x}\right) dx + \epsilon \frac{\partial A}{\partial y} dy \right]^2 + \left[\left(1 + \epsilon \frac{\partial B}{\partial y}\right) dy + \epsilon \frac{\partial B}{\partial x} dx \right]^2$$

we require invariance of ds^2 . By comparing the coefficient with $ds^2 = dx^2 + dy^2$:

$$dx^2: \frac{\partial A}{\partial x} = 0 \Rightarrow A = A(y)$$

$$dy^2: \frac{\partial B}{\partial y} = 0 \Rightarrow B = B(x)$$

$$dxdy: \frac{\partial A}{\partial y} + \frac{\partial B}{\partial x} = 0 \Rightarrow \frac{dA}{dy} = -\frac{dB}{dx} = \text{constant} = c$$

Therefore we got the transformations:

$$A(y) = cy + a$$

$$B(x) = -cx + b$$

where a, b, c are constants.

a represents a shift in space in y -axis

b represents a shift in space in x -axis

c represents ~~boost~~.

$$2(a) \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu$$

$$= -\frac{1}{4} (A_{\mu,\nu} - A_{\nu,\mu})(A^{\mu,\nu} - A^{\nu,\mu}) - j^\mu A_\mu$$

$$= -\frac{1}{4} \eta^{\alpha\mu} \eta^{\beta\nu} (A_{\beta,\alpha} - A_{\alpha,\beta})(A_{\mu,\nu} - A_{\nu,\mu}) - j^\mu A_\mu$$

~~Sub in to~~

Now consider Euler-Lagrange equations:

$$\frac{\partial}{\partial x^\mu} \left(\frac{\partial L}{\partial A_{\mu,\nu}} \right) - \frac{\partial L}{\partial A_\mu} = 0 \quad 2$$

$$\frac{\partial L}{\partial A_\mu} = 0 \Rightarrow \frac{\partial}{\partial x^\mu} \left(\frac{\partial L}{\partial A_{\mu,\nu}} \right) = 0 \quad X$$

~~$\frac{\partial}{\partial A_{\mu,\nu}}$~~

$$\begin{aligned} \frac{\partial}{\partial A_{\mu,\nu}} &= -\frac{1}{4} \eta^{\alpha\beta} \eta^{\gamma\delta} (\delta_\alpha^\mu \delta_\beta^\nu - \delta_\beta^\mu \delta_\alpha^\nu) (A_{\mu,\nu} - A_{\nu,\mu}) \\ &\quad + (A_{\alpha,\beta} - A_{\beta,\alpha}) (\delta_\mu^\alpha \delta_\nu^\beta - \delta_\nu^\alpha \delta_\mu^\beta) + j^\mu \quad X \\ &= -\frac{1}{4} (\eta^{\mu\alpha} \eta^{\nu\beta} - \eta^{\alpha\mu} \eta^{\beta\nu}) (A_{\mu,\nu} - A_{\nu,\mu}) + \cancel{(\eta^{\mu\alpha} \eta^{\nu\beta} - \eta^{\alpha\mu} \eta^{\beta\nu})} \\ &\quad + (\eta^{\alpha\mu} \eta^{\beta\nu} - \eta^{\alpha\nu} \eta^{\beta\mu}) (A_{\beta,\alpha} - A_{\alpha,\beta}) + j^\mu \\ &= -\frac{1}{4} (A^{\mu,\nu} - A^{\nu,\mu}) + j^\mu \\ &= \frac{1}{4} F^{\mu\nu} + j^\mu \quad X \end{aligned}$$

→

$$\frac{\partial}{\partial x^\mu} \left(\frac{\partial L}{\partial A_{\mu,\nu}} \right) = \frac{\partial}{\partial x^\mu} \left(-F^{\mu\nu} + j^\mu \right) = 0$$

$$\Rightarrow \partial_\mu F^{\mu\nu} = j^\nu$$

$$\partial_\nu F^{\mu\nu} = \partial_\nu j^\nu = 0$$

$$\frac{2}{20}$$