MATH431 — Introduction to Modern Particle Theory: FEEDBACK/SOLUTIONS 4

1. *Poincaré group; Pauli-Lubanski vector*

$$\mathrm{d}s^2 \; = \; \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 \tag{1}$$

(a)

$$g_{\mu\nu} \; = \; \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \, , \qquad g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \, .$$

(b) The line element is invariant under 3 translations ($\mathrm{d}t$, $\mathrm{d}x$ and $\mathrm{d}y$), 2 boosts ($\mathrm{d}t\mathrm{d}x$, $\mathrm{d}t\mathrm{d}y$) and 1 rotation ($\mathrm{d}x\mathrm{d}y$). The generators associated with the transformations are: $P_0 = i\partial_t$, $P_1 = -i\partial_x$ and $P_2 = -i\partial_y$, the generators of translations. K_1 and K_2 are the boost generators and J_3 is the generator of rotations in the (x, y) plane.

(c)

$$W^\mu \; = \; -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma \, ,$$

$$W^0 \; = \; -\frac{1}{2}\epsilon^{0ijk} J_{ij} P_k \; = \; \frac{1}{2}\epsilon_{0ijk} J_{ij} P_k \; = \; \frac{1}{2}\epsilon_{kij} J_{ij} P_k \; = \; J_k P_k \, ,$$

$$\begin{aligned} W^i \; = \; & -\frac{1}{2}\epsilon^{i0jk} J_{0j} P_k - \frac{1}{2}\epsilon^{ij0k} J_{j0} P_k - \frac{1}{2}\epsilon^{ijk0} J_{jk} P_0 \; = \\ & -\frac{1}{2}\epsilon^{i0jk} J_{0j} P_k - \frac{1}{2}\epsilon^{i0jk} J_{0j} P_k + \frac{1}{2}\epsilon^{0ijk} J_{jk} P_0 \; = \\ & +\frac{1}{2}\epsilon^{0ijk} J_{0j} P_k + \frac{1}{2}\epsilon^{0ijk} J_{0j} P_k + \frac{1}{2}\epsilon^{0ijk} J_{jk} P_0 \; = \\ & -\epsilon_{0ijk} J_{0j} P_k - \frac{1}{2}\epsilon_{0ijk} J_{jk} P_0 \; = \; \epsilon_{ijk} K_j P_k - J_i P_0 \, . \end{aligned}$$

In the case of the one+two dimensional line element Eq. (1), embedded in our usual one+three dimensions,

$$\vec{K} \; = \; (K_1, K_2, 0) \quad \text{and} \quad \vec{\mathrm{J}} \; = \; (0, 0, \mathrm{J}_3) \, .$$

For $m = 0$ we can take, without loss of generality, that $P^\mu = (p, 0, p, 0)$, $P_\mu = (p, 0, -p, 0)$. Then

$$\begin{aligned} W^0 \; &= \; 0 \, , \\ W^1 \; &= \; \epsilon_{1jk} K_j P_k \; = \; \epsilon_{123} K_2 P_3 + \epsilon_{132} K_3 P_2 \; = \; 0 \, , \\ W^2 \; &= \; \epsilon_{2jk} K_j P_k \; = \; \epsilon_{213} K_1 P_3 + \epsilon_{231} K_3 P_1 \; = \; 0 \, , \\ W^3 \; &= \; -J_3 P_0 + \epsilon_{3jk} K_j P_k \; = \; -J_3 P_0 + \epsilon_{312} K_1 P_2 + \epsilon_{321} K_2 P_1 \; = \\ &= \; -J_3 P_0 - K_1 P_0 \; = \; -(J_3 + K_1) P_0 \, . \end{aligned}$$

For $m > 0$ we can evaluate the Pauli-Lubanski vector most conveniently in the rest frame, $P^\mu = (m, 0, 0, 0)$. Then

$$\begin{aligned} W^0 &= 0\,, \\ W^1 &= 0\,, \\ W^2 &= 0\,, \\ W^3 &= -J_3 m\,. \end{aligned}$$

2. We consider the line element on a two dimensional surface

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\,.$$

(a) The corresponding metric is

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}, \quad \text{and its inverse} \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix}.$$

(b) Now we want to determine the infinitesimal transformations

$$\theta^\mu \to \theta^\mu + \epsilon\,\zeta^\mu(\theta, \phi)\,.$$

For this we want to find the two functions

$$A(\theta,\phi) = \zeta^1(\theta,\phi) \quad \text{and} \quad B(\theta,\phi) = \zeta^2(\theta,\phi)\,,$$

such that $\mathrm{d}s^2$ remains invariant under the transformations

$$\begin{aligned} \theta &\to \theta + \epsilon\,A(\theta,\phi)\,, \\ \phi &\to \phi + \epsilon\,B(\theta,\phi)\,, \\ \text{with which} \qquad \sin\theta &\to \sin(\theta + \epsilon A) \approx \sin\theta + \epsilon\,A\cos\theta \\ \text{and} \qquad \sin^2\theta &\to \sin^2\theta + 2\epsilon A \sin\theta\cos\theta + \mathcal{O}(\epsilon^2)\,. \end{aligned}$$

Keeping terms to first order in ϵ we get the transformations

$$\begin{aligned} \mathrm{d}\theta &\to (1 + \epsilon\frac{\partial A}{\partial\theta})\mathrm{d}\theta + \epsilon\frac{\partial A}{\partial\phi}\mathrm{d}\phi\,, \\ \mathrm{d}\phi &\to \epsilon\frac{\partial B}{\partial\theta}\mathrm{d}\theta + (1 + \epsilon\frac{\partial B}{\partial\phi})\mathrm{d}\phi\,; \\ \mathrm{d}\theta^2 &\to (1 + 2\epsilon\frac{\partial A}{\partial\theta})\mathrm{d}\theta^2 + 2\epsilon\frac{\partial A}{\partial\phi}\mathrm{d}\theta\mathrm{d}\phi\,, \\ \mathrm{d}\phi^2 &\to (1 + 2\epsilon\frac{\partial B}{\partial\phi})\mathrm{d}\phi^2 + 2\epsilon\frac{\partial B}{\partial\theta}\mathrm{d}\theta\mathrm{d}\phi\,, \\ \sin^2\theta\,\mathrm{d}\phi^2 &\to \sin^2\theta\,(1 + 2\epsilon\frac{\partial B}{\partial\phi})\,\mathrm{d}\phi^2 + \sin^2\theta\,2\epsilon\frac{\partial B}{\partial\theta}\,\mathrm{d}\theta\,\mathrm{d}\phi + 2\epsilon\,A\sin\theta\,\cos\theta\,\mathrm{d}\phi^2\,. \end{aligned}$$

We now demand that $\mathrm{d}s^2$ remains invariant:

$$\mathrm{d}\theta^2 + \sin\theta \mathrm{d}\,\phi^2 \;\;\rightarrow\;\; \mathrm{d}\theta^2 + \sin\theta\,\mathrm{d}\phi^2 + \underbrace{\cdots\cdots\cdots\cdots\cdots}_{\text{terms must vanish}}.$$

Thus we obtain the following constraints:

$$\begin{aligned}
\mathrm{d}\theta^2 : \frac{\partial A}{\partial\theta} &= 0 \Rightarrow A = A(\phi) = f'(\phi) = \frac{\mathrm{d}f}{\mathrm{d}\phi}\,, \\
\mathrm{d}\phi^2 : \sin^2\theta \frac{\partial B}{\partial\phi} + A\sin\theta\cos\theta &= 0 \Rightarrow \frac{\partial B}{\partial\phi} = -\frac{\mathrm{d}f}{\mathrm{d}\phi}\frac{\cos\theta}{\sin\theta} \Rightarrow B = -f(\phi)\frac{\cos\theta}{\sin\theta} + g(\theta)\,, \\
\mathrm{d}\theta\mathrm{d}\phi : \frac{\partial A}{\partial\phi} + \sin^2\theta\frac{\partial B}{\partial\theta} &= 0 \Rightarrow f'' + \sin^2\theta\left(\frac{f(\phi)}{\sin^2\theta} + g'(\theta)\right) = 0 \\
&\Rightarrow f''(\phi) + f(\phi) = -\sin^2\theta\, g'(\theta) = \mathit{constant} = c\,.
\end{aligned}$$

Solving the two differential equations for f and g we get

$$\begin{aligned}
f'' + f = c \;\;&\Rightarrow\;\; f = a\sin\phi + b\cos\phi + c\,, \\
\sin^2\theta\,\frac{\mathrm{d}g}{\mathrm{d}\theta} = -c \;\;&\Rightarrow\;\; g = c\frac{\cos\theta}{\sin\theta} + d\,.
\end{aligned}$$

For A and B we thus arrive at

$$\begin{aligned}
A &= f'(\phi) \;=\; a\cos\phi - b\sin\phi\,, \\
B &= -\frac{\cos\theta}{\sin\theta}(a\sin\phi + b\cos\phi + c) + c\frac{\cos\theta}{\sin\theta} + d \;=\; -\frac{\cos\theta}{\sin\theta}(a\sin\phi + b\cos\phi) + d\,.
\end{aligned}$$

We are left with the three parameters a, b and d, which in the following we denote as α, β and γ, respectively:

$$\begin{aligned}
A &= \alpha\cos\phi + \beta\sin\phi\,, \\
B &= -\frac{\cos\theta}{\sin\theta}(\alpha\sin\phi - \beta\cos\phi) + \gamma\,.
\end{aligned}$$

Hence we are left with three degrees of freedom needed to parametrise the transformation.

(c) We now want to find the generators associated with each degree of freedom. The generators are given by summing

$$J \;\equiv\; \zeta^\mu \frac{\partial}{\partial\theta^\mu}\,.$$

As A is transforming θ and B ϕ, we readily read off the three operators connected to α, β, γ from the results for A and B above:

$$\begin{aligned}
\alpha: \quad J_1 &= \cos\phi\frac{\partial}{\partial\theta} - \frac{\cos\theta}{\sin\theta}\sin\phi\frac{\partial}{\partial\phi}\,, \\
\beta: \quad J_2 &= \sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\theta}{\sin\theta}\cos\phi\frac{\partial}{\partial\phi}\,, \\
\gamma: \quad J_3 &= \frac{\partial}{\partial\phi}\,.
\end{aligned}$$

From this we calculate (or may have guessed by now) the commutation relations among the generators J_i (check it!):

$$[J_i, J_j] \;=\; i\,\epsilon_{ijk} J_k \,.$$

This is the $SU(2)$ algebra.

(d) The metric $\mathrm{d}s^2$ is the metric on the surface of a sphere. The transformations are generated by the rotations in (θ, ϕ) of a vector fixed at the origin. The generators associated with the three degrees of freedom are the operators for the three components of the angular momentum, J_i. (They obey the $SU(2)$ algebra, $[J_i, J_j] \;=\; i\,\epsilon_{ijk} J_k\,$.)