from the previous lectures ...

Classification of elementary particles

$SU(2)_I$ – Isospin – global continuous approximate symmetry

additional conserved charges: Exact Baryon & Lepton numbers; Approximate Strangeness number

Weak interaction violates strangeness

$\rightarrow$ classification by Isospin is not sufficient to classify hadronic states

$\rightarrow$ need a larger symmetry group S such that

$$SU(2)_I \subset G \leftarrow SU(2)_I \text{ is a subgroup of } G$$

$$G = ? \rightarrow G = SU(3)_{flavour} \rightarrow \text{Gellmann \& Neeman} \rightarrow \text{the eightfold way}$$

Unitary groups $SU(n)$

simple unitary group of rank n where the rank is the number of mutualy commuting diagonal generators.

$$\begin{aligned} \text{Unitary}: &\quad U^\dagger U = I \quad \Rightarrow \quad U^\dagger = U^{-1} \\ \text{Simple}: &\quad \mathrm{Det}\, U = 1 \end{aligned}$$

NxN simple unitary matrix has $N^2 - 1$ D.O.F.

For $SU(2)$ we have 3–matrices & 3 coefficients

For $SU(3)$ we have 8–matrices & 8 coefficients

Such that $\mathrm{Det}\, U = \mathrm{e}^{i \mathrm{Tr} H} = 1$

$U = \mathrm{e}^{i\vec{\alpha}\cdot\vec{\lambda}} \quad \leftarrow \quad \vec{\lambda}$ are hermitian matrices with $\mathrm{Tr}\lambda_j = 0 \qquad \lambda_1, \cdots, \lambda_8$

The number of $SU(2)$ group generators is three

The number of $SU(3)$ group generators is eight

For $SU(2)$ we have only one diagonal hermitian matrix with trace $H_{2\times 2} = 0$

$$\rightarrow \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The other two matrices are $\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

For $SU(2) \quad \rightarrow \quad$ only one diagonal matrix

Find the $\vec{\lambda}$ matrices in $SU(3)$

<u># of diagonal matrices:</u>

$$\lambda = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \quad \text{with} \quad \mathrm{Tr}\lambda = 0 \Rightarrow \lambda = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -\alpha-\beta \end{pmatrix}$$

$\rightarrow$ <u>Two diagonal matrices</u>

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The diagonal matrices provide the maximal set of mutually commuting operators whose eigenvalues characterise the elementary particles.

The other $\vec{\lambda}$ matrices are not diagonal

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$(\lambda_1,\ \lambda_2,\ \lambda_3)$ form an $SU(2)$ subgroup of $SU(3)$.

subgroup: a subgroup of generators that satisfy commutation relations among themselves, *e.g,* $[\lambda_1, \lambda_2] = i\lambda_3$, etc.

in $SU(2)$: $\quad \psi \to U\psi \quad , U \to 2 \times 2 \text{ matrix}, \ \psi -$ a 2 component vector

take $\frac{1}{2}\tau_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$:

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The spinors $\binom{1}{0}$ & $\binom{0}{1}$ are eigenvectors with eigenvalues $+\frac{1}{2}$ & $-\frac{1}{2}$

These are the eigenstates of τ_3.

In $SU(2)$ we cannot characterise these states with an additional eigenvalue.

We can characterise the spin exactly only in one direction, say along the z–axis

<u>In $SU(3)$</u> the analog of τ_3 is λ_3 $\quad\rightarrow\quad$ $\frac{1}{2}\lambda_3 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$

The three eigenvectors of $\frac{1}{2}\lambda_3$ are:

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Similarl to $SU(2)$ the states can be characterised by the eigenvalues of $(\frac{1}{2}\lambda_3)$

The eigenvectors of $\frac{1}{2}\lambda_3$ are also eigenvectors of

$$\lambda_8 \quad \to \quad \frac{1}{2}\lambda_8 = \begin{pmatrix} \frac{1}{2\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{2\sqrt{3}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{3}} \end{pmatrix}$$

where we normalise the generators to have the same normalisation given by

$$\mathrm{Tr}(\frac{1}{2}\lambda_3)^2 = \mathrm{Tr}(\frac{1}{2}\lambda_8)^2 = \frac{1}{2}$$

$$\frac{1}{2}\begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} \end{pmatrix} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} = \frac{1}{\sqrt{12}} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$\frac{1}{2}\begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -\frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

There are no additional diagonal matrices among $\lambda_1, \cdots, \lambda_8$.

Therefore, these are eigenvectors of λ_3 and λ_8 only.

$\Rightarrow$ Representations of $SU(3)$ are characterised by eigenvalues of λ_3 and λ_8.

In $SU(2)$ we classified particles according to the eigenvalues of τ_3.

Graphical description

$$\text{We define} \qquad T_3 = \frac{1}{2}\lambda_3 \quad , \quad Y = \frac{1}{\sqrt{3}}\lambda_8$$

In $SU(2)$ we had a doublet with 2 eigenvalues $\frac{1}{2}$, $-\frac{1}{2}$

Graph	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$T_3 = \frac{1}{2}\lambda_3$	$T_3 = -\frac{1}{2}$	$T_3 = +\frac{1}{2}$

Using $\frac{1}{2}(\tau_1 \pm i\tau_2)$ we can move from $\binom{1}{0} \leftrightarrow \binom{0}{1}$

$$\tau_+ = \frac{1}{2}(\tau_1 + i\tau_2) = \frac{1}{2}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\tau_- = \frac{1}{2}(\tau_1 - i\tau_2) = \frac{1}{2}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\tau_+\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\tau_-\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\tau_+ \downarrow = \uparrow \quad , \quad \tau_+ \uparrow = 0 \quad , \quad \tau_- \uparrow = \downarrow \quad , \quad \tau_- \downarrow = 0$$

The proton and the neutron form an Isospin doublet

$\rightarrow$ Therefore, $T_3(P) = \frac{1}{2}$, $T_3(N) = -\frac{1}{2}$

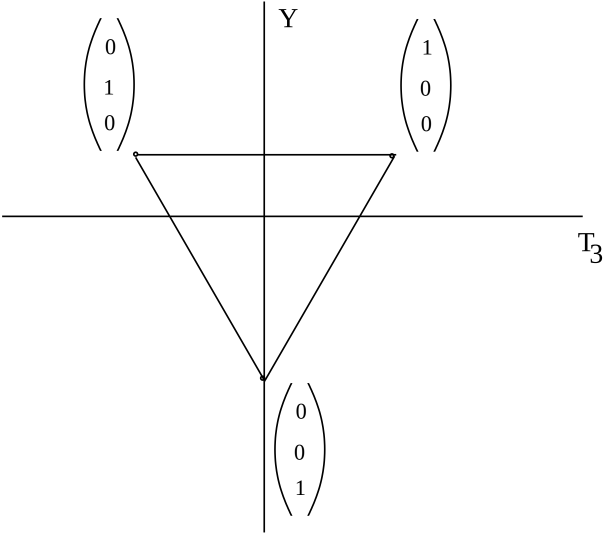
For $SU(3)$ we characterise the states by T_3 & Y.

$$\rightarrow \qquad (T_3, Y) \text{ plane}:$$

$$(T_3, Y): \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (\frac{1}{2}, \frac{1}{3})$$

$$(T_3, Y): \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (-\frac{1}{2}, \frac{1}{3})$$

$$(T_3, Y): \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = (0, -\frac{2}{3})$$



$\rightarrow$ graphical representation of the fundamental triplet representation of $SU(3)$