MATH 431 — May 2019: Solutions

All problems are similar to homework problems or material covered in the lectures.

1.

a. The Lagrangian is a function of the coordinates, velocities and possibly of time. Under $q \to q + \delta q = q + \epsilon h$, where ϵ is an infinitesimal constant, we have $\delta L = \frac{d}{dt}(\epsilon\Lambda)$.

But
$$\delta L = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\delta \dot q = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\frac{d}{dt}\delta q$$

From the Euler-Lagrange EOM we have $\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot q}$ and therefore $\delta L = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\delta q\right)$.

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\delta q - \epsilon\Lambda\right) = 0$$

With the definition of Q as
$$\epsilon Q = \frac{\partial L}{\partial \dot q}\delta q - \epsilon\Lambda$$
we note that Q is a constant of the motion.

b. With $\delta q = \epsilon \dot q$ and $L = L(q, \dot q)$,

$$\delta L = \frac{\partial L}{\partial q}\epsilon \dot q + \frac{\partial L}{\partial \dot q}\epsilon \ddot q = \epsilon\frac{dL}{dt} \quad \Rightarrow \quad \Lambda = L$$

We also note that q at time $t' = t + \epsilon$ is given by

$$q(t+\epsilon) = q + \epsilon \dot q = q + \delta q.$$

Hence, the symmetry correspond to invariance under time translation.

$$\epsilon Q = \frac{\partial L}{\partial \dot q}\epsilon \dot q - \epsilon L \Rightarrow Q = \dot q\frac{\partial L}{\partial \dot q} - L = H$$

i.e. the conversed current coincides with the Hamiltonian. The derivation shows that when the Langrangian does not depend explicitely on time, *i.e.* $\frac{\partial L}{\partial t} = 0$, corresponds to time translation invariance, and consevation of the total energy H. In energy conserving systems time is a cyclic coordinate and the Lagrangian does not depend explicitely on the time coordinate, but only via the implicit dependence of the coordinates and velocities on time. Hence for such systems we have $L = L(q, \dot q)$.

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2$$

2a.

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \quad , \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix}$$

b.

$$\theta^\mu \rightarrow \theta^\mu + \epsilon\zeta^\mu(\theta,\phi)$$

We want to find functions

$$\begin{aligned} A(\theta,\phi) &= \zeta^1(\theta,\phi) \\ \text{and } B(\theta,\phi) &= \zeta^2(\theta,\phi) \end{aligned}$$

such that ds^2 remains invariant under the transformations.

$$\begin{aligned} \theta &\rightarrow \theta + \epsilon A(\theta,\phi) \\ \phi &\rightarrow \phi + \epsilon B(\theta,\phi) \\ \sin\theta &\rightarrow \sin(\theta+\epsilon A) \approx \sin\theta + \epsilon A\cos\theta \\ \sin^2\theta &\rightarrow \sin^2\theta + 2\epsilon A\sin\theta\cos\theta + O(\epsilon^2) \end{aligned}$$

Keeping terms to first order in ϵ

$$\begin{aligned} d\theta &\rightarrow (1+\epsilon\frac{\partial A}{\partial\theta})d\theta + \epsilon\frac{\partial A}{\partial\phi}d\phi \\ d\phi &\rightarrow \epsilon\frac{\partial B}{\partial\theta}d\theta + (1+\epsilon\frac{\partial B}{\partial\phi})d\phi \\ d\theta^2 &\rightarrow (1+2\epsilon\frac{\partial A}{\partial\theta})d\theta^2 + 2\epsilon\frac{\partial A}{\partial\phi}d\theta d\phi \\ d\phi^2 &\rightarrow (1+2\epsilon\frac{\partial B}{\partial\phi})d\phi^2 + 2\epsilon\frac{\partial B}{\partial\theta}d\theta d\phi \\ \sin^2\theta d\phi^2 &\rightarrow \sin^2\theta(1+2\epsilon\frac{\partial B}{\partial\phi})d\phi^2 + \sin^2\theta 2\epsilon\frac{\partial B}{\partial\theta}d\theta d\phi + 2\epsilon A\sin\theta\cos\theta d\phi^2 \end{aligned}$$

we demand that ds^2 remains invariant.

$$d\theta^2 + \sin\theta d\phi^2 \rightarrow d\theta^2 + \sin\theta d\phi^2 + \underbrace{\cdots\cdots\cdots\cdots\cdots}_{\text{terms that vanish}}$$

demanding that the additional terms vanish we obtain the following constraints

$$\begin{aligned} d\theta^2 &: \frac{\partial A}{\partial\theta} = 0 \Rightarrow A = A(\phi) = f'(\phi) = \frac{df}{d\phi} \\ d\phi^2 &: \sin^2\theta\frac{\partial B}{\partial\phi} + A\sin\theta\cos\theta = 0 \Rightarrow \frac{\partial B}{\partial\phi} = -\frac{df}{d\phi}\frac{\cos\theta}{\sin\theta} \Rightarrow B = -f(\phi)\frac{\cos\theta}{\sin\theta} + g(\theta) \\ d\theta d\phi &: \frac{\partial A}{\partial\phi} + \sin^2\theta\frac{\partial B}{\partial\theta} = 0 \Rightarrow f'' + \sin^2\theta\left(\frac{f(\phi)}{\sin^2\theta} + g'(\theta)\right) = 0 \end{aligned}$$

3. (a) Two-particle states are defined by

$$|\mathbf{p}_1,\mathbf{p}_2\rangle \;=\; a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)|0\rangle\,,$$

with $|\mathbf{p}_1,\mathbf{p}_2\rangle = |\mathbf{p}_2,\mathbf{p}_1\rangle$ as $[a^\dagger(\mathbf{p}_1),a^\dagger(\mathbf{p}_2)]=0$.
We also know that

$$[a(\mathbf{p}_1),a^\dagger(\mathbf{p}_2)] = (2\pi)^3 2p_1^0\delta(\mathbf{p}_1-\mathbf{p}_2)\,.$$

Hence

$$\begin{aligned}
\langle\mathbf{p}_1',\mathbf{p}_2'|\mathbf{p}_1,\mathbf{p}_2\rangle &= \langle 0|a(\mathbf{p}_1')a(\mathbf{p}_2')a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)|0\rangle\\
&= \langle 0|a(\mathbf{p}_1')\{a^\dagger(\mathbf{p}_1)a(\mathbf{p}_2') + (2\pi)^3 2p_1^0\delta(\mathbf{p}_1-\mathbf{p}_2')\}a^\dagger(\mathbf{p}_2)|0\rangle\\
&= \langle 0|a(\mathbf{p}_1')a^\dagger(\mathbf{p}_1)\{a^\dagger(\mathbf{p}_2)a(\mathbf{p}_2') + (2\pi)^3 2p_1^0\delta(\mathbf{p}_2-\mathbf{p}_2')\}|0\rangle\\
&\quad + (2\pi)^3 2p_1^0\delta(\mathbf{p}_1-\mathbf{p}_2')\langle 0|a(\mathbf{p}_1')a^\dagger(\mathbf{p}_2)|0\rangle\\
&= (2\pi)^3 2p_2^0\delta(\mathbf{p}_2-\mathbf{p}_2')\langle 0|\{a^\dagger(\mathbf{p}_1)a(\mathbf{p}_1') + (2\pi)^3 2p_1^0\delta(\mathbf{p}_1-\mathbf{p}_1')\}|0\rangle\\
&\quad + (2\pi)^3 2p_1^0\delta(\mathbf{p}_1-\mathbf{p}_2')\langle 0|\{a^\dagger(\mathbf{p}_2)a(\mathbf{p}_1') + (2\pi)^3 2p_2^0\delta(\mathbf{p}_2-\mathbf{p}_1')\}|0\rangle\\
&= (2\pi)^6(2p_1^0)(2p_2^0)\{\delta(\mathbf{p}_1-\mathbf{p}_1')\delta(\mathbf{p}_2-\mathbf{p}_2') + \delta(\mathbf{p}_1-\mathbf{p}_2')\delta(\mathbf{p}_2-\mathbf{p}_1')\}\,.
\end{aligned}$$

(b) The number operator is

$$N \;:=\; \frac{1}{(2\pi)^3}\int\frac{\mathrm{d}^3\mathbf{p}'}{2p'^0}\,a^\dagger(\mathbf{p}')a(\mathbf{p}')\,.$$

With this

$$\begin{aligned}
[N,a^\dagger(\mathbf{p})] &= \left[\frac{1}{(2\pi)^3}\int\frac{\mathrm{d}^3\mathbf{p}'}{2p'^0}\,a^\dagger(\mathbf{p}')a(\mathbf{p}'),a^\dagger(\mathbf{p})\right]\\
&= \frac{1}{(2\pi)^3}\int\frac{\mathrm{d}^3\mathbf{p}'}{2p'^0}\,[a^\dagger(\mathbf{p}')a(\mathbf{p}'),a^\dagger(\mathbf{p})]\\
&= \frac{1}{(2\pi)^3}\int\frac{\mathrm{d}^3\mathbf{p}'}{2p'^0}\,\{a^\dagger(\mathbf{p}')[a(\mathbf{p}'),a^\dagger(\mathbf{p})] + [a^\dagger(\mathbf{p}'),a^\dagger(\mathbf{p})]a(\mathbf{p}')\}\\
&= \frac{1}{(2\pi)^3}\int\frac{\mathrm{d}^3\mathbf{p}'}{2p'^0}\,a^\dagger(\mathbf{p}')2p'^0(2\pi)^3\delta(\mathbf{p}-\mathbf{p}')\\
&= a^\dagger(\mathbf{p})\,.
\end{aligned}$$

So we have

$$\begin{aligned}
Na^\dagger(\mathbf{p}) - a^\dagger(\mathbf{p})N &= a^\dagger(\mathbf{p})\\
\Rightarrow Na^\dagger(\mathbf{p}) &= a^\dagger(\mathbf{p})(N+1)\,,\\
\text{and}\quad N|\mathbf{p}_1\ldots\mathbf{p}_n\rangle &= Na^\dagger(\mathbf{p}_1)\ldots a^\dagger(\mathbf{p}_n)|0\rangle\\
&= a^\dagger(\mathbf{p}_1)(N+1)a^\dagger(\mathbf{p}_2)\ldots a^\dagger(\mathbf{p}_n)|0\rangle\\
&= a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)(N+2)\ldots a^\dagger(\mathbf{p}_n)|0\rangle\\
&= \;\ldots\; = \; a^\dagger(\mathbf{p}_1)\ldots a^\dagger(\mathbf{p}_n)(N+n)|0\rangle\\
&= n\,a^\dagger(\mathbf{p}_1)\ldots a^\dagger(\mathbf{p}_n)|0\rangle \quad (\text{as } a(\mathbf{p})|0\rangle = 0 \text{ and so } N|0\rangle = 0)\\
&= n|\mathbf{p}_1\ldots\mathbf{p}_n\rangle\,.
\end{aligned}$$

4(a).

$$\begin{aligned}\gamma^{5\dagger} =&(i\gamma^0\gamma^1\gamma^2\gamma^3)^\dagger\\ =&-i\gamma^{3\dagger}\gamma^{2\dagger}\gamma^{1\dagger}\gamma^{0\dagger}\\ =&-i(\gamma^0\gamma^3\gamma^0)(\gamma^0\gamma^2\gamma^0)(\gamma^0\gamma^1\gamma^0)\gamma^0\\ =&-i\gamma^0\gamma^3\gamma^2\gamma^1 = i\gamma^0\gamma^2\gamma^3\gamma^1 = -i\gamma^0\gamma^2\gamma^1\gamma^3 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5.\end{aligned}$$

$$\begin{aligned}(\gamma^5)^2 =&-\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = (-1)(-1)^3\gamma^1\gamma^2\gamma^3(\gamma^0)^2\gamma^1\gamma^2\gamma^3 = \gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3\\ =&(-1)^2\gamma^2\gamma^3(\gamma^1)^2\gamma^2\gamma^3 = (-1)\gamma^2\gamma^3\gamma^2\gamma^3\\ =&(-1)(-1)\gamma^3(\gamma^2)^2\gamma^3 = -(\gamma^3)^2 = 1.\end{aligned}$$

(b)

$$\begin{aligned}\gamma^1\gamma^2\gamma^3 =&(\gamma^0)^2\gamma^1\gamma^2\gamma^3 = -i\gamma^0(i\gamma^0\gamma^1\gamma^2\gamma^3) = -i\gamma^0\gamma^5\\ \gamma^0\gamma^2\gamma^3 =&\gamma^0(\gamma^1)^2\gamma^2\gamma^3 = -\gamma^1\gamma^0\gamma^1\gamma^2\gamma^3 = i\gamma^1\gamma^5.\end{aligned}$$

(c)

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}1_4 \Rightarrow \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}1_4,$$

where 1_4 is the 4-dimensional identity matrix, usually not written explicitly. Taking the trace, and using $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, $\mathrm{tr}1_4 = 4$, we get

$$\mathrm{tr}[\gamma_\mu\gamma_\nu] = 4\eta_{\mu\nu}.$$

5. (a) The electromagnetic field strength tensor is given by

$$F_{\mu\nu} \;=\; \partial_\mu A_\nu - \partial_\nu A_\mu \,,$$

and the electromagnetic Lagrangian is

$$\begin{aligned} L_{\text{e.m.}} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\ &= -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\ &= -\frac{1}{4} \eta^{\mu\alpha} \eta^{\nu\beta} (\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial_\mu A_\nu - \partial_\nu A_\mu) \,. \end{aligned}$$

The Euler-Lagrange equations of motion for this Lagrangian are obtained from

$$\frac{\partial}{\partial x^\nu} \left(\frac{\partial L}{\partial(\partial_\nu A_\mu)} \right) - \frac{\partial L}{\partial A_\mu} \;=\; 0 \,.$$

Now

$$\frac{\partial L}{\partial A_\mu} \;=\; 0 \quad\Rightarrow\quad \frac{\partial}{\partial x^\nu} \left(\frac{\partial L}{\partial(\partial_\nu A_\mu)} \right) \;=\; 0 \,.$$

Hence calculate

$$\begin{aligned} \frac{\partial L}{\partial(\partial_\delta A_\epsilon)} &= -\frac{1}{4} \eta^{\alpha\mu} \eta^{\beta\nu} \left[(\delta^\delta_\alpha \delta^\epsilon_\beta - \delta^\delta_\beta \delta^\epsilon_\alpha)(\partial_\mu A_\nu - \partial_\nu A_\mu) \right. \\ &\qquad\qquad \left. + \, (\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\delta^\delta_\mu \delta^\epsilon_\nu - \delta^\delta_\nu \delta^\epsilon_\mu) \right] \\ &= -\frac{1}{4} \left[(\eta^{\delta\mu} \eta^{\epsilon\nu} - \eta^{\epsilon\mu} \eta^{\delta\nu})(\partial_\mu A_\nu - \partial_\nu A_\mu) \right. \\ &\qquad \left. + \, (\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\eta^{\alpha\delta} \eta^{\beta\epsilon} - \eta^{\alpha\epsilon} \eta^{\beta\delta}) \right] \\ &= -\frac{1}{4} \, 4 \left[\partial^\delta A^\epsilon - \partial^\epsilon A^\delta \right] \\ &= -\, F^{\delta\epsilon} \,. \end{aligned}$$

So

$$\frac{\partial}{\partial x^\delta} \left(\frac{\partial L}{\partial(\partial_\delta A_\epsilon)} \right) \;=\; -\frac{\partial}{\partial x^\delta} F^{\delta\epsilon} \;=\; 0 \,,$$

which are the Maxwell equations in the absence of sources.

(b)

$$A_\mu \;\to\; \tilde{A}_\mu \;=\; A_\mu + \partial_\mu \Lambda \,.$$

Now

$$\begin{aligned} \tilde{F}_{\mu\nu} \;=\; \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu &= \partial_\mu (A_\nu + \partial_\nu \Lambda) - \partial_\nu (A_\mu + \partial_\mu \Lambda) \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\mu \partial_\nu \Lambda - \partial_\nu \partial_\mu \Lambda \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu \;=\; F_{\mu\nu} \end{aligned}$$

as the derivatives of the scalar function Λ commute. With $F_{\mu\nu}$ also $L_{\text{e.m.}}$ is invariant under the transformation.

(c) The Lagrangian for a massive photon without sources is given by

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + m^2 A_\mu A^\mu \, .$$

As before the first term is invariant under the transformation considered above. However, the second term transforms as

$$m^2 A_\mu A^\mu \;\; \rightarrow \;\; m^2 \tilde{A}_\mu \tilde{A}^\mu \;\; = \;\; m^2 A_\mu A^\mu + m^2(\partial_\mu \Lambda \partial^\mu \Lambda + \partial_\mu \Lambda A^\mu + A_\mu \partial^\mu \Lambda) \, .$$

The extra terms do not vanish, so the mass term would not be invariant under the transformation, i.e. *break the invariance.* Insisting on the invariance therefore forbids the mass term, i.e. $m^2 = 0$ and the photon has to remain massless.

6a. To show that orbital angular momentum is a constant of the motion we have to show that it commutes with the Hamiltonian, *i.e.*

$$\left[\hat{L}_i, \hat{H}\right] = 0,$$

where the $\hat{L}_i$'s are the components of the angular momentum operator in the x, y and z directions. For a free particle the Hamiltonian is given by $H = \vec{p}^2/(2m)$ and the orbital angular momentum is given by $\vec{L} = \vec{r} \times \vec{p}$. Hence, for example, for L_z (we drop the hats from now on) we have

$$\begin{aligned}
[L_z, p^2] &= [xp_y - yp_z, p_x^2 + p_y^2 + p_z^2] \\
&= [x, p_x^2]p_y - [y, p_y^2]p_x \\
&= (p_x[x, p_x] + [x, p_x]p_x)p_y - (p_y[y, p_y] + [y, p_y]p_y)p_x \\
&= (2i\hbar p_x p_y - 2i\hbar p_y p_x) = 0
\end{aligned}$$

where we used the commutation relations $[x_i, p_j] = i\hbar\delta_{ij}$. Similar results are obtained for L_x and L_y. Hence, the orbital angular momentum commutes with the Hamiltonian and is a contant of the motion.

6b. Similarly to show that the orbital angular momentum is not a constant of the motion for a Dirac particle, we have to show that it does not commute with the Dirac Hamiltonian,

$$H = \vec{\alpha} \cdot \vec{p} + \beta m = \alpha_i p_i + \beta m$$

where summation over i is assumed.

$$[L_z, H] = [x, H]p_y - [y, H]p_x = i\hbar(\alpha_x p_y - \alpha_y p_x) = i\hbar(\vec{\alpha} \times \vec{p})_z$$

hence

$$[\vec{L}, H] = i\hbar\vec{\alpha} \times \vec{p} \neq 0$$

and consequently orbital angular momentum does not commute with the Hamiltonian and is not a constant of the motion.

6c. The total angular momentum for a Dirac particle is

$$\vec{J} = \vec{L} + \vec{S}$$

where $\vec{L}$ is the orbital angular momentum and $\vec{S}$ is the spin angular momentum. we saw in part b that $[\vec{L}, H] = i\hbar\vec{\alpha} \times \vec{p}$ hence we need to find $\vec{S}$ such that

$$[\vec{S}, H] = -i\hbar\vec{\alpha} \times \vec{p}$$

We take

$$\vec{S} = \frac{1}{2}\vec{\Sigma}$$

with

$$\Sigma_j = \begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}$$

where σ_j are the Pauli matrices and the 0 entries are 2×2 zero matrices. It is easy to verify that

$$[\frac{1}{2}\vec{\Sigma}, H] = -i\hbar\vec{\alpha} \times \vec{p}$$

and therefore $[\vec{J}, H] = 0$ and the total angular momentum $\vec{J}$ is a constant of the motion.

7(a). 2 diagonal generators.

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

7b. $D = 8$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

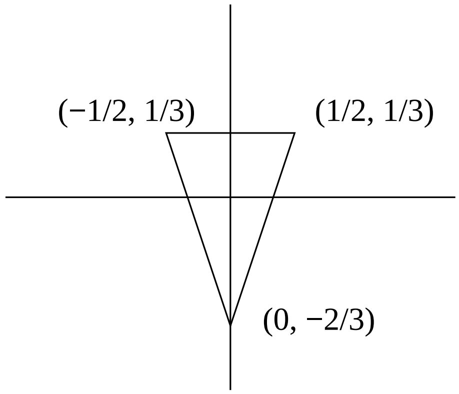
$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

7c.

$$3 = (2, 1/3) + (1, -2/3)$$

under the maximal subgroup $SU(2) \times U(1)$

7d.



7e.

$$3 \times \bar{3} = \{(2, 1/3) + (1, -2/3)\} \times \{(2, -1/3) + (1, 2/3)\} =$$
$$8 + 1 = \{(2, +1) + (3, 0) + (1, 0) + (2, -1)\} + (1, 0)$$

7f.

$$3 \times 3 = \{(2, 1/3) + (1, -2/3)\} \times \{(2, 1/3) + (1, -2/3)\} =$$
$$6 + \bar{3} = \{(3, 2/3) + (2, -1/3) + (1, -4/3)\} + \{(2, -1/3) + (1, 2/3)\}$$