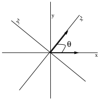
Special relativity and the Lorentz group

Rotations in 2D



Rotations in 2D

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \simeq \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{(for small } \theta\text{)}$$

Similarly in three dimensions $\quad \vec{X}' = R\vec{X} \quad R = \text{3x3 matrix.}$

$$\vec{X}' \cdot \vec{X}' = \vec{X} \cdot \vec{X} \quad \text{or} \quad \vec{X}^T R^T R \vec{X} = \vec{X} \cdot \vec{X} \quad \Longrightarrow \quad R^T R = I$$

The metric in this case is $\quad \eta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Einstein's special relativity follows from two basic tenets:

- c, the speed of light in vacuum is constant in all inertial frames with $c = 3 \cdot 10^8 m/s$ (in these lectures we often set $c = 1$).
- The laws of physics are the same in all inertial frames.

Events are labelled by their time and position in intertial frames.

$$\begin{aligned} \longrightarrow \qquad \text{4vector} \quad X^\mu &= (ct, \vec{X}) \qquad \mu = 0,1,2,3 \\ x^0 = ct \quad \vec{X} &= (x^1, x^2, x^3) = (x, y, z) \end{aligned}$$

The length of the four vector is given by

$$X \cdot X = c^2 t^2 - x^2 - y^2 - z^2 = t^2 - x^2 - y^2 - z^2 \quad \text{with } c = 1$$

written as $X \cdot X = \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} X^\mu X^\nu = \eta_{\mu\nu} X^\mu X^\nu = X_\mu X^\mu \qquad \mu = 0,1,2,3$

Einstein's summation convention, repeated upper and lower indices are summed

The Minkowski metric
$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

general scalar product of two 4–vectors $\rightarrow$ $X \cdot Y = \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} X^{\mu} Y^{\nu}$

using Einstein's summation convention. Different types of objects:

- scalar – no free indices. All indices are summed over.
- vector – one free index (*e.g.* X^{μ})
- tensor – two and more free indices ($g^{\mu\nu}$, $R_{\mu\nu\rho\sigma}$)

Lorentz transformations $\rightarrow$ preserve the scalar product & length of 4–vectors.

$$t^2 - \vec{X}^2 = X \cdot X = \eta_{\mu\nu} X^{\mu} X^{\nu} = X' \cdot X' = \eta_{\mu\nu} X'^{\mu} X'^{\nu} = t'^2 - \vec{X'}^2$$

X and X' are related by a Lorentz transformation : $\quad X^\mu \;\rightarrow\; X'^\mu \;=\; \Lambda^\mu_\nu X^\nu$

In general we write the metric as $g^{\mu\nu}$ and its components can be functions of spacetime. This is the subject of general relativity in which spacetime can be curved. The metric satisfies

$$g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma \;\;\Rightarrow\;\; (g^{\mu\nu})^{-1} \;=\; g_{\mu\nu}$$

In flat spacetime that we are dealing with in this lectures $g^{\mu\nu} \rightarrow \eta^{\mu\nu}$. We distinguish between 4–vectors with upper and lower indices. We write

$$X_\mu = \sum_{\nu=0}^{3} g_{\mu\nu} X^\nu$$

X_μ is a covariant vector and X^μ is a contravariant vector.
For $X^\mu = (t, \vec{X}) \rightarrow X_\mu = (t, -\vec{X})$, for our choice of the Minkowski metric.

Given a vector $X^\mu = (t, \vec{X})$ there are two differential quantities of interest

- 1. $dX^\mu \quad \rightarrow$ differential $\rightarrow$ contra variant four vector
- 2. $\frac{\partial}{\partial X^\mu} = \partial_\mu \quad \rightarrow$ gradient $\rightarrow$ covariant four vector

How do they behave under coordinate transformations $X^\mu \rightarrow X'^\mu$?

- 1. $dX^\mu \rightarrow dX'^\mu \; = \; \frac{\partial X'^\mu}{\partial X^\nu} dX^\nu$
- 2. $\frac{\partial}{\partial X^\mu} \; \rightarrow \frac{\partial}{\partial X'^\mu} \; = \; \frac{\partial X^\nu}{\partial X'^\mu} \frac{\partial}{\partial X^\nu}$

We see that the differential and the grandient transform differently. A four vector that transforms like the differential is a contra–variant vector

$$V'^\mu \;=\; \frac{\partial X'^\mu}{\partial X^\nu} V^\nu$$

whereas four vector that transforms like the gradient is a covariant vector

$$V'_\mu \;=\; \frac{\partial X^\nu}{\partial X'^\mu} V_\nu$$

in the following we will use the notation

$$\partial_\mu = \frac{\partial}{\partial X^\mu}$$

Example of a four vector is the momentum vector $P^\mu = (E, \vec{\mathrm{P}})$ and $P_\mu = (E, -\vec{\mathrm{P}})$. In relativistic quantum mechanics the momentum four vector is proportional to the gradient

$$P_\mu \sim \frac{\partial}{\partial X^\mu}$$

and

$$P_\mu X^\mu = Et - \vec{\mathrm{P}} \cdot \vec{\mathrm{X}}$$

Properties of Lorentz transformations

Lorentz transformations are transformations that preserve the scalar product and the length of Lorentz four vectors, just like the rotations that preserve the length of three vectors in three dimensional spatial space. The only difference is that in three dimensions we are in Euclidean space with its Euclidean metric, whereas Lorentz transformations operate in Minkowski spacetime with its Minkowski metric. The length of four vectors in Minkowski space is given by

$$\eta_{\mu\nu} X^\mu X^\nu$$

which is invariant under Lorentz transformations, *i.e.* there no change in its size and shape. The invariance implies the existence of a symmetry, which is generated by a group, the Lorentz group.

Assume $X^\mu \to X'^\mu$ under some Lorentz transformation, *i.e.*

$$X^\mu \to X'^\mu = \Lambda^\mu{}_\nu X^\nu$$

Hence

$$\eta_{\mu\nu} X^\mu X^\nu \to \eta_{\mu\nu} X'^\mu X'^\nu = \eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta X^\alpha X^\beta = \eta_{\alpha\beta} X^\alpha X^\beta$$

$$\Rightarrow \eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta} \tag{1}$$

or in matrix notation

$$\Lambda^T \eta \Lambda = \eta$$

where η is the 4x4 Minkowski metric and Λ is the 4x4 matrix of Lorentz transformations. The identity in eq. (1) defines the Lorentz transformations.

$$\Rightarrow (\mathrm{Det}\Lambda)^2 = 1 \Rightarrow \mathrm{Det}\Lambda = \pm 1$$

The physical transformations correspond to those that can be continuously connected to the identity, for which $\mathrm{Det}\Lambda = +1$.

- $\mathrm{Det}\Lambda = +1 \rightarrow$ Proper Lorentz tranformations
- $\mathrm{Det}\Lambda = -1 \rightarrow$ Imroper Lorentz tranformations

We can look at the 00 component of the identity $\eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}$

$$\begin{aligned} & \eta_{\mu\nu}\Lambda^{\mu}{}_{0}\Lambda^{\nu}{}_{0} = +1 \\ \Rightarrow \quad & (\Lambda^{0}{}_{0})^2 - \sum_i (\Lambda^{i}{}_{0})^2 = +1 \\ \Rightarrow \quad & (\Lambda^{0}{}_{0})^2 = 1 + \sum_i (\Lambda^{i}{}_{0})^2 \geq 1 \end{aligned}$$

We have that as,

$$\begin{aligned} (\Lambda^{0}{}_{0})^2 \geq 1 \Rightarrow \qquad & \Lambda^{0}{}_{0} \geq +1 \quad \text{orthochronos LT} \\ \text{or} \quad & \Lambda^{0}{}_{0} \leq -1 \quad \text{non orthochronos LT} \end{aligned}$$