MATH431 — Introduction to Modern Particle Theory: FEEDBACK/SOLUTIONS 1

1. In polar coordinates

$$\begin{aligned} x &= r\cos\phi\,, \\ y &= r\sin\phi\,. \end{aligned}$$

The radial speed is therefore $\dot{r}$ and the tangential speed is $r\dot{\phi}$. The kinetic energy is then $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$, and hence the Lagrangian is given by

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - V\,.$$

From this we obtain

$$\begin{aligned} \frac{\partial L}{\partial \dot{r}} &= m\dot{r}\,, \\ \frac{\partial L}{\partial r} &= mr\dot{\phi}^2 - \frac{\partial V}{\partial r}\,, \end{aligned}$$

so the Euler-Lagrange equation for r reads

$$m\frac{d\dot{r}}{dt} - mr\dot{\phi}^2 + \frac{\partial V}{\partial r} = 0\,.$$

Similarly

$$\begin{aligned} \frac{\partial L}{\partial \dot{\phi}} &= mr^2\dot{\phi}\,, \\ \frac{\partial L}{\partial \phi} &= -\frac{\partial V}{\partial \phi}\,. \end{aligned}$$

So the Euler-Lagrange equation for ϕ is

$$m\frac{d}{dt}(r^2\dot{\phi}) + \frac{\partial V}{\partial \phi} = 0\,.$$

If the potential is axisymmetric, $\partial V/\partial\phi = 0$. The EuLa equation for ϕ then states that the angular momentum $mr^2\dot{\phi}$ is constant. If the motion is circular, $\dot{r} = 0$ and the radial equation becomes

$$-m\frac{v^2}{r} = -\frac{\partial V}{\partial r}\,,$$

where $v = r\dot{\phi}$ is the speed. The force $\partial V/\partial r$ is equal to m times the centripetal acceleration $-v^2/r$.

2. (a) In the lecture we have derived

$$\dot{p}_0 \;=\; -\frac{\partial H}{\partial q_0}\,.$$

Therefore, p_0 is constant in time if H does not depend on q_0.

One example for such a system is given in part (b) below, where H does not depend on ϕ. A second example would be a one dimensional harmonic oscillator (in x) embedded in two dimensions such that there is no y dependence and therefore the momentum $p_y =$ *constant*.

(b) An axisymmetric potential does not depend on ϕ, so p_ϕ is a constant of the motion. In polar coordinates

$$\begin{aligned} x &= r\sin\theta\cos\phi\,,\\ y &= r\sin\theta\sin\phi\,,\\ z &= r\cos\theta\,, \end{aligned}$$

the kinetic energy is given by

$$\frac{1}{2}m\left[\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\dot{\phi})^2\right]\,,$$

and with the potential energy V the Lagrangian is given by

$$L \;=\; \frac{1}{2}m\left[\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\dot{\phi})^2\right] - V\,.$$

Now

$$\begin{aligned} p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r}\,,\\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}\,,\\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\dot{\phi}\,. \end{aligned}$$

Hence

$$\begin{aligned} H &= \sum_i p_i\dot{q}_i - L \;=\; m\dot{r}^2 + mr^2\dot{\theta}^2 + mr^2\sin^2\theta\dot{\phi}^2 - \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) + V\\ &= \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) + V\\ &= \frac{{p_r}^2}{2m} + \frac{{p_\theta}^2}{2mr^2} + \frac{{p_\phi}^2}{2mr^2\sin^2\theta} + V\,. \end{aligned}$$

With this we get

$$\dot{p}_\phi \;=\; -\frac{\partial H}{\partial \phi} \;=\; -\frac{\partial V}{\partial \phi} \;=\; 0\,,$$

$\Rightarrow$ the angular momentum about the symmetry axis is conserved.

3. The two-dimensional Harmonic Oscillator

(i) The Lagrangian of the two-dimensional harmonic oscillator is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}k(x^2 + y^2)\,.$$

(ii) The Euler-Lagrange equations are

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= m\ddot{x} + kx = 0\,,\\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} &= m\ddot{y} + ky = 0\,.\end{aligned}$$

(iii) The Hamiltonian is

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}mw^2(x^2 + y^2)\,, \quad \text{with} \quad w = \sqrt{\frac{k}{m}}\,.$$

(iv) In polar coordintes we derive:

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{1}{2}kr^2\,,$$

$$\begin{aligned}p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r}\,,\\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}\,.\end{aligned}$$

Hence we get the Hamiltonian

$$\begin{aligned}H &= \sum_i p_i\dot{q}_i - L = m\dot{r}^2 + mr^2\dot{\phi}^2 - \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{2}mw^2r^2\\ &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{2}mw^2r^2\\ &= \frac{{p_r}^2}{2m} + \frac{{p_\phi}^2}{2mr^2} + \frac{1}{2}mw^2r^2\,.\end{aligned}$$

(v) There are two constants of the motion.
Since the Hamiltonian does not depend explicitly on time, the energy is a constant of the motion with $E = H(q_0, p_0)$. Since it does not depend explicitly on ϕ, also p_ϕ is a constant of the motion, corresponding to conservation of the angular momentum w.r.t. the symmetry axis.