MATH431 — Introduction to Modern Particle Theory: FEEDBACK/SOLUTIONS 9

1. The Lagrangians of the two massive fields are:

$$\begin{aligned} \mathcal{L}_1 &= (\partial_\mu \Phi_1)^\dagger (\partial^\mu \Phi_1) - m_1^2 \Phi_1^\dagger \Phi_1 \,, \\ \mathcal{L}_2 &= (\partial_\mu \Phi_2)^\dagger (\partial^\mu \Phi_2) - m_2^2 \Phi_2^\dagger \Phi_2 \,, \end{aligned}$$

and the total Lagrangian is $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$. Hence $\mathcal{L}$ is in general not invariant under the exchange

$$\Phi_1 \leftrightarrow \Phi_2 \,,$$

unless $m_1^2 = m_2^2$. However, combining the fields into a $SU(2)$ doublet $\Phi = (\Phi_1, \Phi_2)$, the Lagrangian

$$\mathcal{L} = (\partial_\mu \Phi)^\dagger (\partial^\mu \Phi) - \mu^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2$$

preserves the $SU(2)$ symmetry which is spontaneously broken by a choice for the (non-trivial, i.e. $\Phi \neq 0$) vacuum in the case of a symmetry-breaking potential with $\mu^2 < 0$.

2.

$$L = \frac{1}{2}(\partial_\mu \phi_i)^2 - \frac{1}{2}\mu^2 \phi_i^2 - \frac{1}{4}\lambda(\phi_i^2)^2$$

with $\mu^2 < 0$ and $\lambda > 0$.

(a) The first term in the Lagrangian contains the kinetic terms of the three fields which lead to the propagators in the Feynman rules. The second and third term are the quadratic and quartic terms of the scalar potential. The coefficient λ is chosen positive as otherwise there would be no stable vacuum. Only with $\mu^2 < 0$ we get a non-trivial vacuum which allows for spontaneous symmetry breaking. As shown in (b) this leads to mass terms (quadratic in the field) and interaction terms which lead to cubic and quartic vertices in the Feynman rules.

(b)

$$L = \frac{1}{2}\left((\partial_\mu \phi_1)^2 + (\partial_\mu \phi_2)^2 + (\partial_\mu \phi_3)^2\right) - \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2 + \phi_3^2) - \frac{1}{4}\lambda(\phi_1^2 + \phi_2^2 + \phi_3^2)^2 \,.$$

The potential is

$$V = \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2 + \phi_3^2) + \frac{1}{4}\lambda(\phi_1^2 + \phi_2^2 + \phi_3^2)^2 \,.$$

For spontaneous symmetry breaking we need a non-trivial minimum of the potential:

$$\frac{\partial V}{\partial \phi_i} = \left(\mu^2 + \lambda(\phi_1^2 + \phi_2^2 + \phi_3^2)\right)\phi_i = 0 \,,$$

so we require

$$\mu^2 + \lambda(\phi_1^2 + \phi_2^2 + \phi_3^2) \;=\; 0\,.$$

We choose the non-vanishing vacuum expectation value to be

$$\langle \phi_1 \rangle \;=\; \sqrt{-\frac{\mu^2}{\lambda}} \;=\; v$$

so the vacuum (after spontaneous symmetry breaking) is $\Phi_0 = (v, 0, 0)$. We now expand ϕ_1 around the new vacuum (keeping ϕ_2 and ϕ_3):

$$\phi_1(x) \;=\; v + h(x)\,.$$

Inserting this into the Lagrangian we get

$$\frac{1}{2}\left((\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2 + (\partial_\mu\phi_3)^2\right) - \frac{1}{2}\mu^2((v+h(x))^2+\phi_2^2+\phi_3^2) - \frac{1}{4}\lambda((v+h(x))^2+\phi_2^2+\phi_3^2)^2$$

and with $-\mu^2 = v^2\lambda$ we arrive at

$$\begin{aligned} L \;=\; & \frac{1}{2}((\partial_\mu h)^2 + (\partial_\mu\phi_2)^2 + (\partial_\mu\phi_3)^2) + \\ & \frac{v^2\lambda}{2}(h^2(x) + v^2 + 2vh(x) + \phi_2^2 + \phi_3^2) - \frac{\lambda}{4}(6v^2h^2(x) + \ldots) \;=\; \\ & \frac{1}{2}\left((\partial_\mu h)^2 + (\partial_\mu\phi_2)^2 + (\partial_\mu\phi_3)^2\right) - \lambda v^2 h^2(x) + \\ & \qquad\qquad \text{cubic and quartic interaction terms + constant}\,. \end{aligned}$$

The terms quadratic in ϕ_2 and ϕ_3 have cancelled, so there are no mass terms for these fields. Hence the Lagrangian, after SSB, describes two massless 'Goldstone bosons' and one massive scalar Higgs field h with mass $\sqrt{-2\mu^2}$.

3. (a) Setting $u = 1/\alpha_s$, $\mathrm{d}\alpha_s = -\mathrm{d}u/u^2$ the differential equation becomes

$$\frac{\mathrm{d}u}{\mathrm{d}\ln E} = b_0 + \frac{b_1}{u}\,.$$

Truncating at lowest order, the solution is given by $u(E) = u(\mu) + b_0 \ln(E/\mu)$, with a free integration constant $u(\mu)$. After sustituting back $u \to 1/\alpha_s$ this is the proposed solution, with the 'initial condition' $\alpha_s(\mu)$.

(b) The proposed definition is equivalent to the substitution

$$\mu = \Lambda_{\mathrm{QCD}} \exp\left[\frac{1}{b_0\alpha_s(\mu)}\right]$$

with which we obtain the form

$$\alpha_s(E) = \frac{1}{b_0 \ln(E/\Lambda_{\mathrm{QCD}})}\,.$$

At energies $E \to \Lambda_{\mathrm{QCD}}$ the coupling develops a pole, the so-called Landau-pole. For large values of the coupling the perturbative series breaks down, and the Landau pole indicates the non-perturbative region of QCD. This is the region where quarks and gluons are confined into colourless hadrons (baryons and mesons), which are the observed degrees of freedom at low energy scales.
At large energies, the coupling becomes small. This so-called 'asymptotic freedom' of QCD as an $SU(3)$ gauge field theory allows us to perform perturbative calculations on the parton level which in turn reliably describe QCD processes at large momentum transfer, like e.g. jet-production.
The sketch should show a monotonically decreasing positive function, indicating the Landau pole at $E = \Lambda_{\mathrm{QCD}}$ and a small value of α_s at large energies. [Draw the graph!]

(c) The solution valid at the next order of perturbation theory is obtained by inserting our lowest-order solution into the differential equation including the next order term $\sim b_1$, leading to

$$\frac{\mathrm{d}u}{\mathrm{d}\ln E} = b_0 + \frac{b_1}{b_0 \ln(E/\Lambda_{\mathrm{QCD}})}\,.$$

The solution of this differential equation is given by

$$u(E) = \frac{1}{\alpha_s(E)} = b_0 \ln(E/\Lambda_{\mathrm{QCD}}) + (b_1/b_0) \ln\ln(E/\Lambda_{\mathrm{QCD}})\,,$$

with a suitable redefinition of Λ_{QCD} to next order so that the integration constant vanishes.

4. As discussed during the lecture.