**MATH431 - Modern Particle Physics**
**Set Work: Sheet 10;**

**1.** (a) Show that in the Weinberg–Salam model

$$\frac{M_W}{M_Z} = \cos\theta_W$$

where $M_W$ and $M_Z$ are the masses of the charged weak vector bosons and the neutral electroweak vector boson respectively.

(b) Suppose that the Higgs scalar field $\phi(x)$ has weak isospin $T = 3$ and hypercharge $Y = -4$. If the neutral component $\phi_0$ (with $Q_{\text{e.m.}} = 0$) develops a vacuum expectation value $v/\sqrt{2}$, show that

$$M_W^2 = \frac{g^2}{2}\phi^\dagger(T^+T^- + T^-T^+)\phi = 4g^2v^2.$$

Further show that $M_W/M_Z = \cos\theta_W$ just as in the Standard Model.

**2.** The decomposition of the fundamental representation of $SU(5)$ under $SU(3) \times SU(2) \times U(1)$ is given by:

$$5 = (3, 1, -\frac{2}{3}) + (1, 2, 1)$$

Derive the charge assignment of the leptoquark vector bosons in the adjoint (24) representation. Draw the Feynman diagram that couple the fermions to the leptoquark vector bosons. Draw the diagram by which the proton decays.

**3.** (a) Show that a charged weak current of the form

$$\bar{u}_e\gamma^\mu\frac{1}{2}\left(1 - \gamma^5\right)u_\nu$$

involves only left-handed electrons.

(b) Weak decays of leptons:

Explain why the decay $\mu^- \to e^-\gamma$ is not allowed in the Standard Model.

(c) Draw the (lowest order) Feynman diagram for the muon decay in the Standard Model,

$$\mu^- \to \nu_\mu\, e^-\, \bar{\nu}_e\,.$$

Name and explain the different elements of the Feynman diagram and give their corresponding algebraic expressions. (You can leave out the labels for the momenta and spins of the external particles.)

(d) Given that the weak coupling $g$ is of the order of the electromagnetic coupling $e$ ($g\,\sin\theta_W = e$ with the sine of the Weinberg angle $\sin\theta_W \approx 0.5$), why are the weak decays suppressed compared to typical electromagnetic or strong decays?

(e) In analogy to the $\mu$ decay, discuss the possible decays of the $\tau$ lepton.

**4.** (a) Prove that for the group $SU(2)$ the fundamental representation (the representation by complex $2\times2$ matrices) is equivalent to the complex conjugate representation. This means that there exists a unitary $2 \times 2$ matrix $W$ such that $U^\star = W^\dagger\, U\, W$ (unitary equivalence).

*Hints:* The complex conjugate representation can be written as

$$U^\star \;=\; \exp(-\frac{i}{2}\,\theta_a\,\sigma_a^\star)\,,$$

with the three Pauli-matrices in the usual representation,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

First show that $i\sigma_2$ is unitary and that $\sigma_2\,\sigma_a^\star\,\sigma_2 = -\sigma_a$.

(4b) Explain which consequences this has for the construction of mass terms in the Standard Model.