MATH431 - Modern Particle Physics
Set Work: Sheet 4;

1. Prove

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\mathbf{j} = 0, \quad \text{where} \quad \mathbf{j} = -\frac{i\hbar}{2m}\left[\psi^* \nabla \psi - \psi \nabla \psi^*\right]$$

using the standard non-relativistic Schrödinger equation.

2. (i) The Lagrangian density for a real scalar field is given by

$$\mathcal{L}_0 = \tfrac{1}{2}(\partial \phi)^2 - \tfrac{1}{2}m^2\phi^2$$

Show that the Hamiltonian H_0 is given by

$$H_0 = \tfrac{1}{2}\int [\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2] d^3x.$$

(ii) Show that if we take the usual canonical commutation relations,

$$\begin{aligned}
[\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)] &= i\hbar\delta(\mathbf{x} - \mathbf{x}'),\\
[\phi(\mathbf{x}, t), \phi(\mathbf{x}', t)] &= 0,\\
[\pi(\mathbf{x}, t), \pi(\mathbf{x}', t)] &= 0,
\end{aligned}$$

the equations of motion are obtained from

$$i\hbar\dot{\phi} = [\phi, H_0] \quad \text{and} \quad i\hbar\dot{\pi} = [\pi, H_0] \ .$$

(iii) Consider the Lagrangian density

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I,$$

with

$$\mathcal{L}_I = -\frac{\lambda_3}{3!}\phi^3 - \frac{\lambda_4}{4!}\phi^4 \ .$$

Derive the equation of motion from

$$\partial^\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)}\right] - \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$