Problem Solutions for The Standard Model

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Contents

Chapter 1

Field theory review problems

1.1 Identities for Majorana spinors

First, note that

$$\begin{aligned} \gamma_5^T C &= i\gamma_3^T\gamma_2^T\gamma_1^T\gamma_0^T C \\ &= Ci\gamma_3\gamma_2\gamma_1\gamma_0 \\ &= Ci\gamma_0\gamma_1\gamma_2\gamma_3 = C\gamma_5 \end{aligned} \tag{1.1}$$

where in the next to last step we used that distinct γ matrices anticommute and it takes six re-orderings to do the reversal of indices that we made.

To be general, consider $\bar{\psi}_1\gamma_{\mu_1}\ldots\gamma_{\mu_n}\gamma_5^m\bar{\psi}_2$, a product containing $n = 0,1,2$ gamma matrices and $m = 0,1$ factors of γ_5. Reversing the order of the objects, this is

$$\begin{aligned} \bar{\psi}_1\gamma_{\mu_1}\ldots\gamma_{\mu_n}\gamma_5^m\bar{\psi}_2 &= -\psi_2^T(\gamma_5^T)^m\gamma_{\mu_n}^T\ldots\gamma_{\mu_1}^T\bar{\psi}_1^T \\ &= \bar{\psi}_2 C(\gamma_5^T)^m\gamma_{\mu_n}^T\ldots\gamma_{\mu_1}^T\bar{\psi}_1^T \\ &= (-1)^n\bar{\psi}_2\gamma_5^m\gamma_{\mu_n}\ldots\gamma_{\mu_1}C\bar{\psi}_1^T \\ &= (-1)^n\bar{\psi}_2\gamma_5^m\gamma_{\mu_n}\ldots\gamma_{\mu_1}\psi_1 \\ &= (-1)^{n+mn}\bar{\psi}_2\gamma_{\mu_n}\ldots\gamma_{\mu_1}\gamma_5\psi_1\,. \end{aligned} \tag{1.2}$$

This gives the first 4 relations directly. For the last, note that the order of the γ matrices is reversed in the above; this gives a minus sign due to the γ's being combined in a commutator (odd in the order).

Next, we do the same with Hermitian conjugation;

$$\left(\bar{\psi}_1\gamma_{\mu_1}\ldots\gamma_{\mu_n}\gamma_5^m\psi_2\right)^\dagger = \psi_2^\dagger\gamma_5^m\gamma_{\mu_n}^\dagger\ldots\gamma_{\mu_1}^\dagger\beta\psi_1\,. \tag{1.3}$$

Move the β across the gamma matrices, producing one minus sign for each γ_μ or γ_5;

$$= (-1)^{m+n}\psi_2^\dagger\beta\gamma_5^m\gamma_{\mu_n}\dots\gamma_{\mu_1}\psi_1 \tag{1.4}$$

and move the γ_5, if any, across the γ matrices, giving $(-1)^{mn}$;

$$= (-1)^{m+n+mn}\bar{\psi}_2\gamma_{\mu_n}\dots\gamma_{\mu_1}\gamma_5^m\psi_1\,. \tag{1.5}$$

Combining these with the previous results, and remembering the $-$ sign for reversing the order of the γ matrices in the commutator, gives the results quoted. Namely, we have the simple expression,

$$\left(\bar{\psi}_1\gamma_{\mu_1}\dots\gamma_{\mu_n}\gamma_5^m\psi_2\right)^\dagger = (-1)^{m+2(n+mn)}\bar{\psi}_1\gamma_{\mu_1}\dots\gamma_{\mu_n}\gamma_5^m\psi_2\,. \tag{1.6}$$

Only the m matters as the other piece is even.

1.2 $\mathcal{O}$(N) scalar theories

GUY: CHECK IF THIS IS COMPLETE.

1.2.1 $N = 2$ case

The symmetry ensures that terms with odd powers of fields are absent, and terms quadratic in the fields do not contain $\phi_1\phi_2$; and we can re-scale the fields to give canonical kinetic terms; so the most general renormalizable Lagrangian is

$$\frac{1}{2}\left((\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2 + m_1^2\phi_1^2 + m_2^2\phi_2^2\right) + \frac{1}{4}\left(\lambda_1\phi_1^4 + \lambda_{12}\phi_1^2\phi_2^2 + \lambda_2\phi_2^2\right)\,. \tag{1.7}$$

This can be rewritten as

$$\begin{aligned}&\partial_\mu\psi^*\partial^\mu\psi + \frac{m_1^2+m_2^2}{2}\psi^*\psi + \frac{m_1^2-m_2^2}{4}(\psi^2+\psi^{*2})\\ &+\frac{\lambda_1+\lambda_2-\lambda_{12}}{16}\left(\psi^4+\psi^{*4}\right) + \frac{\lambda_1-\lambda_2}{4}\left(\psi^3\psi^* + \psi\psi^{*3}\right) + \frac{3\lambda_1+3\lambda_2+\lambda_{12}}{8}(\psi^*\psi)^2\,.\end{aligned} \tag{1.8}$$

The transformation rules for ψ and ψ^* under rotation by angle θ between ϕ_1 and ϕ_2 are

$$\psi' = \phi_1' + i\phi_2' = (\phi_1\cos(\theta) + \phi_2\sin(\theta)) + i(\phi_2\cos(\theta) - \phi_1\sin(\theta)) = \phi_1 e^{-i\theta} + i\phi_2 e^{-i\theta}\,, \tag{1.9}$$

$$\psi^{*\prime} = \phi_1' - i\phi_2' = (\phi_1\cos(\theta) + \phi_2\sin(\theta)) - i(\phi_2\cos(\theta) - \phi_1\sin(\theta)) = \phi_1 e^{i\theta} - i\phi_2 e^{i\theta}\,, \tag{1.10}$$

or

$$\psi \to e^{-i\theta}\psi \quad \text{and} \quad \psi^* \to e^{i\theta}\psi^* \,. \tag{1.11}$$

$\mathcal{O}(2)$ invariance requires that Lagrangian terms not pick up a phase under such a rotation. This means that ψ and ψ^* must appear an equal number of times in each term. Looking at the Lagrangian, this forces

$$m_1^2 = m_2^2 \,, \quad \lambda_1 = \lambda_2 \,, \quad \lambda_{12} = 2\lambda_1 \,. \tag{1.12}$$

The resulting Lagrangian is

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\left((\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2 + m^2(\phi_1^2+\phi_2^2)\right) + \frac{\lambda}{4}\left(\phi_1^2+\phi_2^2\right)^2 \,, \\ &= \left(\partial_\mu\psi^*\partial^\mu\psi + m^2\psi^*\psi + \lambda(\psi^*\psi)^2\right) \,. \end{aligned} \tag{1.13}$$

$m^2 > 0$: vacuum is $\phi_1 = \phi_2 = 0$ and masses are m.

$m^2 < 0$: the vacuum is whatever field values minimize the potential. This is not unique but by field redefinition (precisely a rotation by some angle θ) we can take it to be $\phi_1 = \sqrt{m^2/\lambda}$ and the masses squared are $m_1^2 = [-2m^2] > 0$ and $m_2^2 = 0$. With this choice, ϕ_2 is the Goldstone boson.

1.2.2 $N = 4$ case

The most general form for the Lagrangian is

$$\mathcal{L} = \sum_i \frac{1}{2}\left((\partial_\mu\phi_i)^2 + m_i^2\phi_i^2\right) + \sum_{i\geq j}\frac{\lambda_{ij}}{4}\phi_i^2\phi_j^2 \,. \tag{1.14}$$

Now we must show that the matrix

$$\Phi \equiv \begin{bmatrix} \phi & \psi^* \\ \psi & -\phi^* \end{bmatrix} \tag{1.15}$$

persists in satisfying

$$\bar{\Phi} \equiv e\Phi^* e = \Phi \,, \tag{1.16}$$

$$\text{Det}\,\Phi = -\frac{1}{2}\phi^{\mathrm{T}}\phi \tag{1.17}$$

under the transformations $\Phi \to U\phi$ and $\phi \to \Phi V$, with U, V special unitary matrices.

The relation involving the determinant is trivial since the determinant of a product of matrices is the product of the determinants:

$$\text{Det}\,U\Phi = (\text{Det}\,U)(\text{Det}\,\Phi) = 1(\text{Det}\,\Phi) \tag{1.18}$$

and similarly for ΦV.

For the property involving $\bar{\Phi}$, note first that $-e^2 = \mathbf{1}$, so

$$\overline{U\Phi} \equiv eU^*\Phi^*e \equiv -eU^*e\, e\Phi^*e \tag{1.19}$$

and it is sufficient to show that $-eU^*e = U$ for $SU(2)$ matrices. This is so because an $SU(2)$ matrix can be written as

$$U = u_0\mathbf{1} + \vec{u}\cdot i\vec{\sigma}\,, \qquad u_0^2 + \vec{u}^2 = 1 \tag{1.20}$$

and because

$$\sigma^* = \sigma \;\;\text{except}\;\; \sigma_2^* = -\sigma_2 \tag{1.21}$$

and

$$\sigma_2\sigma_i\sigma_2 = -\sigma_i\sigma_2^2 = -\sigma_i \;\;\text{except}\;\; \sigma_2\sigma_2\sigma_2 = \sigma_2\,. \tag{1.22}$$

Combining these,

$$\sigma_2\sigma_i^*\sigma_2 = -\sigma_i \qquad (\text{also}\,,\;\; \sigma_2\mathbf{1}\sigma_2 = \mathbf{1})\,. \tag{1.23}$$

Furthermore, $e = i\sigma_2$. Therefore,

$$-eU^*e = \sigma_2U^*\sigma_2 = \sigma_2(u_0\mathbf{1} - \vec{u}\cdot i\vec{\sigma}^*)\sigma_2 = u_0\mathbf{1} + \vec{u}\cdot i\vec{\sigma} = U\,. \tag{1.24}$$

Therefore

$$\overline{U\Phi} \equiv eU^*\Phi^*e = -eU^*e\, e\Phi^*e = Ue\Phi^*e = U\Phi \tag{1.25}$$

as desired. Exactly the same argument goes for V.

Now we consider the case with invariance under only one $SU(2)$. First, observe that of the $SU(2)$ invariant combinations, $\chi^\dagger\chi$, $\bar{\chi}^\dagger\chi$, and $\bar{\chi}^\dagger\bar{\chi}$, there is only one nonzero independent combination;

$$\bar{\chi}^\dagger\bar{\chi} = \chi^\dagger\chi = \frac{\phi_1^2+\phi_2^2+\phi_3^2+\phi_4^2}{2} = \chi^\dagger\chi\,, \quad \text{and} \quad \bar{\chi}^\dagger\chi = 0\,. \tag{1.26}$$

Therefore the quadratic parts of the Lagrangian can be written only using χ; the most general form is

$$\partial_\mu\chi^\dagger\partial^\mu\chi + m^2\chi^\dagger\chi\,. \tag{1.27}$$

Now it remains to compute the most general quartic piece. Besides the obvious term, $\lambda(\chi^\dagger\chi)^2$, one might think of the following alternate terms, involving τ^a:

$$\sum_a(\chi^\dagger\tau^a\chi)^2\,, \quad \sum_a(\bar{\chi}^\dagger\tau^a\chi)^2\,, \quad \sum_a\bar{\chi}^\dagger\tau^a\chi\,\chi^\dagger\tau^a\chi\,. \tag{1.28}$$

However, explicit calculation shows that the first term here equals

$$\begin{aligned}\sum_a (\chi^\dagger \tau^a \chi)^2 &= (\varphi^*\psi+\varphi\psi^*)^2 - (\varphi^*\psi-\varphi\psi^*)^2 + (\varphi^*\varphi-\psi^*\psi)^2 \\ &= (\varphi^*\varphi+\psi^*\psi)^2 = (\chi^\dagger\chi)^2 \,, \end{aligned} \tag{1.29}$$

and so is not independent, while the other two terms turn out to cancel completely; for instance,

$$\sum_a (\bar{\chi}^\dagger \tau^a \chi)^2 = (\psi^2-\varphi^2)^2 - (\psi^2+\varphi^2)^2 + (2\varphi\psi)^2 = 0 \,. \tag{1.30}$$

Therefore, the most general quartic term is

$$\lambda(\chi^\dagger\chi)^2 = \frac{\lambda}{4}(\phi_1^2+\phi_2^2+\phi_3^2+\phi_4^2)^2 \,. \tag{1.31}$$

Note that this expression, in fact the whole Lagrangian, is also invariant under the right acting $SU(2)$. Therefore, renormalizability together with one $SU(2)$ invariance "accidentally" ensures the other $SU(2)$ invariance.

For the choice $m^2 > 0$, the vacuum is that all four ϕ_i are zero, and the masses are all m.

For the choice $m^2 \equiv -\mu^2 < 0$, the vacuum has one scalar field, say ϕ_1, nonzero: its value is that which minimizes the effective potential, namely $\phi_1 = \mu/\lambda$. In this case, of the original $\mathcal{O}(4)$ invariance, the $\mathcal{O}(3)$ subgroup which rotates the remaining 3 fields between themselves, is a symmetry of the vacuum ("unbroken"). This symmetry has 3 generators, while $\mathcal{O}(4)$ has 6; so we expect 3 Goldstone bosons.

Now we work out the spectrum for $m^2 < 0$; define $v = \mu/\lambda$, the expectation value of ϕ_1, and define $\phi_1 \equiv \delta + v$. Then the effective potential term becomes

$$\begin{aligned}&\frac{\lambda}{4}v^4 - \frac{m^2}{2}v^2 + (\lambda v^3 - \mu^2 v)\delta + \left(\frac{3\lambda}{2}v^2 - \frac{\mu^2}{2}\right)\delta^2 + \left(\frac{\lambda^2}{2}v^2 - \frac{\mu^2}{2}\right)(\phi_2^2+\phi_3^2+\phi_4^2) \\ &\quad +(\text{cubic and higher in fields}) \,. \end{aligned} \tag{1.32}$$

We see that our choice of v has ensured the cancellation of the linear term in δ. Evaluating the quadratic coefficients gives $m_1^2 = 2\mu^2$ and $m_2^2 = m_3^2 = m_4^2 = 0$. The other 3 scalars are indeed the massless Goldstone bosons associated with the broken symmetry directions.

1.3 Vacuum energies

Assume that we have made space finite somehow so the spectrum of momentum states is discrete; the summation on momentum states is $\sum_{\mathbf{p}}$ and the volume of space is V. The vacuum energy is expected to scale as volume, that is, to be extensive.

Consider the scalars first. These are just two real scalar fields; compute for one scalar field and multiply by 2 at the end. The Hamiltonian is

$$H = \int d^3\mathbf{x} \left[\frac{1}{2} (\nabla \phi_i)^2 + \frac{\mu^2}{2} \phi_i^2 + \frac{1}{2} \Pi_i^2 \right] , \tag{1.33}$$

with Π the conjugate momentum of Φ.

Momentum space is the best place to solve the theory; define

$$\phi_i(\mathbf{p}) \equiv \frac{1}{\sqrt{V}} \int d^3\mathbf{x}\, e^{i\mathbf{p}\cdot\mathbf{x}} \phi_i(\mathbf{x}) , \qquad \phi_i(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} \phi_i(\mathbf{p}) , \tag{1.34}$$

and $\omega_{\mathbf{p}} \equiv \sqrt{\mathbf{p}^2 + \mu^2}$. One finds by standard techniques that

$$\phi_i(\mathbf{p}) = \frac{1}{\sqrt{2\omega}} \Big(a_i(\mathbf{p}) + a_i^\dagger(\mathbf{p}) \Big) , \quad \Pi_i(\mathbf{p}) = \frac{\sqrt{\omega}}{i\sqrt{2}} \Big(a_i(\mathbf{p}) - a_i^\dagger(\mathbf{p}) \Big) , \tag{1.35}$$

with a and $a^\dagger$ satisfying the commutation relations

$$\Big[a(\mathbf{p}) \, , \, a^\dagger(\mathbf{p}') \Big] = \delta_{\mathbf{p},-\mathbf{p}'} \, . \tag{1.36}$$

Substituting the Fourier transformed versions of ϕ_i, Π_i into the Hamiltonian gives

$$H = \frac{1}{V} \int d^3\mathbf{x} \sum_{\mathbf{p},\mathbf{p}'} \times \left\{ \begin{array}{c} e^{i\mathbf{x}\cdot(\mathbf{p}+\mathbf{p}')} \, \dfrac{\mu^2}{4\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}} (a_i(\mathbf{p}) + a_i^\dagger(\mathbf{p}))(a_i(\mathbf{p}') + a_i^\dagger(\mathbf{p}')) \\ \\ +e^{i\mathbf{x}\cdot(\mathbf{p}+\mathbf{p}')} \, \dfrac{-\mathbf{p}\cdot\mathbf{p}'}{4\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}} (a_i(\mathbf{p}) + a_i^\dagger(\mathbf{p}))(a_i(\mathbf{p}') + a_i^\dagger(\mathbf{p}')) \\ \\ -e^{i\mathbf{x}\cdot(\mathbf{p}+\mathbf{p}')} \, \dfrac{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}}{4} (a_i(\mathbf{p}) - a_i^\dagger(\mathbf{p}))(a_i(\mathbf{p}') - a_i^\dagger(\mathbf{p}')) \end{array} \right\} \tag{1.37}$$

Now do the $\mathbf{x}$ integration, which cancels the $1/V$ and forces $\mathbf{p}' = -\mathbf{p}$. Then $-\mathbf{p}\cdot\mathbf{p}' = \mathbf{p}^2$, and $(\mathbf{p}^2 + m^2)/\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}} = \sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}$ so the terms can be combined. The terms with two a or two $a^\dagger$ cancel and leave

$$H = \sum_{\mathbf{p}} \frac{\omega_{\mathbf{p}}}{2} \left(a_i^\dagger(-\mathbf{p}) a_i(\mathbf{p}) + a_i(\mathbf{p}) a_i^\dagger(-\mathbf{p}) \right) . \tag{1.38}$$

Normally one names $a^\dagger(-\mathbf{p})$ above, $a^\dagger(\mathbf{p})$ instead, even though it emerged in a Fourier transform from $-\mathbf{p}$ rather than $\mathbf{p}$, because it creates a particle of momentum $\mathbf{p}$. We will follow this convention starting now. Rewriting to put a to the right of $a^\dagger$ gives

$$H = \sum_{\mathbf{p}} \omega_{\mathbf{p}} \left(\sum_i a_i^\dagger(\mathbf{p}) a_i(\mathbf{p}) + \frac{1}{2} \sum_i \right) . \tag{1.39}$$

The $a^\dagger a$ term annihilates the vacuum, the vacuum energy arises from the 1/2 factor. The sum on i gives 2 as there are two scalars.

Now we have to compare this to the energy in a fermion. The fermionic Hamiltonian is

$$H = \int d^3\mathbf{x}\, \frac{1}{2}\bar{\psi}(\vec{\gamma}\cdot\vec{\nabla} + m)\psi\,. \tag{1.40}$$

The factor of 1/2 is there because both the upper and lower components of the spinor are the same; so the term really includes a kinetic plus mass term twice, the 1/2 removes this double counting.

e want to evaluate

$$\frac{1}{2}\int d^3\mathbf{x}\bar{\psi}(x)(\gamma_i\partial_i + m)\psi(x)\,. \tag{1.41}$$

Start by Fourier transforming just $\psi(x)$; the ∂_i acts on the $e^{\pm i\mathbf{p}\cdot\mathbf{x}}$ to give $\pm i\mathbf{p}$. Then Fourier transform $\bar{\psi}$;

$$\begin{aligned} H \;=\; & \frac{1}{2V}\sum_{\mathbf{p},\mathbf{q}}\sum_{\sigma\sigma'}\int d^3\mathbf{x}\left[\bar{u}(\mathbf{p},\sigma')e^{-i\mathbf{q}\cdot\mathbf{x}}b^*(\mathbf{p},\sigma') + \bar{v}(\mathbf{p},\sigma')e^{+i\mathbf{q}\cdot\mathbf{x}}b(\mathbf{p},\sigma')\right]\times \\ & \left[(ip_i\gamma_i + m)u(\mathbf{p},\sigma)e^{i\mathbf{p}\cdot\mathbf{x}}b(\mathbf{p},\sigma) + (-ip_i\gamma_i + m)v(\mathbf{p},\sigma)e^{-i\mathbf{p}\cdot\mathbf{x}}b^*(\mathbf{p},\sigma)\right] \end{aligned} \tag{1.42}$$

The terms involving $b(\mathbf{q})b(\mathbf{p})$ and $b^*(\mathbf{q})b^*(\mathbf{p})$ vanish because the operators obey anticommutation relations. Also we can use the property that u and v are solutions to the Dirac equation,

$$(ip_\mu\gamma^\mu + m)u(\mathbf{p},\sigma) = 0\,, \qquad (ip_\mu\gamma^\mu - m)v(\mathbf{p},\sigma) = 0\,, \tag{1.43}$$

to substitute

$$(ip_i\gamma_i + m)u(\mathbf{p},\sigma) = ip^0\gamma^0 u(\mathbf{p},\sigma)\,, \qquad (-ip_i\gamma_i + m)v(\mathbf{p},\sigma) = -ip^0\gamma^0 v(\mathbf{p},\sigma)\,, \tag{1.44}$$

where $p^0 \equiv \sqrt{\mathbf{p}^2 + m^2}$. Also, $i\gamma^0 \equiv \beta$. Doing also the $\mathbf{x}$ integration, which kills one of the momentum integrals, we get

$$H = \frac{1}{2}\sum_{\mathbf{p}}\sum_{\sigma,\sigma'}\sqrt{\mathbf{p}^2 + m^2}\left(u^\dagger(\mathbf{p},\sigma')u(\mathbf{p},\sigma)b^*b(\mathbf{p},\sigma) - v^\dagger(\mathbf{p},\sigma')v(\mathbf{p},\sigma)bb^*(\mathbf{p},\sigma)\right)\,. \tag{1.45}$$

The spinors are normalized so that $\bar{u}(\mathbf{p},\sigma')u(\mathbf{p},\sigma) = \delta_{\sigma\sigma'}m/E$. Since $u^\dagger u$ is the time component of a 4-vector and equals $\bar{u}u$ for a particle at rest, we have $u^\dagger(\mathbf{p},\sigma)u(\mathbf{p},\sigma) = \delta_{\sigma\sigma'}$. Therefore the Hamiltonian is

$$H = \frac{1}{2}\sum_{\mathbf{p}}\sum_{\sigma}\sqrt{\mathbf{p}^2 + m^2}\Big(b^*b(\mathbf{p},\sigma) - bb^*(\mathbf{p},\sigma)\Big)\,, \tag{1.46}$$

or, anticommuting the bb^* term,

$$H = \sum_{\mathbf{p}} \sum_{\sigma} \sqrt{\mathbf{p}^2 + m^2} \left(-\frac{1}{2} + b^* b(\mathbf{p}, \sigma) \right) . \tag{1.47}$$

The ground state energy, after taking $\sum_\sigma = 2$, looks like the bosonic one but with the substitution $\mu^2 \to m^2$ and with the overall sign reversed.

If we evaluate the ground state energy in a momentum cutoff normalization,

$$\sum_{\mathbf{p}} \to V \int^{\mathbf{p}^2 = \Lambda^2} \frac{d^3 p}{(2\pi)^3} , \tag{1.48}$$

we find

$$E_{\text{vac}} = V \frac{1}{2\pi^2} \int_0^\Lambda p^2 dp \left(\sqrt{p^2 + \mu^2} - \sqrt{p^2 + m^2} \right) \simeq \frac{V}{8\pi^2} \left(\Lambda^2 (\mu^2 - m^2) + O(\mu^4, m^4) \right) . \tag{1.49}$$

In the $\mu = m$ case the vacuum energy vanishes. This fact is one of the motivations for the theory of supersymmetry, but we will not discuss that today.

1.4 Symmetries and Yukawa interactions

We have already seen how this symmetry constrains the purely scalar part of the Lagrangian, in the $N = 2$ section of problem 1.2 above. As shown there, the allowed purely scalar terms are

$$\mathcal{L}_\phi = \frac{-1}{2} \left(\partial_\mu \phi \partial^\mu \phi + \partial_\mu \chi \partial^\mu \chi + m^2 (\phi^2 + \chi^2) \right) - \frac{\lambda}{4} \left(\phi^2 + \chi^2 \right)^2 . \tag{1.50}$$

Now, for the terms with fermions. The set of all terms we can think of, after performing a field redefinition as in the notes to eliminate a $\bar\psi \gamma^\mu \gamma^5 \partial_\mu \psi$ term, is

$$\bar\psi \gamma^\mu \partial_\mu \psi + b_1 \bar\psi \psi + b_2 \bar\psi \gamma^5 \psi + c_1 \phi \bar\psi \psi + c_2 \chi \bar\psi \psi + c_3 \phi \bar\psi \gamma^5 \psi + c_4 \chi \bar\psi \gamma^5 \psi . \tag{1.51}$$

We need to see how each transforms under infinitesimal transformation by angle θ, under which the fields pick up

$$\begin{aligned} \psi &\to \psi + i\theta \gamma^5 \psi , \\ \bar\psi &\to \bar\psi + i\theta \bar\psi \gamma^5 , \\ \phi &\to \phi + 2\theta \chi , \\ \chi &\to \chi - 2\theta \phi . \end{aligned} \tag{1.52}$$

The various terms then gain the following extra pieces,

$$\begin{aligned}
\bar{\psi}\psi &\quad \text{gains} \quad 2i\theta\bar{\psi}\gamma^5\psi\,, \\
\bar{\psi}\gamma^5\psi &\quad \text{gains} \quad 2i\theta\bar{\psi}\psi\,, \\
\phi\bar{\psi}\psi &\quad \text{gains} \quad 2\theta\Big(\chi\bar{\psi}\psi + i\phi\bar{\psi}\gamma^5\psi\Big)\,, \\
\chi\bar{\psi}\psi &\quad \text{gains} \quad 2\theta\Big(-\phi\bar{\psi}\psi + i\chi\bar{\psi}\gamma^5\psi\Big)\,, \\
\phi\bar{\psi}\gamma^5\psi &\quad \text{gains} \quad 2\theta\Big(\chi\bar{\psi}\gamma^5\psi + i\phi\bar{\psi}\psi\Big)\,, \\
\chi\bar{\psi}\gamma^5\psi &\quad \text{gains} \quad 2\theta\Big(-\phi\bar{\psi}\gamma^5\psi + i\chi\bar{\psi}\psi\Big)\,.
\end{aligned} \tag{1.53}$$

No term can compensate for the shift in the first two terms, so $b_1 = b_2 = 0$. The remaining terms can cancel in pairs if $c_4 = ic_1$ and $c_3 = -ic_2$. Further, Hermiticity demands that c_1 and c_2 are real, and c_3 and c_4 are imaginary; so this is consistent. Therefore, there are two real parameters for the Yukawa interactions, and the remaining allowed terms are

$$\mathcal{L}_\psi = -\frac{1}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi - c_1\Big(\phi\bar{\psi}\psi + i\chi\bar{\psi}\gamma^5\psi\Big) - c_2\Big(\chi\bar{\psi}\psi - i\phi\bar{\psi}\gamma^5\psi\Big)\,. \tag{1.54}$$

Because $\bar{\psi}\psi$ is purely real and $\bar{\psi}\gamma^5\psi$ is purely imaginary, this can be succinctly rewritten in terms of one complex coefficent $c = \sqrt{2}(c_1 + ic_2)$ and the complex field $\Phi \equiv (\phi + i\chi)/\sqrt{2}$,

$$\mathcal{L}_\psi = -\frac{1}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi - \Big(c\Phi\bar{\psi}P_{\mathrm{L}}\psi + \text{c.c.}\Big)\,, \tag{1.55}$$

where (c.c.) means complex conjugate, and is the same as taking twice the real part of the first term. We see that, in general, Yukawa couplings are complex. For the gauged version, all that changes is that the derivatives must be made covariant derivatives, with

$$\begin{aligned}
D_\mu\Phi &= (\partial_\mu + 2igA_\mu)\Phi\,, \\
D_\mu P_{\mathrm{L}}\psi &= (\partial_\mu - igA_\mu)P_{\mathrm{L}}\psi\,,
\end{aligned} \tag{1.56}$$

and therefore,

$$D_\mu P_{\mathrm{R}}\psi = (\partial_\mu + igA_\mu)P_{\mathrm{R}}\psi\,. \tag{1.57}$$

1.5 Spinor identities

To appear

1.6 Fermion mass matrix diagonalization

To appear

1.7 A Dirac matrix identity

To appear

1.8 More useful identities

To appear

1.9 Fiertz rearrangements

To appear

Chapter 2

Standard model: general features

2.1 Anomaly cancellation and charge assignments

To appear

2.2 Muon decay

The symmetries which must be obeyed by any decay (or other process) involving a muon are electric charge conservation and the separate conservation of B and each L number. Anomalies mean that $B + L$ may be violated, but $L_e - L_\mu$, $L_\mu - L_\tau$, and $B - L$ must be separately conserved.

For the first reaction, the total electric charge of the initial state is -1 and this is unchanged in the final state. The total $L_e = 0$, $L_\mu = 1$, $L_\tau = 0$, and $B = 0$ in the initial and final states. Therefore there is no conservation law violated in the decay.

However, for the second reaction, $L_e - L_\mu$ is -1 in the initial state and $+1$ in the final state; therefore this anomaly-free, absolutely conserved quantity would be violated by such a decay. Within the standard model, the decay rate should be identically zero.

2.3 Right handed neutrinos

2.3.1 Lagrangian

Lorentz invariance demands that fermionic fields appear in even numbers in all terms, and renormalizability restricts to 2 fermionic fields. (A term in the Lagrangian with a product of 4 fermionic fields would be dimension 6, which is non-renormalizable.) The only terms allowed for N by renormalizability are terms with either $\bar{N}$ and N or with N and the bar of some other fermion, with either a derivative, a mass, or a Higgs field. Gauge invariance demands that the charges of the fields involved add to zero for any term. Since N is the only singlet, the kinetic and mass terms cannot mix it with other fermionic fields. Therefore the mass and kinetic terms must be of form

$$-\frac{1}{2}\bar{N}_m\gamma^\mu\partial_\mu N_m - \frac{1}{2}M_m\bar{N}_m N_m\,. \tag{2.1}$$

As argued in the notes, we are free to pick these terms to be diagonal and free of γ^5 terms. Unlike all other fermions in the standard model, the mass term is allowed, because the right and left handed components of N transform in the same way under gauge transformation (namely, not at all).

To appear in a Yukawa interaction with the Higgs field, N must interact with a fermion with the same gauge charges as the Higgs field; this excludes the colored fields and the ($SU(2)$ singlet) right electron field E_R. However, the left handed lepton field L has the same transformation properties as $\tilde{\phi}$, so the combination

$$k_{mn}\bar{L}_m P_{\mathrm{R}} N_n \tilde{\phi} + c.c. \tag{2.2}$$

is allowed. (The [antifundamental] $SU(2)$ index of $\bar{L}P_{\mathrm{R}}$ is contracted against the [fundamental] $SU(2)$ index of $\tilde{\phi}$ in the above term.)

2.3.2 Conservation laws

The left-handed component of L and the right-handed component of E are assigned lepton number 1. That is, before the introduction of N, the Lagrangian was invariant under the symmetry

$$L \to e^{i\theta_L\gamma^5}L\,, \quad E \to e^{-i\theta_L\gamma^5}E\,, \tag{2.3}$$

and the associated conserved number was called lepton number. In fact, after choosing a basis for L and E which diagonalizes the Yukawa matrix f_{mn}, the Lagrangian was invariant under such a rotation of each generation separately (that is, of L_1 and E_1, of L_2 and E_2, and of L_3 and E_3). The three conserved numbers are called electron number, muon number, and tau number.

If N is to be uncharged under this transformation then the new Yukawa term clearly violates this symmetry. If we assign the right-handed component of N a lepton number of 1, that is, if N transforms as

$$N \to e^{-i\theta_L\gamma^5}N\,, \tag{2.4}$$

then the Yukawa interaction term preserves the symmetry. If for some reason the mass term vanishes, $M = 0$, then total lepton number is conserved; but if the new Yukawa matrix k_{mn} cannot be diagonalized by rotating only the N fields, which generically happens, then the individual lepton numbers are separately violated.

Assigning N the transformation property above, the mass term is clearly not invariant (since γ^5 and β do not commute). Therefore, if the mass term is nonzero, there is no conserved lepton number of any kind.

2.3.3 Mass eigenstates

The neutrino mass terms are

$$-\frac{1}{2}M\bar{N}N - k\bar{L}P_R N\tilde{\phi} + \text{h.c.} \tag{2.5}$$

which, on substituting

$$\tilde{\phi} \to \begin{bmatrix} v/\sqrt{2} \\ 0 \end{bmatrix}, \tag{2.6}$$

taking k to be real (which we can do by an appropriate field rotation), and rewriting $vk/\sqrt{2} \equiv m$, and expanding out the Hermitian conjugate, becomes

$$-\frac{1}{2}\left(M\bar{N}N + 2m\bar{\nu}P_R N + 2m\bar{N}P_L\nu\right) . \tag{2.7}$$

Using relations from last homework,

$$\bar{\nu}P_R N = \bar{N}P_R\nu\,, \qquad \bar{N}P_L\nu = \bar{\nu}P_L N\,, \tag{2.8}$$

we can make this expression symmetric in ν and N:

$$-\frac{1}{2}\left(M\bar{N}N + m\bar{\nu}N + m\bar{N}\nu\right) . \tag{2.9}$$

Here we also used $P_R + P_L = \mathbf{1}$. We can rewrite this, if we feel like it, as

$$-\frac{1}{2}\begin{bmatrix}\bar{\nu} & \bar{N}\end{bmatrix}\begin{bmatrix} 0 & m \\ m & M \end{bmatrix}\begin{bmatrix} \nu \\ N \end{bmatrix}, \tag{2.10}$$

where the matrix in the middle is the "mass matrix". This is the form required to apply the manipulations which follow Eq. (1.3.51), where we diagonalize this matrix by performing a unitary transformation on the fermionic fields. Define

$$\cos\theta_\nu = \frac{1}{2}\text{arctan}\frac{2m}{M}\,, \quad \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \nu \\ N \end{bmatrix}, \tag{2.11}$$

which is the matrix which diagonalizes the mass matrix. In terms of ψ_1 and ψ_2 the free Lagrangian is

$$-\frac{1}{2}\left(\bar{\psi}_1\partial\!\!\!/\psi_1+\bar{\psi}_2\partial\!\!\!/\psi_2-m_1\bar{\psi}_1\psi_1+m_2\bar{\psi}_2\psi_2\right) . \tag{2.12}$$

Here

$$m_2=\frac{M}{2}+\sqrt{\frac{M^2}{4}+m^2}\simeq M\,,\qquad m_1=-\frac{M}{2}+\sqrt{\frac{M^2}{4}+m^2}\simeq\frac{m^2}{M}\,, \tag{2.13}$$

where the approximate expressions are valid for $M\gg m$ (a regime which is physically interesting). Finally, a rotation of $\psi_1\to i\gamma^5\psi_1$ flips the sign of m_1 in the Lagrangian, so both masses are positive.

The lepton-boson interaction terms are obtained by writing $\nu=i\gamma^5\cos\theta_\nu\psi_1-\sin\theta_\nu\psi_2$ and $\bar{\nu}=i\cos\theta_\nu\bar{\psi}_1\gamma^5-\sin\theta_\nu\bar{\psi}_2$. The Higgs interactions are

$$-\frac{1}{2}kH(\bar{\nu}N+\bar{N}\nu)=\frac{kH}{2}\left(\frac{\sin 2\theta_\nu}{2}(\bar{\psi}_1\psi_1+\bar{\psi}_2\psi_2)-\frac{\cos 2\theta_\nu}{2}(\bar{\psi}_2 i\gamma^5\psi_1+\text{h.c.})\right) . \tag{2.14}$$

In the large M limit, the dominant Higgs interaction is to couple ψ_1 to ψ_2; the Higgs coupling to ψ_1 is $\sim km/M$, that is, it scales as the mass of the particle "as usual".

The coupling to the A field is still zero, since neither ν nor N have any coupling. The coupling to Z^μ is from the $\bar{\nu}D\!\!\!\!/\,\nu$ term in the Lagrangian; the gauge coupling part gives (defining $e_Z=\sqrt{g_2^2+g_1^2}=e/\sin\theta_W\cos\theta_W$)

$$\begin{aligned}&-\frac{1}{2}\,\frac{ie_Z}{2}\,Z_\mu\bar{\nu}\gamma^\mu\gamma^5\nu\\ =\;&-\frac{1}{2}\,\frac{ie_Z}{2}\,Z_\mu\left(\cos^2\theta_\nu\bar{\psi}_1\gamma^\mu\gamma^5\psi_1+\sin^2\theta_\nu\bar{\psi}_2\gamma^\mu\gamma^5\psi_2+(-i\sin\theta_\nu\cos\theta_\nu\bar{\psi}_2\gamma^\mu\psi_1+\text{h.c.})\right)\end{aligned} \tag{2.15}$$

and the coupling to the W boson becomes, for instance,

$$\frac{ig_2}{\sqrt{2}}W_\mu^-\bar{e}\gamma^\mu P_L\nu=\frac{ig_2}{\sqrt{2}}W_\mu^-\left(i\cos\theta_\nu\bar{e}\gamma^\mu P_L\psi_1+\sin\theta_\nu\bar{e}\gamma^\mu P_L\psi_2\right) . \tag{2.16}$$

Note that in both cases, in the limit of large M, ψ_1 couples with full strength and ψ_2 approximately decouples. In this limit, at energies too low to produce the ψ_2 particle, the theory looks like the normal electroweak theory but with a tiny mass added for the ν field; a mass which is quadratic in the Higgs field. We will see another way of understanding this when we discuss high dimension operators and effective field theories.

2.4 Two Higgs doublet models

2.4.1 Lagrangian

Note that this second Higgs field has the same charge assignments as $\tilde{\phi}$. The electric charge assignments of the components are

$$\psi = \begin{bmatrix} \chi \\ \xi \end{bmatrix} \quad \text{have charges} \quad \begin{bmatrix} 0 \\ -1 \end{bmatrix}. \tag{2.17}$$

The full covariant derivative acting on ψ is

$$\begin{aligned} D_\mu \psi &= \left(\partial_\mu - iW^a_\mu \frac{g_2 \tau^a}{2} + iB_\mu \frac{g_1}{2}\right)\psi \\ &= \begin{bmatrix} \partial_\mu - ig_2 W^3_\mu/2 + iB_\mu g_1/2 & -ig_2/2(W^1_\mu - iW^2_\mu) \\ -ig_2/2(W^1_\mu + iW^2_\mu) & \partial_\mu + ig_2 W^3_\mu/2 + iB_\mu g_1/2 \end{bmatrix} \begin{bmatrix} \chi \\ \xi \end{bmatrix}. \end{aligned} \tag{2.18}$$

Now for the potential. Note that the combination $\psi^\dagger \phi$ has net hypercharge $+1$ and so is not $SU_L(2) \times U_Y(1)$ covariant. However the combination

$$\psi^T e \phi = (\psi^\dagger e \phi^*)^* = (\psi^\dagger \tilde{\phi})^*$$

is $SU_L(2) \times U_Y(1)$ invariant since $\tilde{\phi}$ and ψ have the same charges. It is also complex, which means many terms will be allowed complex coefficients.

Define, as in the problem statement, the quantities

$$a \equiv \phi^\dagger \phi\,, \quad b \equiv \psi^\dagger \psi\,, \quad c \equiv \phi^T e \psi = \tilde{\phi}^\dagger \psi\,. \tag{2.19}$$

The most general available quadratic potential is

$$m_1^2 a + m_2^2 b + m_3^2 (c + c^*) - i m_4^2 (c - c^*)\,, \tag{2.20}$$

which has 4 real parameters. We could rewrite the last two terms as $Mc + M^* c^*$ with M a complex parameter. (In fact it is possible to re-define fields in terms of a new linear combination of ψ and $\tilde{\phi}$ to eliminate the cross-term but we will not pursue this here.)

One might think that the combination $\psi^\dagger \phi \psi^T \phi^*$, which is $SU_L(2) \times U_Y(1)$ invariant, would be an extra possible term in the potential, besides those quadratic in a, b, and c. However, it is not independent,

$$\psi^\dagger \phi \psi^T \phi^* = ab - c^* c\,, \tag{2.21}$$

as can be shown by explicit calculation; so we don't have to consider it separately. Similarly, $\sum_a |\psi^\dagger \tau^a \tilde{\phi}|^2 = |\psi^\dagger \tilde{\phi}|^2 = c^* c$ is not independent. The most general potential for the two Higgs fields is

$$\begin{aligned}\lambda_1\, a^2 + \lambda_2\, b^2 + \lambda_3\, ab + \lambda_4\, c^* c + \lambda_5\, (c^2{+}(c^*)^2) - i\lambda_6\, (c^2{-}(c^*)^2) \\ +\lambda_7\, a(c{+}c^*) - i\lambda_8\, a(c{-}c^*) + \lambda_9\, b(c{+}c^*) - i\lambda_{10}\, b(c{-}c^*)\,. \end{aligned} \tag{2.22}$$

As written the λ_i are all real. If we wrote terms of form ac the coefficient would be complex and the addition of the complex conjugate would be required.

Note that many of these terms can be eliminated by suitable choices of additional symmetries. Not doing so, and writing down the most general possible Yukawa interactions, allows flavor changing neutral currents severely in violation with experiment unless the coefficients are very carefully tuned.

2.4.2 Masses

To see if the electromagnetic group is broken by the VEV's, we have to see how the VEV's rotate under an electromagnetic transformation. These transformation properties are given by the electromagnetic charges; the lower component of ϕ and the upper component of ψ are charge zero, so they are not rotated and the VEV's are unchanged. Therefore, electromagnetism will still be unbroken. Breaking electromagnetism would require that part of the VEV of χ lie in the lower entry. Whether this happens depends on the details of the effective potential, which means that it can be used to constrain the form of the effective potential (specifically, the terms involving c above).

Observe that, dropping all fluctuations from ψ and ϕ,

$$D_\mu \psi \to \frac{i}{2\sqrt{2}} \begin{bmatrix} g_2 W^3 - g_1 B & g_2(W^1 - iW^2) \\ g_2(W^1 + iW^2) & -g_2 W^3 - g_1 B \end{bmatrix} \begin{bmatrix} u + iw \\ 0 \end{bmatrix}, \tag{2.23}$$

where there should be a μ subscript on each W or B; or

$$\begin{aligned}(D_\mu\psi)^\dagger D^\mu \psi &= \frac{u^2+w^2}{8} \begin{bmatrix} g_2 W^3 - g_1 B & g_2(W^1 - iW^2) \end{bmatrix} \begin{bmatrix} g_2 W^3 - g_1 B \\ g_2(W^1 + iW^2) \end{bmatrix} \\ &= \frac{u^2+w^2}{8} \left(g_2^2 (W^1_\mu W^\mu_1 + W^2_\mu W^\mu_2) + (g_2^2 + g_1^2) Z_\mu Z^\mu + 0 A_\mu A^\mu \right). \end{aligned} \tag{2.24}$$

We see that the mixing angle and mass relation will be unchanged from the standard model, $m_W = m_Z / \cos\theta_W$ with $\tan\theta_W = g_1/g_2$. The mass of the W boson is no longer $m_W^2 = g_2^2 v^2/4$, but is $m_W^2 = g_2^2(v^2 + u^2 + w^2)/4$; the Z boson mass depends on the same combination of VEV's.

2.4.3 Yukawa couplings

The field ψ is in general allowed Yukawa coupling matrices which are identical in form to those allowed for Higgs fields with the substitution $\phi \to \tilde{\psi}$ and $\tilde{\phi} \to \psi$. That is, we can write down

$$p_{mn} \bar{L}_m P_R E_n \tilde{\psi} + q_{mn} \bar{Q}_m P_R D_n \tilde{\psi} + r_{mn} \bar{Q}_m P_R U_n \psi + \text{h.c.}\,, \tag{2.25}$$

with p, q, r arbitrary 3 by 3 matrices. There is no *a priori* reason why these matrices should be in any way related to the matrices f, g, h of the ϕ Higgs field. In general this allows flavor changing neutral currents which are grossly in conflict with observations. However, under the global symmetry proposed, $\tilde{\phi} \to e^{i\theta}\tilde{\phi}$ and $\tilde{\psi} \to e^{i\theta}\tilde{\psi}$. Therefore, the Yukawa interactions involving the tilde quantities are forbidden, since they get rotated by a phase of 2θ under the transformation. Only f_{mn}, h_{mn}, and r_{mn} are allowed. Also note that only the scalar interaction terms m_1^2, m_2^2, and λ_1 through λ_4 are consistent with this symmetry, greatly simplifying the analysis of the scalar potential.

These restrictions turn out to be enough to eliminate flavor changing neutral currents. Note that the relation between a Yukawa coupling and a fermionic mass only involves the VEV of the particular participating scalar, so the relation between Yukawa couplings and couplings to the Higgs particle is no longer universal. (Also there are now 5 Higgs particles, three neutral and two charged.)

The model with this symmetry is called the 2 Higgs doublet model and is a commonly proposed alternative to the Standard Model Higgs sector. The two Higgs fields are called (ϕ) the down-type Higgs and (ψ) the up-type Higgs.

2.5 Adjoint Higgs fields

2.5.1 Electromagnetism

The t_i are the $J_{x,y,z}$ operators for spin 1 particles in the z spin state basis, so they surely satisfy the Lie algebra of the group SU(2). Explicit verification is straightforward.

The electric charges can be read off from the action of T_3 on the field: the charges are

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} = \begin{bmatrix} +1 \\ 0 \\ -1 \end{bmatrix}. \tag{2.26}$$

Because the field which takes on a VEV has electric charge zero, the VEV respects electromagnetic symmetry, which remains unbroken. The relation $\tan\theta_W = g_1/g_2$ will still hold.

2.5.2 Mass spectrum

The new field will change the W mass without changing the Z mass:

$$D_\mu \psi_{\min} = \frac{ig_2(u+iw)}{2} \begin{bmatrix} W_\mu^1 - iW_\mu^2 \\ 0 \\ W_\mu^1 + iW_\mu^2 \end{bmatrix} , \tag{2.27}$$

and so

$$(D^\mu \psi_{\min})^\dagger D_\mu \psi_{\min} = \frac{g_2^2(u^2+w^2)}{2} (W_1^\mu W_\mu^1 + W_2^\mu W_\mu^2) \,. \tag{2.28}$$

Note that no new mass term for either W^3 or B appears; so the Z, A mixing is unchanged from its standard model form, and the Z mass is the same as in the Standard Model, $m_Z^2 = (g_2^2 + g_1^2)v^2/4$. However, the W mass is

$$m_W^2 = g_2^2 \frac{v^2}{4} + g_2^2(u^2 + w^2) \,. \tag{2.29}$$

This violates the relation between W and Z masses observed in the Standard Model. The experimental value of the W mass is known to 0.05% accuracy and is slightly higher than the Standard Model prediction. Requiring that it be higher by at most 0.1% places a limit, $(u^2 + w^2) < v^2/4000 = (4\mathrm{GeV})^2$, which is quite severe. The scalar potential generically contains a $\psi^a \phi^\dagger \tau^a \phi$ term; when the Higgs field takes on a VEV, this induces a linear term in the ψ field potential, which will force it to take on a VEV as well. This makes it difficult to include a ψ field in a hypothetical extension of the standard model unless its mass is very large, to keep the ψ field VEV small.

2.6 Gauged $B - L$ coupling

2.6.1 Right handed neutrino

First note that, as the Higgs boson is neutral under F, all fermions participating in a Yukawa interaction have the same charge–or rather, $P_R E$ has the charge of $P_L L$ and $P_R U$ and $P_R D$ have the charge of $P_L Q$, to ensure the Yukawa terms are neutral under F. Therefore, since $P_L L$ has charge -1, so must $P_R N$. The mass term $M\bar{N}N$ couples right to left and is only permitted if N is neutral under everything; so this term is forbidden by the new gauge coupling.

2.6.2 Charge assignments

Gauge invariance of the Yukawa interactions requires that, writing the q_i as the charges of the left handed components,

$$q_L = -q_E = -q_N \,, \qquad q_Q = -q_U = -q_D \,. \tag{2.30}$$

Therefore in the following we will only keep track of q_L and q_Q.

The first anomaly cancellation condition to try is $(2, 2, 1')$. This gets contributions only from L and Q, and demands $q_Q = -q_L/3$. (we will not give the complete details since they are so similar to the first problem.) This fixes the charges of all particles already. Once again, the $(3, 3, 1')$ anomaly condition doesn't give a new constraint as it automatically vanishes, because $q_Q = -q_D = -q_U$.

The $(1', 1', 1')$ anomaly cancellation condition gives

$$2q_L^3 + q_E^3(+q_N^3) + 6q_Q^3 + 3q_D^3 + 3q_U^3 = 0\,. \tag{2.31}$$

Here we put the q_N term in parenthesis because we have not decided whether we want an N particle or not. Putting in that $-q_D = -q_U = q_Q$, and that $-q_E = q_L$, the quark terms cancel, leaving

$$q_L^3 = -(q_N^3)\,. \tag{2.32}$$

We see that if there is no N particle, the anomaly condition fails unless $q_L = 0$. (In this case all the charges are zero, which is boring. We want $q_L = -1$ as in the problem statement.) We need the N particle.

Once we have it, the $(1', 1', 1')$ anomaly condition is satisfied. The charges of the particles are

$$q_L = -q_N = -q_E = -1\,, \qquad q_Q = -q_D = -q_U = \frac{1}{3}\,. \tag{2.33}$$

The $(1, 1, 1')$ anomaly condition is

$$-2\left(\frac{-1}{2}\right)^2 + (-1)^2 + (0)^2 + \frac{1}{3}6\left(\frac{1}{6}\right)^2 - \frac{1}{3}3\left(\frac{-2}{3}\right)^2 - \frac{1}{3}3\left(\frac{1}{3}\right)^2 = 0\,, \tag{2.34}$$

which is identically true. Similarly the $(1, 1', 1')$ anomaly cancellation condition is

$$2(-1)^2\frac{-1}{2} - (1)^2(-1) - (1)^2 0 + 6\left(\frac{1}{3}\right)^2\frac{1}{6} - 3\left(\frac{1}{3}\right)^2\frac{2}{3} - 3\left(\frac{1}{3}\right)^2\frac{-1}{3} = 0\,, \tag{2.35}$$

which is also true. Therefore all anomalies cancel and this theory is allowed. Note that the charge assignments are just -1 for leptons and $+1/3$ for quarks, or $+1$ for baryons; the $U(1)'$ field couples to $B - L$, which is not anomalous.

No Yukawa terms are allowed to the χ field, since no $SU_c(3) \times SU_L(2) \times U_Y(1)$ invariant pairings of Standard Model fermions are possible and the χ field is a singlet under $SU_c(3) \times SU_L(2) \times U_Y(1)$. The absence of such terms is actually the same as the absence of mass terms in the standard model.

The baryon and lepton numbers remain conserved, though the Yukawa couplings now possible to the N bosons violate the separate lepton numbers. As before, B and L have $SU_L(2) \times U_Y(1)$ anomalies, but $B - L$ is non-anomalous.

2.6.3 Coulomb-like interactions

A massless F boson does not acquire a mass or mix with the A or Z due to the Higgs mechanism, since the Higgs boson is neutral under $U(1)'$. Therefore, for $\mu < 0$, the F is exactly massless. The leptons have charge -1, and the baryons charge +1, under this field. In particular the neutron does have a charge. Its interaction with the electron is attractive, since they have opposite charge (like the proton and electron under ordinary electromagnetism).

The most obvious constraint on g_4 is from spectroscopy; for instance, the deuteron's energy levels would be shifted with respect to hydrogen's by order g_4^2/e^2. Surely such a shift at the 10^{-6} level would have been observed, placing a conservative limit of $g_4^2 < 10^{-7}$.

Much stronger constraints can be achieved by thinking about the physics of the Sun. If $g_4^2 \ll e^2$, the Sun actually emits *more* energy into F photons than into A photons, in gross contradiction with solar modeling and the age of the Sun. This is because a more weakly interacting particle is emitted from deeper within the Sun, where it is hotter. Solar modeling only permits a small fraction of the Sun's energy to be lost to the F photon, which requires the entire Sun to be approximately transparent to F particles. This requires roughly $g_4^2 < 10^{-20}e^2$ (we will not provide details). This is a pretty strong constraint.

It is possible, however, to come up with an even stronger constraint. Consider the Earth. It is made up largely of neutron rich elements (Si and O have as many neutrons as protons; Fe has more neutrons). If the g_4 charge is larger than gravitational strength, then the heavy isotopes on the surface would be pushed away unless the Earth holds a balancing number of neutrinos (which would be attracted to the Earth by their $B-L$ interactions with the neutrons–the $B-L$ charge of electrons and protons is exactly zero because ordinary electromagnetism ensures that the Earth is charge neutral to high precision). Now the density of neutrons in the Earth is of order $10^{30}/m^3$. Since neutrinos are massless, to hold them at such a density would require a Fermi momentum of about $10^{10}/m \sim$ 1KeV. The kinetic energy of the typical neutrino would then be order 1KeV. To keep such energetic neutrinos trapped to the Earth requires a $B-L$ trapping potential of 1KeV; but this is much more than the gravitational or chemical binding energy of neutron rich atoms on the surface or in the atmosphere of the Earth. Therefore the atmosphere and surface of the Earth would be stripped off (in fact it would be energetically unfavorable for the Earth to form–it would explode).

From this we conclude that the F mediated interaction between neutrons must be weaker than gravitational,

$$\frac{g_4^2}{4\pi} < G_N m_n^2 \simeq 10^{-38}\,, \tag{2.36}$$

so $g_4^2 < 10^{-37}$, a rather severe limit. However the actual limit is even tighter. For such a small g_4^2, neutrinos are not bound to the Earth at all, and the planet has a $B-L$ charge equal to the number of neutrons it contains. Then the $B-L$ interaction causes a species dependent shift in the acceleration towards the Earth's center. Extremely precise searches for such isotopic shifts have been performed, and the limits are at the 10^{-11} level. Therefore the actual limit on g_4^2 is 11 orders of magnitude better,

$$g_4^2 < 10^{-48}\,. \tag{2.37}$$

Probably even better limits are available by looking for corrections to Newtonian behavior in the solar system, owing to the different neutron to proton ratios of, say, the Earth and the Moon. We believe such constraints replace 10^{-11} above with 10^{-14}.

The moral is: constraints on massless or very light new particles are often *extremely* strong.

2.6.4 Mass spectrum

The gauge boson masses arise from the Higgs and χ kinetic terms. Fortunately only the F couples to χ and it does not couple to ϕ, so there are no mixing terms. The F boson mass is $m_F^2 = g_4^2\mu^2$ (see last problem for how this term arises). The other masses are unaffected. There is no mixing between Z and F bosons.

2.7 Colored scalar fields

To appear

2.8 Adjoint representation fermions

To appear

Chapter 3

Cross sections and lifetimes

No problems were presented in this chapter so no solutions are provided either.

Chapter 4

Elementary boson decays

4.1 W width at finite fermion mass

To appear

4.2 Decay of the top quark

4.2.1 Interaction Hamiltonian

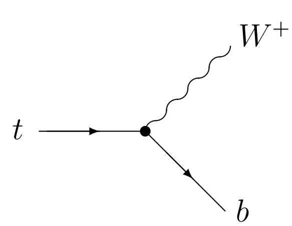
The top quark decays into a quark (almost always a b quark) and a W boson. It takes one insertion of the interaction Hamiltonian, namely the term in the interaction Hamiltonian,

$$H_I \supset \frac{g_2}{2\sqrt{2}} V^\dagger_{mn} \bar{d}_m W^-_\mu \gamma^\mu (1+\gamma^5) u_n \,. \tag{4.1}$$

The relevant term is when n is the 3 (top quark) index, that is, the terms involving $V^\dagger_{mt}$. Of these we see that the b quark term is totally dominant, the width to go to some other quark will be suppressed by $(V_{ts}/V_{tb})^2 < 0.005$, so we will take $V_{tb} = 1$ for current purposes.

4.2.2 Matrix element

The relevant Feynman diagram is



and the matrix element is therefore

$$e_W \bar{u}(p_b, \sigma_b) \not{\epsilon} (1+\gamma^5) u(p_t, \sigma_t) \,. \tag{4.2}$$

4.2.3 Spin summed $\mathcal{M}^2$

Squaring and summing (averaging) over final (initial) spins,

$$\frac{e_W^2}{2} \sum_{\sigma_b, \sigma_t, \lambda} \text{tr}\; u\bar{u}(p_b, \sigma_b) \not{\epsilon} (1+\gamma^5) u\bar{u}(p_t, \sigma_t) \not{\epsilon}^* (1+\gamma^5) \tag{4.3}$$

gives

$$\frac{e_W^2}{2} \left(\eta^{\mu\nu} + \frac{p_W^\mu p_W^\nu}{M_W^2} \right) \text{tr}\,(-i\not{p}_b + m_b)\gamma_\mu (1+\gamma^5)(-i\not{p}_t + m_t)\gamma_\nu (1+\gamma^5) \,. \tag{4.4}$$

We should drop the small m_b, which means that m_t will also not contribute, since that would involve an odd number of gamma matrices.

4.2.4 Reducing $\mathcal{M}^2$

Now we do the trace. The $(1+\gamma^5)$ can be combined into a $2(1+\gamma^5)$, and the γ^5 does not contribute as only terms symmetric in $\mu \leftrightarrow \nu$ will contribute. The trace gives,

$$\begin{aligned} |\mathcal{M}|^2 &= 4e_W^2 \left(\eta_{\mu\nu} + \frac{p_{W\mu} p_{W\nu}}{M_W^2} \right) \left(p_b^\mu p_t^\nu + p_b^\nu p_t^\mu - p_b \cdot p_t \eta^{\mu\nu} \right) \,, \\ &= 4e_W^2 \left(2p_b \cdot p_t - 2\frac{p_b \cdot p_W p_t \cdot p_W}{p_W^2} - 3p_b \cdot p_t \right) \\ &= \frac{g_2^2}{2} \left(-p_b \cdot p_t + 2\frac{p_b \cdot p_W p_t \cdot p_W}{M_W^2} \right) \,. \end{aligned} \tag{4.5}$$

Both terms are positive. Now work in the center of mass frame, and use that $\mathbf{p}_W = -\mathbf{p}_b$, and $|\mathbf{p}_b| = p_b^0 \equiv p_b$; the expression becomes

$$|\mathcal{M}|^2 = \frac{g_2^2}{2} \left(p_b m_t + 2\frac{m_t E_W (p_b E_W + p_b^2)}{M_W^2} \right) \,. \tag{4.6}$$

4.2.5 Final integration

The width is

$$\Gamma_t = \int \frac{d^3 p_W d^3 p_b}{(2\pi)^6 8 p_t^0 p_W^0 p_b^0} (2\pi)^4 \delta^4 (p_t - p_W - p_b) |\mathcal{M}|^2 \,. \tag{4.7}$$

Energy conservation will give

$$\begin{aligned} E_W + E_b &= m_t \\ E_W^2 &= (m_t - p_b)^2 \\ p_b^2 + M_W^2 &= m_t^2 - 2m_t p_b + p_b^2 \\ E_b = p_b &= \frac{m_t^2 - M_W^2}{2m_t}\,, \quad \text{and} \quad E_W = \frac{m_t^2 + M_W^2}{2m_t}\,. \end{aligned} \tag{4.8}$$

The phase space integral gives

$$\int \frac{d^3p_W d^3p_b}{(2\pi)^6 8 p_t^0 p_W^0 p_b^0}(2\pi)^4\delta^4(p_t - p_W - p_b)|\mathcal{M}|^2 = \frac{1}{8\pi}\int \frac{p_b^2 dp_b}{p_b E_W m_t}\delta(p_b + E_W - m_t)|\mathcal{M}|^2\,, \tag{4.9}$$

with $E_W \equiv \sqrt{p_b^2 + M_W^2}$. We need to know the p_b dependence of the argument of the delta function;

$$\frac{d}{dp_b}(p_b + E_W - m_t) = 1 + \frac{p_b}{E_W} = \frac{m_t}{E_W} \quad \Rightarrow \quad \int \frac{p_b dp_b}{E_W m_t}\delta(p_b + E_W - m_t)|\mathcal{M}|^2 = \frac{p_b}{m_t^2}|\mathcal{M}|^2\,. \tag{4.10}$$

Therefore the result of the p_b integration is

$$\Gamma = \frac{1}{8\pi}\frac{p_b}{m_t^2}|\mathcal{M}|^2 = \frac{g_2^2}{16\pi}\frac{p_b^2}{m_t}\left(1 + 2\frac{m_t E_W}{M_W^2}\right) = \frac{\alpha_2(m_t^2 - M_W^2)^2(2M_W^2 + m_t^2)}{16 m_t^3 M_W^2}\,. \tag{4.11}$$

Inserting values gives

$$\Gamma = 1.5 \text{ GeV} \tag{4.12}$$

which converts into a lifetime of

$$\tau \simeq 4.4 \times 10^{-25} \text{ sec}\,. \tag{4.13}$$

Even at quite relativistic speeds, since $c = 3 \times 10^8$m/s, a top quark can propagate less than 10^{-15} meters, which is obviously not detectable (about 100 microns is the actual limit).

4.3 Gamma matrix identities

First,

$$\not{k}\not{k} = k_\mu k_\nu \gamma^\mu \gamma^\nu\,. \tag{4.14}$$

But $k_\mu k_\nu$ is obviously symmetric, so this can be rewritten as

$$\not{k}\not{k} = k_\mu k_\nu \frac{1}{2}\Big(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\Big) = k_\mu k_\nu \frac{1}{2} 2\eta^{\mu\nu} = k^2\,. \tag{4.15}$$

Second,

$$
\begin{aligned}
\not{k}\not{p}\not{k} &= \left(\not{k}\not{p}+\not{p}\not{k}\right)\not{k} - \not{p}\not{k}\not{k}\,, \\
&= 2k_\mu p_\nu\left(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu\right)\not{k} - \not{p}k^2\,, \\
&= 2k_\mu p_\nu \eta^{\mu\nu}\not{k} - k^2\not{p}\,, \\
&= 2p\cdot k\,\not{k} - k^2\,\not{p}\,,
\end{aligned} \tag{4.16}
$$

as desired.

Third,

$$
\gamma^\mu\gamma_\mu \equiv \gamma^\mu\gamma^\nu\eta_{\mu\nu}\,, \tag{4.17}
$$

but symmetry of $\eta_{\mu\nu}$ allows this to be rewritten as

$$
= \frac{1}{2}\left(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu\right)\eta_{\mu\nu} = \frac{1}{2}2\eta^{\mu\nu}\eta_{\mu\nu} = \eta^\mu{}_\mu = 4\,. \tag{4.18}
$$

An easier way to see this is to note that the square of each gamma matrix is the identity, and there are 4 gamma matrices.

Fourth,

$$
\begin{aligned}
\gamma^\mu\not{k}\gamma_\mu &= k_\nu\gamma^\mu\gamma^\nu\gamma_\mu \\
&= k_\nu\left(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu\right)\gamma_\mu - k_\nu\gamma^\nu\gamma^\mu\gamma_\mu \\
&= k_\nu 2\eta^{\mu\nu}\gamma_\mu - k_\nu\gamma^\nu 4 \\
&= 2\not{k} - 4\not{k} = -2\not{k}\,.
\end{aligned} \tag{4.19}
$$

We can do this faster by noting that

$$
\gamma^\mu\not{k} = 2k^\mu - \not{k}\gamma^\mu\,, \tag{4.20}
$$

essentially the identity we were using above; then

$$
\gamma^\mu\not{k}\gamma_\mu = 2k^\mu\gamma_\mu - \not{k}\gamma^\mu\gamma_\mu = 2\not{k} - 4\not{k} = -2\not{k}\,. \tag{4.21}
$$

Fifth,

$$
\begin{aligned}
\gamma^\mu\not{p}\not{k}\gamma_\mu &= 2p^\mu\not{k}\gamma_\mu - \not{p}\gamma^\mu\not{k}\gamma_\mu\,, \\
&= 2p^\mu\not{k}\gamma_\mu + 2\not{p}\not{k}\,, \\
&= 2\not{k}\not{p} + 2\not{p}\not{k}\,, \\
&= 2k_\mu p_\nu\left(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu\right), \\
&= 2k_\mu p_\nu 2\eta^{\mu\nu} = 4k\cdot p\,.
\end{aligned} \tag{4.22}
$$

Sixth,

$$
\begin{aligned}
\gamma^\mu \not{p}\not{k}\not{q}\gamma_\mu &= 2p^\mu \not{k}\not{q}\gamma_\mu - \not{p}\gamma^\mu \not{k}\not{q}\gamma_\mu \,, \\
&= 2\not{k}\not{q}\not{p} - 4\not{p}\; k\cdot q \,, \\
&= 2\not{k}\not{q}\not{p} - 4k\cdot q\; \not{p} \,, \\
&= 2\not{k}\not{q}\not{p} - 2\Big(\not{q}\not{k} + \not{k}\not{q}\Big)\not{p} \,, \\
&= -2\not{q}\not{k}\not{p} \,.
\end{aligned} \tag{4.23}
$$

In passing from the first to second line we used the fifth identity.

Chapter 5

Leptonic weak interactions: decays

5.1 Z decay including a Higgs

To appear

5.2 Neutron lifetime

The key as stated above is to make the $m_p \simeq m_n \gg (m_p - m_n) \sim m_e$ approximation as early and often as possible. Define $m_n - m_p \equiv \Delta$.

$$\Gamma_n = \int \frac{d^3\mathbf{p}_p d^3\mathbf{p}_e d^3\mathbf{p}_\nu}{(2\pi)^9 16 m_n m_p p_e^0 p_\nu^0} (2\pi)^4 \delta^3(\mathbf{p}_e + \mathbf{p}_\nu + \mathbf{p}_p)\, \delta(\Delta - p_e^0 - p_\nu^0) |\mathcal{M}|^2 \,. \tag{5.1}$$

Here

$$\begin{aligned} |\mathcal{M}|^2 &= \frac{g_w^4 |V_{ud}|^2}{64 m_W^4} \frac{1}{2} P^{\mu\nu} Q_{\mu\nu} \,, \\ P^{\mu\nu} &= -\operatorname{tr}(-i\not{p}_p + m_p)\gamma^\mu (g_V + g_A\gamma^5)(-i\not{p}_n + m_n)\gamma^\nu (g_V + g_A \gamma^5) \,, \\ Q_{\mu\nu} &= -\operatorname{tr}(-i\not{p}_e - m_e)\gamma_\mu (1+\gamma^5)(-i\not{p}_\nu)\gamma^\nu (1+\gamma^5) \,. \end{aligned} \tag{5.2}$$

The factor of $\frac{1}{2}$ is the average over the neutron spin state. In this problem we should use $g_V = g_A = 1$, but we will leave them as free as long as possible because it turns out that $g_A \neq 1$ is the key difference between the "up and down quark" treatment provided here and the actual behavior of a neutron.

In evaluating $P^{\mu\nu}$ we should take advantage of the fact that the neutron and proton momenta are almost perfectly timelike; $|\vec{p}_n| \sim \Delta$ but $p_n^0 \simeq m_n$. Therefore, terms involving a single γ^5 are suppressed, since they contain $\epsilon_{\mu\nu\alpha\beta}p_n^\alpha p_p^\beta$; at least one of the p must be a spatial component. Separating the term with masses and the term with $\not{p}$'s,

$$\begin{aligned} P^{\mu\nu} &= -m_p m_n \,\mathrm{tr}\, \gamma^\mu(g_V{+}g_A)\gamma^\nu(g_V{+}g_A) \;+\mathrm{tr}\, \not{p}_p\gamma^\mu(g_V{+}g_A)\not{p}_n\gamma^\nu(g_V{+}g_A) \\ &\simeq 4m_p^2 g^{\mu\nu}(g_A^2{-}g_V^2) + 4(g_V^2{+}g_A^2)m_p^2\left[2g^{0\mu}g^{0\nu} + g^{\mu\nu}\right] . \end{aligned} \tag{5.3}$$

Similarly, $Q_{\mu\nu}$ reduces to

$$\begin{aligned} Q_{\mu\nu} &= (\text{antisymm}) + 8\left[(p_\nu)_\mu(p_e)_\nu + (p_\nu)_\nu(p_e)_\mu - g_{\mu\nu}p_\nu\cdot p_e\right] , \\ P^{\mu\nu}Q_{\mu\nu} &\simeq 2^6 m_p^2\left(g_V^2(2p_e^0p_\nu^0 + p_\nu\cdot p_e) + g_A^2(2p_e^0p_\nu^0 - p_\nu\cdot p_e)\right) , \\ |\mathcal{M}|^2 &\simeq \frac{g_w^4|V_{ud}|^2}{2m_W^4}m_p^2p_\nu^0p_e^0\left[g_V^2\left(2+\frac{p_\nu\cdot p_e}{p_\nu^0p_e^0}\right) + g_A^2\left(2-\frac{p_\nu\cdot p_e}{p_\nu^0p_e^0}\right)\right] . \end{aligned} \tag{5.4}$$

The $\simeq$ signs mean that $O(\Delta/m_p)$ corrections have been neglected. Note for what follows that $p_\nu\cdot p_e = -p_\nu^0p_e^0 + \mathbf{p}_e\cdot\mathbf{p}_\nu = p_\nu^0p_e^0(-1+\mathbf{v}_\nu\cdot\mathbf{v}_e)$.

The $m_pm_np_e^0p_\nu^0$ factor in $|\mathcal{M}|^2$ cancels the same factor in the denominator of the phase space measure, leaving

$$\begin{aligned} \Gamma_n = \frac{g_w^4|V_{ud}|^2}{32m_W^4}\int\frac{d^3\mathbf{p}_ed^3\mathbf{p}_\nu d^3\mathbf{p}_p}{(2\pi)^9}(2\pi)^4\delta^3(\mathbf{p}_e{+}\mathbf{p}_\nu{+}\mathbf{p}_p)\delta(\Delta{-}p_e^0{-}p_\nu^0)\times \\ \left[g_V^2(1{+}\mathbf{v}_\nu\cdot\mathbf{v}_e) + g_A^2(3{-}\mathbf{v}_\nu\cdot\mathbf{v}_e)\right] . \end{aligned} \tag{5.5}$$

The $\mathbf{p}_p$ proton momentum only appears in the integration and the momentum conserving delta function. The other two particles can "push off" against the proton arbitrarily hard without the proton "kick" costing any energy, because we have dropped the $\mathbf{p}_p^2/2m_p$ term which would appear in the energy. Therefore we can do the $\mathbf{p}_p$ integration which performs the momentum conserving delta function. The angular integrations of the other particles are totally independent. Each gives (4π), and $\mathbf{v}_\nu\cdot\mathbf{v}_e$ averages to zero over angles, leaving the integrals over the magnitudes of the momenta;

$$\Gamma_n = \frac{(g_V^2{+}3g_A^2)g_w^4(4\pi)^2|V_{ud}|^2}{m_W^4 32(2\pi)^5}\int_0 dp_\nu dp_e p_\nu^2p_e^2\delta(\Delta{-}p_\nu{-}\sqrt{p_e^2{+}m_e^2}) . \tag{5.6}$$

Use the delta function to perform the p_ν integration, and change variables from p_e to $E_e = \sqrt{p_e^2{+}m_e^2}$. Note that at this point p_ν and p_e are independent variables, so the integral with the δ function has unit Jacobian, and just gives a θ function which keeps $E_e \leq \Delta$;

$$\Gamma_n = \frac{g_w^4(g_V^2{+}3g_A^2)|V_{ud}|^2}{64\pi^3m_W^4}\int_{m_e}^{\Delta} dE E\sqrt{E^2{-}m_e^2}(\Delta-E)^2 . \tag{5.7}$$

The integral is

$$\int_{m_e}^{\Delta} dE E\sqrt{E^2-m_e^2}(\Delta - E)^2 = \frac{(2\Delta^4 - 9\Delta^2 m^2 - 8m^4)\sqrt{\Delta^2-m^2} + 15m^4\Delta \ln \frac{\Delta+\sqrt{\Delta^2-m^2}}{m}}{60} . \tag{5.8}$$

Taking $g_V^2 = g_A^2 = 1$, $|V_{ud}|^2 = 0.95$, replacing $g_w^4/16m_W^4$ with $2G_F^2$ and using the other numbers quoted we get a width of $\Gamma \simeq 4.744 \times 10^{-28}$ GeV. This corresponds to a lifetime of $1/\Gamma = 2.108 \times 10^{27}/$ GeV, which is 1387 seconds. This is longer than the actual life time, mostly because the W boson does *not* couple to a neutron–proton like it couples to a down quark–up quark. The amplitude of the axial coupling g_A turns out to be enhanced by approximately 1.267. As we see above, the width goes as $(g_V^2+3g_A^2)$. This enhances the cross section by $(1+3g_A^2)/4 \simeq 1.454$ which explains most of the discrepancy; the corrected half life is 950 seconds. There is still some discrepancy from the experimental value, due, for instance, to the fact that the electron wave function is enhanced by the Coulomb interaction with the proton; the matrix element should be multiplied by $|\psi^2(0)|/|\psi^2(\infty)|$, ψ the wave function of the electron in the presence of the proton's Coulomb field (which depends on energy and must be evaluated inside the integral over E_e). This is touched on in the discussion of the "distorted wave Born approximation" in the notes, but you weren't expected to do it.

5.3 Higgs decay to $Wf\bar{f}$

5.3.1 Matrix element

The widths with W^+ and with W^- in the final state are equal because of CP symmetry; the two decay final states transform into each other under CP symmetry, and the Higgs boson is CP even. Now CP is not an exact symmetry of the standard model, but it is only violated due to the phase in the CKM matrix, which does not enter into this calculation and in fact can only matter in quite high order phenomena.

The matrix element is

$$\mathcal{M} = \frac{2e_W M_W^2}{v}\,\epsilon^\alpha\,\eta_{\alpha\mu}\frac{\eta^{\mu\nu} + q^\mu q^\nu/M_W^2}{q^2 + M_W^2}\bar{u}(p_f,\sigma_f)\gamma_\nu(1+\gamma^5)v(p_{\bar{f}},\sigma_{\bar{f}})\,. \tag{5.9}$$

Here, the HWW vertex contributed the $2M_W^2\eta_{\alpha\mu}/v$, the fraction is the propagator, and the $Wf\bar{f}$ vertex contributed the $e_W\gamma_\nu(1+\gamma^5)$.

5.3.2 Squared matrix element

The spin summed $\mathcal{M}^2$ is,

$$\begin{aligned}\mathcal{M}^2 &= \frac{4M_W^4 e_W^2}{v^2}\,\frac{1}{(q^2+M_W^2)^2}\,\left(\eta_{\mu\alpha}+\frac{q_\mu q_\alpha}{M_W^2}\right)\sum_\epsilon \epsilon^\alpha \epsilon^{\beta *}\left(\eta_{\beta\nu}+\frac{q_\beta q_\nu}{M_W^2}\right)\times \\ &\quad \times \sum_{\sigma\sigma'}(-1)\bar{u}(p_f,\sigma)\gamma^\mu(1+\gamma^5)v(p_{\bar f},\sigma')\bar{v}(p_{\bar f},\sigma')\gamma^\nu(1+\gamma^5)u(p_f,\sigma)\,. \end{aligned} \tag{5.10}$$

The factor (-1) arose from moving β across γ^ν when conjugating the second gamma matrix expression.

Now, we use $\sum_\epsilon \epsilon^\alpha\epsilon^{\beta *} = \eta^{\alpha\beta} + p_W^\alpha p_W^\beta / M_W^2$, and the usual tricks with the spin sums on the gamma matrices, to get

$$\begin{aligned}\mathcal{M}^2 &= \frac{4M_W^4 e_W^2}{v^2}\,\frac{1}{(q^2+M_W^2)^2}\,\left(\eta_{\mu\alpha}+\frac{q_\mu q_\alpha}{M_W^2}\right)\left(\eta^{\alpha\beta}+\frac{p_W^\alpha p_W^\beta}{M_W^2}\right)\left(\eta_{\beta\nu}+\frac{q_\beta q_\nu}{M_W^2}\right)\times \\ &\quad \times(-1)\,\mathrm{tr}\,(-i\not{p}_f)\gamma^\mu(1+\gamma^5)(-i\not{p}_{\bar f})\gamma^\nu(1+\gamma^5)\,, \end{aligned} \tag{5.11}$$

where we have ignored m_f in the last expression. We can move the $(1+\gamma^5)$ across the two gamma matrices to get $(1+\gamma^5)^2 = 2(1+\gamma^5)$. The first line is symmetric in $\mu \leftrightarrow \nu$, so the γ^5 term, which will be antisymmetric, will not contribute. Therefore, we get

$$\begin{aligned}\mathcal{M}^2 &= \frac{4M_W^4 e_W^2}{v^2}\,\frac{1}{(q^2+M_W^2)^2}\,8\Big(p_{f\mu}p_{\bar f\nu}+p_{\bar f\mu}p_{f\nu}-\eta_{\mu\nu}p_f\cdot p_{\bar f}\Big)\times \\ &\quad \times\left(\eta_{\mu\alpha}+\frac{q_\mu q_\alpha}{M_W^2}\right)\left(\eta^{\alpha\beta}+\frac{p_W^\alpha p_W^\beta}{M_W^2}\right)\left(\eta_{\beta\nu}+\frac{q_\beta q_\nu}{M_W^2}\right)\,. \end{aligned} \tag{5.12}$$

We can contract the indices here, but it will only do work we have to undo in a minute, so we will not.

5.3.3 Integration on fermionic momenta

The width can be written as,

$$\Gamma_{H\to W^+ f\bar f} = \frac{1}{2p_H^0}\int \frac{d^3p_W d^3p_f d^3p_{\bar f}}{(2\pi)^9 8p_W^0 p_f^0 p_{\bar f}^0}(2\pi)^4\delta^4(p_H-p_W-p_f-p_{\bar f})|\mathcal{M}|^2\,. \tag{5.13}$$

For our purposes it is convenient to introduce an extra integration over q, which is forced to equal what it should by an extra delta function;

$$(2\pi)^4\delta^4(p_H-p_W-p_f-p_{\bar f}) = \int\frac{d^4q}{(2\pi)^4}\,(2\pi)^4\delta^4(p_H-p_W-q)\,(2\pi)^4\delta^4(q-p_f-p_{\bar f})\,. \tag{5.14}$$

Now we write the problem as

$$\begin{aligned}
\Gamma &= \frac{16M_W^4 e_W^2}{p_H^0 v^2}\int \frac{d^3p_W d^4q}{2p_W^0(2\pi)^7}\,(2\pi)^4\delta^4(p_H - p_W - q)\frac{1}{(q^2+M_W^2)^2}\times \\
&\quad \times\left(\eta_{\mu\alpha}+\frac{q_\mu q_\alpha}{M_W^2}\right)\left(\eta^{\alpha\beta}+\frac{p_W^\alpha p_W^\beta}{M_W^2}\right)\left(\eta_{\beta\nu}+\frac{q_\beta q_\nu}{M_W^2}\right)I_{\mu\nu}(q)\,, \\
I^{\mu\nu}(q) &\equiv \int \frac{d^3p_f d^3p_{\bar f}}{4p_f^0 p_{\bar f}^0(2\pi)^6}\,(2\pi)^4\delta^4(q-p_f-p_{\bar f})\,\Big(p_{f\mu}p_{\bar f\nu}+p_{\bar f\mu}p_{f\nu}-\eta_{\mu\nu}p_f\cdot p_{\bar f}\Big)\,. \qquad (5.15)
\end{aligned}$$

At this point we recognize this last quantity as almost one which was dealt with in the notes. We know that it must be of form,

$$I^{\mu\nu}(q) = Aq^\mu q^\nu + B\eta^{\mu\nu}\,, \qquad (5.16)$$

and the coefficients can be gotten by contraction with $q^\mu q^\nu$ and with $\eta^{\mu\nu}$. Note that $q\cdot p_f = p_f^2 + p_f\cdot p_{\bar f}$. But $p_f^2 = 0 = p_{\bar f}^2$ since both are lightlike (massless fermion approximation). Therefore,

$$q\cdot p_f = q\cdot p_{\bar f} = p_f\cdot p_{\bar f} = q^2/2\,. \qquad (5.17)$$

Using these, we can simplify the contractions of $I^{\mu\nu}$ against $q^\mu q^\nu$ and against $\eta^{\mu\nu}$:

$$\begin{aligned}
q_\mu q_\nu I^{\mu\nu} = Aq^4 + Bq^2 &= \int \frac{d^3p_f d^3p_{\bar f}}{4p_f^0 p_{\bar f}^0(2\pi)^6}\,(2\pi)^4\delta^4(q-p_f-p_{\bar f})\,\left(\frac{q^2}{2}\frac{q^2}{2}+\frac{q^2}{2}\frac{q^2}{2}-q^2\frac{q^2}{2}\right) \\
&= 0\,, \\
\eta_{\mu\nu}I^{\mu\nu} = Aq^2 + 4B &= \int \frac{d^3p_f d^3p_{\bar f}}{4p_f^0 p_{\bar f}^0(2\pi)^6}\,(2\pi)^4\delta^4(q-p_f-p_{\bar f})\,p_f\cdot p_{\bar f}(1+1-4) \\
&= -q^2\int \frac{d^3p_f d^3p_{\bar f}}{4p_f^0 p_{\bar f}^0(2\pi)^6}\,(2\pi)^4\delta^4(q-p_f-p_{\bar f})\,. \qquad (5.18)
\end{aligned}$$

This is the integral $I(q)$ from the notes, and it is $\theta(-q^2)/8\pi$. Therefore, we find that

$$I^{\mu\nu}(q) = \frac{-\theta(-q^2)}{24\pi}\Big(q^2\eta^{\mu\nu} - q^\mu q^\nu\Big)\,. \qquad (5.19)$$

5.3.4 Total width

Return to our expression for the integration over the fermionic momenta.

This now allows some simplification of Eq. (5.15). Note in particular that $\eta^{\mu\nu} - q^\mu q^\nu/q^2$ is a projection operator which “kills” q_μ;

$$q_\mu\Big(q^2\eta^{\mu\nu} - q^\mu q^\nu\Big) = q^2 q^\nu - q^2 q^\nu = 0\,. \qquad (5.20)$$

This helps a lot to simplify the previous expression. Namely, in $(\eta^{\mu\alpha} + q^\mu q^\alpha / M_W^2)$, the $q^\mu q^\alpha$ term gives zero, so those two objects just behave as $\eta^{\mu\alpha}$ and $\eta^{\beta\nu}$. The width becomes,

$$\begin{aligned} \Gamma &= \frac{2M_W^4 e_W^2}{3\pi v^2 p_H^0} \int \frac{d^3 p_W d^4 q}{2p_W^0 (2\pi)^7} (2\pi)^4 \delta^4(p_H - p_W - q) \frac{1}{(q^2 + M_W^2)^2} \times \\ &\quad \times \left(-3q^2 - \frac{p_W^2 q^2 - (p_W \cdot q)^2}{M_W^2} \right) \theta(-q^2) \,. \end{aligned} \tag{5.21}$$

Note, though, that $p_W^2 = -M_W^2$ since the W is a real particle. The quantity on the last line is $(-2q^2 + (p_W \cdot q)^2 / M_W^2)$, which is manifestly positive (as it should be).

The energy-momentum delta function neatly performs $d^4 q$. It enforces that $q = p_H - p_W$, which incidentally implies that $q^2 = p_H^2 + p_W^2 - 2p_H \cdot p_W$, which in the rest frame of the Higgs boson is

$$-q^2 = M_H^2 + M_W^2 - 2M_H p_W^0 \,. \tag{5.22}$$

Therefore, the step function, $\theta(-q^2)$, becomes the step function $\theta(M_H^2 + M_W^2 - 2M_H p_W^0)$. That is, it enforces the kinematic constraint,

$$p_W^0 \le \frac{M_H^2 + M_W^2}{2M_H} \qquad \text{or} \qquad |\mathbf{p}_W| \le \frac{M_H^2 - M_W^2}{2M_H} \,. \tag{5.23}$$

The angular integration is trivial, returning 4π. The remaining integral is,

$$\Gamma = \frac{2M_W^4 e_W^2}{3\pi v^2 M_H} \frac{1}{4\pi^2} \int_0^{\max} \frac{p_W^2 dp_W}{p_W^0} \frac{-2q^2 + (p_W \cdot q)^2 / M_W^2}{(q^2 + M_W^2)^2} \,. \tag{5.24}$$

Next,

$$p_W \cdot q = p_W \cdot (p_H - p_W) = -M_H p_W^0 + M_W^2 \,, \tag{5.25}$$

and

$$(M_W^2 + q^2)^2 = (2M_H p_W^0 - M_H^2)^2 = M_H^2 (2p_W^0 - M_H)^2 \,. \tag{5.26}$$

Also, we can rewrite

$$\int_0^{\frac{M_H^2 - M_W^2}{2M_H}} \frac{p_W^2 dp_W}{p_W^0} = \int_{M_W}^{\frac{M_H^2 + M_W^2}{2M_H}} dp_W^0 \sqrt{(p_W^0)^2 - M_W^2} \,. \tag{5.27}$$

For ease of notation we will now write $p_W^0 = E$. We should also multiply by a factor of 9 for the number of possibilities for the $f\bar{f}$ pair (3 leptons plus 3 colors each of 2 quarks, just as in the W boson width), and by a factor of 2 for the W^- final state possibility. Therefore the integration we want to perform is,

$$\frac{3e_W^2 M_W^4}{\pi^3 M_H \nu^2} \int_{M_W}^{\frac{M_H^2 + M_W^2}{2M_H}} dE \sqrt{E^2 - M_W^2} \frac{3M_W^2 + 2M_H^2 - 6EM_H + E^2 M_H^2 / M_W^2}{M_H^2 (2E - M_H)^2} \,. \tag{5.28}$$

We can make this a little bit neater. $(M_W^2/v^2) = g_2^2/4$, and $e_W^2 = g_2^2/8$. Also, it is convenient to define $x = M_H/M_W$. Then the above becomes

$$\frac{3M_H\alpha^2}{2\pi\sin^4\Theta_W}\int_1^{\frac{x^2+1}{2x}} dy \frac{3+2x^2+x^2y^2-6xy}{x^4(2y-x)^2}\sqrt{y^2-1}\,. \tag{5.29}$$

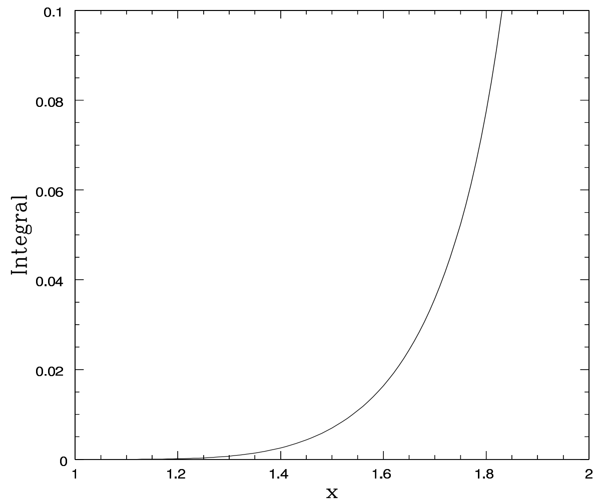
Doing this integral analytically is not simple. However, we can observe the following properties:

1. The integral vanishes for $x = 1$, or $M_H = M_W$, which is the point where there is no energy available for the $f\bar{f}$ pair.

2. As $x \to 1$, the integral vanishes as $(3/5)(x-1)^5$.

3. The integral goes to infinity as $x \to 2$. This is the point where the virtual W boson propagator is achieving enough energy to be on-shell, so its propagator can "blow up." Handling this apparent divergence will be the subject of a couple of lectures when we treat scattering via the Z boson resonance; basically it means that you should treat the problem as a decay to two real W bosons.

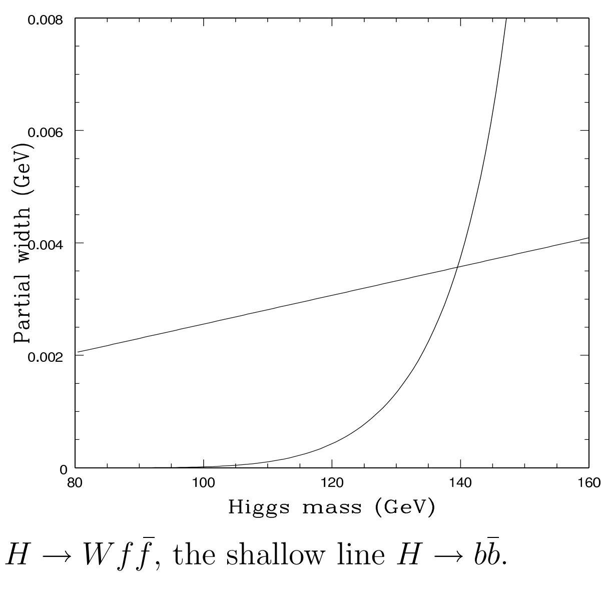
The numerical behavior of the integral is shown here:



Taking the values we are given, calling the integral $I(x)$, the two widths,

$$\begin{aligned} \Gamma_{H\to b\bar{b}} &= \frac{3m_b^2}{8\pi v^2} M_H \simeq 2.56 \times 10^{-5} M_H \qquad \text{and} \\ \Gamma_{H\to Wf\bar{f}} &= \frac{3\alpha^2}{2\pi \sin^4 \Theta_W} M_H I(x) \simeq 5.45 \times 10^{-4} M_H I(x)\,, \end{aligned} \tag{5.30}$$

equal when $I(x) = .047$, which is at $x = 1.74$ or $M_H = 1.74 M_W = 140$ GeV. Plotting it,



The steep line is $H \to Wf\bar{f}$, the shallow line $H \to b\bar{b}$.

5.4 The miracle of Lorentz invariance

To appear

Chapter 6

Leptonic weak interactions: collisions

6.1 Crossing symmetry

We have that the matrix element for Møller scattering, $e^-e^- \to e^-e^-$, is

$$|\overline{\mathcal{M}}|^2 = 2e^4 \left(\frac{s^2+u^2}{t^2} + \frac{s^2+t^2}{u^2} + \frac{2s^2}{tu} \right) . \tag{6.1}$$

Now write the incoming momenta for that process as p, p' and the outgoing momenta as k, k'. To get Bhabha scattering, one should exchange a final state e^- and an initial state e^-, say, by flipping $p' \to -k'$ and $k' \to -p'$. Under this transformation,

$$\begin{aligned} s = -2p\cdot p' \quad &\to \quad +2p\cdot k' = u\,, \\ t = +2p\cdot k \quad &\to \quad +2p\cdot k = t\,, \\ u = +2p\cdot k' \quad &\to \quad -2p\cdot p' = s\,, \end{aligned} \tag{6.2}$$

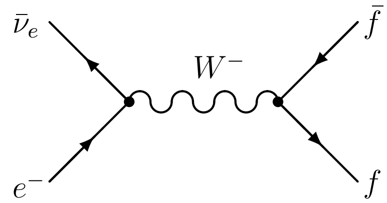
and there is a factor of $(-1)^2$ from moving two fermionic lines between initial and final states. Therefore, the Bhabha matrix element is,

$$|\overline{\mathcal{M}}|^2 = 2e^4 \left(\frac{s^2+u^2}{t^2} + \frac{u^2+t^2}{s^2} + \frac{2u^2}{ts} \right) = 2e^4 \left(\frac{s^2}{t^2} + \frac{t^2}{s^2} + \left[\frac{u}{t} + \frac{u}{s} \right]^2 \right) . \tag{6.3}$$

Going from $|\mathcal{M}|^2$ to $d\sigma/dudt$ introduces the additional factor of $1/(16\pi s^2)$ which takes this expression to the desired one (note that $e^4/16\pi = \pi\alpha^2$).

6.2 Electron-neutrino scattering

The single diagram is W boson exchange,



The matrix element is,

$$\mathcal{M} = e_W^2 U_{mn}^* \bar{v}(p')\gamma^\mu(1+\gamma^5)u(p)\frac{\eta_{\mu\nu} + \frac{(p+p')_\mu(p+p')_\nu}{M_W^2}}{(p+p')^2 + M_W^2}\bar{u}_n(k)\gamma^\nu(1+\gamma^5)v_m(k')\,. \tag{6.4}$$

We can drop the $(p+p')$ term in the propagator, in the limit of massless e^- and ν_e, because it gives a term proportional to

$$\bar{v}(p')(\not{p}' + \not{p})(1+\gamma^5)u(p) = \bar{v}(p')\not{p}(1+\gamma^5)u(p) = \bar{v}(p')(1-\gamma^5)\not{p}u(p) = 0\,, \tag{6.5}$$

where we used the Dirac equation twice.

Squaring and summing over external states, NOTING that there is no sum over the spin of the neutrino because there is only one available spin of neutrino, gives

$$|\bar{\mathcal{M}}|^2 = \frac{e_W^4 U_{mn}^* U_{mn}}{2(s-M_W^2)^2}\,\mathrm{tr}\,\not{p}'\gamma^\mu(1+\gamma^5)\not{p}\gamma^\nu(1+\gamma^5)\,\mathrm{tr}\,\not{k}\gamma_\mu(1+\gamma^5)\not{k}'\gamma_\nu(1+\gamma^5)\,. \tag{6.6}$$

The traces give,

$$\begin{aligned}\mathrm{tr}\,\not{p}'\gamma^\mu(1+\gamma^5)\not{p}\gamma^\nu(1+\gamma^5) &= 8\Big(p^\mu p'^\nu + p'^\mu p^\nu - \eta^{\mu\nu}p\cdot p' + i\epsilon^{\mu\nu\alpha\beta}p_\alpha p'_\beta\Big)\,,\\ \mathrm{tr}\,\not{k}\gamma_\mu(1+\gamma^5)\not{k}'\gamma_\nu(1+\gamma^5) &= 8\Big(k_\mu k'_\nu + k'_\mu k_\nu - \eta_{\mu\nu}k\cdot k' - i\epsilon_{\mu\nu\alpha\beta}k^\alpha k'^\beta\Big)\,.\end{aligned} \tag{6.7}$$

Contracting these gives,

$$64\Big(2p\cdot k p'\cdot k' + 2p\cdot k' p'\cdot k + (4-4)p\cdot p' k\cdot k' - 2(p\cdot k p'\cdot k' - p\cdot k' p'\cdot k)\Big) = 256 p\cdot k' p'\cdot k = 64u^2\,. \tag{6.8}$$

Next, write $e_W^4 = G_F^2 M_W^4/2$, and combine; the matrix element is,

$$|\bar{\mathcal{M}}|^2 = 16 G_F^2 |U_{mn}|^2 u^2 \frac{M_W^4}{(s-M_W^2)^2}\,. \tag{6.9}$$

Using the relation from the notes, that

$$\frac{1}{|v_1 - v_2| 2p^0 p'^0} = \frac{1}{2s} \tag{6.10}$$

and that the rest of the phase space is,

$$(2\pi)^4\delta^4(p+p'-k-k')\frac{d^3kd^3k'}{(2\pi)^6 2k^0 2k'^0} = \frac{-1}{8\pi s}\delta(s+t+u)dtdu\,, \tag{6.11}$$

gives

$$\frac{d\sigma}{dudt} = -\frac{G_F^2}{\pi}|U_{mn}|^2\frac{u^2}{s^2}\left(\frac{M_W^2}{s-M_W^2}\right)^2\delta(s+t+u)\,, \tag{6.12}$$

as desired.

Now consider the case $f = \mu$, $\bar{f} = \bar{\nu}_\mu$. This is related to the case where ν_μ is an in particle and ν_e is an out particle by exchanging $\bar{\nu}_e \to \bar{\nu}_\mu$ to $\nu_\mu \to \nu_e$, that is, switching both neutrinos between the initial and final states. This makes the exchange, $p' \leftrightarrow -k'$, which as we have seen makes the exchanges, $s \to u$, $u \to s$ in the matrix element (though not in the treatment of the final state phase space factors). This modifies the cross-section we just found to

$$\frac{d\sigma}{dudt} = \frac{G_F^2}{\pi}\left(\frac{M_W^2}{u-M_W^2}\right)^2\delta(s+t+u)\,. \tag{6.13}$$

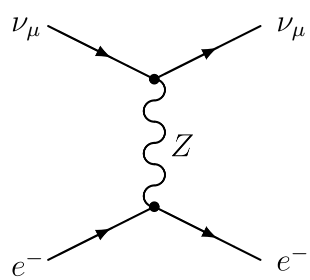
6.3 Supernova neutrinos

This problem solution is incomplete.

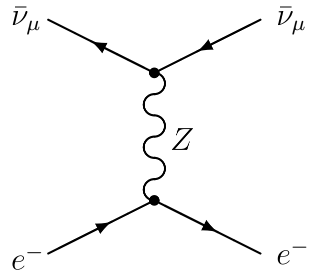
6.3.1 Diagrams

The first two have a single diagram with Z exchange; the other two have W exchange as well, in either the t or s channel.

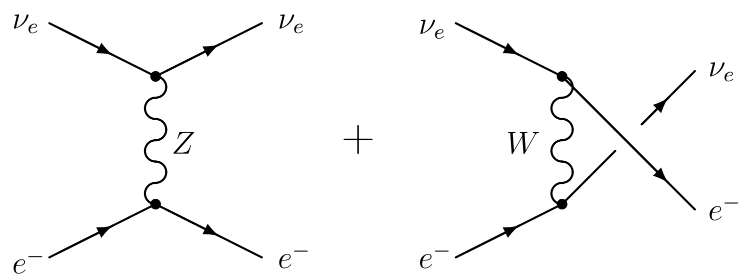
$\nu_\mu e^- \to \nu_\mu e^-$:



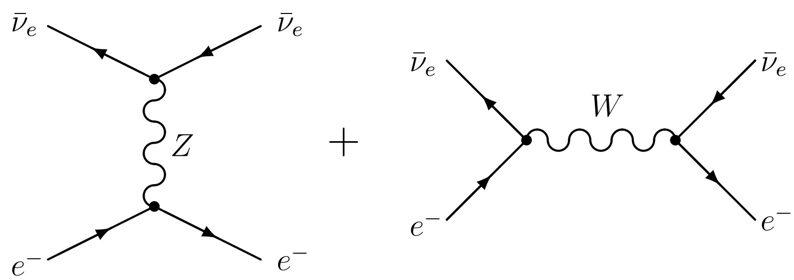
$\bar{\nu}_\mu e^- \to \bar{\nu}_\mu e^-$:



$\nu_e e^- \to \nu_e e^-$:



and $\bar{\nu}_e e^- \to \bar{\nu}_e e^-$:



We see already that the ν_e and $\bar{\nu}_e$ diagrams will be a potentially larger pain than the ν_μ and $\bar{\nu}_\mu$ diagrams.

6.3.2 Cherenkov radiation

In order to Čerenkov radiate, the velocity must exceed $1/n$, requiring

$$\frac{p}{E} > \frac{1}{n}\,; \quad n^2p^2 > E^2\,; \quad (n^2-1)E^2 > n^2m^2\,; \quad E > \frac{n}{\sqrt{n^2-1}}m = 1.51m \tag{6.14}$$

For an electron this requires 775 KeV; for a hydrogen nucleus it requires 1400 MeV, which is far more than is available.

Note however that the reaction $\bar{\nu}_e + p \to e^+ n$ is promising, as the e^+ can carry enough energy. In fact, the SNO detector looks for a similar reaction in heavy water (actually,

$\nu_e + n \to e^- + p$, which is possible because there are weakly bound neutrons available in heavy water).

6.3.3 Kinematics

The lab-frame, total momentum 4-vector is $(m + \omega, \omega, 0, 0)$. The velocity of the center of mass is $v = p_l/p_l^0 = \omega/(m + \omega)$. The gamma factor for this velocity, which is the same as the velocity of the boost between the center of mass and lab frames, is

$$\gamma = \frac{1}{\sqrt{1 - v^2}} = \frac{m + \omega}{\sqrt{2m\omega + m^2}}\,. \tag{6.15}$$

In the center of mass frame, the initial 4-momentum is $(p_c^0, -p_c, 0, 0)$, with $p_c^0 = \gamma m$ and $p_c = \gamma v m$. The final 4-momentum is $(p_c^0, p_c \cos\varphi, p_c \sin\varphi, 0)$ (not worrying about the irrelevant azimuthal angle). Boosting back, the lab frame momentum is

$$\begin{bmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_c^0 \\ p_c \cos\varphi \\ p_c \sin\varphi \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma(p_c^0 + v p_c \cos\varphi) \\ \gamma(v p_c^0 + p_c \cos\varphi) \\ p_c \sin\varphi \\ 0 \end{bmatrix}. \tag{6.16}$$

The tangent of the angle with respect to the neutrino's axis (which I have been writing at the top entry in the momentum vector) is the ratio of the second to first entries in this momentum vector:

$$\tan\theta = \frac{p_x(\text{lab})}{p_z(\text{lab})} = \frac{p_c \sin\varphi}{\gamma(p_c \cos\varphi + v p_c^0)}\,. \tag{6.17}$$

Using the relation between p^0 and p, found above, $p = vp^0$, and dividing by p, and substituting the determined value of γ, we find

$$\tan\theta = \frac{\sin\varphi}{\cos\varphi + 1}\,\frac{\sqrt{2m\omega + m^2}}{m + \omega}\,. \tag{6.18}$$

This relation is exact, we did not have to make any small m/ω expansion.

For the energy of the electron it is best to work entirely in the lab frame. Write the final electron 4-momentum in the lab frame as $(E, p_z, p_x, 0)$, and the final neutrino 4-momentum is, by 4-momentum conservation, $(m + \omega - E, \omega - p_z, -p_x, 0)$. These satisfy the mass shell conditions,

$$E^2 = m^2 + p_z^2 + p_x^2\,, \qquad (m + \omega - E)^2 = (\omega - p_z)^2 + p_x^2\,. \tag{6.19}$$

Taking the difference,

$$2E(m + \omega) = 2m(m + \omega) + 2\omega p_z\,. \tag{6.20}$$

Divide by $2(m+\omega)$, move m to the other side, write $p_z = p\cos\theta = \cos\theta\sqrt{E^2-m^2}$, and square both sides;

$$(E-m)^2 = \frac{\omega^2\cos^2\theta}{(m+\omega)^2}(E^2-m^2)\,. \tag{6.21}$$

Now cancel a factor of $(E-m)$, and regroup terms;

$$E\left(1-\frac{\omega^2\cos^2\theta}{(m+\omega)^2}\right) = m\left(1+\frac{\omega^2\cos^2\theta}{(m+\omega)^2}\right)\,, \qquad E = m\left(\frac{(m+\omega)^2+\omega^2\cos^2\theta}{(m+\omega)^2-\omega^2\cos^2\theta}\right)\,. \tag{6.22}$$

Note again that the result is exact, we have not needed $\omega \gg m$.

For $\theta = 0$, in the large ω limit, the ω^2 factors in the denominator cancel and this reduces to

$$E(\theta=0) = m\frac{m^2+2m\omega+2\omega^2}{m^2+2m\omega} \simeq \omega\,. \tag{6.23}$$

For $\theta \simeq \sqrt{m/\omega}$, then $\cos^2\theta \simeq 1-\theta^2 \simeq 1-m/\omega$; the numerator is unchanged at leading order, the denominator becomes approximately $m^2+2m\omega+m\omega$, so $E \simeq (2/3)\omega$. Therefore the range of energies is of order a factor of 2 (with the estimate made for θ, we would say a range from 1 to 2/3.)

6.3.4 Effective interaction

We have,

$$\begin{aligned}
\bar{\nu}_e\gamma^\mu(1+\gamma^5)e\bar{e}\gamma_\mu(1+\gamma^5)\nu_e &= \bar{\nu}_e\gamma^\mu(1+\gamma^5)e\bar{e}(1-\gamma^5)\gamma_\mu\nu_e \\
&= -\frac{1}{2}\left[\bar{e}\gamma^\nu(1+\gamma^5)e\right]\bar{\nu}_e\gamma^\mu(1+\gamma^5)\gamma_\nu\gamma_\mu\nu_e \\
&= -\frac{1}{2}\left[\bar{e}\gamma^\nu(1+\gamma^5)e\right]\bar{\nu}_e\gamma^\mu\gamma_\nu\gamma_\mu(1+\gamma^5)\nu_e \\
&= -\frac{1}{2}\left[\bar{e}\gamma^\nu(1+\gamma^5)e\right]\bar{\nu}_e(-2\gamma_\nu)(1+\gamma^5)\nu_e \\
&= \bar{e}\gamma^\nu(1+\gamma^5)e\;\bar{\nu}_e\gamma_\nu(1+\gamma^5)\nu_e\,,
\end{aligned} \tag{6.24}$$

as desired. In the first transformation we used that the quantity in square brackets is a scalar and may be moved to the outside. We also used $\gamma^\mu\gamma_\nu\gamma_\mu = -2\gamma_\nu$. Renaming the dummy index ν to μ gives the desired result.

The two terms present now look the same except for the coefficients on γ^5 and 1 in the electron part of the Z exchange parts. Therefore they can be added, defining h_V and h_A as defined in the problem.

To get the matrix element is now trivial; replace $\bar{\nu}$, ν, etc with the appropriate $\bar{u}$ and u's;

$$\mathcal{M} = \frac{iG_F}{\sqrt{2}} \bar{u}(q')\gamma^\mu(1+\gamma^5)u(q)\ \bar{u}(p')\gamma_\mu(h_V+h_A\gamma^5)u(p) \tag{6.25}$$

for ν scattering; for $\bar{\nu}$ scattering the first term is $\bar{v}(q)\gamma^\mu(1+\gamma^5)v(q')$ instead.

6.3.5 Cross-section

When we square and average over spins, we must remember there is only one factor of (1/2) in the spin averaging, since a neutrino only has one spin state. The matrix element squared for neutrino scattering is

$$\begin{aligned} \frac{1}{2}|\mathcal{M}|^2 &= \frac{G_F}{4} \operatorname{tr}(-i\not{q}')\gamma^\mu(1+\gamma^5)(-i\not{q})\gamma^\nu(1+\gamma^5) \\ &\qquad \operatorname{tr}(-i\not{p}'+m)\gamma_\mu(h_V+h_A\gamma^5)(i-\not{p}+m)\gamma_\nu(h_V+h_A\gamma^5)\,. \end{aligned} \tag{6.26}$$

The case of antineutrinos differs by $q \leftrightarrow q'$. Cancel off the $-i$'s, which puts $+i$ in front of each m. The first trace is

$$2\operatorname{tr}\not{q}'\gamma^\mu\not{q}\gamma^\nu(1+\gamma^5) = 8\left(q'^\mu q^\nu + q^\mu q'^\nu - g^{\mu\nu} q\cdot q' - i\epsilon^{\mu\nu\alpha\beta}q_\alpha q'_\beta\right)\,. \tag{6.27}$$

For the case of antineutrinos, the sign of the last term flips, since it is odd in $q \leftrightarrow q'$; the other term, which is even, stays the same.

The second trace is

$$-m^2\operatorname{tr}(h_V+h_A\gamma^5)\gamma_\mu(h_V+h_A)\gamma_\nu + \operatorname{tr}\not{p}'\gamma_\mu\not{p}\gamma_\nu(h_V^2+h_A^2+2h_Vh_A\gamma^5) \tag{6.28}$$

which is

$$4(h_A^2-h_V^2)m^2 g_{\mu\nu} + 4\left((h_V^2+h_A^2)\left[p'_\mu p_\nu + p_\mu p'_\nu - g_{\mu\nu}p\cdot p'\right] - 2h_Vh_A i\epsilon_{\mu\nu}{}^{\lambda\sigma}p_\lambda p'_\sigma\right)\,. \tag{6.29}$$

Contracting these two expressions, as usual the symmetric part of one expression vanishes against the antisymmetric part of the other; so there are two terms. The $g_{\mu\nu}$ type terms in the symmetric combination turn out to give $2+2-4=0$ times $q\cdot q' p\cdot p'$, since $g_{\mu\nu}g^{\mu\nu}=4$; there is only a contribution from the p's and q's contracting against each other. The two terms are

$$\begin{aligned} \frac{1}{2}\sum_\sigma |\mathcal{M}|^2 &= 16G_F^2\left[m^2(h_V^2-h_A^2)q\cdot q' + (h_V^2+h_A^2)(p\cdot q p'\cdot q' + p\cdot q' p'\cdot q)\right] \\ &\quad \pm 16G_F^2(2h_Vh_A)(p\cdot q p'\cdot q' - p\cdot q' p'\cdot q)\,, \end{aligned} \tag{6.30}$$

where the upper term comes from the symmetric parts and the lower term from the antisymmetric ones; the $\pm$ is for neutrinos and antineutrinos respectively. The sign of the last term is a bit tricky; there is a -1 from i^2 and a -2 from contracting the $\epsilon^{\mu\nu\alpha\beta}$ with $\epsilon_{\mu\nu\lambda\sigma}$.

Now we use $h_V^2+h_A^2 \pm 2h_V h_A = (h_V \pm h_A)^2$ to combine the two terms;

$$\frac{1}{2}\sum_\sigma |\mathcal{M}|^2 = 16G_F^2\left[m^2(h_V^2-h_A^2)q\cdot q' + (h_V\pm h_A)^2 p\cdot q p'\cdot q' + (h_V\mp h_A)^2 p\cdot q' p'\cdot q\right] . \quad (6.31)$$

This is the desired result; the power of 2 we would not tell you, is 16.

Now we need a center of mass frame differential cross-section in the small m limit. The small m limit means we drop the m^2 term above and take $p\cdot q = p'\cdot q' = s/2$ and $p\cdot q' = p'\cdot q = -u/2$ so the matrix element becomes

$$\frac{1}{2}\sum_\sigma |\mathcal{M}|^2 = 4G_F^2\left[(h_V \pm h_A)^2 s^2 + (h_V \mp h_A)^2 u^2\right] . \quad (6.32)$$

In the center of mass frame, in terms of $\omega_{\rm cm} = \sqrt{s}/2$, $s^2 = 16\omega_{\rm cm}^4$ and $u^2 = 4\omega_{\rm cm}^4(1-\cos\varphi)^2$. (You would expect u to involve $1+\cos\varphi$, but we defined the angle with respect to the initial neutrino, rather than electron, direction.) Therefore the cross-section is

$$\begin{aligned}\sigma &= \frac{1}{4v_{\rm rel}\omega_{\rm cm}^2}\int \frac{d3p'd^3q'}{(2\pi)^6\, 2(q')^0 2(p')^0}(2\pi)^4\delta^3(p'+q')\delta(2\omega_{\rm cm}-(p')^0-(q')^0)\times \\ &\quad \times 64G_F^2\omega_{\rm cm}^4\left[(h_V\pm h_A)^2 + \frac{1}{4}(h_V\mp h_A)^2(1-\cos\varphi)^2\right] . \end{aligned} \quad (6.33)$$

The relative velocity is 2. Think about it! Each particle has velocity 1, and they are moving at each other, that is, in opposite directions. The δ^3 performs the d^3q' integration, forcing $(q')^0 = (p')^0$. The d^3p integration can be written as $d\phi\, d\cos\varphi\, p'^2 dp'$, and the ϕ integration is trivial and gives 2π. So far, then, we have

$$\begin{aligned}\sigma &= \frac{1}{8\omega_{\rm cm}^2(2\pi)}\int_{-1}^{1} d\cos\varphi \int_0^\infty p'^2 dp' \frac{1}{4p'^2}\delta(2\omega_{\rm cm}-2p')\times \\ &\quad \times 64G_F^2\omega_{\rm cm}^4\left[(h_V\pm h_A)^2 + \frac{1}{4}(h_V\mp h_A)^2(1-\cos\varphi)^2\right] . \end{aligned} \quad (6.34)$$

The delta function gives $\int dp'\delta(2\omega_{\rm cm}-2p') = (1/2)$, forcing $p' = \omega_{\rm cm}$ (which we already used to simplify the matrix element). Combining the remaining terms, we indeed get

$$\frac{d\sigma}{d\cos\varphi} = \frac{G_F^2\omega_{\rm cm}^2}{2\pi}\left[(h_V\pm h_A)^2 + \frac{1}{4}(h_V\mp h_A)^2(1-\cos\varphi)^2\right] . \quad (6.35)$$

Each term is positive definite, and the second term carries all the $\cos\varphi$ dependence. It is maximum when $\cos\varphi = -1$, which is at $\varphi = \pi$.

Now perform the $\cos\varphi$ integration; $\int 1 = 2$, $\int \cos^2\varphi = 2/3$, and $\int \cos\varphi = 0$. Therefore the total cross-section is

$$\sigma = \frac{G_F^2 \omega_{\rm cm}^2}{\pi}\left[(h_V \pm h_A)^2 + \frac{1}{3}(h_V \mp h_A)^2 \right] . \tag{6.36}$$

It remains to re-express $\omega_{\rm cm}^2$ in terms of lab frame quantities; $\omega_{\rm cm}^2 = s/4$, and as we saw earlier, $s = 2m\omega + m^2 \simeq 2m\omega$. Therefore $\omega_{\rm cm}^2 = m\omega/2$. The cross-section is therefore

$$\sigma = \frac{G_F^2 m\omega}{2\pi}\left[(h_V \pm h_A)^2 + \frac{1}{3}(h_V \mp h_A)^2 \right] . \tag{6.37}$$

Now for the ratios; we have

$$\begin{aligned} \text{electrons}: \qquad & h_V = \frac{1}{2} + 2\sin^2\theta_W \simeq 1 \,, \qquad & h_A = \frac{1}{2} \,; \\ \text{muons or taus}: \qquad & h_V = -\frac{1}{2} + 2\sin^2\theta_W \simeq 0 \,, \qquad & h_A = -\frac{1}{2} \,; \end{aligned} \tag{6.38}$$

so

$$\begin{aligned} \sigma_{\nu_e e} &\propto \left(1+\frac{1}{2}\right)^2 + \frac{1}{3}\left(1-\frac{1}{2}\right)^2 &= \frac{9}{4} + \frac{1}{12} = \frac{7}{3} \,; \\ \sigma_{\bar\nu_e e} &\propto \left(1-\frac{1}{2}\right)^2 + \frac{1}{3}\left(1+\frac{1}{2}\right)^2 &= \frac{1}{4} + \frac{9}{12} = \frac{3}{3} \,; \\ \sigma_{\nu_\mu e} &\propto \left(0+\frac{1}{2}\right)^2 + \frac{1}{3}\left(0-\frac{1}{2}\right)^2 &= \frac{1}{4} + \frac{1}{12} = \frac{1}{3} \,; \\ \sigma_{\bar\nu_\mu e} &\propto \left(0-\frac{1}{2}\right)^2 + \frac{1}{3}\left(0+\frac{1}{2}\right)^2 &= \frac{1}{4} + \frac{1}{12} = \frac{1}{3} \,; \end{aligned} \tag{6.39}$$

and the ν_τ results are the same as the ν_μ results. Therefore the cross sections are in ratio

$$7 : 3 : 1 : 1 : 1 : 1 \,. \tag{6.40}$$

Incidentally, this result is central to the first proof that the neutrinos from the Sun are definitely oscillating into an "active" type of neutrino other than the electron neutrino. Super-Kamiokande looks for the scattering of neutrinos off of electrons, which detects both electron neutrinos and other types, but is about 7 times more sensitive to the electron type neutrinos. SNO detects the scattering of electron neutrinos off of Deuterium, $\nu_e D \to e^- pp$, which ONLY the electron neutrino can induce. (You can tell these apart from $\nu e \to \nu e$ electrons from the very different angular distribution of the outgoing neutrinos; the former are very "forward," as we saw in this problem; the latter are almost isotropic and are slightly more common in the *backwards* direction.) There was a clear discrepancy between the neutrino flux the two measured, because most of the neutrinos were ν_μ or ν_τ and therefore contributed, somewhat, in Super-K but not in SNO. Now SNO has also measured the rate of $\nu D \to \nu np$, which is equally sensitive to all three types of neutrinos (it is purely Z boson exchange), and confirmed that there are the expected extra number of ν_μ and ν_τ.

6.4 Higgsstrahlung

6.4.1 Matrix element

The phase space integration for the cross section is, taking the relative velocity to be $\simeq 2$,

$$\sigma = \frac{1}{2s}\int \frac{d^3\mathbf{p}_Z d^3\mathbf{p}_H}{(2\pi)^6 2p_Z^0 2p_H^0}(2\pi)^4\delta^3(\mathbf{p}_Z+\mathbf{p}_H)\delta(\sqrt{s}-p_Z^0-p_H^0)|\mathcal{M}|^2\,. \tag{6.41}$$

Here we are in the center of mass frame, and have written the energy and momentum integrations separately. We will also define the total momentum $p_Z+p_H = p_{e^-}+p_{e^+}\equiv q$, and the incoming electron and positron momenta as p and p' respectively. Note that $q^2=-s$.

The matrix element is

$$\mathcal{M} = -\frac{e_Z M_Z^2}{v}\epsilon_\nu^*\left(\frac{\eta^{\mu\nu}+\frac{q^\mu q^\nu}{M_Z^2}}{s-M_Z^2}\right)\bar{v}(p',\sigma')\gamma_\mu(g_V+g_A\gamma^5)u(p,\sigma)\,. \tag{6.42}$$

Squaring this and averaging over electron and positron spins and summing over final state Z particle polarizations gives

$$\begin{aligned}\frac{1}{4}\sum_{\sigma,\lambda}|\mathcal{M}|^2 &= \frac{(g_2^2+g_1^2)^2M_Z^2}{16(s-M_Z^2)^2}\left(\eta^{\alpha\beta}+\frac{p_Z^\alpha p_Z^\beta}{M_Z^2}\right)\left(\eta_{\alpha\mu}+\frac{q_\alpha q_\mu}{M_Z^2}\right)\left(\eta_{\beta\nu}+\frac{q_\beta q_\nu}{M_Z^2}\right)\times\\ &\quad\times \mathrm{tr}\,\not{p}\gamma^\mu(g_V+g_A\gamma^5)\not{p}'\gamma^\nu(g_V+g_A\gamma^5)\,.\end{aligned} \tag{6.43}$$

Now note that the quantities from Z boson propagators and external states is symmetric in μ,ν. Therefore the contribution from a single γ^5, which will be antisymmetric in μ and ν, vanishes. The trace yields

$$\mathrm{tr}\,\not{p}\gamma^\mu(g_V+g_A\gamma^5)\not{p}'\gamma^\nu(g_V+g_A\gamma^5) = 4(g_V^2+g_A^2)(-\eta^{\mu\nu}p\cdot p'+p^\mu p'^\nu+p^\nu p'^\mu)+\text{antisymm}\,. \tag{6.44}$$

As usual, $q_\mu=(p+p')_\mu$ and q_ν vanish against this combination, so the Z boson propagators can be simplified by dropping those parts. If you don't believe me,

$$(p+p')_\mu\,(p^\mu p'^\nu+p^\nu p'^\mu-\eta^{\mu\nu}p\cdot p') = (p\cdot p'p'^\nu+p\cdot p'p^\nu-p\cdot p'(p+p')^\nu)=0\,. \tag{6.45}$$

The matrix element is therefore

$$\begin{aligned}\frac{1}{4}\sum_{\sigma,\lambda}|\mathcal{M}|^2 &= \frac{(g_2^2+g_1^2)^2M_Z^2(g_V^2+g_A^2)}{4(s-M_Z^2)^2}\left(-p\cdot p'+2\frac{p_Z\cdot p\;p_Z\cdot p'}{M_Z^2}\right)\,,\\ &= \frac{(g_2^2+g_1^2)^2M_Z^2(g_V^2+g_A^2)}{8(s-M_Z^2)^2}\left(s+\frac{(t-M_Z^2)(u-M_Z^2)}{M_Z^2}\right)\,.\end{aligned} \tag{6.46}$$

6.4.2 Total cross-section

We will use Eq. (6.41) and Eq. (6.46). The spatial delta function performs the $\mathbf{p}_H$ integration. The relation between the energies, masses, $\mathbf{p}_Z^2$, and s, are (defining $E_{\rm CM} \equiv \sqrt{s}$)

$$\begin{aligned} E_Z^2 &= E_H^2 - 2E_H E_{\rm CM} + E_{\rm CM}^2 \qquad \text{gives} \\ E_H &= \frac{E_{\rm CM}}{2} + \frac{M_H^2 - M_Z^2}{2E_{\rm CM}}\,, \\ E_Z &= \frac{E_{\rm CM}}{2} - \frac{M_H^2 - M_Z^2}{2E_{\rm CM}}\,, \\ \mathbf{p}_Z^2 &= \frac{(E_{\rm CM}^2 - M_Z^2 - M_H^2)^2 - 4M_H^2 M_Z^2}{4E_{\rm CM}^2}\,. \end{aligned} \tag{6.47}$$

The Jacobian for the delta function is

$$\frac{d}{dp}\left(\sqrt{p^2{+}M_Z^2} + \sqrt{p^2{+}M_H^2} - E_{\rm CM}\right) = \frac{p}{E_H} + \frac{p}{E_Z} = \frac{pE_{\rm CM}}{E_H E_Z}\,. \tag{6.48}$$

Its inverse appears if we use the delta function to perform the momentum integration; that neatly cancels the factors of E_H and E_Z in the denominator. The cross-section has become

$$\sigma = \frac{1}{32\pi^2 E_{\rm CM}^2}\frac{p}{E_{\rm CM}}\int d\Omega|\mathcal{M}|^2\,. \tag{6.49}$$

It remains only to evaluate t and u, and to do the angular integral. We have

$$t \equiv -(p_1 - p_Z)^2 = -p_1^2 + 2p_1\cdot p_2 - p_2^2 = 0 - E_{\rm CM}E_Z + E_{\rm CM}|\mathbf{p}_Z|\cos\theta + M_Z^2\,, \tag{6.50}$$

and

$$u \equiv -(p_1 - p_H)^2 = -E_{\rm CM}E_H - E_{\rm CM}|\mathbf{p}_Z|\cos\theta + M_H^2 = -E_{\rm CM}E_Z - E_{\rm CM}|\mathbf{p}_Z|\cos\theta + M_Z^2\,, \tag{6.51}$$

where we define θ as the angle between e^- and Z (and hence $\pi - \theta$ is the angle between e^- and H). Checking (as we should have to begin with) and using $s = E_{\rm CM}^2$, we do indeed find $(s + t + u) = M_Z^2 + M_H^2$. The matrix element is therefore,

$$\frac{1}{4}\sum_{\sigma,\lambda}|\mathcal{M}|^2 = \frac{(g_2^2{+}g_1^2)^2 M_Z^2 (g_V^2{+}g_A^2)}{8(s - M_Z^2)^2}\left(E_{\rm CM}^2 + \frac{E_{\rm CM}^2 E_Z^2 - E_{\rm CM}^2\mathbf{p}_Z^2\cos^2\theta + \cos\theta\ldots}{M_Z^2}\right)\,. \tag{6.52}$$

We have not bothered writing the argument of $\cos\theta$, because it vanishes when we integrate over angles. On integration over angles, there is an overall 4π, and $\cos^2\theta \to 1/3$, $\cos\theta \to 0$.

The final answer is messy:

$$\sigma = \frac{(g_V^2{+}g_A^2)(g_2^2{+}g_1^2)^2}{64\pi}\frac{p}{E_{\rm CM}(E_{\rm CM}^2{-}M_Z^2)^2}\left(M_Z^2 + E_Z^2 - \frac{1}{3}p_Z^2\right)\,. \tag{6.53}$$

Here $g_A^2 = 1/16$ and $g_V^2 = (\sin^2\theta_W - 1/4)^2 \simeq .00035$. Maybe someone can figure out a tighter way of writing this expression.

6.4.3 Experimental limit

The luminosity of the LEP II accelerator was about $10^{32}/(\text{cm}^2\text{s})$. Running for a few months, say, 10^7 seconds, at center of mass energy $\sqrt{s}$=205 GeV, would LEP have been able to detect a 110 GeV Higgs boson, if the condition for detection is a sample of 10 events?

We will be sloppy and take $M_Z = 91$ GeV without decimal. Substituting in the explicit numbers, we find (all in GeV)

$$\begin{aligned}
E_{\text{CM}} &= 205 \\
E_Z &= 93.2 \\
E_H &= 111.8 \\
p_Z &= 20 \\
\sigma &= \frac{(.0629)(.302)}{201}\frac{20}{2.33\times 10^{11}}(16800) = 1.36\times 10^{-10}\ \text{GeV}^{-2}\,.
\end{aligned} \tag{6.54}$$

Under the described conditions, something with a cross-section of 10^{-38} cm^2 could have been observed. Using that 1 GeV^{-1} is approximately 2×10^{-14} cm, we get the above cross section to be, in centimeters squared,

$$\sigma \simeq 5.3\times 10^{-38}\ \text{cm}^2\,. \tag{6.55}$$

It looks like we would have expected about 53 events. This is enough provided that there is a sufficiently clean signal.

In fact, the LEP people claimed an exclusion up to 114 GeV on the Higgs mass, on the basis of running up to 208 GeV (though only at the highest energy for a very short while). If you stick in the numbers, that sounds hard to believe–there would only be 3 GeV of extra energy, and most of the data is where there was even less. However, some Z bosons are produced a few GeV of energy below the supposed mass, as we will soon see; this becomes important when you are so close to being energetically excluded. There were a few claimed events, reconstructing to $M_H = 115$ GeV, immediately before the machine was shut down for good; but there is no consensus that this was a genuine observation of the Higgs boson.

6.5 Resonances

The cross-section is,

$$\sigma = \frac{16\pi}{s}\frac{2j+1}{(2s_1+1)(2s_2+1)}\frac{\Gamma_{\Upsilon\to ee}}{\Gamma_\Upsilon} = \frac{12\pi}{m_\Upsilon^2}2.8\times 10^{-5}\,. \tag{6.56}$$

Put this into barns; a barn is 10^{-24} cm^2, and $1/\mathrm{GeV}^2 = 0.000388$ barns. Therefore,

$$\sigma = 3.7 \text{ nb}\,, \tag{6.57}$$

where nb is nano-barns.

The downward correction due to initial state radiation is by about,

$$\exp\left(\frac{-\alpha}{\pi}\log\frac{m_\Upsilon^2}{m_e^2}\left[\log\frac{m_\Upsilon^2}{\Gamma_\Upsilon^2}-\frac{3}{2}\right]\right) \simeq \exp -.55 \simeq 0.58\,, \tag{6.58}$$

so about 40% of the cross-section is lost to initial state radiation.

6.6 Compton scattering

To appear

Chapter 7

Effective Lagrangians

7.1 Running b quark mass

Rewrite the first beta function as

$$\frac{\mu\partial}{\partial\mu}\frac{1}{g_3^2} = \frac{-46}{3}\frac{1}{16\pi^2} \tag{7.1}$$

which is solved by

$$\frac{1}{\alpha_3} \equiv \frac{4\pi}{g_3^2} = \frac{23}{6\pi}\left(\ln\mu - \ln\Lambda_{QCD}\right) . \tag{7.2}$$

From the data, $\alpha_3(\mu = 91\ \text{GeV})$, we get

$$\ln\Lambda_{QCD} = \ln(91\ \text{GeV}) - \frac{6\pi}{23\times 0.118} \Rightarrow \Lambda_{QCD} = .0876\ \text{GeV} . \tag{7.3}$$

Another way of writing the QCD beta function expression is,

$$\frac{\mu\partial}{\partial\mu}\ln g_3^2 = \frac{-23}{24\pi^2}g_3^2 , \qquad d\ln g^2 = \frac{-23}{24\pi^2}g_3^2 d\ln\mu . \tag{7.4}$$

This lets us rearrange the other equation as follows:

$$\begin{aligned}
\frac{\partial}{\partial\ln\mu}h_{33} &= \frac{-8g_3^2h_33}{16\pi^2} &(7.5)\\
\frac{\partial}{\partial\ln\mu}\ln h_{33} &= \frac{-8g_3^2}{16\pi^2} &(7.6)\\
d\ln h_{33} &= -\frac{g_3^2 d\ln\mu}{2\pi^2} &(7.7)\\
d\ln h_{33} &= -\frac{g_3^2}{2\pi^2}\frac{24\pi^2}{-23g_3^2}d\ln g_3^2 &(7.8)\\
d\ln h_{33} &= \frac{12}{23}d\ln g_3^2 . &(7.9)
\end{aligned}$$

That means, simply, that

$$\frac{h_{33}(\mu_1)}{h_{33}(\mu_2)} = \left(\frac{g_3^2(\mu_1)}{g_3^2(\mu_2)}\right)^{\frac{12}{23}} . \tag{7.10}$$

Therefore,

$$\frac{h_{33}(120\ \text{GeV})}{h_{33}(4.24\ \text{GeV})} = \left(\frac{g_3^2(120\ \text{GeV})}{g_3^2(4.24\ \text{GeV})}\right)^{\frac{12}{23}} = \left(\frac{\ln(4.24\ \text{GeV}/\Lambda_{QCD})}{\ln(120\ \text{GeV}/\Lambda_{QCD})}\right)^{\frac{12}{23}} = 0.723 \tag{7.11}$$

This is the rescaling of the effective b mass, leading to

$$m_b(\mu = 120\ \text{GeV}) = 0.723 \times 4.24\ \text{GeV} = 3.07\ \text{GeV}\,. \tag{7.12}$$

7.2 Hypercharge beta function

To appear

7.3 WW scattering and unitarity

7.3.1 Polarization vectors

The longitudinal polarization vector is given, up to an arbitrary phase, by

$$\epsilon^\mu(\mathbf{p}, \lambda_3) = \left(\frac{|\mathbf{p}|}{M_W}\,,\, \frac{p^0}{M_W}\hat{\mathbf{p}}\right) , \tag{7.13}$$

with $\hat{\mathbf{p}}$ the unit vector in the direction of $\mathbf{p}$. This satisfies $\epsilon^2 = (-\mathbf{p}^2 + (p^0)^2)/M_W^2 = 1$, $\epsilon \cdot p = (-|\mathbf{p}|p^0 + |\mathbf{p}|p^0)/M_W = 0$, and is orthogonal to the other (transverse) polarization vectors.

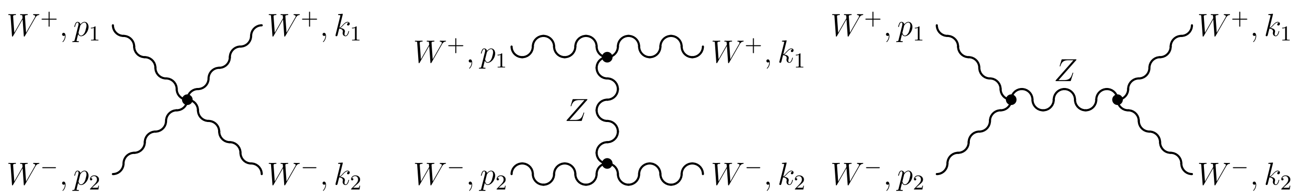
The four polarization vectors are therefore,

$$\begin{aligned}
\epsilon^\mu(\mathbf{p}_1, \lambda_3) &= \left(\frac{p}{M_W}\,,\, \frac{E}{M_W}\,,\, 0\,,\, 0\right) ,\\
\epsilon^\mu(\mathbf{p}_2, \lambda_3) &= \left(\frac{p}{M_W}\,,\, -\frac{E}{M_W}\,,\, 0\,,\, 0\right) ,\\
\epsilon^\mu(\mathbf{k}_1, \lambda_3) &= \left(\frac{p}{M_W}\,,\, \frac{E}{M_W}\cos\theta\,,\, \frac{E}{M_W}\sin\theta\,,\, 0\right) ,\\
\epsilon^\mu(\mathbf{k}_2, \lambda_3) &= \left(\frac{p}{M_W}\,,\, -\frac{E}{M_W}\cos\theta\,,\, -\frac{E}{M_W}\sin\theta\,,\, 0\right) .
\end{aligned} \tag{7.14}$$

For future reference, and using that all the momenta will have the same magnitude in the rest frame, we find $\epsilon(\mathbf{p}, \lambda_3) \cdot \epsilon(\mathbf{k}, \lambda_3) = (E^2\cos\theta - p^2)/M_W^2$.

7.3.2 Matrix element

The three diagrams are, the four-W interaction and t and s channel Z exchange:



Let us work out the matrix element for each, in turn. For the first, it is,

$$\mathcal{M}_1 = -g_2^2 \left[2\eta_{\mu\nu}\eta_{\lambda\rho} - \eta_{\mu\lambda}\eta_{\nu\rho} - \eta_{\mu\rho}\eta_{\nu\lambda}\right] \epsilon_1^\mu \epsilon_4^\nu \epsilon_2^\lambda \epsilon_3^\rho \,. \tag{7.15}$$

Here ϵ_1 means the one for p_1, ϵ_3 means the one for k_1, and we will use this notation throughout what follows. Note: the factor of 1/4 in the Feynman rule is canceled by the four ways the operators in the vertex can attach to the external states; the initial state W^+ can be annihilated by either of the W^+ in the vertex, the final state W^+ can be created by either of the W^- operators. NOTE that the W^+ operators destroy the incoming W^+ and create the outgoing W^-, that is why the 2 term in [ ] connects the incoming W^+ with the outgoing W^-.

Evaluating the external polarization contractions, gives

$$\begin{aligned}
\mathcal{M}_1 &= -g_2^2 \left(2\epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3 - \epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 - \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4\right) \,, \\
&= -\frac{g_2^2}{M_W^4} \Big(2(p^2 + E^2 \cos\theta)^2 - (-p^2 + E^2\cos\theta)^2 - (p^2 + E^2)^2\Big) \,, \\
&= \frac{g_2^2}{M_W^4} \Big(E^4(1 - \cos^2\theta) + p^2 E^2 (2 - 6\cos\theta)\Big) \,.
\end{aligned} \tag{7.16}$$

This term is $O(E^4/M_W^4)$, and its sign can depend on the angle $\cos\theta$. Substituting $p^2 = E^2 - M_W^2$ to find like-order terms, this becomes,

$$g_2^2 \left[(3 - 6\cos\theta - \cos^2\theta)\frac{E^4}{M_W^4} + (6\cos\theta - 2)\frac{E^2}{M_W^2}\right] \,. \tag{7.17}$$

The next contribution is from the t-channel Z boson exchange, which gives,

$$\begin{aligned}
\mathcal{M}_2 &= -g_2^2 \left[\epsilon_1 \cdot (p_1 - 2k_1)\epsilon_3^\mu + \epsilon_3 \cdot (k_1 - 2p_1)\epsilon_1^\mu + \epsilon_1 \cdot \epsilon_3 (p_1 + k_1)^\mu\right] \times \\
&\quad \times \left[\epsilon_4 \cdot (2p_2 - k_2)\epsilon_2^\nu + \epsilon_2 \cdot (2k_2 - p_2)\epsilon_4^\nu + \epsilon_2 \cdot \epsilon_4 (-p_2 - k_2)^\nu\right] \times \\
&\quad \times \frac{\eta_{\mu\nu} + \frac{(p_1 - k_1)_\mu (p_1 - k_1)_\nu}{M_W^2}}{M_W^2 + (p_1 - k_1)^2} \,.
\end{aligned} \tag{7.18}$$

Note that we have M_W^2 rather than M_Z^2 appearing here; that is because we have made the approximation that $g_1 = 0$, in which case the two masses are the same.

Now we have to evaluate this mess. A few tricks help; $\epsilon_1 \cdot p_1 = 0$ and similarly for the other combinations, and $(p_1 + k_1) \cdot (p_1 - k_1) = p_1^2 - k_1^2 = 0$. Therefore, we can show that the term in the propagator which is not $\eta_{\mu\nu}$, contributes zero;

$$[\epsilon_1 \cdot (-2k_1)\epsilon_3^\mu + \epsilon_3 \cdot (-2p_1)\epsilon_1^\mu + \epsilon_1 \cdot \epsilon_3(p_1 + k_1)^\mu] (p_1 - k_1) = -2\epsilon_1 \cdot k_1 \epsilon_3 \cdot p_1 + 2\epsilon_3 \cdot p_1 \epsilon_1 \cdot k_1 + 0 = 0 \, . \tag{7.19}$$

The first bracket is,

$$-2k_1 \cdot \epsilon_1 \epsilon_3^\mu - 2p_1 \cdot \epsilon_3 \epsilon_1^\mu + \epsilon_1 \cdot \epsilon_3 (p_1 + k_1)^\mu = \frac{2Ep(1 - \cos\theta)}{M_W}(\epsilon_1 + \epsilon_3)^\mu + \frac{(E^2 \cos\theta - p^2)}{M_W^2}(p_1 + k_1)^\mu \, . \tag{7.20}$$

The other bracket is similarly,

$$\frac{2Ep(\cos\theta - 1)}{M_W}(\epsilon_2 + \epsilon_4)_\mu + \frac{p^2 - E^2\cos\theta}{M_W^2}(p_2 + k_2)_\mu \, . \tag{7.21}$$

Contracting, we get,

$$\frac{g_2^2}{M_W^4(2p^2(1 - \cos\theta) + M_W^2)} \Big(-8E^2p^2(1 - \cos\theta)^2(2p^2 + E^2(1 + \cos\theta)) + 4E^2p^2(1 - \cos\theta)(3 + \cos\theta)(p^2 - E^2\cos\theta) + 4E^2p^2(1 - \cos\theta)(3 + \cos\theta)(p^2 - E^2\cos\theta) - 2(E^2\cos\theta - p^2)^2(2E^2 + p^2(1 + \cos\theta)) \Big) , \tag{7.22}$$

which on re-arranging a bit, becomes, (writing $\cos\theta = c$ in what follows)

$$\frac{g^2}{M_W^4(2E^2(1-c) + M^2(2c-1))} \Big(-2E^6(1-c)^2(3+c) + 2E^4M_W^2(1-c)(3 - 10c - c^2) - 2E^2M_W^4(1-2c)(1-5c) + 2M_W^6(1+c) \Big) \, . \tag{7.23}$$

For future reference, the most singular parts of this are

$$-g^2\frac{E^4}{M_W^4}(1-c)(3+c) + g^2\frac{E^2}{M_W^2}\left(\frac{3}{2} - \frac{15}{2}c\right) \, . \tag{7.24}$$

Finally, the s-channel diagram gives,

$$\begin{aligned} \mathcal{M}_3 \;=\; & \frac{+g^2}{4E^2 - M_W^2} \left((p_1 - p_2)_\mu \epsilon_1 \cdot \epsilon_2 + \epsilon_2 \cdot (-2p_1 - p_2)\epsilon_{1\mu} + \epsilon_1 \cdot (p_1 + 2p_2)\epsilon_{2\mu} \right) \\ & \times \left((k_1 - k_2)^\mu \epsilon_3 \cdot \epsilon_4 + \epsilon_3 \cdot (k_1 + 2k_2)\epsilon_4^\mu + \epsilon_4 \cdot (-2k_1 - k_2)\epsilon_3^\mu \right) \, . \end{aligned} \tag{7.25}$$

The first bracket simplifies to,

$$\left(\frac{p^2+E^2}{M_W^2}(p_2-p_1)+\frac{4Ep}{M_W}(\epsilon_1-\epsilon_2)\right)\ . \tag{7.26}$$

The second bracket is similarly,

$$\left(\frac{p^2+E^2}{M_W^2}(k_2-k_1)+\frac{4Ep}{M_W}(\epsilon_3-\epsilon_4)\right)\ . \tag{7.27}$$

Contracting therefore gives,

$$\frac{g^2}{M_W^4(4E^2-M_W^2)}\left(4p^2(E^2+p^2)^2c-32E^2p^2(p^2+E^2)c+64E^4p^2c\right)\ . \tag{7.28}$$

The leading order in E^2 terms here are,

$$g^2c\left(4\frac{E^4}{M_W^2}+\frac{E^2}{M_W^2}\right)\ . \tag{7.29}$$

7.3.3 The problem

Summing the three terms, we find that the leading in E behavior cancels,

$$g^2\frac{E^4}{M_W^4}\left(3-6c-c^2-3+2c+c^2-4c\right)=0\,, \tag{7.30}$$

but the next to leading behavior,

$$g^2\frac{E^2}{M_W^2}\left(6c-2+\frac{3}{2}-\frac{15}{2}c+c\right)=g^2\frac{E^2}{M_W^2}\left(-\frac{1}{2}-\frac{1}{2}c\right)\ , \tag{7.31}$$

does not cancel.

The matrix element squared should not exceed $(16\pi)^2$, otherwise the cross-section, $\sim \frac{1}{16\pi s}|\mathcal{M}|^2$, will exceed the unitarity bound, $16\pi/s$. This is the case unless the matrix element's divergent behavior is only at very small angles, that is, for singularities near $c=1$. Since our matrix element is a simple polynomial in c, this is not the case.

Roughly, this means we get in trouble when

$$E^2=\frac{16\pi M_W^2}{g^2}\simeq 120M_W^2\,, \tag{7.32}$$

which happens for $E\simeq 900$ GeV. Beyond this energy, the scattering cross-section is larger than phase shift analysis allows it to be.

7.3.4 Higgs to the Rescue

The s channel Higgs diagram contributes,

$$-\frac{4M_W^4}{v^2}\frac{1}{M_H^2 - 4E^2}\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 \, , \tag{7.33}$$

where the first fraction is the coupling squared and the second is the Higgs propagator. Note that $4M_W^2/v^2 = g_2^2$.

The diagram is clearly only $O(E^2/M_W^2)$, so we will only evaluate it to this accuracy. Contracting polarization tensors, we find,

$$g_2^2 \frac{E^2}{M_W^2} \, . \tag{7.34}$$

Similarly, the t channel diagram is,

$$-\frac{4M_W^4}{v^2}\frac{1}{M_H^2 + 2p^2(1-c)}\epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 = -g_2^2 \frac{(1-c)}{2}\frac{E^2}{M_W^2} \, . \tag{7.35}$$

Adding these contributions, the divergent part is

$$g^2 \frac{E^2}{M_W^2}\left(\frac{1}{2} + \frac{1}{2}c\right) \, , \tag{7.36}$$

which exactly cancels the divergent part of the matrix element from Z exchange and the quartic interaction.

Therefore, the matrix element is finite and unitarity is safe, PROVIDED that the Higgs boson exists and has a mass below about 900 GeV.

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