from the previous lecture ...

in four dimensional Minkowski spacetime

$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2,$

<u>we have</u>

$4 -$ translations dt, dx, dy, dz

$3 -$ rotations $dxdy, dxdz, dydz$

$3 -$ boosts $dtdx, dtdy, dtdz$

The group that describes this set of symmetries is the Poincare group. The total number of generator of the Poincare group is 10. Our aim is to find the algebra of the Poincare group and its invariants. This will give us the labels of elementary particles. The translation symmetries are important. They relate to the momentum operator that generates translations. We will therefore obtain momentum and mass from the relativistic invariant $P_\mu P^\mu = m^2$, where P^μ is the particle momentum four vector. Mass is the second label of particle states.

let us look at the transformation of the function $\phi(x)$ in one dimension

$$\begin{aligned} x &\rightarrow x + a \\ \phi(x) &\rightarrow \phi(x + a) \end{aligned}$$

we are looking for an operator that induces this transformation.

$$\phi(x + a) = U(a)\phi(x)$$

$$\begin{aligned} \phi(x) \rightarrow \phi(x + a) &= \sum_n \frac{a^n}{n!} \left(\frac{\partial^n}{\partial x^n} \phi(x) \right) \Big|_{a=0} \\ &= \underbrace{\left(\sum_n \frac{a^n}{n!} \frac{\partial^n}{\partial x^n} \right)}_{\text{operator}} \phi(x) = \mathrm{e}^{a \frac{\partial}{\partial x}} \phi(x) \end{aligned}$$

$$
\begin{aligned}
U(a) = \mathrm{e}^{a\frac{\partial}{\partial x}} &\leftarrow \quad \text{is the operator} \\
U(a) = \mathrm{e}^{a\frac{\partial}{\partial x}} &\simeq 1 + a\frac{\partial}{\partial x} + \cdots = 1 + i(-ia\frac{\partial}{\partial x}) + \cdots \\
&= 1 + iaP + \cdots
\end{aligned}
$$

i.e $P = -i\frac{\partial}{\partial x}$ is the operator that induces the translation in x. The complex factor i arises because we require that $U(a)$ is a unitary operator.

$$
\begin{aligned}
U &= I + iaP \\
U^\dagger &= I - iaP \\
U^{-1} &= I - iaP \\
U^\dagger U &= (I - iaP)(I + iaP) = I + a^2P^2 \simeq I
\end{aligned}
$$

In four spacetime dimensions

$$U(a^\mu) \simeq I + ia^\mu P_\mu$$

The generators of the Poincare group are:

$$
\begin{aligned}
P_\mu &= i\partial_\mu && \leftarrow \text{4 translations} && (1)\\
L_{\mu\nu} &= i(X_\mu\partial_\nu - X_\nu\partial_\mu) && \leftarrow \text{3 rotations} + \text{3 boosts} && (2)
\end{aligned}
$$

The most general transformation consistent with the Poincare group.

$$X^\mu \to X'^\mu = \Lambda^\mu_\nu X^\nu + a^\mu$$

We want to find:

- the commutation relations among the generators of the group
- the maximal set of commuting operators $\rightarrow$ physical labels

Example : Rotations

$$[L_i, L_j] = i\epsilon_{ijk}$$

Casimir operator L^2

$$\left[L^2, L_i\right] = 0 \quad \rightarrow \quad \text{2 mutually commuting operators } L^2,\ L_z.$$

$$L^2|j, m\rangle = j(j+1)|j, m\rangle$$

$$L_z|j, m\rangle = m|j, m\rangle$$

the states are labelled by their eigenvalues under the commuting operators.

We saw that $P_\mu = i\frac{\partial}{\partial X^\mu}$ $\leftarrow$ 4 generators

1 $\Rightarrow \quad [P_\mu, P_\nu] = 0$

the order does not matter if we perform two successive translations.

$$\begin{aligned} X^\mu \to X'^\mu &= X^\mu + a^\mu \quad \text{1. translate by } a^\mu \\ X'^\mu \to X''^\mu &= X'^\mu + b^\mu \quad \text{2. translate by } b^\mu \\ &= X^\mu + a^\mu + b^\mu \end{aligned}$$

$$\begin{aligned} X^\mu \to X'^\mu &= X^\mu + b^\mu \quad \text{1. translate by } b^\mu \\ X'^\mu \to X''^\mu &= X'^\mu + a^\mu \quad \text{2. translate by } a^\mu \\ &= X^\mu + a^\mu + b^\mu \end{aligned}$$

The order of the translations does not matter, hence the commutator of the two operations (generators) commutes.

(2) $$[P_\mu, X^\nu] = \left[i\frac{\partial}{\partial X^\mu}, X^\nu\right] = i\left[\frac{\partial}{\partial X^\mu}, X^\nu\right] = i\delta^\nu{}_\mu$$

(3) $$[P_\mu, K_i] = \,? \qquad , \qquad [P_\mu, J_i] = \,?$$

We perform successive Poincare transcriptions

$$\begin{aligned}
1.\quad X^\mu \;\to\; X'^\mu &= \Lambda_1{}^\mu{}_\nu X^\nu + a_1{}^\mu \quad \text{first transformation}\\
2.\quad X'^\mu \;\to\; X''^\mu &= \Lambda_2{}^\mu{}_\nu X'^\nu + a_2{}^\mu \quad \text{second transformation}\\
&= \Lambda_2{}^\mu{}_\nu \left(\Lambda_1{}^\nu{}_\lambda X^\lambda + a_1{}^\nu\right) + a_2{}^\mu\\
&= \underbrace{\Lambda_2{}^\mu{}_\nu \Lambda_1{}^\nu{}_\lambda X^\lambda}_{\text{2 successive L.T.}} + \underbrace{\Lambda_2{}^\mu{}_\nu a'^\nu + a_2{}^\mu}_{\text{translation that includes a L.T.}}
\end{aligned}$$

we symbolise:

$$U(\Lambda_2, a_2)U(\Lambda_1, a_1)$$

We perform the first transformation 1. and then the second 2.

we require:

$$U(\Lambda_2, a_2)U(\Lambda_1, a_1) = U(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2) \tag{3}$$

where $U(\lambda, a)$ is given by

$$U(\Lambda, a) = 1 + i\vec{\alpha}\cdot\vec{J} - i\vec{\beta}\cdot\vec{K} + ia^\mu P_\mu \tag{4}$$

Here, α_i, β_i and a^μ are infinitesimal parameters. Hence, (4) is an expansion of $U(\Lambda, a)$ to first order in the infinitesimal parameters. Inserting (4) into (3) and keeping terms to second order in the inifinitesimal parameters, we derive the commutation relations. The generator of translations is given in eq. (1), whereas the generators of boosts and rotations are given in eq. (2). The translation generators satisfy the communation relations given in 1, whereas the generators of translations and boosts satisfy the commutation relations

$$[L_{\mu\nu}, L_{\rho\sigma}] = i\eta_{\nu\rho}L_{\mu\sigma} - i\eta_{\mu\rho}L_{\nu\sigma} - i\eta_{\nu\sigma}L_{\mu\rho} + i\eta_{\mu\rho}L_{\nu\sigma} \tag{5}$$

which are the commutation relations of the $SO(1,3)$ Lie algebra. The most general representation of the generators of the $SO(1,3)$ algebra that obeys eq. (5) is given by

$$J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}$$

where $S_{\mu\nu}$ obeys eq. (5) and commutes with $L_{\mu\nu}$.

$$[J_{\mu\nu}, P_\rho] = -i\eta_{\mu\rho}P_\nu + i\eta_{\nu\rho}P_\mu \tag{6}$$

In terms of J_i and K_i the commutation relations become

$$[J_i, P_j] = i\epsilon_{ijk}P_k$$

$$[K_i, P_j] = iH\delta_{ij} \qquad \text{where} \quad \boxed{H = P_0}$$

$$[J_i, H] = 0 \quad , \quad [P_i, H] = 0 \quad , \quad [K_i, H] = iP_i$$