MATH 431 — May 2018: Solutions

All problems are similar to homework problems or material covered in the lectures.

1.

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2$$

a.

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \quad , \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix}$$

b.

$$\theta^\mu \to \theta^\mu + \epsilon\zeta^\mu(\theta, \phi)$$

We want to find functions

$$A(\theta, \phi) = \zeta^1(\theta, \phi)$$
$$\text{and } B(\theta, \phi) = \zeta^2(\theta, \phi)$$

such that ds^2 remains invariant under the transformations.

$$\theta \to \theta + \epsilon A(\theta, \phi)$$
$$\phi \to \phi + \epsilon B(\theta, \phi)$$
$$\sin\theta \to \sin(\theta + \epsilon A) \approx \sin\theta + \epsilon A\cos\theta$$
$$\sin^2\theta \to \sin^2\theta + 2\epsilon A\sin\theta\cos\theta + O(\epsilon^2)$$

Keeping terms to first order in ϵ

$$d\theta \to (1 + \epsilon\frac{\partial A}{\partial\theta})d\theta + \epsilon\frac{\partial A}{\partial\phi}d\phi$$
$$d\phi \to \epsilon\frac{\partial B}{\partial\theta}d\theta + (1 + \epsilon\frac{\partial B}{\partial\phi})d\phi$$
$$d\theta^2 \to (1 + 2\epsilon\frac{\partial A}{\partial\theta})d\theta^2 + 2\epsilon\frac{\partial A}{\partial\phi}d\theta d\phi$$
$$d\phi^2 \to (1 + 2\epsilon\frac{\partial B}{\partial\phi})d\phi^2 + 2\epsilon\frac{\partial B}{\partial\theta}d\theta d\phi$$
$$\sin^2\theta d\phi^2 \to \sin^2\theta(1 + 2\epsilon\frac{\partial B}{\partial\phi})d\phi^2 + \sin^2\theta 2\epsilon\frac{\partial B}{\partial\theta}d\theta d\phi + 2\epsilon A\sin\theta\cos\theta d\phi^2$$

we demand that ds^2 remains invariant.

$$d\theta^2 + \sin\theta d\phi^2 \to d\theta^2 + \sin\theta d\phi^2 + \underbrace{\cdots\cdots\cdots\cdots\cdots}_{\text{terms that vanish}}$$

demanding that the additional terms vanish we obtain the following constraints

$$
\begin{aligned}
d\theta^2 &: \frac{\partial A}{\partial \theta} = 0 \Rightarrow A = A(\phi) = f'(\phi) = \frac{df}{d\phi} \\
d\phi^2 &: \sin^2\theta \frac{\partial B}{\partial \phi} + A \sin\theta\cos\theta = 0 \Rightarrow \frac{\partial B}{\partial \phi} = -\frac{df}{d\phi}\frac{\cos\theta}{\sin\theta} \Rightarrow B = -f(\phi)\frac{\cos\theta}{\sin\theta} + g(\theta) \\
d\theta d\phi &: \frac{\partial A}{\partial \phi} + \sin^2\theta \frac{\partial B}{\partial \theta} = 0 \Rightarrow f'' + \sin^2\theta\left(\frac{f(\phi)}{\sin^2\theta} + g'(\theta)\right) = 0
\end{aligned}
$$

2. (a) In the lecture we have derived

$$
\dot{p}_0 = -\frac{\partial H}{\partial q_0} .
$$

Therefore, p_0 is constant in time if H does not depend on q_0.

One example for such a system is given in part (b) below, where H does not depend on ϕ. A second example would be a one dimensional harmonic oscillator (in x) embedded in two dimensions such that there is no y dependence and therefore the momentum $p_y = \mathit{constant}$.

(b) In polar coordinates

$$
\begin{aligned}
x &= r\sin\theta\cos\phi , \\
y &= r\sin\theta\sin\phi , \\
z &= r\cos\theta ,
\end{aligned}
$$

the kinetic energy is given by **Bookwork**

$$
\frac{1}{2}m\left[\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\dot{\phi})^2\right] ,
$$

and with the potential energy V the Lagrangian is given by

$$
L = \frac{1}{2}m\left[\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\dot{\phi})^2\right] - V .
$$

Now

$$
\begin{aligned}
p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r} , \\
p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} , \\
p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\dot{\phi} .
\end{aligned}
$$

Hence

$$H = \sum_i p_i\dot q_i - L = m\dot r^2 + mr^2\dot\theta^2 + mr^2\sin^2\theta\dot\phi^2 - \frac{m}{2}\left(\dot r^2 + r^2\dot\theta^2 + r^2\sin^2\theta\dot\phi^2\right) + V$$

$$= \frac{m}{2}\left(\dot r^2 + r^2\dot\theta^2 + r^2\sin^2\theta\dot\phi^2\right) + V$$

$$= \frac{{p_r}^2}{2m} + \frac{{p_\theta}^2}{2mr^2} + \frac{{p_\phi}^2}{2mr^2\sin^2\theta} + V\,.$$

With this we get

$$\dot p_\phi \;=\; -\frac{\partial H}{\partial\phi} \;=\; -\frac{\partial V}{\partial\phi} \;=\; 0\,.$$

An axisymmetric potential does not depend on ϕ, so p_ϕ is a constant of the motion, and the angular momentum about the symmetry axis is conserved.

3. (a) Two-particle states are defined by

$$|\mathbf{p}_1,\mathbf{p}_2\rangle \;=\; a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)|0\rangle\,,$$

with $|\mathbf{p}_1,\mathbf{p}_2\rangle = |\mathbf{p}_2,\mathbf{p}_1\rangle$ as $[a^\dagger(\mathbf{p}_1), a^\dagger(\mathbf{p}_2)] = 0$.

We also know that **Bookwork**

$$[a(\mathbf{p}_1), a^\dagger(\mathbf{p}_2)] = (2\pi)^3 2p_1^0\delta(\mathbf{p}_1 - \mathbf{p}_2)\,.$$

Hence

$$\begin{aligned}
\langle\mathbf{p}_1',\mathbf{p}_2'|\mathbf{p}_1,\mathbf{p}_2\rangle &= \langle 0|a(\mathbf{p}_1')a(\mathbf{p}_2')a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)|0\rangle \\
&= \langle 0|a(\mathbf{p}_1')\{a^\dagger(\mathbf{p}_1)a(\mathbf{p}_2') + (2\pi)^3 2p_1^0\delta(\mathbf{p}_1 - \mathbf{p}_2')\}a^\dagger(\mathbf{p}_2)|0\rangle \\
&= \langle 0|a(\mathbf{p}_1')a^\dagger(\mathbf{p}_1)\{a^\dagger(\mathbf{p}_2)a(\mathbf{p}_2') + (2\pi)^3 2p_1^0\delta(\mathbf{p}_2 - \mathbf{p}_2')\}|0\rangle \\
&\quad + (2\pi)^3 2p_1^0\delta(\mathbf{p}_1 - \mathbf{p}_2')\langle 0|a(\mathbf{p}_1')a^\dagger(\mathbf{p}_2)|0\rangle \\
&= (2\pi)^3 2p_2^0\delta(\mathbf{p}_2 - \mathbf{p}_2')\langle 0|\{a^\dagger(\mathbf{p}_1)a(\mathbf{p}_1') + (2\pi)^3 2p_1^0\delta(\mathbf{p}_1 - \mathbf{p}_1')\}|0\rangle \\
&\quad + (2\pi)^3 2p_1^0\delta(\mathbf{p}_1 - \mathbf{p}_2')\langle 0|\{a^\dagger(\mathbf{p}_2)a(\mathbf{p}_1') + (2\pi)^3 2p_2^0\delta(\mathbf{p}_2 - \mathbf{p}_1')\}|0\rangle \\
&= (2\pi)^6(2p_1^0)(2p_2^0)\{\delta(\mathbf{p}_1 - \mathbf{p}_1')\delta(\mathbf{p}_2 - \mathbf{p}_2') + \delta(\mathbf{p}_1 - \mathbf{p}_2')\delta(\mathbf{p}_2 - \mathbf{p}_1')\}\,.
\end{aligned}$$

(b) The number operator is

$$N \;:=\; \frac{1}{(2\pi)^3}\int\frac{\mathrm{d}^3\mathbf{p}'}{2p'^0}\,a^\dagger(\mathbf{p}')a(\mathbf{p}')\,.$$

With this

$$\begin{aligned}
[N, a^\dagger(\mathbf{p})] &= \left[\frac{1}{(2\pi)^3}\int\frac{\mathrm{d}^3\mathbf{p}'}{2p'^0}\,a^\dagger(\mathbf{p}')a(\mathbf{p}'), a^\dagger(\mathbf{p})\right] \\
&= \frac{1}{(2\pi)^3}\int\frac{\mathrm{d}^3\mathbf{p}'}{2p'^0}\,[a^\dagger(\mathbf{p}')a(\mathbf{p}'), a^\dagger(\mathbf{p})] \\
&= \frac{1}{(2\pi)^3}\int\frac{\mathrm{d}^3\mathbf{p}'}{2p'^0}\,\{a^\dagger(\mathbf{p}')[a(\mathbf{p}'), a^\dagger(\mathbf{p})] + [a^\dagger(\mathbf{p}'), a^\dagger(\mathbf{p})]a(\mathbf{p}')\} \\
&= \frac{1}{(2\pi)^3}\int\frac{\mathrm{d}^3\mathbf{p}'}{2p'^0}\,a^\dagger(\mathbf{p}')2p'^0(2\pi)^3\delta(\mathbf{p} - \mathbf{p}') \\
&= a^\dagger(\mathbf{p})\,.
\end{aligned}$$

So we have

$$\begin{aligned}
Na^\dagger(\mathbf{p}) - a^\dagger(\mathbf{p})N &= a^\dagger(\mathbf{p}) \\
\Rightarrow Na^\dagger(\mathbf{p}) &= a^\dagger(\mathbf{p})(N+1)\,, \\
\text{and} \quad N|\mathbf{p}_1 \dots \mathbf{p}_n\rangle &= Na^\dagger(\mathbf{p}_1)\dots a^\dagger(\mathbf{p}_n)|0\rangle \\
&= a^\dagger(\mathbf{p}_1)(N+1)a^\dagger(\mathbf{p}_2)\dots a^\dagger(\mathbf{p}_n)|0\rangle \\
&= a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)(N+2)\dots a^\dagger(\mathbf{p}_n)|0\rangle \\
&= \;\dots\; = \; a^\dagger(\mathbf{p}_1)\dots a^\dagger(\mathbf{p}_n)(N+n)|0\rangle \\
&= n\, a^\dagger(\mathbf{p}_1)\dots a^\dagger(\mathbf{p}_n)|0\rangle \quad (\text{as } a(\mathbf{p})|0\rangle = 0 \;\text{ and so } N|0\rangle = 0) \\
&= n|\mathbf{p}_1 \dots \mathbf{p}_n\rangle\,.
\end{aligned}$$

4.

$$\begin{aligned}
&\gamma^\mu \partial_\mu \psi + im\psi \;=\; 0\,, \qquad (\partial_\mu \psi^\dagger)\gamma^{\mu\dagger} - im\psi^\dagger \;=\; 0 \\
\Rightarrow \qquad &(\partial_\mu \psi^\dagger)\gamma^0\gamma^\mu\gamma^0 - im\psi^\dagger \;=\; 0 \qquad \text{and} \qquad (\partial_\mu \bar\psi)\gamma^\mu - im\bar\psi \;=\; 0\,.
\end{aligned}$$

(a) With this

$$\partial_\mu(\bar\psi\gamma^\mu\psi) \;=\; (\partial_\mu\bar\psi)\gamma^\mu\psi + \bar\psi\gamma^\mu(\partial_\mu\psi) \;=\; (im\bar\psi)\psi + \bar\psi(-im\psi) \;=\; 0\,.$$

(b) Similarly, and with $\{\gamma^5, \gamma^\mu\} = 0$ we get

$$\partial_\mu(\bar\psi\gamma^\mu\gamma^5\psi) = (\partial_\mu\bar\psi)\gamma^\mu\gamma^5\psi + \bar\psi\gamma^\mu\gamma^5(\partial_\mu\psi) = (im\bar\psi)\gamma^5\psi - \bar\psi\gamma^5(\gamma^\mu\partial_\mu\psi) = 2im\bar\psi\gamma^5\psi\,.$$

(c) From the Dirac equation for the spinors $\bar u_f$ and u_i we have

$$0 \;=\; \bar u_f(\not{p}_f - m)\gamma^\mu u_i \;=\; \bar u_f\gamma^\mu(\not{p}_i - m)u_i$$

$$\begin{aligned}
\Rightarrow \qquad 2m\bar u_f\gamma^\mu u_i \;&=\; \bar u_f(\not{p}_f\gamma^\mu + \gamma^\mu\not{p}_i)u_i \\
\text{and} \qquad \not{p}_f\gamma^\mu + \gamma^\mu\not{p}_i \;&=\; \gamma^\nu\gamma^\mu p_{f_\nu} + \gamma^\mu\gamma^\nu p_{i_\nu}\,.
\end{aligned}$$

Now

$$\begin{aligned}
\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \;&=\; 2g^{\mu\nu}\,, \\
\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu \;&=\; -2i\sigma^{\mu\nu}
\end{aligned}$$

$$\Rightarrow \qquad \gamma^\mu\gamma^\nu \;=\; g^{\mu\nu} - i\sigma^{\mu\nu} \qquad \text{and} \qquad \gamma^\nu\gamma^\mu \;=\; g^{\mu\nu} + i\sigma^{\mu\nu}\,.$$

So we get

$$\not{p}_f\gamma^\mu + \gamma^\mu\not{p}_i \;=\; g^{\mu\nu}(p_f+p_i)_\nu + i\sigma^{\mu\nu}(p_f-p_i)_\nu \;=\; (p_f+p_i)^\mu + i\sigma^{\mu\nu}(p_f-p_i)_\nu\,,$$

and finally have derived the Gordon decomposition

$$\bar{u}_f \gamma^\mu u_i \;=\; \frac{1}{2m}\,\bar{u}_f \left[(p_f + p_i)^\mu + i\sigma^{\mu\nu}(p_f - p_i)_\nu\right] u_i\,.$$

5. (a) The electromagnetic field strength tensor is given by

$$F_{\mu\nu} \;=\; \partial_\mu A_\nu - \partial_\nu A_\mu\,,$$

and the electromagnetic Lagrangian is

$$\begin{aligned}
L_{\text{e.m.}} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\
&= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\
&= -\frac{1}{4}\eta^{\mu\alpha}\eta^{\nu\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial_\mu A_\nu - \partial_\nu A_\mu)\,.
\end{aligned}$$

The Euler-Lagrange equations of motion for this Lagrangian are obtained from

$$\frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial(\partial_\nu A_\mu)}\right) - \frac{\partial L}{\partial A_\mu} \;=\; 0\,.$$

Now

$$\frac{\partial L}{\partial A_\mu} \;=\; 0 \quad\Rightarrow\quad \frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial(\partial_\nu A_\mu)}\right) \;=\; 0\,.$$

Hence calculate

$$\begin{aligned}
\frac{\partial L}{\partial(\partial_\delta A_\epsilon)} &= -\,\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}\left[(\delta^\delta_\alpha\delta^\epsilon_\beta - \delta^\delta_\beta\delta^\epsilon_\alpha)(\partial_\mu A_\nu - \partial_\nu A_\mu)\right. \\
&\qquad\qquad \left. +\,(\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\delta^\delta_\mu\delta^\epsilon_\nu - \delta^\delta_\nu\delta^\epsilon_\mu)\right] \\
&= -\,\frac{1}{4}\left[(\eta^{\delta\mu}\eta^{\epsilon\nu} - \eta^{\epsilon\mu}\eta^{\delta\nu})(\partial_\mu A_\nu - \partial_\nu A_\mu)\right. \\
&\qquad \left. +\,(\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\eta^{\alpha\delta}\eta^{\beta\epsilon} - \eta^{\alpha\epsilon}\eta^{\beta\delta})\right] \\
&= -\,\frac{1}{4}\,4\left[\partial^\delta A^\epsilon - \partial^\epsilon A^\delta\right] \\
&= -\,F^{\delta\epsilon}\,.
\end{aligned}$$

So

$$\frac{\partial}{\partial x^\delta}\left(\frac{\partial L}{\partial(\partial_\delta A_\epsilon)}\right) \;=\; -\frac{\partial}{\partial x^\delta}F^{\delta\epsilon} \;=\; 0\,,$$

which are the Maxwell equations in the absence of sources.

(b)

$$A_\mu \;\rightarrow\; \tilde{A}_\mu \;=\; A_\mu + \partial_\mu \Lambda\,.$$

Now

$$\begin{aligned}\tilde{F}_{\mu\nu} \;=\; \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu \;&=\; \partial_\mu (A_\nu + \partial_\nu \Lambda) - \partial_\nu (A_\mu + \partial_\mu \Lambda) \\ &=\; \partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\mu \partial_\nu \Lambda - \partial_\nu \partial_\mu \Lambda \\ &=\; \partial_\mu A_\nu - \partial_\nu A_\mu \;=\; F_{\mu\nu}\end{aligned}$$

as the derivatives of the scalar function Λ commute. With $F_{\mu\nu}$ also $L_{\text{e.m.}}$ is invariant under the transformation.

(c) The Lagrangian for a massive photon without sources is given by

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + m^2 A_\mu A^\mu\,.$$

As before the first term is invariant under the transformation considered above. However, the second term transforms as

$$m^2 A_\mu A^\mu \;\rightarrow\; m^2 \tilde{A}_\mu \tilde{A}^\mu \;=\; m^2 A_\mu A^\mu + m^2 (\partial_\mu \Lambda \partial^\mu \Lambda + \partial_\mu \Lambda A^\mu + A_\mu \partial^\mu \Lambda)\,.$$

The extra terms do not vanish, so the mass term would not be invariant under the transformation, i.e. *break the invariance*. Insisting on the invariance therefore forbids the mass term, i.e. $m^2 = 0$ and the photon has to remain massless.

6a. To show that orbital angular momentum is a constant of the motion we have to show that it commutes with the Hamiltonian, *i.e.*

$$\left[\hat{L}_i, \hat{H}\right] = 0,$$

where the $\hat{L}_i$'s are the components of the angular momentum operator in the x, y and z directions. For a free particle the Hamiltonian is given by $H = \vec{p}^2/(2m)$ and the orbital angular momentum is given by $\vec{L} = \vec{r} \times \vec{p}$. Hence, for example, for L_z (we drop the hats from now on) we have

$$\begin{aligned}[L_z, p^2] &= [xp_y - yp_z, p_x^2 + p_y^2 + p_z^2] \\ &= [x, p_x^2]p_y - [y, p_y^2]p_x \\ &= (p_x[x, p_x] + [x, p_x]p_x)p_y - (p_y[y, p_y] + [y, p_y]p_y)p_x \\ &= (2i\hbar p_x p_y - 2i\hbar p_y p_x) = 0\end{aligned}$$

where we used the commutation relations $[x_i, p_j] = i\hbar\delta_{ij}$. Similar results are obtained for L_x and L_y. Hence, the orbital angular momentum commutes with the Hamiltonian and is a contant of the motion.

6b. Similarly to show that the orbital angular momentum is not a constant of the motion for a Dirac particle, we have to show that it does not commute with the Dirac Hamiltonian,

$$H = \vec{\alpha} \cdot \vec{p} + \beta m = \alpha_i p_i + \beta m$$

where summation over i is assumed.

$$[L_z, H] = [x, H]p_y - [y, H]p_x = i\hbar(\alpha_x p_y - \alpha_y p_x) = i\hbar(\vec{\alpha} \times \vec{p})_z$$

hence

$$[\vec{L}, H] = i\hbar\vec{\alpha} \times \vec{p} \neq 0$$

and consequently orbital aangular momentum does not commute with the Hamiltonian and is not a constant of the motion.

6c. The total angular momentum for a Dirac particle is

$$\vec{J} = \vec{L} + \vec{S}$$

where $\vec{L}$ is the orbital angular momentum and $\vec{S}$ is the spin angular momentum. we saw in part b that $[\vec{L}, H] = i\hbar\vec{\alpha} \times \vec{p}$ hence we need to find $\vec{S}$ such that

$$[\vec{S}, H] = -i\hbar\vec{\alpha} \times \vec{p}$$

We take

$$\vec{S} = \frac{1}{2}\vec{\Sigma}$$

with

$$\Sigma_j = \begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}$$

where σ_j are the Pauli matrices and the 0 entries are 2×2 zero matrices. It is easy to verify that

$$[\frac{1}{2}\vec{\Sigma}, H] = -i\hbar\vec{\alpha} \times \vec{p}$$

and therefore $[\vec{J}, H] = 0$ and the total angular momentum $\vec{J}$ is a constant of the motion.

7(a). 2 diagonal generators.

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

7b. $D = 8$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

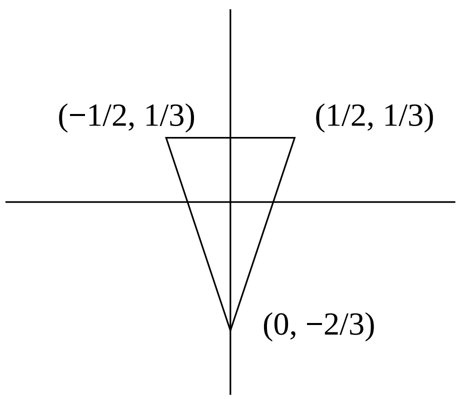
$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

7c.

$$3 = (2, 1/3) + (1, -2/3)$$

under the maximal subgroup $SU(2) \times U(1)$

7d.



7e.

$$3 \times \bar{3} = \{(2, 1/3) + (1, -2/3)\} \times \{(2, -1/3) + (1, 2/3)\} =$$
$$8 + 1 = \{(2, +1) + (3, 0) + (1, 0) + (2, -1)\} + (1, 0)$$

7f.

$$3 \times 3 = \{(2, 1/3) + (1, -2/3)\} \times \{(2, 1/3) + (1, -2/3)\} =$$
$$6 + \bar{3} = \{(3, 2/3) + (2, -1/3) + (1, -4/3)\} + \{(2, -1/3) + (1, 2/3)\}$$