from the previous lectures ...

Classification of elementary particles

$SU(2)_I$ – Isospin – global continuous $SU(2)$ symmetry, which is exact if we ignore E&M interactions. Isospin symmetry is approximate in nature versus E&M which is exact.

In the 1950's a slew of particles (resonances) were discovered

All the particles that interacted strongly formed families of Isospin interactions

Example: Proton and neutron form a doublet of Isospin

$s = \frac{1}{2} \qquad m = 939\mathrm{MeV};\ 938\mathrm{MeV} \qquad e - \mathrm{charge} + 1,\ 0$

$$
\begin{aligned}
\mathrm{Strong} &\sim 10^{-24}\ \text{sec} \\
\mathrm{E\&M} &\sim 10^{-18}\ \text{sec} \\
\mathrm{Weak} &\sim 10^{-8}\ \text{sec}
\end{aligned}
$$

Hadrons – strongly interacting

baryons. spin $= n + \frac{1}{2}$ $\quad n = 0, 1, \ldots$

mesons. spin $= n$ $\quad n = 0, 1, \ldots$

Leptons – not strongly interacting

charged $\rightarrow$ $e, \ \mu, \ , \tau$

neutral $\rightarrow$ $\nu_e, \ \nu_\mu, \ \nu_\tau$

gauge bosons. spin 1 $\quad \gamma; \ W^\pm, \ Z; \ G$

All interactions respect the familiar conservation laws

additional conservation laws must be imposed *e.g.* $P \ \rightarrow \ e^+ \ \pi^0$ charge, angular momentum and energy are conserved but $\tau_P \geq 10^{32}\text{years}$.

$\Rightarrow$ inroduce $B(P) = +1, \ B(\bar{P}) = -1, \ B(L) = 0 = B(M)$

$$\Rightarrow \quad P \ \nrightarrow \ e^+ \ \pi^0$$

Similarly for leptons

$$\begin{array}{lclr} \mu^- & \to & e^- \ \bar{\nu}_e \ \nu_\mu & \text{allowed} \\ \mu^- & \to & e^- \ \gamma & \text{forbidden} \end{array}$$

Introduce $L_e, \ L_\mu, \ L_\tau$

$$\begin{aligned} L_e &= +1 \ \ \text{for} \ \ e^- \text{ and } \nu_e \\ L_e &= -1 \ \ \text{for} \ \ e^+ \text{ and } \bar{\nu}_e \\ L_e &= \ \ 0 \ \ \text{for everyone else} \end{aligned}$$

Therefore, for the process $\mu^- \ \to \ e^- \ \bar{\nu}_e \ \nu_\mu$

$$\begin{array}{cccccc} & \mu^- & \to & e^- & \bar{\nu}_e & \nu_\mu \\ \text{spin} & \frac{1}{2} & \to & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \text{charge} & -1 & \to & -1 & 0 & 0 \\ L_\mu & +1 & \to & 0 & 0 & +1 \\ L_e & 0 & \to & +1 & -1 & 0 \end{array} \qquad J_f = j_1 + j_2 - j_3 = \frac{1}{2}$$

Baryon & Lepton numbers are seen to be exactly conserved charges in nature.

Some conservation laws may be approximate, *e.g.*

$$K^+ \quad \rightarrow \quad \pi^+\pi^0 \quad \rightarrow \ \sim \ 20\% \ \text{ branching ratio}$$

$$\tau(K^+) \quad \sim \quad 10^{-8}\text{sec} \quad \rightarrow \quad \text{weak decay?}$$

$$\text{why not strong } K^+ \text{ decay?}$$

Gellmann & Nishijima $\rightarrow$ a new additive conserved quantum number.

S – strangeness

$$S(P) = S(\pi) = 0$$

$$S(K^+) = S(K^0) = +1$$

$$S(\Lambda,\ \Sigma) = -1$$

$$S(\Xi) = -2$$

Strong & electromagnetic interactions conserve strangeness

Weak interaction violates strangeness

$\rightarrow$ classification by Isospin is not sufficient to classify hadronic states

$\rightarrow$ need a larger symmetry group S such that

$$SU(2)_I \subset G \leftarrow SU(2)_I \text{ is a subgroup of } G$$

$$G = ? \rightarrow G = SU(3)_{flavour} \rightarrow \text{Gellmann \& Neeman} \rightarrow \text{the eightfold way}$$

Unitary groups $SU(n)$

simple unitary group of rank n where the rank is the number of mutualy commuting diagonal generators.

$$\text{Unitary}: \qquad U^\dagger U \;\; \Rightarrow \;\; U^\dagger \;=\; U^{-1}$$

$$\text{Simple}: \qquad \mathrm{Det}\, U \;=\; 1$$

<u>Examples</u> : $n = 1 \quad , \quad \psi \to U\psi \; , \; U^\dagger = U^{-1} \to U = \mathrm{e}^{i\alpha} \Rightarrow U^\dagger = \mathrm{e}^{-i\alpha} \;=\; U^{-1}$

$$n = 2 \quad , \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \to \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad , \quad U^\dagger = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \quad , \quad U^{-1} = \frac{1}{(AD - BC)} \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}$$

$$U^\dagger = U^{-1} \;\Rightarrow\; A^* = \frac{D}{\mathrm{DetU}} \;,\; D^* = \frac{A}{\mathrm{DetU}} \;,\; C^* = -\frac{B}{\mathrm{DetU}} \;,\; A^* = -\frac{C}{\mathrm{DetU}}$$

<u>Unitary matrices:</u> $U^\dagger = U^{-1} \quad \Rightarrow \quad U^\dagger U = I$

$$\mathrm{Det}(U^{-1}) = (\mathrm{Det}\,U)^{-1} \quad ; \quad \mathrm{Det}(U^\dagger) = (\mathrm{Det}\,U)^*$$

$$U^\dagger = U^{-1} \Rightarrow (\mathrm{Det}\,U)^* = \frac{1}{\mathrm{Det}\,U} \Rightarrow |\mathrm{Det}\,U| = +1$$

Since $|\mathrm{Det}\,U| = 1$ we get from equations (**??**) that

$$|A| = |D| \qquad \& \qquad |B| = |C|$$

The most general 2x2 unitary matrix can therefore be written as

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{i\alpha} & 0 \\ 0 & \mathrm{e}^{i\beta} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathrm{e}^{i\gamma} & 0 \\ 0 & \mathrm{e}^{i\delta} \end{pmatrix}$$

which is the most general solution of the equations in (**??**).

$\#$ of D.O.F. in U is 4: $\theta +$ three of $\alpha,\ \beta,\ \gamma,\ \delta.$

The phases appearing in U are $\alpha+\gamma$, $\alpha+\delta$, $\beta+\gamma$, $\beta+\delta$.

in a 2x2 unitary matrix there are 4 D.O.F.

generalisation: An NxN unitary matrix has N^2 D.O.F.

Theorem: A unitary matrix U can be written as $U = \mathrm{e}^{iH}$, where H is hermitian ($H^\dagger = H$).

Assume an hermitian NxN matrix: $H = S + iA$,

where S is a real symmetric matrix and A is a real antisymmetric matrix.

$$\text{The } \# \text{ of D.O.F. in } H: \qquad S \ \rightarrow\ N + \left(\frac{N^2-N}{2}\right) = \frac{N(N+1)}{2}$$

where N is the number of diagonal terms

$$A \ \rightarrow\ \frac{N^2-N}{2} = \frac{N(N-1)}{2}$$

$$\underline{\text{Hence}}\text{: } \# \text{ of D.O.F. in } H: \frac{N(N+1)}{2} + \frac{N(N-1)}{2} = N^2$$

A unitary NxN matrix also has N^2 degrees of freedom.

In $U(2)$ write the most general 2×2 unitary matrix

we need 4 independent hermitian matrices:

The space of 2×2 hermitian matrices is spanned by the <u>basis</u>:

$$H = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \quad \vec{\tau} \right]$$

Where $\vec{\tau}$ are the Pauli matrices : $\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\tau_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\text{general } H\text{–matrix} \quad H = \beta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \vec{\alpha} \cdot \vec{\tau}$$

$$\text{general } U(2) \text{ matrix :} \qquad U = \mathrm{e}^{i(\beta I + \vec{\alpha} \cdot \vec{\tau})}$$

The S in an $SU(2)$ matrix stands for Simple, which means that $\mathrm{Det}\, U = 1$

If $U = \mathrm{e}^{iH}$ then $\mathrm{Det}\, U = \mathrm{e}^{i \mathrm{Tr} H}$

In general: if $\mathrm{Det}A \neq 0 \Rightarrow \mathrm{Det}A = \mathrm{Det}PAP^{-1} = \mathrm{Det}A_D = \lambda_1 \cdots \lambda_n$

If $\mathrm{Det}A \neq 0 \Rightarrow \mathrm{Tr}A = \mathrm{Tr}PAP^{-1} = \mathrm{Tr}A_D = \lambda_1 + \cdots + \lambda_n$

Hence, if $U = \mathrm{e}^A$

$$
\begin{aligned}
\mathrm{Det}\,U &= \mathrm{Det}\ \mathrm{e}^A = \\
&= \mathrm{Det}(I + A + \frac{A^2}{2} + \cdots) = \\
&= \mathrm{Det}(I + PAP^{-1} + \frac{PAP^{-1}PAP^{-1}}{2} + \cdots) = \\
&= \mathrm{Det}(I + A_D + \frac{A_D^2}{2} + \cdots) = \\
&= \mathrm{Det}\begin{pmatrix} \mathrm{e}^{\lambda_1} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \mathrm{e}^{\lambda_n} \end{pmatrix} = \\
&= \mathrm{e}^{\lambda_1} \cdots \mathrm{e}^{\lambda_n} = \mathrm{e}^{\lambda_1 + \cdots + \lambda_n} = \mathrm{e}^{\mathrm{Tr}A}
\end{aligned}
$$

$$\text{Hence,}\quad \det U = \mathrm{e}^{i\mathrm{Tr}H} \quad \Leftrightarrow \quad \mathrm{Det}\, U = 1 \quad \Rightarrow \quad \mathrm{Tr} H = 0$$

$$\Rightarrow \quad \beta = 0 \quad , \quad \mathrm{Tr} I = 2 \quad , \quad \mathrm{Tr}\tau_j = 0$$

Therefore, the most general $SU(2)$ matrix with $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$.

$$U = \mathrm{e}^{i\vec{\alpha}\cdot\vec{\tau}} \simeq I + i\vec{\alpha}\cdot\vec{\tau}$$

Find the most general hermitian 3x3 matrix $H_{3\times 3}$ $\#(\text{D.O.F.}) = 9$.

Requiring $\mathrm{Det}\, U = 1 \quad \Rightarrow \quad \mathrm{Tr} H = 0 \quad \Rightarrow \quad \#(\mathrm{D.O.F.}) = 8$