MATH431 Modern Particle Physics Solutions 6

1

$$\psi(r,\theta,\phi) = R(r)\begin{pmatrix} 1 \\ 0 \\ ia\cos\theta \\ iae^{i\phi}\sin\theta \end{pmatrix}$$

Normalize

$$\int d^3\mathbf{r}\psi^\dagger\psi = 1 = \int 4\pi r^2 dr(|R|^2(1+a^2))$$

$$\Rightarrow \int_0^\infty r^2|R|^2 dr = [4\pi(1+a^2)]^{-1}$$

(a.)

$$L_z = -i\frac{\partial}{\partial\phi} \Rightarrow L_z\psi = R\begin{pmatrix} 0 \\ 0 \\ 0 \\ iae^{i\phi}\sin\theta \end{pmatrix} \not\propto \psi$$

so ψ is not an eigenstate of L_z.

(b.)

$$\langle L_z\rangle = \int d^3\mathbf{r}\psi^\dagger L_z\psi = \int 2\pi r^2 d\cos\theta|R|^2 a^2\sin^2\theta$$

$$\int_{-1}^1 d\cos\theta(1-\cos^2\theta) = 2 - \frac{2}{3} = \frac{4}{3} \Rightarrow \langle L_z\rangle = \frac{8\pi}{3}a^2\cdot\frac{1}{4\pi(1+a^2)}$$

$$\Rightarrow \langle L_z\rangle = \frac{2a^2}{3(1+a^2)}$$

In H-atom , $v/c \sim \alpha \Rightarrow \langle L_z\rangle = O(v^2/c^2)$. This is a relativistic effect - spin–orbit interaction.

(c.)

$$S_z = \frac{1}{2}\hbar\begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\Rightarrow S_z\psi = \frac{1}{2}\hbar R\begin{pmatrix} 1 \\ 0 \\ ia\cos\theta \\ -iae^{i\phi}\sin\theta \end{pmatrix}$$

$$(L_z+S_z)\psi = R\begin{pmatrix} 1 \\ 0 \\ ia\cos\theta \\ iae^{i\phi}\sin\theta \end{pmatrix} = \frac{1}{2}\psi \Rightarrow J_z = +\frac{1}{2}$$

2. (a) Operating with $\gamma^\nu \partial_\nu$ from the left on the Dirac equation we have

$$\begin{aligned} \gamma^\nu \partial_\nu \left(i\gamma^\mu \partial_\mu - m\right) \psi(x) &= \\ i\frac{1}{2}\left(\gamma^\nu \gamma^\mu + \gamma^\mu \gamma^\nu\right) \partial_\nu \partial_\mu \psi - m\gamma^\nu \partial_\nu \psi &= \\ i\eta^{\mu\nu} \partial_\nu \partial_\mu \psi + im^2 \psi &= \\ i\left(\partial^\mu \partial_\mu + m^2\right) I\psi &= 0 \end{aligned}$$

where I is the 4×4 identity matrix.

(b)

$$\gamma^\mu \partial_\mu \psi + im\psi = 0 \quad , \quad (\partial\psi^\dagger)\gamma^{\mu^\dagger} - im\psi^\dagger = 0$$

$$\begin{aligned} &\Rightarrow (\partial\psi^\dagger)\gamma^0\gamma^\mu\gamma^0 - im\psi^\dagger = 0 \\ &\Rightarrow (\partial_\mu\bar{\psi})\gamma^\mu - im\bar{\psi} = 0 \end{aligned}$$

(c)

- $\partial_\mu(\bar{\psi}\gamma^\mu\psi) = (\partial_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu(\partial_\mu\psi)$
  $= (im\bar{\psi})\psi + \bar{\psi}(-im\psi) = 0$

- $\partial_\mu(\bar{\psi}\gamma^\mu\gamma^5\psi) = (\partial_\mu\bar{\psi})\gamma^\mu\gamma^5\psi + \bar{\psi}\gamma^\mu\gamma^5(\partial_\mu\psi)$
  $= (im\bar{\psi})\gamma^5\psi - \bar{\psi}\gamma^5(\gamma^\mu\partial_\mu\psi) = 2im\bar{\psi}\gamma^5\psi$

3.

$$\bar{u}_f(\not{p}_f - m)\gamma^\mu u_i = \bar{u}_f\gamma^\mu(\not{p}_i - m)u_i = 0 \qquad \text{(Dirac eq.)}$$

$$\Rightarrow 2m\bar{u}_f\gamma^\mu u_i = \bar{u}_f(\not{p}_f\gamma^\mu + \gamma^\mu\not{p}_i)u_i$$

$$\not{p}_f\gamma^\mu + \gamma^\mu\not{p}_i = \gamma^\nu\gamma^\mu p_{f_\nu} + \gamma^\mu\gamma^\nu p_{i_\nu}$$

$$\begin{aligned} \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu &= 2g^{\mu\nu} \\ \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu &= -2i\sigma^{\mu\nu} \end{aligned}$$

Hence
$\gamma^\mu\gamma^\nu = g^{\mu\nu} - i\sigma^{\mu\nu}; \gamma^\nu\gamma^\mu = g^{\mu\nu} + i\sigma^{\mu\nu}$

$$\Rightarrow \not{p}_f\gamma^\mu + \gamma^\mu\not{p}_i = g^{\mu\nu}(p_f + p_i)_\nu + i\sigma^{\mu\nu}(p_f - p_i)_\nu = (p_f + p_i)^\mu + i\sigma^{\mu\nu}(p_f - p_i)_\nu$$

$$\Rightarrow \bar{u}_f\gamma^\mu U_i = \frac{1}{2m}\bar{u}_f[(p_f + p_i)^\mu + i\sigma^{\mu\nu}(p_f - p_i)_\nu]u_i$$

4. We consider an electron in a constant magentic field $\vec{B} = (0, 0, B)$ with $B > 0$.
(a.)
The vector potential

$$A^\mu = (0, 0, Bx, 0)$$

(b.)

$$(i\partial_0 - m)\phi = \vec{\sigma} \cdot (\vec{p} - e\vec{A})\chi$$
$$(i\partial_0 + m)\chi = \vec{\sigma} \cdot (\vec{p} - e\vec{A})\phi$$

where, as usual, $\vec{p} = -i\nabla$.

(c.) Assuming a solution of the form

$$\phi(x) = \phi(\vec{x})e^{-iEt}, \chi(x) = \chi(\vec{x})e^{-iEt}$$

Inserting into the equations from (b.) these equations become

$$(E - m)\phi(\vec{x}) = \vec{\sigma} \cdot (\vec{p} - e\vec{A})\chi(\vec{x})$$
$$(E + m)\chi(\vec{x}) = \vec{\sigma} \cdot (\vec{p} - e\vec{A})\phi(\vec{x})$$

Substituting $\chi(\vec{x})$ from the second equation into the first and repeating the steps that we too in class when deriving the gyromagnetic factor from the Dirac equation, we get

$$(E^2 - m^2)\phi(\vec{x}) = [(\vec{p} - e\vec{A})^2 - e\vec{\sigma} \cdot \vec{B}]\phi(\vec{x})$$
$$= [\vec{p}^2 + e^2B^2x^2 - 2ep_yBx - e\sigma_zB]\phi(\vec{x})$$

Since p_x, p_y commute with x, we can seach for solutions of the form

$$\phi(\vec{x}) = e^{i(p_y y + p_z z)} f(x)$$

where p_y and p_z are c-numbers and $f(x)$, as $\phi(\vec{x})$, is a two component spinor. The equation for $f(x)$ becomes

$$[-\frac{d^2}{dx^2} + (p_y - eBx)^2 - eB\sigma_z]f(x) = (E^2 - m^2 - p_z^2)f(x)$$

$f(x)$ can be taken to be an eigenfunction of σ_z with eigenvalues $\sigma = \pm 1$, $\sigma_z f = \sigma f$. Then

$$[-\frac{d^2}{dx^2} + \frac{1}{2}(2e^2B^2)(x - \frac{p_y}{eB})^2]f(x) = (E^2 - m^2 - p_z^2 + eB\sigma)f(x)$$

This is formally identical to the Schrödinger equation of an harmonic oscillator with frequency $2|e|B$. The energy levels are therefore given by

$$E^2 - m^2 - p_z^2 + eB\sigma = (n + \frac{1}{2})2|e|B$$

or

$$E = [m^2 + p_z^2 + (2n+1+\sigma)|e|B]^{\frac{1}{2}}$$

Observe that there is a continuous degeneracy in p_x and p_y, as well as a discrete degeneracy

$$E(n, p_z, \sigma = +1) = E(n+1, p_z, \sigma = -1).$$

In the nonrelativistic limit $p_z << m^2$, $(2n+1)|e|B << m^2$ the nonrelativistic limit therefore gives

$$E(n, p_z, \sigma) \simeq m + \frac{p_z^2}{2m} + \left(n + \frac{1+\sigma}{2}\right)\omega_B$$

with $\omega_B = |e|B/m$. These are the Landau levels of nonrelativistic quantum mechanics.

4.

$$\begin{aligned}
\gamma^{5\dagger} =& (i\gamma^0\gamma^1\gamma^2\gamma^3)^\dagger \\
=& -i\gamma^{3\dagger}\gamma^{2\dagger}\gamma^{1\dagger}\gamma^{0\dagger} \\
=& -i(\gamma^0\gamma^3\gamma^0)(\gamma^0\gamma^2\gamma^0)(\gamma^0\gamma^1\gamma^0)\gamma^0 \\
=& -i\gamma^0\gamma^3\gamma^2\gamma^1 = i\gamma^0\gamma^2\gamma^3\gamma^1 = -i\gamma^0\gamma^2\gamma^1\gamma^3 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5.
\end{aligned}$$

For the second part, it's best to do for each μ in turn:

$$\begin{aligned}
\gamma^5\gamma^0 =& i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 = -i\gamma^0\gamma^1\gamma^2\gamma^0\gamma^3 = i\gamma^0\gamma^1\gamma^0\gamma^2\gamma^3 \\
=& -i\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^0\gamma^5 \Rightarrow \{\gamma^5, \gamma^0\} = 0, \\
\gamma^5\gamma^1 =& i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1 = -i\gamma^0\gamma^1\gamma^2\gamma^1\gamma^3 = i\gamma^0\gamma^1\gamma^1\gamma^2\gamma^3 \\
=& -i\gamma^1\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^1\gamma^5 \Rightarrow \{\gamma^5, \gamma^1\} = 0.
\end{aligned}$$

It is clear that the other two calculations will be similar.

5. N.B. Of course the question was wrong–should have said $(\gamma^0)^2 = 1$, $(\gamma^i)^2 = -1$, $i = 1, 2, 3$.

$$\begin{aligned}
\gamma^0\gamma^1\gamma^2 =& -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3 = i\gamma^5\gamma^3 \\
\gamma^0\gamma^1\gamma^3 =& -\gamma^0\gamma^1\gamma^2\gamma^2\gamma^3 = \gamma^0\gamma^1\gamma^2\gamma^3\gamma^2 = -i\gamma^5\gamma^2.
\end{aligned}$$

6.

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}1_4 \Rightarrow \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}1_4,$$

where 1_4 is the 4-dimensional identity matrix, usually not written explicitly. Taking the trace, and using $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, $\mathrm{tr}1_4 = 4$, we get

$$\mathrm{tr}[\gamma_\mu\gamma_\nu] = 4\eta_{\mu\nu}.$$

Now

$$\begin{aligned}
\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma =& -\gamma_\nu\gamma_\mu\gamma_\rho\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma \\
=& \gamma_\nu\gamma_\rho\gamma_\mu\gamma_\sigma - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma \\
=& -\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu + 2\eta_{\mu\sigma}\gamma_\nu\gamma_\rho - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma \\
\Rightarrow \gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma + \gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu =& 2\eta_{\mu\sigma}\gamma_\nu\gamma_\rho - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma.
\end{aligned}$$

Taking the trace and using

$$\mathrm{tr}[\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu] = \mathrm{tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma],$$

together with $\mathrm{tr}[\gamma_\mu\gamma_\nu = 4\eta_{\mu\nu}$, we find

$$\mathrm{tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] = 4[\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}]$$

7.

The number operator is

$$N := \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{p}'}{2p'^0}\, a^\dagger(\mathbf{p}')a(\mathbf{p}')\,.$$

With this

$$\begin{aligned}
[N, a^\dagger(\mathbf{p})] &= \left[\frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{p}'}{2p'^0}\, a^\dagger(\mathbf{p}')a(\mathbf{p}'), a^\dagger(\mathbf{p})\right] \\
&= \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{p}'}{2p'^0}\, [a^\dagger(\mathbf{p}')a(\mathbf{p}'), a^\dagger(\mathbf{p})] \\
&= \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{p}'}{2p'^0}\, \{a^\dagger(\mathbf{p}')[a(\mathbf{p}'), a^\dagger(\mathbf{p})] + [a^\dagger(\mathbf{p}'), a^\dagger(\mathbf{p})]a(\mathbf{p}')\} \\
&= \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{p}'}{2p'^0}\, a^\dagger(\mathbf{p}')2p'^0(2\pi)^3\delta(\mathbf{p}-\mathbf{p}') \\
&= a^\dagger(\mathbf{p})\,.
\end{aligned}$$

So we have

$$\begin{aligned}
Na^\dagger(\mathbf{p}) - a^\dagger(\mathbf{p})N &= a^\dagger(\mathbf{p}) \\
\Rightarrow Na^\dagger(\mathbf{p}) &= a^\dagger(\mathbf{p})(N+1)\,, \\
\text{and} \quad N|\mathbf{p}_1 \ldots \mathbf{p}_n\rangle &= Na^\dagger(\mathbf{p}_1)\ldots a^\dagger(\mathbf{p}_n)|0\rangle \\
&= a^\dagger(\mathbf{p}_1)(N+1)a^\dagger(\mathbf{p}_2)\ldots a^\dagger(\mathbf{p}_n)|0\rangle \\
&= a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)(N+2)\ldots a^\dagger(\mathbf{p}_n)|0\rangle \\
&= \ldots = a^\dagger(\mathbf{p}_1)\ldots a^\dagger(\mathbf{p}_n)(N+n)|0\rangle \\
&= n\, a^\dagger(\mathbf{p}_1)\ldots a^\dagger(\mathbf{p}_n)|0\rangle \quad (\text{as } a(\mathbf{p})|0\rangle = 0 \ \text{ and so } N|0\rangle = 0) \\
&= n|\mathbf{p}_1 \ldots \mathbf{p}_n\rangle\,.
\end{aligned}$$