from the previous lecture ...

The Dirac equation

$$i\hbar\frac{1}{c}\frac{\partial}{\partial t}\Psi(\vec{x},t) = (-i\hbar\vec{\alpha}\cdot\vec{\nabla} + \beta mc)\Psi(\vec{x},t) \quad (1)$$

The wave function $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$ is a 4–component vector (not a four spacetime vector)

spin $\uparrow$ spin $\downarrow$ particles spin $\uparrow$ spin $\downarrow$ anti–particles

consequences:

1. The Dirac equation predicts the existence of anti–particles
2. Time and space derivatives are linear

$$\text{Setting} \quad \hbar = c = 1 \quad \Rightarrow \quad (i\gamma^\mu \partial_\mu - m)\Psi = (i\not{\partial} - m)\Psi = 0$$

Defining identity
$$\gamma^0 \gamma^\mu \gamma^0 = \gamma^{\mu\dagger} \quad \mu = 0, 1, 2, 3$$

together with
$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}$$

are the two properties that define the Dirac γ–matrices

representation
$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \qquad i = 1, 2, 3$$

where σ_i are the Pauli matrices

Solutions of the Dirac equation $\rightarrow$

4–component objects $\rightarrow$ spinors (not 4–vectors)

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

Each component obeys the KGE by construction

$$E^2 = H^2 = \vec{p}^2 + m^2 \quad \Rightarrow \quad E^2 I \psi = (\vec{p}^2 + m^2) I \psi$$

Spin of the Dirac particles

How do we prove that the Dirac equation correspond to spin 1/2 particles?

<u>Show</u>: exist an operator $\vec{S}$ such that $\vec{J} = \vec{L} + \vec{S}$ is a constant of the motion,

$$\text{and} \qquad \vec{S}^2|s\rangle = s(s+1)|s\rangle = \frac{3}{4}I|s\rangle \qquad (\hbar = 1)$$

<u>Note</u>: $\vec{L} = \vec{r} \times \vec{p}$ is not a constant of the motion:

$$\begin{aligned} H &= \beta m + \vec{\alpha}\cdot\vec{p} = \beta m + \alpha_i p_i \\ \vec{L} &= L_x\hat{i} + L_y\hat{j} + L_z\hat{k} \\ e.g. \qquad L_z &= (xp_y - yp_x) \end{aligned}$$

$$[L_z, H] = [x, H]\, p_y - [y, H]\, p_x = i\alpha_x py - i\alpha_y p_x = i(\vec{\alpha} \times \vec{p})_z \qquad (\hbar = 1)$$

in general: $\left[\vec{L}, H\right] = i\vec{\alpha} \times \vec{p} \neq 0$

$\Rightarrow$ we need $\left[\vec{S}, H\right] = -i\vec{\alpha} \times \vec{p}$

$$\text{This is true if} \quad \vec{S} = \frac{1}{2}\Sigma \quad \text{with} \quad \Sigma_j = \begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}$$

we want :

$$\begin{aligned} & [S_j, \alpha_i p_i + \beta m] = -i(\vec{\alpha} \times \vec{P})_j \\ = \; & [S_j, \alpha_i]\, p_i + [S_j, \beta]\, m \end{aligned}$$

$$[S_j, \beta] = \begin{pmatrix} A & \\ & A \end{pmatrix} \begin{pmatrix} I & \\ & -I \end{pmatrix} - \begin{pmatrix} I & \\ & -I \end{pmatrix} \begin{pmatrix} A & \\ & A \end{pmatrix} = 0$$

$$\rightarrow \quad S_j = \begin{pmatrix} A & \\ & A \end{pmatrix}$$

$$
\begin{aligned}
[S_j, \alpha_i] &= \sim \alpha_k \sim \begin{pmatrix} A & \\ & A \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \begin{pmatrix} A & \\ & A \end{pmatrix} \\
&= \begin{pmatrix} 0 & \sigma_j\sigma_i - \sigma_i\sigma_j \\ \sigma_j\sigma_i - \sigma_i\sigma_j & 0 \end{pmatrix} = -i \begin{pmatrix} 0 & 2\sigma_k \\ 2\sigma_k & 0 \end{pmatrix} = -2i\alpha_k
\end{aligned}
$$

Defining $S_j = \frac{1}{2}\begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}$ we get the desired result

Then : $\vec{S}^2 = \frac{1}{4}\left(\Sigma_x^2 + \Sigma_y^2 + \Sigma_z^2\right) = \frac{3}{4} I \quad \Rightarrow \quad \text{spin} = \frac{1}{2}$

Furthermore : $[L_i, S_j] = 0$

Magnetic moment of the Dirac equation

In an electromagnetic field we make the usual minimal substitutions

$$H \;\to\; H - eV \quad , \quad \vec{p} \;\to\; \vec{p} - e\vec{A}$$

where e is the electric charge. In the Dirac equation we obtain

$$H - eV \;=\; \vec{\alpha}\cdot(\vec{p} - e\vec{A}) + \beta m$$

Note: We no longer get the KG equation when squaring

$$\begin{aligned}(H - eV)^2 &= \sum_{j,k} \alpha_j \alpha_k (p_j - eA_j)(p_k - eA_k) + m^2 \\ &= (\vec{p}^2 - e\vec{A})^2 + m^2 - \sum_{j \neq k} (\alpha_j \alpha_k p_j A_k + \alpha_j \alpha_k A_j p_k)\end{aligned}$$

$$\underline{\text{For } j \neq k}: \ \alpha_j\alpha_k = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} = \begin{pmatrix} \sigma_j\sigma_k & 0 \\ 0 & \sigma_j\sigma_k \end{pmatrix} = i\epsilon_{jkl}\begin{pmatrix} \sigma_l & 0 \\ 0 & \sigma_l \end{pmatrix}$$

$$= i\epsilon_{jkl}\Sigma_l$$

$$\text{where we used} \qquad [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$$

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}$$

$$\Rightarrow \qquad \sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$$

$$P_jA_kf = (-i\nabla_jA_k)f = A_k(-i\nabla_jf) - i(\nabla_jA_k)f = (A_kP_j - i(\nabla_jA_k))f$$

$$\epsilon_{jkl}\Sigma_l\nabla_jA_k = \Sigma_l\epsilon_{ljk}\nabla_jA_k = \Sigma_l(\vec{\nabla}\times\vec{A})_l = \vec{\Sigma}\cdot\vec{B}$$

we get :

$$-e\sum_{j\neq k}(\alpha_j\alpha_k p_j A_k + \alpha_j\alpha_k A_j p_k)$$

$$= -e\sum_{j\neq k}(\alpha_j\alpha_k A_j p_k + \alpha_j\alpha_k A_k p_j) - e\vec{\Sigma}\cdot\vec{B}$$

$$= -e\sum_{j\neq k}(\alpha_j\alpha_k + \alpha_k\alpha_j)A_j p_k - e\vec{\Sigma}\cdot\vec{B} = -e\vec{\Sigma}\cdot\vec{B}$$

Hence :

$$(H - eV)^2 = (\vec{p} - e\vec{A})^2 + m^2 - e\vec{\Sigma}\cdot\vec{B}$$

$$(H - eV) = m\left(1 + \frac{(\vec{p} - e\vec{A})^2 - e\vec{\Sigma}\cdot\vec{B}}{m^2}\right)^{\frac{1}{2}}$$

$$\text{NR limit} \quad \simeq \quad m + \frac{1}{2m}(\vec{p} - e\vec{A})^2 - \frac{e}{2m}\vec{\Sigma}\cdot\vec{B}$$

This correspond to a magnetic moment

$$\mu = \frac{e}{m}\vec{S} = g_e(\frac{e}{2m})\vec{S}$$

where $g_e = 2$ (experiment $\Rightarrow$ 2.0023193...) where the (0.0023193...) are quantum field theory corrections.

Great success of Dirac equation.

(g-2) of the muon is of great contemporary interest and substantial experimental effort to measure it.