from the previous lecture ...

The Dirac equation $$i\hbar\frac{1}{c}\frac{\partial}{\partial t}\Psi(\vec{x},t) = (-i\hbar\vec{\alpha}\cdot\vec{\nabla} + \beta mc)\Psi(\vec{x},t)$$

Charge conjugation: $\psi \rightarrow \psi^C = C\psi^* \rightarrow$ charge conjugation $C = i\gamma_2$

Parity invariance $\psi(\vec{r},t) \rightarrow \psi^P(\vec{r},t) = P\psi(-\vec{r},t) = \gamma_0\psi(-\vec{r},t)$

KGE invariant under parity transformation $\phi(\vec{r},t) \rightarrow \phi^P(\vec{r},t) = \phi(-\vec{r},t)$

Since $\phi(\vec{r},t)$ is a scalar under LT$\phi'(\vec{r}',t') = \phi(\vec{r},t)$

In the case of the Dirac equation the scalar is $\Phi = \bar{\psi}\psi = \psi^\dagger\gamma^0\psi$

Similarly, j^μ is a true vector $j^{P0}(\vec{r},t) = j^0(-\vec{r},t)$, $\vec{j}^P(\vec{r},t) = -\vec{j}(-\vec{r},t)$

We define the matrix

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I_{2\times 2} \\ I_{2\times 2} & 0 \end{pmatrix} \qquad \text{(in our representation)}$$

We will see that weak interactions involve the axial current

$$J_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi = \psi^\dagger\gamma^0\gamma^\mu\gamma^5\psi$$

under parity transformations $\qquad \psi \to \psi^P = \gamma^0\psi(-\vec{r}, t)$

$$J_A^{P\mu} = \psi^\dagger(-\vec{r}, t)\gamma^{0\dagger}\gamma^0\gamma^\mu\gamma^5\gamma^0\psi(-\vec{r}, t)$$

$$\begin{aligned} \text{now} \quad \{\gamma^\mu, \gamma^5\} &= 0 \quad \text{for} \mu = 0, 1, 2, 3 \\ (\gamma_5)^2 &= I \end{aligned}$$

<u>Hence</u> $\quad J_A^{P0}(\vec{r}, t) = -J_A{}^0(-\vec{r}, t) \quad , \quad \vec{J}_A^P(\vec{r}, t) = \vec{J}_A(-\vec{r}, t)$

As expected for an axial vector.

Similarly : $\Phi_P = \bar{\psi}\gamma^5\psi = \psi^\dagger\gamma^0\gamma^5\psi$ a pseudo scalar

$$\begin{aligned}
\Phi_P^P(\vec{r},t) &= \underbrace{\psi^\dagger(-\vec{r},t)\gamma^{0\dagger}}_{\psi^{P\dagger}}\gamma^0\gamma^5\underbrace{\gamma^0\psi(-\vec{r},t)}_{\psi^P} \\
&= -\bar{\psi}(-\vec{r},t)\gamma^5\psi(-\vec{r},t) \\
&= -\Phi_P(-\vec{r},t)
\end{aligned}$$

Massless Dirac particles

The Dirac equation : $$H\psi = i\hbar\frac{\partial\psi}{\partial t} = \left(-i\hbar\vec{\alpha}\cdot\vec{\nabla} + \beta m\right)\psi$$

where

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$$

The positive energy free particle solutions are

$$\psi = u(E, \vec{p})\mathrm{e}^{-i(Et - \vec{p}\cdot\vec{x})}$$

For $m = 0 \quad \Rightarrow \quad E = |\vec{p}|$ and $u = \binom{\phi}{\chi}$ gives

$$EIu = \vec{\alpha}\cdot\vec{p}u = \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix} u$$

$$\underline{\text{hence}}: \quad \begin{pmatrix} |\vec{p}| & -\vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & |\vec{p}| \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0 \quad \Rightarrow \quad \begin{array}{l} \vec{\sigma}\cdot\vec{p}\chi = |\vec{p}|\phi \\ \vec{\sigma}\cdot\vec{p}\phi = |\vec{p}|\chi \end{array} \tag{1}$$

$$\Rightarrow \quad \chi = \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\phi \quad \& \quad \phi = \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\chi \quad \Rightarrow \quad \chi = \left(\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\right)^2 \chi$$

$$\Rightarrow \quad \left(\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\right)^2 = I$$

$$\Rightarrow \quad \Lambda = \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} \quad \text{is the helicity operator with} \quad \Lambda^2 = I$$

$$\text{and eigenvalues } \Lambda = \pm 1 \;\rightarrow\; \text{spin} \frac{\text{along}}{\text{against}} \;\vec{p} \;\rightarrow\; \frac{\text{right}}{\text{left}} \;\; \text{handed}$$

Helicity operator: projection of spin on $\frac{\vec{p}}{|\vec{p}|}$.

$\Rightarrow$ for massless particles with $m = 0$ the 2 components spinors χ and ϕ are eigenstates of the helicity operator.

For massless particles helicity is a Lorentz invariant.

Note that if ψ represents a massless particle them

$$\gamma^5 \psi = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} \Lambda\phi \\ \phi \end{pmatrix} = \begin{pmatrix} \Lambda\phi \\ \Lambda\chi \end{pmatrix} = \Lambda\psi \qquad (\Lambda^2 = 1)$$

Hence, γ^5 is the helicity operator for massless particles (minus helicity for massless anti–particles).

In the case of massless particles we can decompose the Dirac equation into two equations for the two helicity eigenstates.

We can introduce the basis

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad ; \quad \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \quad ; \quad \gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

This basis is called the chiral basis. In this basis

$$\left(\gamma^0|\vec{p}| - \vec{\gamma}\cdot\vec{p}\right)\begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} 0 & |\vec{p}| - \vec{\sigma}\cdot\vec{p} \\ |\vec{p}| + \vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0$$

Hence, in this basis,

$$\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\chi = +1\chi \quad ; \quad \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\phi = -1\phi$$

Therefore, in this basis χ and ϕ are the eigenstates of the helicity operator with eigenvalues $+1$ and -1 respectively. We denote,

$$\chi = \psi_R \quad ; \quad \phi = \psi_L$$

ψ_L and ψ_R are two component spinors that tranform as $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations of the Lorentz group.

A Dirac spinor can be written as

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \qquad \psi_L,\ \psi_R \text{ are called Weyl spinors}$$

We define the operators

$$P_{L,R} = \left(\frac{1 \pm \gamma^5}{2} \right)$$

In the chiral basis we have

$$\begin{aligned} P_L &= \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad , \qquad {P_L}^2 = P_L \\ P_R &= \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad , \qquad {P_R}^2 = P_R \end{aligned}$$

and

$$
\begin{aligned}
P_R P_L &= P_L P_R = 0 \\
P_L + P_R &= \frac{1+\gamma^5}{2} + \frac{1-\gamma^5}{2} = 1 \\
P_L \gamma^\mu &= \gamma^\mu P_R \quad ; \quad P_R \gamma^\mu = \gamma^\mu P_L
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
P_L \psi &= \begin{pmatrix} \psi_L \\ 0 \end{pmatrix} = \psi_L \\
P_R \psi &= \begin{pmatrix} 0 \\ \psi_R \end{pmatrix} = \psi_R
\end{aligned}
$$

Furthermore, from $P_R P_L = 0$ we have $P_R \psi_L = 0$, $P_L \psi_R = 0$.

The weak interactions were observed experimentally to have the vector minus axial–vector form, $(V - A)$, *i.e.*

$$(J_V^\mu - J_A^\mu)_{fi} = \bar{\psi}_f \gamma^\mu (1 - \gamma^5) \psi_i$$

If i is a massless particle then $(1 - \gamma^5)\psi_i$ vanishes for helicity $+1$, *i.e.* only left–handed fields interact. The same applies to particle f since

$$\begin{aligned} \bar{\psi}_f \gamma^\mu (1 - \gamma^5) \psi_i &= \psi_f^\dagger \gamma^0 (1 + \gamma^5) \gamma^\mu \psi_i \\ &= \psi_f^\dagger (1 - \gamma^5) \gamma^0 \gamma^\mu \psi_i = \left[(1 - \gamma^5) \psi_f \right]^\dagger \gamma^0 \gamma^\mu \psi_i \end{aligned}$$

This is non–vanishing only if the f–particle is a left–handed field.

In the Standard Model only left–handed fields interact via the weak interactions.

The Langrangian density that gives the Dirac equation of motion

$$
\begin{aligned}
&\mathcal{L}_D = \bar{\psi} i\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi = \\
&\bar{\psi}(P_L^2 + P_R^2) i\gamma^\mu \partial_\mu \psi - m\bar{\psi}(P_L^2 + P_R^2)\psi = \\
&\psi^\dagger P_R \gamma^0 i\gamma^\mu \partial_\mu P_R \psi + \psi^\dagger P_L \gamma^0 i\gamma^\mu \partial_\mu P_L \psi - m\psi^\dagger P_R \gamma^0 P_L \psi - m\psi^\dagger P_L \gamma^0 P_R \psi = \\
&\psi_R^\dagger \gamma^0 i\gamma^\mu \partial_\mu \psi_R + \psi_L^\dagger \gamma^0 i\gamma^\mu \partial_\mu \psi_L - m(\psi_R^\dagger \gamma^0 \psi_L + \psi_L^\dagger \gamma^0 \psi_R) = \\
&\bar{\psi}_R i\gamma^\mu \partial_\mu \psi_R + \bar{\psi}_L i\gamma^\mu \partial_\mu \psi_L - m(\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R).
\end{aligned}
$$

The first two terms are the kinetic terms, whereas the last two are the Dirac mass terms.

We see that the kinetic terms containing the derivatives involve $L \leftrightarrow L$ and $R \leftrightarrow R$ terms, whereas the mass terms involve $L \leftrightarrow R$ and $R \leftrightarrow L$ terms. This is a crucial result for modern particle physics.