

MATH431 Modern Particle Physics Solutions 10

1a The vacuum expectation value of the electroweak Higgs field ϕ :

$$\langle \phi_0 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

The Standard Model $SU(2) \times U(1)_Y$ quantum numbers of the component of the Higgs field with non-trivial VEV are:

$$\phi : T = \frac{1}{2} \quad , \quad T_3 = -\frac{1}{2} \quad , \quad Y = 1$$

and its electric charge

$$Q_{\text{e.m.}}(\phi) = T_3 + \frac{1}{2}Y = -\frac{1}{2} + \frac{1}{2} = 0.$$

Hence, the electromagnetic symmetry remains unbroken by the VEV of the Higgs field.

The Lagrangian density of the Higgs field is given by:

$$\mathcal{L} = \left| \left(i\partial_\mu - ig\vec{T} \cdot \vec{W}_\mu - ig'\frac{Y}{2}B_\mu \right) \phi \right|^2 - V(\phi)$$

where

$$V(\phi) = \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$$

The relevant term for the gauge boson masses:

$$\begin{aligned} & \left| \left(g\frac{\vec{\tau}}{2} \cdot \vec{W}_\mu + g'\frac{1}{2}B_\mu \right) \phi \right|^2 = \\ & \left| \left(\frac{g}{2} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot W_\mu^1 + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cdot W_\mu^2 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot W_\mu^3 \right] + \frac{g'}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot B_\mu \right) \phi \right|^2 = \\ & \frac{1}{4} \left| \left[g \begin{pmatrix} W_\mu^3 & W_\mu^2 - iW_\mu^1 \\ W_\mu^2 + iW_\mu^1 & -W_\mu^3 \end{pmatrix} + g' \begin{pmatrix} B_\mu & 0 \\ 0 & B_\mu \end{pmatrix} \right] \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \right|^2 = \\ & \frac{1}{8} \left| \begin{pmatrix} gW_\mu^3 + g'B_\mu & g(W_\mu^2 - iW_\mu^1) \\ g(W_\mu^2 + iW_\mu^1) & -gW_\mu^3 + g'B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \right|^2 = \\ & \frac{1}{8} v^2 \left(g(W_\mu^2 + iW_\mu^1), (-gW_\mu^3 + g'B_\mu) \right) \begin{pmatrix} g(W_\mu^2 - iW_\mu^1) \\ -gW_\mu^3 + g'B_\mu \end{pmatrix} = \\ & \frac{1}{4} g^2 v^2 W^{\mu+} W^{\mu-} + \frac{1}{8} v^2 (g^2 + g'^2) \frac{(-gW_\mu^3 + g'B_\mu)^2}{(\sqrt{g^2 + g'^2})^2} = \\ & \left(\frac{1}{2} g v \right)^2 W^{\mu+} W^{\mu-} + \frac{1}{2} v^2 \frac{(g^2 + g'^2)}{4} \left(\frac{-gW_\mu^3 + g'B_\mu}{\sqrt{g^2 + g'^2}} \right)^2 = \\ & M_{W^\pm}^2 W^{\mu+} W^{\mu-} + \frac{1}{2} M_Z^2 Z^2 \end{aligned}$$

reading off from the last two lines we have

$$M_{W^\pm} = \frac{gv}{2} \quad ; \quad M_Z = \frac{1}{2}v (g^2 + g'^2)^{\frac{1}{2}}$$

hence

$$\frac{M_W}{M_Z} = \frac{g}{(g^2 + g'^2)^{\frac{1}{2}}} = \cos \theta_W$$

1b We will derive the general case for Higgs representations and then substitute. The general term for the gauge bosons masses has the form

$$\begin{aligned} & \sum_j \left| \left(-ig \vec{T}_j \cdot \vec{W}_\mu - ig' \frac{Y_j}{2} B_\mu \right) \phi_j \right|^2 = \\ & \sum_j \left| \left(-ig [T_j^1 W_\mu^1 + T_j^2 W_\mu^2 + T_j^3 W_\mu^3] - ig' \frac{Y_j}{2} B_\mu \right) \phi_j \right|^2 = \\ & \sum_j \left| \left(g [T_j^+ W_\mu^- + T_j^- W_\mu^+] + \left[g T_j^3 W_\mu^3 + \frac{g'}{2} Y_j B_\mu \right] \right) \phi_j \right|^2 \end{aligned}$$

where in the last line we used

$$T_j^\pm = \frac{1}{\sqrt{2}} (T_j^1 \pm iT_j^2) \quad , \quad W_\mu^\pm = \frac{1}{\sqrt{2}} (W_\mu^1 \pm iW_\mu^2).$$

which yields

$$T^1 W_\mu^1 + T^2 W_\mu^2 = T^+ W_\mu^- + T^- W_\mu^+$$

M_W^2 is the coefficient of $W^+ W^-$,

$$\begin{aligned} M_W^2 &= \left\langle \sum_j g^2 \phi_j^\dagger (T_j^+ T_j^- + T_j^- T_j^+) \phi_j \right\rangle = \\ & g^2 \left\langle \sum_j \phi_j^\dagger (\vec{T}_j^2 - T_{j3}^2) \phi_j \right\rangle = \\ & g^2 \left\langle \sum_j \phi_{j0}^\dagger (T_j(T_j + 1) - T_{j3}^2) \phi_{j0} \right\rangle = \\ & \frac{g^2}{2} \sum_j v_j^2 (T_j(T_j + 1) - T_{j3}^2) \end{aligned}$$

where we used

$$T_j^+ T_j^- + T_j^- T_j^+ = T_j^1 T_j^1 + T_j^2 T_j^2 = \vec{T}_j^2 - T_{j3}^2 = T_j(T_j + 1) - T_{j3}^2$$

and where in the last line correspond to the quantum eigenvalues of the electrically neutral component of the Higgs representations. We require that $Q(\phi_{j0}) = 0$, which ensures that electric charge remains unbroken by the vacuum expectation values of the Higgs fields.

$$\Rightarrow T_{j3} + \frac{Y_j}{2} = 0 \Rightarrow T_{j3} = -\frac{Y_j}{2}$$

$$\Rightarrow M_W^2 = \frac{g^2}{2} \sum_j \left(T_j(T_j + 1) - \frac{Y_j^2}{2} \right) v_j^2$$

M_Z^2 is obtained from the matrix element of the second term

$$\sum_j \left| \left[gT_j^3 W_\mu^3 + \frac{g'}{2} Y_j B_\mu \right] \phi_j \right|^2$$

using $T_{j3} = -\frac{Y_j}{2}$ we get

$$\begin{aligned} & \sum_j \left| \frac{Y_j}{2} [-gW_\mu^3 + g'B_\mu] \phi_j \right|^2 = \\ & \frac{1}{8} \sum_j (y_j^2 v_j^2) \frac{|[-gW_\mu^3 + g'B_\mu]|^2}{(g'^2 + g^2)} (g'^2 + g^2) \\ & \Rightarrow M_Z^2 = \frac{(g'^2 + g^2)}{4} \sum_j v_j^2 y_j^2 \\ & \Rightarrow \frac{M_W^2}{M_Z^2} = \frac{\frac{g^2}{2} \sum_j v_j^2 \left(T_j(T_j + 1) - \frac{Y_j^2}{4} \right)}{\frac{(g^2 + g'^2)}{4} \sum_j v_j^2 Y_j^2} = \\ & \cos^2 \theta_W \frac{\sum_j v_j^2 \left(T_j(T_j + 1) - \frac{Y_j^2}{4} \right)}{\frac{1}{2} \sum_j v_j^2 Y_j^2} \\ & \Rightarrow \rho^2 = \left(\frac{M_W}{M_Z \cos \theta_W} \right)^2 = \frac{\sum_j v_j^2 \left(T_j(T_j + 1) - \frac{Y_j^2}{4} \right)}{\frac{1}{2} \sum_j v_j^2 Y_j^2} \end{aligned}$$

For $T_j = 3$ and $Y = 4$ we have

$$\rho^2 = \frac{v^2 (12 - 4)}{v^2 \frac{1}{2} 16} = 1$$

For any number of doublets $T_j = \frac{1}{2}$, $Y_j = 1$

$$\rho^2 = \frac{\sum_j v_j^2 \left(\frac{3}{4} - \frac{1}{4} \right)}{\frac{1}{2} \sum_j v_j^2} = \frac{\sum_j v_j^2}{\sum_j v_j^2} = 1$$

For a triplet $T_j = 1$ $Y = 2$

$$\rho^2 = \frac{v^2}{\frac{1}{2}v^2} \frac{(2-1)}{4} \neq 1$$

Experimentally, $\rho \approx 1.0000\dots$ is highly constrained.

(3a)

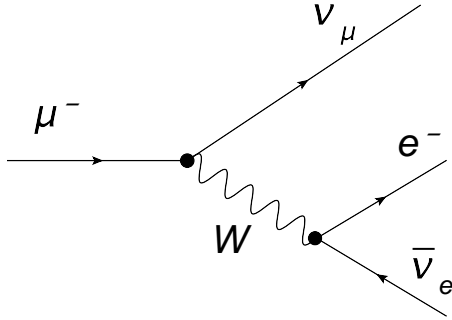
$$\bar{u}_e \gamma^\mu \frac{1}{2} (1 - \gamma^5) u_\nu = \bar{u}_e \frac{1}{2} (1 + \gamma^5) \gamma^\mu u_\nu ,$$

which contains the left-handed electron field \bar{u}_e^L , as

$$\bar{u}_e^L = u_e^{L\dagger} \gamma^0 = u_e^\dagger \frac{1}{2} (1 - \gamma^5) \gamma^0 = \bar{u}_e \frac{1}{2} (1 + \gamma^5) .$$

(3b) The decay $\mu \rightarrow e\gamma$ is allowed by kinematics and w.r.t. charge conservation, but is forbidden in the Standard Model as it would violate the separate conservation of the lepton numbers N_μ and N_e . Empirically, $N_{e,\mu,\tau}$ are conserved and the decay $\mu \rightarrow e\gamma$ is not observed.

(3c) Feynman diagram for $\mu^- \rightarrow \nu_\mu e^- \bar{\nu}_e$ in the Standard Model to lowest order:



The solid ‘external’ lines stand for the incoming and outgoing spin-1/2 particles, the muon, electron, muon-neutrino and electron anti-neutrino. The wavy internal line is the propagator of the W^- boson, the carrier of the weak interaction. It mediates the transition from the initial state μ^- to the final state ν_μ and creates the e^- and $\bar{\nu}_e$ in the final state. The solid dots denote the vertices of this weak interaction.

The algebraic expressions for the different elements are:

- incoming μ^- : spinor u
- outgoing ν_μ : spinor \bar{u}
- outgoing e^- : spinor \bar{u}
- outgoing $\bar{\nu}_e$: spinor v
- vertices: $-i \frac{g}{\sqrt{2}} \gamma^\alpha \frac{1}{2} (1 - \gamma^5)$ and $-i \frac{g}{\sqrt{2}} \gamma^\beta \frac{1}{2} (1 - \gamma^5)$, with weak coupling constant g
- propagator: $i \eta_{\alpha\beta} / (q^2 - m_W^2)$, where q is the four-momentum transfer and m_W is the mass of the W boson

- (3d) The momentum transfer squared, q^2 , is of the order of (but limited by) m_μ^2 , which is very small compared to m_W^2 . Therefore the propagator is well approximated by $-i\eta_{\alpha\beta}/m_W^2$. This is a very strong suppression factor, which would not be present if the W would be massless.
- (3e) The τ lepton is much heavier than the μ or the e , therefore it can decay into both:

$$\tau^- \rightarrow \nu_\tau e^- \bar{\nu}_e, \quad \tau^- \rightarrow \nu_\tau \mu^- \bar{\nu}_\mu.$$

The decays are very similar, with only small differences due to the different masses of the final state particles. In addition, as the τ is also heavier than light hadrons, it can decay in many hadronic final states like pions (the ν_τ must always be there due to N_τ conservation). In these cases the W initially couples to a quark pair which then hadronises. One example is the decay

$$\tau^- \rightarrow \nu_\tau \pi^- \pi^0.$$

4. (a) *First step:*

$$i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so $i\sigma_2$ is unitary.

Second step: Prove by explicit matrix multiplication that $\sigma_2 \sigma_a^* \sigma_2 = -\sigma_a$ ($i = 1, 2, 3$). For $i = 1$ we have e.g.

$$\sigma_2 \sigma_1^* \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -\sigma_1.$$

Third step: Choose $W = i\sigma_2$, and with $\sigma_2^* \sigma_2 = \mathbb{I} = \sigma_2 \sigma_2$ and the result of step two we write:

$$\begin{aligned} W^\dagger U^* W &= -i\sigma_2^\dagger U^* i\sigma_2 = \sigma_2 \exp\left(-\frac{i}{2}\theta_a \sigma_a^*\right) \sigma_2 \\ &= \sigma_2 \left(\mathbb{I} - \frac{i}{2}\theta_a \sigma_a^* - \frac{1}{4} \frac{1}{2!} \theta_a \theta_a - \frac{i}{8} \frac{1}{3!} \theta_a \theta_a \theta_b \sigma_b^* - \dots \right) \sigma_2 \\ &= \left(\mathbb{I} + \frac{i}{2}\theta_a \sigma_a - \frac{1}{4} \frac{1}{2!} \theta_a \theta_a + \frac{i}{8} \frac{1}{3!} \theta_a \theta_a \theta_b \sigma_b - \dots \right) = U. \end{aligned}$$

Taking the complex conjugate of

$$W^\dagger U^* W = U$$

we now also have

$$W^{\dagger\star} U W^{\star} = U^{\star}$$

and as $W = i\sigma_2 = W^{\star}$ we arrive at the desired relation

$$U^{\star} = W^{\dagger} U W.$$

- (b) In the Standard Model, fermion and gauge boson masses are obtained in a gauge invariant way through electroweak symmetry breaking which is mediated by a Higgs potential. Because of the unitary equivalence between the fundamental and the complex conjugate representations of $SU(2)$, gauge invariant mass terms for both up- and down quarks (which are grouped together in $SU(2)$ doublets) can be constructed from only one complex Higgs doublet. In other words, it is due to this special property of $SU(2)$ that the Higgs sector in the Standard Model is the minimal one resulting in only one physical Higgs boson.