from the previous lecture ...

Pauli–Lubanski vector $\qquad W_\sigma = -\frac{1}{2}\epsilon_{\sigma\mu\nu\lambda} J^{\mu\nu} P^\lambda$

Casimir invariants of the Poincare group : $\quad P_\mu P^\mu = m^2 \quad$ and $\quad W_\mu W^\mu$

If $m \neq 0$ take $P^\mu = (m, 0, 0, 0) \quad \Rightarrow \quad -W_\mu W^\mu = m^2 \vec{J}^2$

$$-W_\mu W^\mu |\vec{P}, S\rangle = m^2 j(j+1)|\vec{P}, S\rangle$$

massive particles of spin j have $\underline{2j+1}$ degrees of freedom

If $m = 0$ there is no rest frame. Choose $P^\mu = (w, 0, 0, w)$.

The "Little Group": $\Lambda^\mu{}_\nu P^\nu = P^\mu$

$A,\ B,$ and $L^z = (xP^y - yP^x) \qquad \Rightarrow \quad$ non–compact $\quad \rightarrow \quad$ infinite dimensional

We demand that A and B annihilate the physical massless states

$$\begin{aligned}\hat{A}|\vec{P}, a, b\rangle &= a|\vec{P}, a, b\rangle \\ \hat{B}|\vec{P}, a, b\rangle &= b|\vec{P}, a, b\rangle\end{aligned}$$

with $a = b = 0$ for physical massless states. Therefore,

$$-W_\mu W^\mu = 0$$

for physical massless states. This agrees well with

$$\lim_{m\to 0} W_\mu W^\mu = \lim_{m\to 0} -m^2(j(j+1)) = 0$$

that we found in the massive case. For massless states with $a,\ b = 0$ the little group is $SO(2)$ or $U(1)$. The generator of rotations in the $x - y$ plane is J^3. Hence, the representations are labeled by the eigenvalue h of J^3, which is the angular momentum in the direction of propagation.

$\rightarrow$ helicity : projection of the spin on the direction of momentum.

The helicity is quantised. The proof is based on topological properties of the Lorentz group.

$\rightarrow$ massless states are labeled by their helicity.

$$h = \frac{\vec{P}}{|\vec{P}|} \cdot \vec{J}$$

where $\frac{\vec{P}}{|\vec{P}|}$ is a unit vector in the direction of the momentum. Helicity is quantised. For massless states there are only two helicity states.

$$h = 0,\ \pm\frac{1}{2},\ \pm 1,\ \pm\frac{3}{2},\ \pm 2,\ \cdots$$

Photon	m^2	$=$	0	two polarisation states $h = \pm 1$
Graviton	m^2	$=$	0	two polarisation states $h = \pm 2$

For massive particles there are $(2j + 1)$ helicity states.

For massive states we can go from $-h$ to $+h$ polarisation states by a Lorentz transformation, *i.e.* by a boost.

For massless particles this is not possible as $c = 1$ in all inertial frames.

Helicity is a Lorentz invariant of massless states.

Lagrangian & Hamiltonian Mechanics

Newtonian mechanics : specify position and velocity at $t = t_i$

$$m\frac{d^2X}{dt^2} = F(X) \Rightarrow X = X(t = t_f) \;;$$
$$\dot{X} = \dot{X}(t = t_f)$$

In Newtonian mechanics we specify initial conditions for the position and velocity, and solve the second order differential Newton equation for the position and velocity as a function of time.

Modern particle physics : specify energy and momentum at $t = t_i$

Energy and momentum are constants of the motion. Extract quantities that remain constant in the initial and final time and measure them experimentally.

In modern particle physics calculations are done in the framework of quantum field theories. A bridge between the "old" Newtonian mechanics and the "modern" particle physics is provided by the classical Lagrangian & Hamiltonian formulations of classical mechanics.

$$\underline{\text{Newton}} : \vec{F} = m\vec{a} \quad \Rightarrow \qquad -\vec{\nabla} V(\vec{X}) = m\frac{d^2\vec{X}}{dt^2} \qquad \text{for conserved forces}$$

$$\vec{v} = \frac{d\vec{X}}{dt} \qquad \vec{a} = \frac{d^2\vec{X}}{dt^2}$$

$$\vec{P} = m\frac{d\vec{X}}{dt} \qquad \vec{F} = \frac{d\vec{P}}{dt}$$

Form dynamical functions of $\vec{X}$ and $\vec{v}$

$$\underline{\text{Examples}} : \qquad \vec{L} = \vec{X} \times \vec{P}$$

$$E = \frac{1}{2}m\vec{v}^2 + V(\vec{X}, t) = T + V(\vec{X})$$

Constants of the motion

Energy is conserved if $V = V(\vec{X}) \neq V(t)$ and $\vec{F} = -\vec{\nabla} V(\vec{X})$, which follows from

$$\begin{aligned}
\frac{dE}{dt} &= \frac{d}{dt}\left(\frac{1}{2}m\vec{v}^2 + V(\vec{X})\right) = m\vec{v}\cdot\frac{d\vec{v}}{dt} + \frac{d}{dt}V(\vec{X}) \\
&= m\vec{v}\cdot\dot{\vec{v}} + \vec{\nabla}V(\vec{X})\frac{d\vec{X}}{dt} = \left(m\dot{\vec{v}} + \vec{\nabla}V\right)\cdot\vec{v} = 0
\end{aligned}$$

where the last equality follows from Newton's equation. Take

$$\begin{aligned}
L = T - V &= \frac{1}{2}mv^2 - V(\vec{X}) \\
&= \frac{1}{2}m\dot{\vec{X}}^2 - V(x, y, z) \\
&= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z)
\end{aligned}$$

$$\text{for } x \quad \frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} \qquad \frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x}$$

$$\text{we get} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = m\ddot{x} + \frac{\partial V}{\partial x} = 0$$

This is a very important result and generalises to many mechanical systems and modern field theories.

For a conserving mechanical system with n–degrees of freedom $q_1, \; q_2, \cdots, q_n$ with potential $V(q_1, q_2, \cdots q_n)$.

The 2^{nd} order Euler–Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

The motion of the physical system is solved by specifying $2n$ boundary conditions:

$$q_1, \; \cdots \; , q_n \; , \; \dot{q}_1, \; \cdots \; , \dot{q}_n, \qquad \text{at } t = t_0$$

We define the conjugate momentum

$$p_i = \frac{\partial L}{\partial \dot{q}_i}(\vec{q}, \dot{\vec{q}}, t)$$

Cyclic coordinate a coordinate that does not appear explicitly in L

$$L = L(q_1, \cdots, q_{n-1}, \dot{q}_1, \cdots, \dot{q}_n, t) \Rightarrow \frac{\partial L}{\partial q_n} = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) = 0 \Rightarrow \frac{d}{dt}(p_n) = 0 \Rightarrow p_n = \text{constant} \qquad (1)$$

$\Rightarrow$ Aim $\rightarrow$ find cyclic coordinates $\rightarrow$ constants of the motion

Lagrange formulation $\rightarrow$ system described in terms of n 2^{nd} order diff. eqs.

An alternative formulation is provided by the Hamiltonian formulation

Hamilton $\rightarrow$ describe the system in terms of 1^{st} order differential equations.
Still need to specify $2n$ boundary conditions
the price is that we have $2n$ first order differential equations, rather than n second order equations.
<u>Hamilton</u> change variables from configuration space to phases space

$$
\begin{array}{ll}
\underline{\text{Configuration space :}} & (q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n), \\
\underline{\text{Phase space :}} & (q_1, \cdots, q_n, p_1, \cdots, p_n), \\
\text{where} & p_i = \dfrac{\partial L}{\partial \dot{q}_i}
\end{array}
$$