Yuhang Tia MATH 431 Class Test

1(a) $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dx^2 + dy^2$

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(b) $x \to x + \epsilon A(x,y)$
$y \to y + \epsilon B(x,y)$

$$dx \to dx + \epsilon \left(\frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy \right)$$

$$\cancel{dy \to dy + \epsilon \Big(\frac{\partial A}{}}$$

$$dy \to dy + \epsilon \left(\frac{\partial B}{\partial x} dx + \frac{\partial B}{\partial y} dy \right)$$

$$\cancel{ds^2 \to \Big[\Big(1 + \epsilon \frac{\partial A}{\partial x}}$$

$$ds^2 \to \left[\left(1 + \epsilon \frac{\partial A}{\partial x}\right) dx + \epsilon \frac{\partial A}{\partial y} dy \right]^2 + \left[\left(1 + \epsilon \frac{\partial B}{\partial x}\right) dx + \epsilon \frac{\partial B}{\partial y} dy \right]^2$$

We require the invariances of ds^2. By comparing the equation above to $ds^2 = dx^2 + dy^2$:

dx^2 : $\frac{\partial A}{\partial x} = 0 \Rightarrow A = A(y)$

dy^2 : $\frac{\partial B}{}$

$\longrightarrow$

$$ds^2 \to \left[\left(1+\epsilon\frac{\partial A}{\partial x}\right)dx + \epsilon\frac{\partial A}{\partial y}dy\right]^2 + \left[\left(1+\epsilon\frac{\partial B}{\partial y}\right)dy + \epsilon\frac{\partial B}{\partial x}dx\right]^2$$

~~we~~ we require invariance of ds^2. By comparing the coefficient with $ds^2 = dx^2 + dy^2$:

$$dx^2: \quad \frac{\partial A}{\partial x} = 0 \Rightarrow A = A(y)$$

$$dy^2: \quad \frac{\partial B}{\partial y} = 0 \Rightarrow B = B(x)$$

$$dxdy: \quad \frac{\partial A}{\partial y} - \frac{\partial B}{\partial x} = 0 \Rightarrow \frac{dA}{dy} = \frac{dB}{dx} = \text{constant} = c$$

Therefore we got the transformations:

$$A(y) = cy + a$$
$$B(x) = cx + b$$

where a, b, c are constants.

a represents a shift in space in y-axis
b represents a shift in space in x-axis
c represents boost.

2(a) $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu$$

$$= -\frac{1}{4}\left(A_{\mu,\nu} - A_{\nu,\mu}\right)\left(A^{\mu,\nu} - A_{\nu,\mu}\right) - j^\mu A_\mu$$

$$= -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}\left(A_{\beta,\alpha} - A_{\alpha,\beta}\right)\left(A_{\mu,\nu} - A_{\nu,\mu}\right) - j^\mu A_\mu$$

~~Sub in to~~

Now consider Euler-Lagrange equations:

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial A_{\mu,\nu}}\right) - \frac{\partial L}{\partial A_\mu} = 0$$

$$\frac{\partial L}{\partial A_\mu} = 0 \Rightarrow \frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial A_{\mu,\nu}}\right) = 0$$

$\cancel{\frac{\partial}{\partial A_{p,q}}}$

$$\frac{\partial}{\partial A_{p,q}} = -\frac{1}{4}\eta^{\alpha\beta}\eta^{\beta\nu}\left(\delta^p_\alpha \delta^q_\beta - \delta^p_\beta \delta^q_\alpha\right)\left(A_{\mu,\nu} - A_{\nu,\mu}\right)$$
$$+ \left(A_{\alpha,\beta} - A_{\beta,\alpha}\right)\left(\delta^p_\mu \delta^q_\nu - \delta^p_\nu \delta^q_\mu\right) + j^\mu$$

$$= -\frac{1}{4}\left(\eta^{p\mu}\eta^{q\nu} - \eta^{q\mu}\eta^{p\nu}\right)\left(A_{\mu,\nu} + A_{\nu,\mu}\right) \cancel{+\left(\eta^{\alpha p}\eta^{\beta q} - \eta^{\alpha q}\eta^{\beta p}\right)}$$
$$+ \left(\eta^{\alpha p}\eta^{\beta q} - \eta^{\alpha q}\eta^{\beta p}\right)\left(A_{\beta,\alpha} - A_{\alpha,\beta}\right) + j^\mu$$

$$= -\frac{4}{4}\left(A^{q,p} - A^{p,q}\right) + j^\mu$$

$$= \cancel{-\frac{1}{4}} - F^{pq} + j^\mu$$

$\longrightarrow$

$$\therefore \frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial A_{\mu,\nu}}\right) = \frac{\partial}{\partial x^\mu}\left(-F^{\mu\nu} + j^\mu\right) = 0$$

$$\Rightarrow \partial_\mu F^{\mu\nu} = j^\nu \quad \text{¤}$$