MATH431 — Introduction to Modern Particle Theory: FEEDBACK/SOLUTIONS 8

1. (a) The fundamental triplet representation decomposes into

$$3 \;=\; (2, 1/3) + (1, -2/3)$$

under $SU(2)_I \times U(1)_Y$. The proton and neutron form an isospin doublet with $SU(2)_I$ charges $+1/2$ and $-1/2$, repectively. Then, with the electric charges $+1$ and 0, the combination of T_3 and Y which give the electric charges are given by

$$+1 \;=\; \alpha\frac{1}{2} + \beta\frac{1}{3}\,, \quad 0 \;=\; \alpha(-\frac{1}{2}) + \beta\frac{1}{3}\,,$$

which is solved with $\alpha = +1$ and $\beta = +3/2$.

(b) The decomposition of the sextet, octet and decuplet of $SU(3)$ in terms of $SU(2) \times U(1)$ can be derived from the graphical construction as done in class:

$$\begin{aligned} 6 &= \{(3, 2/3) + (2, -1/3) + (1, -4/3)\}\,, \\ 8 &= \{(2, +1) + (3, 0) + (1, 0) + (2, -1)\}\,, \\ 10 &= \{(4, +1) + (3, 0) + (2, -1) + (1, -2)\}\,. \end{aligned}$$

With $Q = T_3 + \frac{3}{2}Y$ from part (a), we then get the electric charges of the states in the multiplet decompositions,

$$\begin{aligned} 6 &= \{(2, 1, 0) + (0, -1) + (-2)\}\,, \\ 8 &= \{(2, 1) + (1, 0, -1) + (0) + (-1, -2)\}\,, \\ 10 &= \{(3, 2, 1, 0) + (1, 0, -1) + (-1, -2) + (-3)\}\,. \end{aligned}$$

2. (a) There are $5-1 = 4$ traceless, diagonal 5×5 matrices. Knowing the traceless, diagonal $SU(3)$ matrices we already have two diagonal $SU(5)$ generators. A third and forth are then readily constructed from lower dimensional 3×3 and 4×4 identity matrices in the same way we have we have constructed the traceless, diagonal $SU(3)$ matrices. So we get a set of four traceless, diagonal 5×5 generators (conventional numbering)

$$\lambda_3 \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\,, \quad \lambda_8 \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\,,$$

$$\lambda_{15} \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\,, \quad \lambda_{24} \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix}\,.$$

This basis corresponds to the decomposition into a maximal subgroup, $SU(5) \rightarrow SU(4) \times U(1)$. Another basis is given by

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_{15} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad \lambda_{24} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix}.$$

This basis corresponds to the decomposition into another maximal subgroup, $SU(5) \rightarrow SU(3) \times SU(2) \times U(1)$.

(b) The dimension of the group is $D = 24$.

(c) The fundamental representation is the 5. The decomposition under $SU(3) \times SU(2) \times U(1)$ is

$$5 = (3, 1, 1/3) + (1, 2, -1/2),$$

i.e. a $SU(3)$ triplet and $SU(2)$, $U(1)$ singlet with $U(1)$ charges $1/3$, plus a $SU(3)$, $U(1)$ singlet and $SU(2)$ doublet with $-1/2$. ($U(1)$ charges up to an overall normalisation.)

(d) Equivalently the five-bar decomposes as ($\bar{2} = 2$)

$$\bar{5} = (\bar{3}, 1, -1/3) + (1, 2, 1/2).$$

Hence the product is

$$5 \times \bar{5} = \{(3, 1, 1/3) + (1, 2, -1/2)\} \times \{(\bar{3}, 1, -1/3) + (1, 2, 1/2)\}.$$

From the hadron classifications done in class we know the decompositions

$$3 \times \bar{3} = 8 + 1, \quad 2 \times 2 = 3 + 1.$$

Hence in total we get

$$5 \times \bar{5} = 24 + 1 = \{(8, 1, 0) + (1, 3, 0) + (1, 1, 0) + (3, 2, 5/6) + (\bar{3}, 2, -5/6)\} + (1, 0).$$

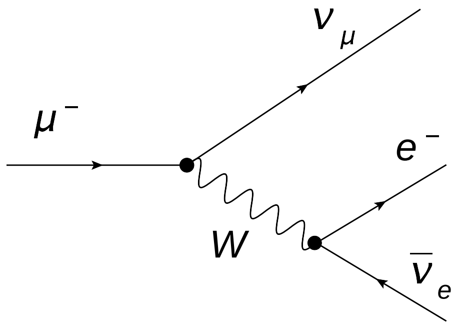
3. (a) With $\{\gamma^5, \gamma^\mu\} = 0$ we can write

$$\bar{u}_e \gamma^\mu \tfrac{1}{2}\left(1-\gamma^5\right) u_\nu \;=\; \bar{u}_e \tfrac{1}{2}\left(1+\gamma^5\right)\gamma^\mu u_\nu\,,$$

which contains the left-handed electron field $\bar{u}_e^L$, as

$$\bar{u}_e^L \;=\; u_e^{L\,\dagger}\gamma^0 \;=\; u_e^\dagger \tfrac{1}{2}\left(1-\gamma^5\right)\gamma^0 \;=\; \bar{u}_e \tfrac{1}{2}\left(1+\gamma^5\right)\,.$$

(b) Feynman diagram for $\mu^- \to \nu_\mu\, e^-\, \bar{\nu}_e$ in the Standard Model to lowest order:



The solid 'external' lines stand for the incoming and outgoing spin-1/2 particles, the muon, electron, muon-neutrino and electron anti-neutrino. The wavy internal line is the propagator of the W^- boson, the carrier of the weak interaction. It mediates the transition from the initial state μ^- to the final state ν_μ and creates the e^- and $\bar{\nu}_e$ in the final state. The solid dots denote the vertices of this weak interaction.
The algebraic expressions for the different elements are:

– incoming μ^-: spinor u

– outgoing ν_μ: spinor $\bar{u}$

– outgoing e^-: spinor $\bar{u}$

– outgoing $\bar{\nu}_e$: spinor v

– vertices: $-i\frac{g}{\sqrt{2}}\gamma^\alpha \frac{1}{2}(1-\gamma^5)$ and $-i\frac{g}{\sqrt{2}}\gamma^\beta \frac{1}{2}(1-\gamma^5)$, with weak coupling constant g

– propagator: $i\eta_{\alpha\beta}/(q^2 - m_W^2)$, where q is the four-momentum transfer and m_W is the mass of the W boson

(c) The momentum transfer squared, q^2, is of the order of (but limited by) m_μ^2, which is very small compared to m_W^2. Thefore the propagator is well approximated by $-i\eta_{\alpha\beta}/m_W^2$. This is a very strong suppression factor (in the amplitude, $\sim 1/m_W^4$ in the decay rate!), which would not be present if the W would be massless.

(d) The τ lepton is much heavier than the μ or the e, therefore it can decay into both:

$$\tau^- \to \nu_\tau\, e^-\, \bar{\nu}_e\;, \quad \tau^- \to \nu_\tau\, \mu^-\, \bar{\nu}_\mu\,.$$

The decays are very similar, with only small differences due to the different masses of the final state particles. In addition, as the τ is also heavier than light hadrons, it can decay in many hadronic final states like pions (the ν_τ must always be there due to N_τ conservation). In these cases the W intially couples to a quark pair which then hadronises. One example is the decay

$$\tau^- \to \nu_\tau\, \pi^-\, \pi^0\,.$$