PAPER CODE NO.
MATH 431

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UNIVERSITY OF LIVERPOOL

May 2021 EXAMINATIONS

Introduction to Modern Particle Physics

TIME ALLOWED: 1 hour

INSTRUCTIONS TO CANDIDATES: In this paper bold-face quantities like $\mathbf{x}$ represent three-dimensional vectors.

Full marks can be obtained for complete answers to TWO questions. Only the best TWO answers will be counted.

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1. The Dirac wave function for the ground state of the hydrogen atom has the form (spin-up state, standard Dirac matrix representation)

$$\psi_\uparrow(r,\theta,\phi) \;=\; R(r)\begin{pmatrix} 1 \\ 0 \\ ia\cos\theta \\ iae^{i\phi}\sin\theta \end{pmatrix},$$

where $a = (1-\sqrt{1-\alpha^2})/\alpha \approx \alpha/2$ (with $\alpha \approx 1/137$ the QED coupling) and R is a function of the radial variable r only.

(a) Investigate whether $\psi_\uparrow$ is an eigenstate of L_z. [4 marks]

(b) Calculate the expectation value of L_z and discuss the result. What would happen in the non-relativistic limit? [8 marks]

(c) Show that $\psi_\uparrow$ is an eigenstate of J_z and find its eigenvalue. [8 marks]

Hint: Don't forget to normalise the state $\psi_\uparrow$.

2. Consider the simple unitary group $SU(4)$.

(a) How many diagonal generators of the Lie algebra are there? Write down a representation of the diagonal generators in terms of 4×4 hermitian matrices.

[3 marks]

(b) What is the dimension of the group? Write down a representation of all generators in terms of 4×4 hermitian matrices.

[3 marks]

(c) What is the fundamental representation of $SU(4)$? Write down its decomposition in terms of a maximal subgroup.

[3 marks]

(d) Discuss the interpretation of the decomposition of the $SU(4)$ group, and its fundamental representation, in terms of the Standard Model subgroups.

[3 marks]

(e) Find the product and the decomposition under the maximal subgroup of the fundamental times the anti-fundamental representations of $SU(4)$.

[4 marks]

(f) Find the product and the decomposition under the maximal subgroup of the fundamental times the fundamental representations of $SU(4)$.

[4 marks]

3. Consider the $SU(2)_W \times U(1)_Y$ Weinberg–Salam model of electroweak symmetry breaking.

(a) The Higgs bosons of the model reside in a complex doublet representation of $SU(2)_W$. Write down the weak and hypercharge charges of the components of the Higgs doublet.

[2 marks]

(b) Calculate the electric charges of the components in part (a).

[2 marks]

(c) Write down the Lagrangian for the Higgs field, including the kinetic and potential term.

[4 marks]

(d)

Assuming a VEV of the form

$$\phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

use the Lagrangian to show that

$$\frac{M_W}{M_Z} = \cos\theta_W$$

where M_W and M_Z are the masses of the charged weak vector bosons and the neutral electroweak vector boson respectively.

[12 marks]