from the previous lecture ...

$J_k^{\pm} = (J_k \pm iK_K) \;\rightarrow\;$ the generators of the Lorentz Tranformations,

$$\left[J_i^+, J_j^+\right] = i\epsilon_{ijk}J_k^+$$
$$\left[J_i^-, J_j^-\right] = i\epsilon_{ijk}J_k^-$$
$$\left[J_i^+, J_j^-\right] = 0$$

The J_i^+ and J_i^- generate two disjoint generators of an $SU(2)$ algebra, $SU(2) \times SU(2)^{\dagger}$.

Each representation of the Lorentz group is labelled by the indecis of the two disjoint $SU(2)$ algebras (j_1, j_2).

Each representation has $(2j_1 + 1) \otimes (2j_2 + 1)$ components.

As $\vec{J} = \vec{J}_+ + \vec{J}_-$ spin is given by $j_1 + j_2$.

examples:	(j_1, j_2)	spin	components	
a	(0,0)	0	1	singlet
b	$(\frac{1}{2}, 0)$	$\frac{1}{2}$	2	Weyl spinor
c	$(0, \frac{1}{2})$	$\frac{1}{2}$	2	Weyl spinor
d	$(\frac{1}{2}, 0) + (0, \frac{1}{2})$	$\frac{1}{2}$	4	Dirac spinor
e	$(\frac{1}{2}, \frac{1}{2})$	1,0	4	vector

write $X^\mu = (t, x, y, z) \;\rightarrow\;$ under rotation t is a singlet and (x, y, z) is a triplet.

the generator of rotations is given by $\vec{J} = \vec{J}_+ + \vec{J}_-$,

under rotations a Lorentz four vector decomposes into a singlet and a triplet.

in the $(\frac{1}{2}, \frac{1}{2})$ representation, each spin $\frac{1}{2}$ state has two components $\uparrow$ and $\downarrow$

we have four possibilities $\uparrow\uparrow$, $\uparrow\downarrow$, $\downarrow\uparrow$, $\downarrow\downarrow$

we can form three symmetric combinations and one asymmetric

$$
\begin{array}{lll}
3 \;\leftrightarrow \uparrow\uparrow;\; \dfrac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow);\; \downarrow\downarrow & & \text{spin} = 1 \;;\; j_3 = (+1, 0, -1) \\
1 \;\leftrightarrow \qquad \dfrac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow); & & \text{spin} = 0 \;;\; j_3 = 0
\end{array}
$$

the triplet is symmetric with respect to exchange of the two spin states, whereas the singlet is asymmetric.

The Poincare group

We saw that spin is a label of representations of the Lorentz group, which correspond to elementary particles in nature. However, we cannot yet classify elementary particles which also have mass. The reason is that the Lorentz transformations are not the most general infinitesimal transformations of the infinitesimal relativistic line element. We should write down the most general transformations of a relativistic line element

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \tag{1}$$

So far we discussed rotations & boosts that are symmetries of the Lorentz group. We want to write the most general set of transformations that keep the line element in eq. (1) invariant.

To determine the most general set of transformations we can look at the two dimensional case

$$dS^2 = c^2 dt^2 - dx^2 \tag{2}$$

The most general inifintesimal transformations are given by

$$\begin{aligned} t &\rightarrow t + \epsilon T(t, x) \\ x &\rightarrow x + \epsilon R(t, x) \end{aligned}$$

where ϵ is an infinitesimal parameters. We want to find the functions T and R such that the line element remains invariant

$$\begin{aligned} dt &\rightarrow dt + \epsilon(\frac{\partial T}{\partial t} dt + \frac{\partial T}{\partial x} dx) \\ dx &\rightarrow dx + \epsilon(\frac{\partial R}{\partial t} dt + \frac{\partial R}{\partial x} dx) \end{aligned}$$

$$ds^2 \to [(1+\epsilon\frac{\partial T}{\partial t})dt + \epsilon\frac{\partial T}{\partial x}dx]^2 - [(1+\epsilon\frac{\partial R}{\partial x})dx + \epsilon\frac{\partial R}{\partial t}dt]^2$$

we require invariance of ds^2. Expanding to first order in ϵ

$$ds^2 \;\to\; dt^2 - dx^2 + 2\epsilon\left(\frac{\partial T}{\partial t}dt^2 - \frac{\partial R}{\partial x}dx^2 + (\frac{\partial T}{\partial x} - \frac{\partial R}{\partial t})dxdt\right) + O(\epsilon^2)$$

we impose that the coefficients of the addtional terms vanish. These yield the constraints on the functions T and R.

$$\begin{aligned}
dt^2 \quad &: \quad \frac{\partial T}{\partial t} = 0 \Rightarrow T = T(x) \\
dx^2 \quad &: \quad \frac{\partial R}{\partial x} = 0 \Rightarrow R = R(t) \\
dxdt \quad &: \quad \frac{\partial T}{\partial x} - \frac{\partial R}{\partial t} = 0 \Rightarrow \frac{dT}{dx} = \frac{dR}{dt} = \text{constant} = c
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow T(x) &= cx + a \\
R(t) &= ct + b
\end{aligned}
$$

We obtained three degrees of freedom that are respresented by the three constants of the motion a, b , c. We can check what are the three constants of motion.

a. choose $c = b = 0$; $a \neq 0 \quad \Rightarrow \quad x \to x$; $t \to t + \epsilon a$

   hence this case corresponds to a translation in time

b. choose $a = c = 0$; $b \neq 0 \quad \Rightarrow \quad t \to t$; $x \to x + \epsilon b$

   hence this case corresponds to a translation in space

c. choose $a = b = 0$; $c \neq 0 \quad \Rightarrow \quad t \to t + \epsilon cx$; $x \to \epsilon ct + x$

   the third transformation corresponds to a boost

If we had taken the line element to be $ds^2 = dt^2 + dx^2$ we would have obtained orinary rotations.

$$\begin{aligned} t &\rightarrow t + \epsilon c x \\ x &\rightarrow x - \epsilon c t \end{aligned}$$

or in matrix form

$$\begin{pmatrix} t \\ x \end{pmatrix} \rightarrow \begin{pmatrix} t \\ x \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

that we have seen before. Hence, the case of the infinitesimal line element $ds^2 = dt^2 + dx^2$ corresponds to rotaions in two dimensions. The boost transformation is a generalisation of rotations to four dimensional Minkowski spacetime.

Returning to four dimensional Minkowski spacetime in eq. (1)
$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$,

$$\underline{\text{we have}} \qquad \begin{array}{ll} 4 - \text{translations} & dt, dx, dy, dz \\ 3 - \text{rotations} & dxdy, dxdz, dydz \\ 3 - \text{boosts} & dtdx, dtdy, dtdz \end{array}$$

The group that describes this set of symmetries is the Poincare group. The total number of generator of the Poincare group is 10. Our aim is to find the algebra of the Poincare group and its invariants. This will give us the labels of elementary particles. The translation symmetries are important. They relate to the momentum operator that generates translations. We will therefore obtain momentum and mass from the relativistic invariant $P_\mu P^\mu = m^2$, where P^μ is the particle momentum four vector. Mass is the second label of particle states.