

Math 431 Particle Theory Class Test 15.4.21

1a. $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dx^2 + dy^2$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1b. Want to find functions

$$A(x, y) = \xi^1(x, y)$$

$$B(x, y) = \xi^2(x, y)$$

such that ds^2 remains invariant under the transformations:

$$\begin{aligned} x &\rightarrow x + \epsilon A(x, y) \\ y &\rightarrow y + \epsilon B(x, y) \end{aligned}$$

Keeping terms to 1st order in ϵ

$$dx \rightarrow \left(1 + \epsilon \frac{dA}{dx}\right) dx + \epsilon \frac{dA}{dy} dy$$

$$dy \rightarrow \left(1 + \epsilon \frac{dB}{dy}\right) dy + \epsilon \frac{dB}{dx} dx$$

$$dx^2 \rightarrow \left(1 + 2\epsilon \frac{dA}{dx}\right) dx^2 + 2\epsilon \frac{dA}{dy} dx dy$$

$$dy^2 \rightarrow \left(1 + 2\epsilon \frac{dB}{dy}\right) dy^2 + 2\epsilon \frac{dB}{dx} dx dy$$

Demanding ds^2 invariant so

$$dx^2 + dy^2 \rightarrow dx^2 + dy^2 + \underbrace{0}_{\text{vanishing terms}}$$

$$dx^2: \frac{dA}{dx} = 0 \Rightarrow A = A(y) = f'(y) = \frac{df}{dy}$$

$$dy^2: \frac{dB}{dy} = 0 \Rightarrow B = B(x) = g'(x) = \frac{dg}{dx}$$

$$dx dy: \frac{\partial A}{\partial y} + \frac{\partial B}{\partial x} = 0 \Rightarrow \frac{\partial A}{\partial y} = -\frac{\partial B}{\partial x} \quad \text{?}$$

Only holds if $\frac{\partial A}{\partial y} = \text{constant} = -\frac{\partial B}{\partial x}$

$$\text{Since } A = A(y); \quad \frac{\partial A}{\partial y} = \frac{dA}{dy} (= f''(y))$$

$$\frac{dA}{dy} = c \quad \text{likewise} \quad \frac{dB}{dx} = -c$$

$$A = cy + d \quad \text{and} \quad B = -cx + e$$

where d, e constants of integration.

We are left with three parameters, c, d and e which we denote as α, β and γ respectively;

$$A = \alpha y + \beta \quad \text{?}$$

$$B = \gamma - \alpha x \quad \text{?}$$

Lorentz boosts.

X

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$$2a. \quad L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu \quad \begin{matrix} F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \\ F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \end{matrix}$$

$$L = -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu) (\partial^\mu A^\nu - \partial^\nu A^\mu) - j^\mu A_\mu$$

$$L = -\frac{1}{4} \eta^{\alpha\mu} \eta^{\beta\nu} (A_{\beta,\alpha} - A_{\alpha,\beta}) (A_{\mu,\nu} - A_{\nu,\mu}) - j^\mu A_\mu$$

Euler-Lagrange equations:

$$\frac{d}{dx^\nu} \left(\frac{\partial L}{\partial A_{\mu,\nu}} \right) - \frac{\partial L}{\partial A_\mu} = 0 \quad 2$$

$$\frac{\partial L}{\partial A_\mu} = -j^\mu \quad \Rightarrow \quad \frac{d}{dx^\nu} \left(\frac{\partial L}{\partial A_{\mu,\nu}} \right) = j^\mu \quad 4$$

$$\begin{aligned} \frac{\partial L}{\partial A_{p,q}} &= -\frac{1}{4} \eta^{\alpha\mu} \eta^{\beta\nu} (\delta_\alpha^p \delta_\beta^q - \delta_\beta^p \delta_\alpha^q) (A_{\mu,\nu} - A_{\nu,\mu}) \\ &\quad + (A_{\alpha\beta} - A_{\beta\alpha}) (\delta_\mu^p \delta_\nu^q - \delta_\nu^p \delta_\mu^q) \end{aligned}$$

$$\begin{aligned} &= -\frac{1}{4} (\eta^{\rho\mu} \eta^{q\nu} - \eta^{q\mu} \eta^{\rho\nu}) (A_{\mu,\nu} - A_{\nu,\mu}) \\ &\quad + (\eta^{\alpha\rho} \eta^{\beta q} - \eta^{\alpha q} \eta^{\beta\rho}) (A_{\beta,\alpha} - A_{\alpha,\beta}) \end{aligned}$$

$$= -4 \cdot \frac{1}{4} (A^{q,p} - A^{p,q})$$

$$= -F^{pq} \quad 4$$

$$\Rightarrow \frac{d}{dx^\nu} \left(\frac{\partial L}{\partial A_{\mu,\nu}} \right) = \frac{d}{dx^\nu} (-F^{\mu\nu}) = j^\mu$$

$$\Rightarrow \frac{d}{dx^\mu} (-F^{\nu\mu}) = j^\nu \quad 1$$

$$\Rightarrow \partial_\mu F^{\mu\nu} = j^\nu$$

$$A_\mu \rightarrow \tilde{A}_\mu = A_\mu + d_\mu \Delta$$

$$F_{\mu\nu} \rightarrow \tilde{F}_{\mu\nu}$$

$$\begin{aligned}\tilde{F}_{\mu\nu} &= d_\mu \tilde{A}_\nu - d_\nu \tilde{A}_\mu = d_\mu (A_\nu - d_\nu \Delta) - d_\nu (A_\mu - d_\mu \Delta) \\ &= d_\mu A_\nu - d_\mu d_\nu \Delta - d_\nu A_\mu + d_\nu d_\mu \Delta \\ &= d_\mu A_\nu - d_\nu A_\mu \\ &= F_{\mu\nu}\end{aligned}$$

Therefore, L is also invariant under transformation and j^ν is conserved.
 Since $\tilde{j}^\nu = d_\mu \tilde{F}^{\mu\nu}$.

$$2b. \quad L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu + \frac{1}{2} m^2 A_\mu A^\mu$$

$$= -\frac{1}{4} \eta^{\alpha\mu} \eta^{\beta\nu} (A_{\beta,\alpha} - A_{\alpha,\beta}) (A_{\mu,\nu} - A_{\nu,\mu}) \\ - j^\mu A_\mu + \frac{1}{2} m^2 A_\mu \eta^{\mu\rho} A_\rho$$

~~$$\frac{\partial L}{\partial A_\mu} = -j^\mu + \frac{1}{2} m^2 \eta^{\mu\rho} A_\rho + \frac{1}{2} m^2 \eta^{\mu\rho} A_\mu$$~~

~~$$= -j^\mu + \frac{1}{2} m^2 (A^\mu + A^\mu)$$~~

$$L = -\frac{1}{4} \eta^{\alpha\mu} \eta^{\beta\nu} (A_{\beta,\alpha} - A_{\alpha,\beta}) (A_{\mu,\nu} - A_{\nu,\mu}) \\ + A_\mu \left(\frac{1}{2} m^2 A^\mu - j^\mu \right)$$

$$\cancel{\frac{d}{dx^\mu}} \partial_\nu (-F^{\mu\nu}) = \cancel{\frac{d}{dx^\mu}} j^\mu - \frac{1}{2} m^2 A^\mu$$

$$\partial_\mu F^{\mu\nu} = j^\nu - \frac{1}{2} m^2 A^\nu$$

$$\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = j^\nu - \frac{1}{2} m^2 A^\nu$$

$$\partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = j^\nu - \frac{1}{2} m^2 A^\nu$$

$$\frac{\partial L}{\partial A_\mu} = -j^\mu + \frac{1}{2} m^2 A^\mu + \frac{1}{2} m^2$$

$$\bullet \quad \frac{\partial}{\partial x^\nu} (-F^{\mu\nu}) - \frac{\partial}{\partial A_\mu} \left(-j^\mu A_\mu + \frac{1}{2} m^2 A_\mu A^\mu \right) = 0 \quad 2$$

Lorentz gauge $\partial_\mu A^\mu = 0$

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