MATH431.

Q1

(a) $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dx^2 + dy^2$

$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

inverse $= (g_{\mu\nu})^{-1} = g^{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $x \to x + \epsilon A(x, y)$

$y \to y + \epsilon B(x, y)$

$dx \to dx + \epsilon \left(\frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy \right)$

$dy \to dy + \epsilon \left(\frac{\partial B}{\partial y} dy + \frac{\partial B}{\partial x} dx \right)$

$$ds^2 \to \left[\left(1+\epsilon\frac{\partial A}{\partial x}\right)dx + \epsilon\frac{\partial A}{\partial y}dy\right]^2 + \left[\left(1+\epsilon\frac{\partial B}{\partial x}\right)dx + \epsilon\frac{\partial B}{\partial y}dy\right)\right]^2$$

we require invariance of ds^2.

Expanding to first order in 'ϵ'.

$$dx^2 \to \left(1+2\epsilon\frac{\partial A}{\partial x}\right)dx^2 + 2\epsilon\left(\frac{\partial A}{\partial y}\right)dx\,dy$$

$$dy^2 \to \left(1+2\epsilon\frac{\partial B}{\partial y}\right)dy^2 + 2\epsilon\left(\frac{\partial B}{\partial x}\right)dx\,dy.$$

we then find the following

$$ds^2 \to \underset{①}{\left[\left(1+\epsilon\frac{\partial A}{\partial x}\right)dx + \epsilon\frac{\partial A}{\partial y}dy\right]^2} + \underset{②}{\left[\left(1+\epsilon\frac{\partial B}{\partial y}\right)dy + \epsilon\frac{\partial B}{\partial x}dx\right]^2}.$$

(to check).

$$① \left[\left(1+\epsilon\frac{\partial A}{\partial x}\right)dx + \frac{\epsilon\partial A}{\partial y}dy\right]\left[\left(1+\epsilon\frac{\partial A}{\partial x}\right)dx + \epsilon\frac{\partial A}{\partial y}dy\right]$$

$$= \left(1+\epsilon\frac{\partial A}{\partial x}\right)^2 dx^2 + 2\left(1+\epsilon\frac{\partial A}{\partial x}\right)dx\,\epsilon\frac{\partial A}{\partial y}dy + \epsilon^2\left(\frac{\partial A}{\partial y}\right)^2 dy^2.$$

$$= (dx^2)\left(1+2\epsilon\frac{\partial A}{\partial x} + \underbrace{\epsilon^2\frac{\partial A^2}{\partial x^2}}_{2^{nd}\text{ order term.}}\right) + 2\epsilon\frac{\partial A}{\partial y}dy + \ldots O(\epsilon^2)$$

$$= (dx)^2 + 2\epsilon \frac{\partial A}{\partial x} dx^2 + 2\epsilon \frac{\partial A}{\partial y} dy \cancel{dx}$$

$$② \left[\left(1+\epsilon\frac{\partial B}{\partial y}\right)dxy + \epsilon\frac{\partial B}{\partial y}\cancel{dx}\right]\left[\left(1+\epsilon\frac{\partial B}{\partial y}\right)dxy + \epsilon\frac{\partial A}{\partial y}\cancel{dx}\right]$$

$$\Rightarrow \left(1+\epsilon\frac{\partial B}{\partial y}\right)^2 \cancel{dx}^2 + 2\left(1+\epsilon\frac{\partial B}{\partial y}\right)\cancel{dy}\,\epsilon\frac{\partial A}{\partial y}\cancel{dy} + \epsilon^2\frac{\partial B}{\partial y}^2\cancel{dy}^2$$

$$\Rightarrow (dy)^2 + 2\epsilon\frac{\partial B}{\partial y}dy^2 + 2\epsilon\frac{\partial B}{\partial x}dx$$

$$ds^2 \to dx^2 + dy^2 + 2\epsilon\left(\frac{\partial A}{\partial x}dx^2 + \frac{\partial B}{\partial y}dy^2 + \left(\frac{\partial A}{\partial y} + \frac{\partial B}{\partial x}\right)dxdy\right) + O(\epsilon^2).$$

terms of additional coefficients vanish.

$$dx^2: \frac{\partial A}{\partial x} = 0 \Rightarrow A = A(y)$$

$$dy^2: \frac{\partial B}{\partial y} = 0 \Rightarrow B = B(x)$$

$$dxdy: \left(\frac{\partial A}{\partial y} + \frac{\partial B}{\partial x}\right) = 0 \Rightarrow \frac{\partial A}{\partial y} = -\frac{\partial B}{\partial x} = \text{constant} = c.$$

$$A(y) = cy + a, \quad B(x) = cx + b$$

We obtain 3 degrees of freedom represented by the constants a, b, c.

- $a = c = 0$; $y \to y \cancel{+z} \epsilon b$ - $y \to y + \epsilon b$
  $b \neq 0$ (translation in space)

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- We end up from the line element with 2 translations in space and 1 rotation. translation in x and y and rotation in xy plane.

Q2

(a) $L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^{\mu} A_{\mu}$

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} \eta^{\alpha\mu} \eta^{\beta\nu} (A_{\beta,\alpha} - A_{\alpha,\beta})(A_{\mu,\nu} - A_{\nu,\mu})$$

$$= -\frac{1}{4} (A_{\mu,\nu} - A_{\nu,\mu})(A^{\mu\nu} - A_{\nu\mu})$$

From Euler Lagrange eqn.

$$\frac{\partial}{\partial x^{\nu}} \left(\frac{\partial L}{\partial A_{\mu,\nu}} \right) - \frac{\partial L}{\partial A_{\mu}} = 0$$

$$\frac{\partial L}{\partial A_{p,q}} = -\frac{1}{4} \eta^{\alpha\mu} \eta^{\beta\nu} (\delta^{p}_{\alpha} \delta^{q}_{\beta} - \delta^{p}_{\beta} \delta^{q}_{\alpha})(A_{\mu,\nu} - A_{\nu,\mu})$$
$$+ (A_{\alpha,\beta} - A_{\beta,\alpha})(\delta^{p}_{\mu} \delta^{q}_{\nu} - \delta^{p}_{\nu} \delta^{q}_{\mu})$$

$$= -\frac{1}{4} (\eta^{p\mu} \eta^{q\nu} - \eta^{q\mu} \eta^{p\nu})(A_{\mu,\nu} + A_{\nu,\mu})$$
$$+ (\eta^{\alpha p} \eta^{\beta q} - \eta^{\alpha q} \eta^{\beta p})(A_{\beta,\alpha} - A_{\alpha,\beta})$$

$$= -4 \times \frac{1}{4} (A^{q,p} - A^{p,q}) = -F^{pq}$$

$$\frac{\partial}{\partial x^{\nu}} \left(\frac{\partial L}{\partial A_{\mu,\nu}} \right) - \left(\frac{\partial L}{\partial A_{\mu}} \right) = -\frac{1}{4} \frac{\partial}{\partial x^{\nu}} \left(-F^{pq} \right) = -j^{\mu}$$

j^{μ} comes from $\left(-\frac{\partial L}{\partial A_{\mu}} \right)$ term. $\therefore \frac{\partial}{\partial x^{\nu}} = \partial_{\mu} \Rightarrow \partial_{\mu} F^{\mu\nu} = j^{\nu}$

hence show $j^{\mu} = 0$.

given a symmetry transformation, we can show that the four current is conserved.

i.e show $\partial_\mu j^{\mu} = 0$.

$\partial_\mu j^{\mu} = \nabla \cdot \bar{J} = 0$

(6) $\frac{1}{2} m^2 A_\mu A^\mu$

$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^{\mu} A_\mu + \frac{1}{2} m^2 A_\mu A^\mu$

$$\left(\frac{\partial \mathcal{L}}{\partial A_{\mu,\nu}}\right) = \frac{\partial}{\partial x^\nu}\left(\frac{\partial \mathcal{L}}{\partial A_{\mu,\nu}}\right)$$

$$\frac{\partial}{\partial x^\nu}\left(\frac{\partial \mathcal{L}}{\partial A_{\mu,\nu}}\right) = (A^{q,p} - A^{p,q}) + \frac{\partial}{\partial x^\nu}\left(\frac{1}{2} m^2 A_\mu A^\mu\right)$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

① $= -(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{\partial}{\partial x^\nu}(\frac{1}{2} m^2 A_\mu A^\mu)$

$\Rightarrow \frac{1}{2} m^2 \frac{\partial}{\partial x^\nu} (A_\mu A^\mu)$

$= \ldots$

$\Rightarrow \partial_\mu A^\mu A_\mu + \cancel{\partial_\mu A^\mu A_\mu}$

$\partial_\mu A_\mu A^\mu$

$\Rightarrow (\partial_\mu A^\mu + \partial^\mu A^\mu + m^2 A^\mu)$

(cancels with term in ①)