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Lorentz group

So far we discussed the formal structure of Lagrangian & Hamiltonian mechanics that emanates from classical mechanis and hence provide the foundational and conceptual link from classical → quantum → quantum field theory

(**) An additional important component of quantum field theories is the invariance under Lorentz transformations. Quantum mechanics in the schrödinger formulation is nonrelativistic and the quantum field theories are relativistic

The transition from the non-relativistic to the relativistic quantum field theories was achieved by the Klein-Gordon Eq. & Dirac Eq

that we will study in some detail.

First I discuss some properties of the Lorentz group that underlies special relativity

Lorentz group.

A contravariant 4-vector $X^\mu = (ct, \vec{X})$ $\mu = 0,1,2,3$

In particle physics we often set $c = 1$ $\hbar = 1$

In this convention $m_{electron} = 0.5 \frac{MeV}{c^2} = 0.5 MeV$

etc. $c \neq 1$ may be restored if needed, otherwise all quatities are expressed in energy units.

hence

$$X^0 = t \qquad X^i = \vec{X}$$

the length of the four vector is given by

$$X \cdot X = c^2 t^2 - x^2 - y^2 - z^2 \underset{c=1}{=} t^2 - x^2 - y^2 - z^2$$

This is symbolized by

$$X \cdot X = \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} X^\mu X^\nu = X_\mu X^\mu$$

where

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

is the Minkowski metric.

In general the scalar product of two 4-vectors X^μ; Y^μ is given by $X \cdot Y = \sum_{\mu,\nu} \eta_{\mu\nu} X^\mu Y^\nu$

We will use Einstein's summation convention a down index is summed with an up index and the summation symbol is dropped.

Scalar - no free indecies. - (all are summed in the scalar product).

Vector - one free index (X^μ).

Tensor - two and more ($g^{\mu\nu}$, $R_{\mu\nu\rho\sigma}$, ...)

The Lorentz transformations are the transformation that preserve the scalar product,

In particular they preserve the length of a 4-vector

$$t^2 - \vec{X}^2 = X \cdot X = \eta_{\mu\nu} X^\mu X^\nu = X' \cdot X' = \eta_{\mu\nu} X'^\mu X'^\nu = t'^2 - \vec{X}'^2$$

where X and X' are related by a Lorentz transformation.

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In general we write the metric as $g^{\mu\nu}$ and its components can be functions of space-time $\Rightarrow$ general relativity. curved space

In flat-space $g^{\mu\nu} \to \eta^{\mu\nu}$

$g^{\mu\nu} g_{\nu\sigma} = \delta^{\mu}_{\sigma}$ $(g^{\mu\nu})^{-1} =$

<u>we write:</u> $X_\mu = \sum_{\nu=0}^{3} g_{\mu\nu} X^\nu$

X_μ is a covariant vector.

$X_\mu = (t, -\vec{X})$.

The Lorentz invariant can be written as $X_\mu X^\mu$.

Given a vector $X^\mu = (t, \vec{X})$.

There are two differential quantities of interest.

1. $dX^\mu \to$ differential $\to$ contravariant 4-vector
2. $\frac{\partial}{\partial X^\mu} = \partial_\mu \to$ gradient $\to$ covariant 4-vector.

How do they behave under coordinate transformations $X^\mu \to X'^\mu$.

$$dX^\mu \to dX'^\mu = \frac{\partial X'^\mu}{\partial X^\nu} dX^\nu$$

$$\frac{\partial}{\partial X^\mu} \to \frac{\partial}{\partial X'^\mu} = \frac{\partial X^\nu}{\partial X'^\mu} \frac{\partial}{\partial X^\nu}$$

The differential and the gradient transform differently.

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Vector that transforms like the differential is a contra-variant vector.

$$V'^{\mu} = \frac{\partial X'^{\mu}}{\partial X^{\nu}} V^{\nu}$$

Vector that transforms like the gradient is a covariant vector,

$$V'_{\mu} = \frac{\partial X^{\nu}}{\partial X'^{\mu}} V_{\nu} \qquad \boxed{\partial_{\mu} = \frac{\partial}{\partial X^{\mu}}}$$

Example the momentum vector $P^{\mu} = (E, \vec{P})$

$P_{\mu} = (E, -\vec{P})$.

in relativistic quantum mechanics $P_{\mu} \sim \frac{\partial}{\partial X^{\mu}}$

$P_{\mu} X^{\mu} = Et - \vec{P} \cdot \vec{X}$

Properties of Lorentz transformations

1. Transformation that preserve the scalar product and the length of 4-vectors.

$g_{\mu\nu} X^{\mu} X^{\nu}$ is invariant under Lorentz transformation

no change in size and shape.

The invariance implies the existence of a symmetry

The symmetry is generated by a GROUP.

2. Assume $x^\mu \to x'^\mu$ under some Lorentz trans.

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

$$g_{\mu\nu} x^\mu x^\nu \to g_{\mu\nu} x'^\mu x'^\nu = g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta x^\alpha x^\beta$$

$$= g_{\alpha\beta} \, x^\alpha x^\beta$$

$\Rightarrow (*) \quad g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = g_{\alpha\beta}$ | in matrix notation $\Lambda^T g \Lambda =$

$\longrightarrow$ Defines the Lorentz transformations.

$\Rightarrow (\det \Lambda)^2 = 1 \Rightarrow \det \Lambda = \pm 1$

The physical case $\det \Lambda = 1 \to$ continuously connected to the identity.

$\det \Lambda = 1 \quad \to$ Proper Lorentz transformations

$\det \Lambda = -1 \quad \to$ improper " "

Look at the 00 component of $\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g_{\mu\nu} = g_{\alpha\beta}$

$$g_{\mu\nu} \Lambda^\mu{}_0 \Lambda^\nu{}_0 = -1$$

$$-(\Lambda^0{}_0)^2 + \sum_i (\Lambda^i{}_0)^2 = -1$$

$$(\Lambda^0{}_0)^2 = 1 + \sum_i (\Lambda^i{}_0)^2 \geq 1$$

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$$(\Lambda^0{}_0)^2 \geq 1 \begin{cases} \Lambda^0{}_0 \geq 1 & \text{orthochronos} \\ \Lambda^0{}_0 \leq -1 & \text{non orthochronos} \end{cases}$$

An example of a nonorthochronos Lorentz transformation is given by:

Reflection: $\Lambda^\mu{}_\nu = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

The Lorentz transformations that are continuously connected to the ident.

ⓐ $\det \Lambda = 1$ ← Proper.

ⓑ $\Lambda^0{}_0 \geq 1$ ← orthochronos.

Examples: $\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ $\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$.

ⓐ Rotation: $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & \vec{R} \end{pmatrix}$ $\det \Lambda = \det R$

$\det R = \pm 1$

$\det R = +1 \rightarrow$ Proper.

ⓑ boosts in x-direction $\Lambda = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\det \Lambda = \cosh^2\eta - \sinh^2\eta = 1$$

$$\Lambda^0{}_0 = \cosh\eta \geq 1.$$

ⓒ time inversion $\Lambda = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

$\det \Lambda = -1$ $\Lambda^0{}_0 = -1 \Rightarrow$ improper non-orthochronos.

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(d) Full inversion $\Lambda = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

$\det \Lambda = +1 \quad \Lambda^0{}_0 = -1 \Rightarrow$ Proper non-orthochronos

$\rightarrow$ All L.T. are generated by the above.

The proper orthochronos Lorentz transformations correspond to Rotations & boosts.

$\rightarrow$ 6 parameters = 3 angles + 3 boosts.

L.T. that are proper orthochronos are continuousely connected to the identity

Performing an infinitesimal proper orthochronos L.T.

(**) $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$

identity trans. — an infinitesimal transformation

$\delta^\mu{}_\nu = +1$ for $\mu = \nu$; $= 0$ for $\mu \neq \nu$

calculate to order $O(\omega)$

for continuous transformations the properties of infinitesimal transformations fixes the transformation properties of the Lorentz group.

Finite transformations are obtained by integration

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substitute (**) in $g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = g_{\alpha\beta}$

$$g_{\mu\nu}(\delta^\mu{}_\alpha + W^\mu{}_\alpha)(\delta^\nu{}_\beta + W^\nu{}_\beta) = g_{\alpha\beta}$$

$$g_{\mu\nu}\delta^\mu{}_\alpha \delta^\nu{}_\beta + \underbrace{g_{\mu\nu} W^\mu{}_\alpha \delta^\nu{}_\beta}_{g_{\mu\beta} W^\mu{}_\alpha} + \underbrace{g_{\mu\nu}\delta^\mu{}_\alpha}_{g_{\alpha\nu}} W^\nu{}_\beta + g_{\mu\nu} W^\mu{}_\alpha W^\nu{}_\beta = g_{\alpha\beta}$$

$$g_{\alpha\beta} \quad + \quad W_{\beta\alpha} \quad + \quad W_{\alpha\beta}$$ end of lecture 2

$$\omega_{\beta\alpha} + \omega_{\alpha\beta}$$

The tensor of infinitesimal transformations is antisymmetric.

Example: Rotation group in two dimensions.

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\theta \quad \text{for small } \theta.$$

The number of degrees of freedom in a 4×4 antisymmetric matrix.

For general n: $\dfrac{n^2 - n}{2} = \dfrac{n(n-1)}{2}$

for $n = 4 \Rightarrow 4 \cdot 3/2 = 6 \rightarrow$ 3 Rotation angles + 3 boosts.

$$W_{\mu\nu} = \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & b_3 & -b_2 \\ -a_2 & -b_3 & 0 & b_1 \\ -a_3 & b_2 & -b_1 & 0 \end{pmatrix}$$

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Exercise: show how a finite transformation is obtained from infinitesimal transformations.

Algebraic Properties of the Lorentz Group

we associate an operator $U(\Lambda)$ with the LT Λ.

for the special case $\Lambda = \delta \rightarrow U(\delta) = \mathbb{I}$

we want to find the operator associated with $\Lambda = \delta + \omega$.

to order $O(\omega)$ $\quad U(\delta+\omega) = \mathbb{I} + \frac{1}{2} i \underbrace{J_{\mu\nu}}_{\text{operators}} \underbrace{\omega^{\mu\nu}}_{\text{infinitesimal trans.}} + \dots$

Look at the example of rotations $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & \alpha & -\beta \\ -\alpha & 1 & \gamma \\ \beta & -\gamma & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$A = \mathbb{I} + \alpha \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

operators $J^{\mu\nu}$ $\qquad \alpha, \beta, \gamma \in \omega^{\mu\nu}$

These operators are non-hermitian $\rightarrow$ multiply by i to get hermitian operators

$$A = \mathbb{I} + i\alpha \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + i\beta \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} + i\gamma \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

now the operators are hermitian.

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From the definition of $U(\delta+\omega)$ we defined 16 $J_{\mu\nu}$ operators

only the antisymmetric components are independent → only 6 are independent,

Define the operators

$$K_i = J_{i0} = -J_{0i} \quad i=1,2,3 \qquad \text{(3 operators)}$$

$$J_i = \frac{1}{2}\sum_{jk} \epsilon_{ijk} J_{jk} \qquad \text{(3 operators)}$$

This is a projection of the antisymmetry of ω on J.
These are the six physical generators of the L.G.

$$U(\delta+\omega) = I + i\vec{a}\cdot\vec{K} - i\vec{b}\cdot\vec{J} \qquad \omega^{\mu\nu} = \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & b_3 & -b_2 \\ -a_2 & -b_3 & 0 & b_1 \\ -a_3 & b_2 & -b_1 & 0 \end{pmatrix}$$

Example: Regular Rotations $\left(A = I + i\vec{\alpha}\vec{J}\right)$

The operators $\vec{K}, \vec{J}$ are the basic operators of the L.G.
These are the generators of the L.G.
$\vec{J}$ are the generators of Rotations
K are the generators of boosts.

We will determine the commutation relations by imposing the group property on U

$(*)$ $U(\Lambda_1\Lambda_2) = U(\Lambda_1)\,U(\Lambda_2)$ ← the group property.

for Regular rotations $\Lambda_1 = \theta_1 \quad \Lambda_2 = \theta_2 \quad \Lambda_1\Lambda_2 = \theta_1+\theta_2$

$$\begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix}\begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{pmatrix} = \begin{pmatrix} \cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) \\ -\sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix}.$$

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$$U(\Lambda_1) = I + \frac{i}{2} J_{\mu\nu} W_1^{\mu\nu} \qquad \Lambda_1 = \delta + \omega_1$$

$$U(\Lambda_2) = I + \frac{i}{2} J_{\mu\nu} W_2^{\mu\nu} \qquad \Lambda_2 = \delta + \omega_2$$

$$\Lambda_1\Lambda_2 = (\delta+\omega_1)(\delta+\omega_2) \underset{\text{first order}}{=} \delta + (\omega_1+\omega_2) + O(\omega^2)$$

$$\rightarrow \quad U(\Lambda_1\Lambda_2) = I + \frac{i}{2} J_{\mu\nu}\left(\omega_1^{\mu\nu} + \omega_2^{\mu\nu}\right).$$

substituting into the group property (*) and keeping terms to second order in $O(\omega^2)$ we get the commutation relations.

otations $\rightarrow$

$$[J_i, J_j] = i\epsilon_{ijk} J_k \qquad i,j,k = 1,2,3$$

$$[J_i, K_j] = i\epsilon_{ijk} K_k$$

$$[K_i, K_j] = -i\epsilon_{ijk} J_k.$$

The minus sign in the third relation distinguishes The Lorentz transformations from ordinary Rotations in four dimensions

define the combinations

$$\vec{J}_+ = \frac{1}{2}(\vec{J} + i\vec{K}) \qquad \vec{J}_- = \frac{1}{2}(\vec{J} - i\vec{K})$$

$\vec{J}_+, \vec{J}_-$ are not hermitian $\quad \vec{J}_+^{\,+} = \vec{J}_-$, $\quad \vec{J}_-^{\,+} = \vec{J}_+$

The commutation relations for J_+, J_-

$$[J_i^+, J_j^+] = \frac{1}{4}[J_i + iK_i, J_j + iK_j] = \frac{1}{4} i\epsilon_{ijk}(J_k + iK_k + iK_k + J_k) = i\epsilon_{ijk} J_k^+$$

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similarly. $[J_i^+, J_j^+] = i\epsilon_{ijk} J_k^+$

$$[J_i^-, J_j^-] = i\epsilon_{ijk} J_k^-$$

$$[J_i^-, J_j^+] = 0.$$

we found two disjoint groups of generators each obeying an SU(2) alg.

$$SU(2) \times SU(2)^+$$

Each representation of the Lorentz group is labeled by the indecies of the two disjoint SU(2) algebras (j_1, j_2)

each representation has $(2j_1+1) \otimes (2j_2+1)$ components.

as $\vec{J} = \vec{J}_+ + \vec{J}_-$ spin is given by $j_1 + j_2$.

Examples:

	(j_1, j_2)	spin	components	
ⓐ	$(0, 0)$	0	1	singlet.
ⓑ	$(1/2, 0)$	$1/2$	2	Weyl-spinor
ⓒ	$(0, 1/2)$	$1/2$	2	Weyl spinor
ⓓ	$(1/2,0)+(0,1/2)$	$1/2$	4	Dirac spinor
ⓔ	$(1/2, 1/2)$	$1, 0$	4	Vector.

if we write $x^\mu = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$ under rotation t is a singlet. $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is a triplet

the generator of rotations $\vec{J} = \vec{J}_+ + \vec{J}_-$

under rotations a four vector decomposes into a singlet and a triplet.

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From the point of view of $(\frac{1}{2},\frac{1}{2})$ Rep. $\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow$

$3 \rightarrow \uparrow\uparrow \,;\, \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \,;\, \downarrow\downarrow$ spin = 1 $\quad j_3 = (-1, 0, 1)$
$1 \rightarrow \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$ spin 0 $\quad j_3 = 0$.

The Poincare group.
we saw that spin is a label of representations of the Lorentz group.
However we cannot yet classify elementary particles, which also have mass!

The Lorentz transformation are not the most general.

we should write down all the symmetries of a Relativistic line element,

(*) $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$.

So far we discussed Rotations & boosts that form the symmetries of the Lorentz group.

we want to write the most general set of transformations that keep the line element invariant,

look at the two dimensional case: $ds^2 = -dt^2 + dx^2$

we perform the most general infinitesimal transformations.
$t \rightarrow t + \epsilon T(t, x)$ $\quad\epsilon$ an infinitesimal number.
$x \rightarrow x + \epsilon R(t, x)$.

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A contra-variant 4-vector $X^\mu = (ct, \vec{X})$ $\mu = 0,1,2,3$

I will use the notation with $c = 1$ throughout.

$$X^0 = t \qquad X^i = \vec{X}$$

The Minkowski metric is $\eta^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

The scalar product of two 4-vectors is

$$X \cdot y = \eta_{\mu\nu} X^\mu y^\nu =$$

The Lorentz transformations preserve the scalar product.

In particular the Lorentz transformation preserve the Length of a four vector.

$$X \cdot X = \eta_{\mu\nu} X^\mu X^\nu = X' \cdot X' = \eta_{\mu\nu} X'^\mu X'^\nu$$
$$= t^2 - \vec{X}^2 = \qquad = t'^2 - \vec{X}'^2$$

where X and X' are related by a Lorentz transformation.

Note that I have used the Einstein summation convention

$$\sum_{\mu,\nu=0}^{3} g_{\mu\nu} X^\mu X^\nu = g_{\mu\nu} X^\mu X^\nu$$

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A covariant vector is : $X_\mu = g_{\mu\nu} X^\nu = (+t, -\vec{x})$.

$g_{\mu\nu}$ is the metric tensor $= \eta_{\mu\nu}$ in flat Minkowski space but can be more general.

given a vector $X^\mu = (t, \vec{x})$.

There are two differential quantities of interest.

1. $dX^\mu \rightarrow$ contravariant 4-vector.
2. $\frac{\partial}{\partial X^\mu} = \partial_\mu \rightarrow$ gradient $\rightarrow$ covariant 4-vector.

How do they behave under coordinate transformations.

$$dX^\mu \rightarrow dX'^\mu = \frac{\partial X'^\mu}{\partial X^\nu} dX^\nu$$

$$\frac{\partial}{\partial X^\mu} \rightarrow \frac{\partial}{\partial X'^\mu} = \frac{\partial X^\nu}{\partial X'^\mu} \frac{\partial}{\partial X^\nu}$$

The differential & the gradient transform differently.

vector that transform like the differential is a contravariant vector

$$V'^\mu = \frac{\partial X'^\mu}{\partial X^\nu} V^\nu$$

vectors that transform like the gradient is a covariant vector

$$V'_\mu = \frac{\partial X^\nu}{\partial X'^\mu} V_\nu$$

$$\boxed{\partial_\mu = \frac{\partial}{\partial X^\mu}}$$

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The momentum four vector $P_\mu = (E, -\vec{P})$

$P^\mu = (E, \vec{P})$

$P_\mu \cdot X^\mu = E^2 - \vec{P}^2$

$P_\mu \sim \frac{\partial}{\partial X^\mu}$ in relativistic quantum mechanics.

Properties of Lorentz transformations

1. transformation that preserve the scalar product and the length of 4-vectors.

$g_{\mu\nu} X^\mu X^\nu$ is invariant under Lorentz transformations no change in size and shape.

The invariance implies the existence of a symmetry. The symmetry is generated by a group.

2. Assume $X^\mu \to X'^\mu$ under some Lorentz tran.

$$X^\mu \longrightarrow X'^\mu = \Lambda^\mu{}_\nu X^\nu$$

$$g_{\mu\nu} X^\mu X^\nu \to g_{\mu\nu} X'^\mu X'^\nu = g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta X^\alpha X^\beta$$

$$= g_{\alpha\beta} X^\alpha X^\beta$$

$$\Longrightarrow (*) \; g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = g_{\alpha\beta} \xrightarrow[\text{notation}]{\text{in matrix}} \Lambda^T \underline{g} \Lambda = \underline{g}$$

$\to$ Defines the Lorentz transformations.

$$\Rightarrow (\det \Lambda)^2 = 1 \Rightarrow \det \Lambda = \pm 1$$

The physical case $\det \Lambda = 1$ -> continuously connected to the identity.

$\det \Lambda = 1 \rightarrow$ Proper Lorentz transformations.
$\det \Lambda = -1 \rightarrow$ improper Lorentz transformations.

look at the 00 component of $= \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g_{\mu\nu} = g_{\alpha\beta}$

$$g_{\mu\nu} \Lambda^\mu{}_0 \Lambda^\nu{}_0 = +1$$

$$+(\Lambda^0{}_0)^2 - \sum_i (\Lambda^i{}_0)^2 = +1$$

$$(\Lambda^0{}_0)^2 = 1 + \sum_i (\Lambda^i{}_0)^2 \geq 1$$

$$(\Lambda^0{}_0)^2 \geq 1 \quad \begin{cases} (\Lambda^0{}_0) \geq 1 \\ (\Lambda^0{}_0) \leq -1 \end{cases}$$

Example: Reflection $\Lambda^\mu{}_\nu = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

we are interested in the $\Lambda^0{}_0 \geq 1$ ← orthochronos
The Lorentz transformations that are continuously connected to the identity

ⓐ $\det \Lambda = 1$ ← Proper.
ⓑ $\Lambda^0{}_0 \geq 1$ ← orthochronos.

+

Examples! $\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ $\quad \Lambda^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$

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Further examples:

(a) : Rotation $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & \vec{R} \end{pmatrix}$ $\det \Lambda = \det R$
$\det R = \pm 1$
$\det R = +1 \longrightarrow$ proper.

(b) boosts in x-direction $\Lambda = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$\det \Lambda = \cosh^2\eta - \sinh^2\eta = 1$
$\Lambda^0{}_0 = \cosh\eta \geq 1$

End of lecture 5

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(c) time inversion: $\Lambda = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

$\det \Lambda = -1$ $\Lambda^0{}_0 = -1 \Rightarrow$ improper non-orthochronous

(d) Full inversion $\Lambda = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

$\det \Lambda = +$ $\Lambda^0{}_0 = -1 \Rightarrow$ proper non-orthochronous.

$\rightarrow$ ALL L.T. generated by the above
The proper orthochronous Lorentz transformations correspond to Rotations & boosts.

$\rightarrow$ 6 parameters = 3 angles + 3 boosts.

Lorentz transformations that are proper orthochronos are continuousely connected to the identity.

Performing an infinitisemal proper orthochronos Lorentz transformation,

(**) $\Lambda^{\mu}{}_{\nu} = \underbrace{\delta^{\mu}{}_{\nu}}_{\text{identity trans.}} + \underbrace{\omega^{\mu}{}_{\nu}}_{\text{an infinitesmal transformation}}$

$\delta^{\mu}{}_{\nu} = +1$ for $\mu = \nu$ $= 0$ for $\mu \neq \nu$.

calculate to order $O(\omega)$.

for continuous transformations the properties of infinitesimal transformations fixes the transformation properties of the Lorentz group.

finite transformations are obtained by integration

substitute (**) in $g_{\mu\nu} \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} = g_{\alpha\beta}$.

$$g_{\mu\nu}(\delta^{\mu}{}_{\alpha} + W^{\mu}{}_{\alpha})(\delta^{\nu}{}_{\beta} + \omega^{\nu}{}_{\beta}) = g_{\alpha\beta}.$$

$$g_{\mu\nu}\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta} + \underbrace{g_{\mu\nu} W^{\mu}{}_{\alpha} \delta^{\nu}{}_{\beta}}_{g_{\mu\beta} W^{\mu}{}_{\alpha}} + \underbrace{g_{\mu\nu}\delta^{\mu}{}_{\alpha} W^{\nu}{}_{\beta}}_{g_{\alpha\nu} W^{\nu}{}_{\beta}} + g_{\mu\nu} \omega^{\mu}{}_{\alpha} \omega^{\nu}{}_{\beta} = g_{\alpha\beta}.$$

$$\cancel{g_{\alpha\beta}} + W_{\beta\alpha} + W_{\alpha\beta} + O(\omega^2) = \cancel{g_{\alpha\beta}}$$

$$W_{\beta\alpha} + W_{\alpha\beta} = 0$$

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The infinitesimal transformations are antisymmetric.

Example: Rotation group in two dimensions:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{for small } \theta$$

The number of degrees of freedom in an 4×4 antisymmetric matrix.

For general n: $\quad \dfrac{n^2 - n}{2} = \dfrac{n(n-1)}{2}$

for $n=4 \quad \Rightarrow \quad \dfrac{4 \cdot 3}{2} = 6.$

3 Rotation angles + 3 boosts.

$$W_{\mu\nu} = \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & b_3 & -b_2 \\ -a_2 & -b_3 & 0 & b_1 \\ -a_3 & b_2 & -b_1 & 0 \end{pmatrix}.$$

The algebraic properties of the Lorentz group

we associate an operator $U(\Lambda)$ with the trans. Λ.

for the special case $\Lambda = \delta \longrightarrow U(\delta) = I$

we want to find what is the operator associated with $\Lambda = \delta + \omega$

to order $O(\omega)$, $U(\delta+\omega) = I + \frac{1}{2} i \underbrace{J_{\mu\nu}}_{\text{operators.}} \underbrace{W^{\mu\nu}}_{\text{infinitesimal trans.}} + \ldots$

Look at the Example of Rotations.

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & \alpha & -\beta \\ -\alpha & 1 & \gamma \\ \beta & -\gamma & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

$$A = I + \alpha \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

operators $J^{\mu\nu}$ $\qquad \alpha, \beta, \gamma \in \omega^{\mu\nu}$

These operators are not hermitian $\Rightarrow$ multiply by i to get hermitian operators.

$$A = I + i\alpha \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + i\beta \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} + i\gamma \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

now the operators are hermitian.

from the definition of $U(\delta+\omega)$ we defined 16 $J_{\mu\nu}$ operators only 6 are independent $\rightarrow$ only the antisymmetric components.

define the operators

$$K_i = J_{i0} = -J_{0i} \qquad i = 1, 2, 3.$$

$$J_i = \frac{1}{2} \varepsilon_{ijk} J_{jk}$$

$$U(\delta+\omega) = I + i\vec{a}\cdot\vec{K} - i\vec{b}\cdot\vec{J} \qquad \omega^{\mu\nu} = \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & b_3 & -b_2 \\ -a_2 & -b_3 & 0 & b_1 \\ -a_3 & b_2 & -b_1 & 0 \end{pmatrix}$$

Example of Regular Rotations

$$\left(A = I + i\vec{\alpha}\cdot\vec{J}\right).$$

The operators K, J are the basic operators of the Lorentz group

These are the generators of the Lorentz group

$\vec{J}$ are the generators of Rotations

$\vec{K}$ are the generators of boosts.

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we will determine the commutation relations by imposing the group property on $\bar{U}$.

(*) $U(\Lambda_1\Lambda_2) = \bar{U}(\Lambda_1)\bar{U}(\Lambda_2)$. ← the group property.

for regular rotations $\Lambda_1 = \Theta_1$ $\Lambda_2 = \Theta_2$ $\Lambda_1\Lambda_2 = \Theta_1 + \Theta_2$.

$$U(\Lambda_1) = I + \frac{i}{2} J_{\mu\nu}\, \omega_1^{\mu\nu} \qquad \Lambda_1 = \delta + \omega_1$$
$$U(\Lambda_2) = I + \frac{i}{2} J_{\mu\nu}\, \omega_2^{\mu\nu} \qquad \Lambda_2 = \delta + \omega_2.$$

$$\bar{U}(\Lambda_1\Lambda_2) = I + \frac{i}{2} J_{\mu\nu}\left(\omega_1^{\mu\nu} + \omega_2^{\mu\nu}\right).$$

$$\left[\Lambda_1\Lambda_2 = (I+\omega_1)(I+\omega_2) \underset{\substack{\uparrow \\ \text{first order.}}}{\simeq} I + (\omega_1+\omega_2) + \cdots \right].$$

substituting into the group property (*). and keeping terms to second order in $O(\omega^2)$. we get the commutation relations.

$$[J_i, J_j] = i\epsilon_{ijk} J_k \qquad i,j,k = 1,2,3.$$

$$[J_i, K_j] = i\epsilon_{ijk} K_k$$

$$[K_i, K_j] = -i\epsilon_{ijk} J_k$$

The minus sign in the third relation distinguishes The Lorentz transformations from ordinary Rotations in four dimensions.

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define the combinations:

$$\vec{J}_+ \equiv \frac{1}{2}\left(\vec{J} + i\vec{K}\right)$$
$$\vec{J}_- \equiv \frac{1}{2}\left(\vec{J} - i\vec{K}\right).$$

$\vec{J}_+ , \vec{J}_-$ are not hermitian $\quad \vec{J}_+^{\,+} = \vec{J}_-$, $\vec{J}_-^{\,+} = \vec{J}_+$

The commutation relations for $\vec{J}_+ , \vec{J}_-$

$$[J_i^+, J_j^+] = \frac{1}{4}[J_i + iK_i , J_j + iK_j] = \frac{1}{4} i\epsilon_{ijk} [J_k + iK_k + iK_k + J_k]$$
$$= i\epsilon_{ijk} J_k$$

similarly

$$[J_i^+, J_j^+] = i\,\epsilon_{ijk} J_k^+$$

$$[J_i^-, J_j^-] = i\,\epsilon_{ijk} J_k^-$$

$$[J_i^-, J_j^+] = 0$$

two disjoint groups of generators each obeying an SU(2) algebra.

$SU(2) \times SU(2)^+$

each representation of the Lorentz group is labeled by by the indexes of the two disjoint SU(2) algebras.

(j_1, j_2)

each Rep. has $(2j_1+1) \otimes (2j_2+1)$ components.

as $\vec{J} = \vec{J}_+ + \vec{J}_-$ spin is given by $j_1 + j_2$

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Examples:

(a) $(0,0) \rightarrow$ spin $= 0$ 1 component.

(b) $(\frac{1}{2},0) \rightarrow$ spin $= \frac{1}{2}$ 2 components.

(c) $(0,\frac{1}{2}) \rightarrow$ spin $= \frac{1}{2}$ 2 components. $\rightarrow$ Weyl spinor

(d) $(\frac{1}{2},0) + (0,\frac{1}{2}) \rightarrow$ spin $= \frac{1}{2}$ 4 components. $\rightarrow$ Dirac spinor

(e) $(\frac{1}{2},\frac{1}{2}) \rightarrow$ spin $= 1$ 4 components $\rightarrow$

The Poincare group.

We should write down all the invariances (or symmetries) of a relativistic line element.

$$(*) \qquad ds^2 = c^2dt^2 - dx^2 - dy^2 - dt^2.$$

So far we discussed rotations & boost that form the symmetries of the Lorentz group.

Examining the relativistic line element, ds^2 $(*)$

Additional invariances are noted

$t \rightarrow t + a$

$x \rightarrow x + b$

$y \rightarrow y + c$

$z \rightarrow z + d$.

we have 4-additional parameters

these additional symmetries extend the Lorentz group to the poincare group.

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The total number of parameters of the poincare group.
3 Rotations + 3 boost + 4 translations.

consider the effect of translation on the field $\phi(x)$

in one dimension.

$\phi(x) \rightarrow \phi(x+a)$

what is the operator that reduces this transformation.

$\phi(x+a) = U(a)\ \phi(x)$

U is a unitary operator

Performing a taylor expansion of $\phi(x+a)$

$$\phi(x) \rightarrow \phi(x+a) = \sum \frac{a^n}{n!}\left(\frac{\partial^n}{\partial x^n}\phi(x)\right)\Bigg|_{a=0} =$$

$$= \underbrace{\left(\sum_n \frac{a^n \partial^n}{n!\, \partial x^n}\right)}_{\text{Operator}} \phi(x) = e^{a\frac{\partial}{\partial x}}\ \phi(x).$$

$U(a) = e^{a\frac{\partial}{\partial x}}$ ← is the operator,

$$U(a) = e^{a\frac{\partial}{\partial x}} \simeq 1 + a\frac{\partial}{\partial x} + \cdots = 1 + i\left(-ia\frac{\partial}{\partial x}\right) + \cdots$$

$$= \mathbb{1} + ia\,\underline{P} + \cdots$$

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$$P = -i \frac{\partial}{\partial x}$$

The i factor arises from the hermiticity of U

$U = I + iap.$

$U^{+} = I - iap$

$U^{-1} = 1 - iap$

$U^{+}U = (I + iap)(I - iap) = I - a^2 p^2 \simeq I.$

in four space-time dimensions we have

$$U(a^\mu) \simeq I + i\, a^\mu P_\mu.$$

The generator of the Poincare group are:

$P^\mu = -i\partial_\mu. \leftarrow$ translations. $\rightarrow 4$

$L_{\mu\nu} = i(X_\mu \partial_\nu - X_\nu \partial_\mu) \leftarrow$ Rotations + boosts. $\rightarrow 6.$

They obey the comm. Relations.

$[P^\mu, P^\nu] = 0$

(*) $[L_{\mu\nu}, L_{\rho\sigma}] = i g_{\nu\rho} L_{\mu\sigma} - i g_{\mu\rho} L_{\nu\sigma} - i g_{\nu\sigma} L_{\mu\rho} + i g_{\mu\sigma} L_{\nu\rho}.$

The later one is the Lie algebra of $SO(3,1)$. the most general Rep. of the generators of $SO(3,1)$ that obeys (*) is given by

$$M_{\mu\nu} \equiv L_{\mu\nu} + S_{\mu\nu}.$$

where $S_{\mu\nu}$ obeys (*) and commute with $L_{\mu\nu}$.

Additionally. $[M_{\mu\nu}, P_\rho] = -i g_{\mu\rho} P_\nu + i g_{\nu\rho} P_\mu.$

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Representations of the Poincare group are labelled by two Casimir invariants.
(casimir operator → commutes with all generators of the group).

the two casimir operators of the Poincare group are

$g_{\mu\nu} P^{\mu} P^{\nu}$
$g_{\mu\nu} W^{\mu} W^{\nu}$ where $W^{\mu} = \frac{-i}{2} \epsilon^{\mu\nu\rho\sigma} S_{\rho\sigma} \partial_{\nu}$

The eigenvalues of $P_{\rho} P^{\rho} \equiv m^{2} \rightarrow$ mass
the second casimir is related to spin. $W_{\rho} W^{\rho} \rightarrow -m^{2} s(s+1)$

where s is the spin.

Therefore Reps. of the Poincare group are labelled by two good quantum numbers.

1) Rest mass m_0.
2) spin. s

End of lectures
6+7.