① The vacuum expectation value.

$$\langle \phi_0 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}.$$

$\phi : T = 1/2 \quad T_3 = -1/2 \quad , \quad Y = 1$

$$Q_{em}(\phi) = T_3 + \frac{1}{2} Y = -\frac{1}{2} + \frac{1}{2} = 0.$$

$$\mathcal{L} = \left| \left(i\partial_\mu - g \vec{T} \cdot \vec{W}_\mu - i g' \frac{Y}{2} B_\mu \right) \phi \right|^2 - V(\phi)$$

$$V(\phi) = \mu^2 \phi^+ \phi + \lambda (\phi^+ \phi)^2$$

The relevant term for the gauge boson masses.

$$\left| \left(g \frac{\vec{\tau}}{2} \cdot \vec{W}_\mu + g' \frac{1}{2} B_\mu \right) \phi \right|^2$$

$$= \left| \left(\frac{g}{2} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot W^1_\mu + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} W_2^\mu + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} W_3^\mu \right] + \frac{g'}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} B_\mu \right) \phi \right|^2$$

$$= \frac{1}{4} \left| \left(g \begin{pmatrix} W_3^\mu & W_1^\mu - i W_2^\mu \\ W_1^\mu + i W_2^\mu & -W_3^\mu \end{pmatrix} + g' \begin{pmatrix} B_\mu & 0 \\ 0 & B_\mu \end{pmatrix} \right) \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \right|^2$$

$$= \frac{1}{8} \left| \begin{pmatrix} g W_3^\mu + g' B^\mu & g(W_1^\mu - i W_2^\mu) \\ g(W_1^\mu + i W_2^\mu) & -g W_3^\mu + g' B^\mu \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \right|^2 =$$

$$= \frac{1}{8} v^2 \left(g(W_1^\mu + i W_2^\mu), (-gW_3^\mu + g'B^\mu) \right) \begin{pmatrix} g(W_1^\mu - iW_2^\mu) \\ -gW_3^\mu + g'B^\mu \end{pmatrix}$$

$$= \frac{1}{4} g^2 v^2 W^+ W^- + \frac{v^2}{8} (g^2+g'^2) \left(\frac{-gW_3^\mu + g'B^\mu}{(\sqrt{g^2+g'^2})} \right)^2$$

$$= \left(\frac{1}{2} g v \right)^2 W^{\mu +} W_\mu^- + \frac{1}{2} \frac{v^2(g^2+g'^2)}{4} \left(\frac{-gW_3^\mu + g'B^\mu}{\sqrt{g^2+g'^2}} \right)^2$$

$$M_{W^\pm} = \frac{gv}{2} \qquad M_Z = \frac{1}{2} v (g^2+g'^2)^{1/2}$$

$$\frac{M_W}{M_Z} = \frac{g}{(g^2+g'^2)^{1/2}} = \cos\theta_w.$$

$$\left(g'/g = \tan\theta_w. \right).$$

$$\frac{1}{(1+\tan^2\theta_w)^{1/2}} = \frac{1}{\left(1+\frac{s^2}{c^2}\right)^{1/2}} = c \quad \checkmark$$

(b). We will derive the general case for any ~~(represen)~~ Higgs representation(s) and then substitute.

The general term for the gauge boson masses has the form $\sum_j \left| \left(-ig \vec{T}_j \cdot \vec{W}_\mu - g' \frac{Y_j}{2} B_\mu \right) \phi_j \right|^2 =$

(3) $$\sum_j \left| \left(-ig \left(T_{1j} W_1^\mu + T_{2j} W_2^\mu + T_{3j} W_3^\mu \right) - ig' \frac{Y_j}{2} \right) \phi_j \right|^2.$$

$$T_j^{\pm} = \frac{1}{\sqrt{2}} \left(T_1 \pm i T_2 \right) \qquad \cancel{\#} \quad W_\mu^{\pm} = \frac{1}{\sqrt{2}} \left(W_1 \pm i W_2 \right)$$

$$T_1 W_1 + T_2 W_2 = T^+ W^- + T^- W^+$$

$$= \sum_j \left| \left(g \left(T_j^+ W_\mu^- + T_j^- W_\mu^+ \right) + \left(g T_{3j} W_\mu^3 + \frac{g'}{2} Y_j B_\mu \right) \right) \phi_j \right|^2$$

M_W^2 is the coeficient of $W^+ W^-$

$$\Rightarrow M_W^2 = \left\langle \sum_j g^2 \phi_j^+ \left(T_j^+ T_j^- + T_j^- T_j^+ \right) \phi_j \right\rangle$$

$$= g^2 \sum_j \phi_j^+ \left(T_j^+ T_j^- + T_j^- T_j^+ \right) \phi_j$$

$$T^+ T^- + T^- T^+ = T^1 T^1 + T^2 T^2 = \vec{T}_j^2 - T_{j3}^2$$

$$= g^2 \sum_j \phi_j^+ \left(\vec{T}_j^2 - \hat{T}_{j3}^2 \right) \phi_j .$$

$$= \frac{g^2}{} \sum_j \phi_{j0}^+ \left(T_j (T_j + 1) - (T_{j3})^2 \right) \phi_{j0} =$$

$$= \frac{g^2}{2} \sum_j v_j^2 \left(T_j (T_j + 1) - (T_{j3})^2 \right)$$

(4) We require that $\cancel{Q}\, Q_{\ast}(\phi_{j_0}) = 0$.

$$\Rightarrow \quad T_{j3} + \frac{1}{2} Y_j = 0. \quad \Rightarrow T_{3_j} = \frac{-1}{2} Y_j.$$

$$\Rightarrow M_W^2 = \frac{g^2}{2} \sum_j \left(T_j(T_j+1) - \frac{Y_j^2}{\cancel{2}\,4} \right) v_j^2 .$$

M_Z^2 is obtained from the matrix element of the second term

$$\sum_j \left| \left(g T_{3_j} W_3^{\mu} + \frac{g'}{2} Y_j B_{\mu} \right) \phi_j \right|^2$$

Using $T_{j3} = \frac{-Y_j}{2}$ we get.

$$\sum_j \left| \frac{Y_j}{2} \left(-g W_3^{\mu} + g' B^{\mu} \right) \phi_j \right|^2$$

$$= \frac{1}{8} \sum_j \left(v_j^2 Y_j^2 \right) \Big| \frac{\left(-g W_3^{\mu} + g' B^{\mu}\right)}{\left(g'^2 + g^2\right)} \Big| \left(g'^2 + g^2\right).$$

$$M_Z^2 = \frac{(g'^2 + g^2)}{4} \cdot \sum v_j^2 Y_j^2$$

(5)

~~This~~ Thus.

$$\frac{M_W^2}{M_Z^2} = \frac{g^2/2 \sum_j v_j^2 \left(T_j(T_j+1) - \frac{1}{4} Y_j^2\right)}{\frac{(g^2+g'^2)}{4} \sum v_j^2 Y_j^2}.$$

$$\cancel{\frac{1}{2}\cos\theta_W} = \frac{\cos^2\theta_W}{\cancel{2}} \frac{\sum v_j^2 \left(T_j(T_j+1) - \frac{Y_j^2}{4}\right)}{\frac{1}{2}\sum v_j^2 Y_j^2}$$

$$\rho^2 = \left(\frac{M_W}{M_Z \cos\theta_W}\right)^2$$

$$= \frac{\sum v_j^2 \left(T_j(T_j+1) - \frac{Y_j^2}{4}\right)}{\frac{1}{2}\sum v_j^2 Y_j^2}.$$

FOR $T_j = 3$ $Y = 4$ we have.

$$\rho^2 = \frac{v^2}{v^2} \frac{12 - \cancel{8}4}{\frac{1}{2} 16} = 1.$$

for any number of doublets $T_j = 1/2$ $Y_j = 1$.

$$\rho^2 = \frac{\sum v_j^2 \left(\frac{3}{4} - \frac{1}{4}\right)}{\frac{1}{2}\sum v_j^2} = \cancel{1} \frac{\sum v_j^2}{\sum v_j^2} = 1.$$

for a triplet $T_j = 1$ $Y = 2$

$$\rho^2 = \frac{\cancel{v^2}(2-1)}{\frac{1}{2}\cancel{v^2} \cdot 4} = \frac{1}{2} \neq 1$$

Experimentally $\rho \approx 1.000\ldots$ highly constrained.

(6)

$$5 = (3,1)_{-2/3} + (1,2)_{+1}$$

$$\bar{5} = (\bar{3},1)^{d_L^c}_{2/3} + (1,2)^{L_L}_{-1}$$

$$5\cdot\bar{5} = (8,1)_0 + (1,1)_0 + (3,2)_{-5/3} + (\bar{3},2)_{5/3} + (1,3)_0 + (1,1)_0$$

$$5\cdot 5 = (6,1)_{-4/3} + (\bar{3},1)_{-4/3} + (3,2)_{1/3} + (3,2)_{1/3} + (1,3)_2 + (1,1)_2$$

$$= [(6,1)_{-4/3} + (1,3)_2 + (3,2)_{1/3}] + [\underset{u_L^c}{(\bar{3},1)_{-4/3}} + \underset{Q_L}{(3,2)_{1/3}} + \underset{e_L^c}{(1,1)_2}]$$

$$= 15_S + 10_A.$$

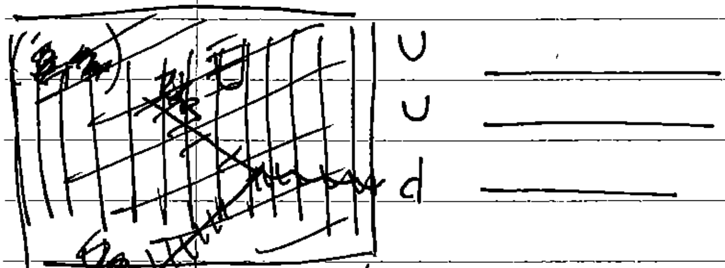
$$(3,2)_{-5/3} + (\bar{3},2)_{5/3}.$$

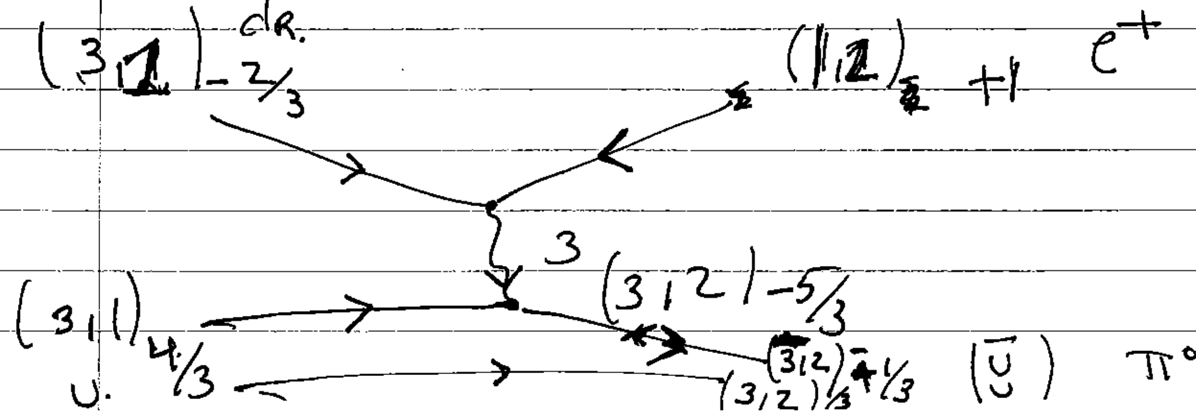
Electric charges.

$$Q = \tfrac{1}{2} + \tfrac{1}{2}(-5/3) = \tfrac{1}{2}\left(-\tfrac{2}{3}\right) = -1/3$$

$$Q = -1/2 - \tfrac{5}{6} = -\tfrac{8}{6} = -4/3.$$

$$Q(X;Y) = (-1/3;-4/3).$$





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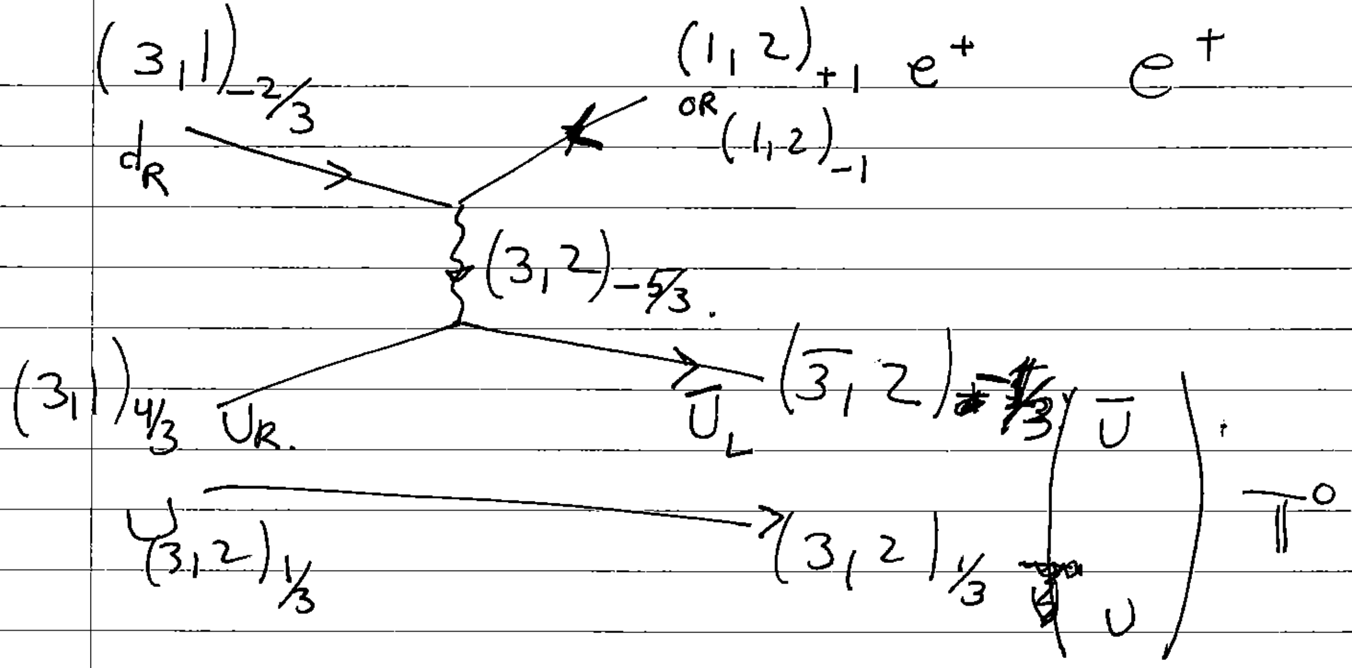
$(3,1)_{-2/3}$
d_R
$(1,2)_{+1}$ e^+
OR $(1,2)_{-1}$
e^+
$(3,2)_{-5/3}$
$(3,1)_{4/3}$ u_R
$\bar{u}_L$ $(\bar{3},2)$
u $(3,2)_{1/3}$
$(3,2)_{1/3}$
$\bar{u}$
u
π^0

$p^+ \to \pi^0 e^+$

b) if we have several Higgs doublets or triplets.
Then The relevant term to the gauge boson masses will have the form.

$$\sum_j^N \left| \left(-ig\, \vec{T}_j \cdot \vec{W}_\mu - g' \frac{Y_j}{2} B_\mu \right) \phi_j \right|^2 =$$

where the sum is on all the higgs doublets and triplets.

$$= \sum_{j=1}^N \left| \left(\frac{g}{\sqrt{2}} \left(T_j^+ W_\mu^+ + T_j^- W_\mu^- \right) + \left(g T_j^3 W_\mu^3 + \frac{g'}{\sqrt{2}} Y_j B_\mu \right) \right) \phi_{j0} \right|^2$$

where $\phi_{j0} = \langle \phi_j \rangle_0$ is The vacuum expectation value of the j^{th} doublet or triplet.
M_W is the coefficient $\sim \; W_\mu^+ W^{\mu -}$
Thus $M_W^2 = \sum_j \frac{g^2}{2} \phi_{j0}^\dagger \left(T_j^+ T_j^- + T_j^- T_j^+ \right) \phi_{j0} =$

$$= \frac{g^2}{2} \sum_j \phi_{j0}^\dagger \left(\hat{T}_j^2 - \hat{T}_{j3}^2 \right) \phi_{j0} =$$

$$= \frac{g^2}{2} \sum_j v_j^2 \left(T_j (T_j + 1) - (T_{j3})^2 \right)$$

now we require that

$Q_{\phi_{j_0}} = 0$ (otherwise $U(1)_{e.m.}$ will be broken).

$$\Rightarrow T_{j3} + \frac{1}{2}Y_j = 0 \Rightarrow T_{j3} = -\frac{1}{2}Y_j$$

$$\Rightarrow M_W^2 = \frac{g^2}{2}\sum_j v_j^2\left(T_j(T_j+1) - \frac{Y_j^2}{4}\right)$$

M_Z^2 ~~(shown)~~ can be found from the matrix element of the second ~~term~~ term related to the non vanishing eigenvalue of the mass matrix. (one eigenvalue has to vanish because we assume invariance under U(1) e.m.)

using $T_{j3} = -\frac{Y_j}{2}$ we get for the ~~secon~~ second term

$$\sum_j \left| (\,) -\frac{Y_j}{2}\phi_{j_0}^{\dagger}\left(gW_\mu^3 - g'B_\mu\right)\phi_{j_0}\right|^2$$

$$\Rightarrow M_Z^2 = \left(\sum_j \frac{1}{2}v_j^2 Y_j^2\right)\left(\sqrt{g'^2+g^2}\right)^2$$

where again Z_μ is normalized.

The vacuum expectation value is

$$\phi_0 = \sqrt{\tfrac{1}{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}.$$

with this choise of ϕ_0 and $T = \frac{1}{2}$ $T_3 = -\frac{1}{2}$ $Y = 1$
The vacuum expectation value is invariant under
The $U(1)_{e.m}$ i.e. $Q|\phi_0\rangle = 0\,|\phi_0\rangle = 0$ and Thus The
The gauge boson associated with this symetry will remain massless, (namely the photon will remain massless).

$$\left(Q|\phi_0\rangle = 0\,|\phi_0\rangle = 0 \Rightarrow \phi^0 \to \phi_0' = e^{i\alpha(x) Q}\phi_0 = \phi_0 \right).$$

To generate the masses we add to the $SU(3)_c \times SU(2)_L \times U(1)$ gauge invariant lagrangian a term.

$$\mathcal{L}_2 = \left| \left(i\partial_\mu - g\,\vec{T}\cdot\vec{W}_\mu - g'\frac{Y}{2}B_\mu \right)\phi \right|^2 - V(\phi)$$

$$V(\phi) = \mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2.$$

The relevant term to the gauge boson masses is.

$$\left| \left(-ig\,\frac{\vec{\tau}}{2}\cdot\vec{W}_\mu - i\frac{g'}{2}B_\mu \right)\phi \right|^2 =$$

$$= \left\| \left(-\frac{ig}{2}\begin{pmatrix} W_\mu^3 & (W_\mu^1 - iW_\mu^2) \\ (W_\mu^1 + iW_\mu^2) & -W_\mu^3 \end{pmatrix} - \frac{ig'}{2}\begin{pmatrix} B_\mu & \\ & B_\mu \end{pmatrix} \right) \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} \right\|^2 =$$

$$= \frac{1}{8}(0, v) \begin{pmatrix} gW_\mu^3 + g' B_\mu & g(W_\mu^1 - iW_\mu^2) \\ g(W_\mu^1 + iW_\mu^2) & -gW_\mu^3 + g' B_\mu \end{pmatrix} \begin{pmatrix} gW_\mu^3 + g' B_\mu & g(W_\mu^1 - iW_\mu^2) \\ g(W_\mu^1 + iW_\mu^2) & -gW_\mu^3 + g' B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

$$= \frac{1}{8} g^2 v^2 \left[(W_\mu^1)^2 + (W_\mu^2)^2 \right] + \frac{1}{8} v^2 (-gW_\mu^3 + g' B_\mu)(-gW^{\mu 3} + g' B^\mu)$$

$$= \left(\frac{v}{2} g\right)^2 W_\mu^+ W^{\mu -} + \frac{1}{8} v^2 (W_\mu^3, B_\mu) \begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix} \begin{pmatrix} W^{\mu 3} \\ B^\mu \end{pmatrix}$$

$$\Rightarrow m_W = \frac{1}{2} v g$$

The (~~remaining~~) second term can be written in the form.

$$\frac{1}{8} v^2 \left[g^2 (W_\mu^3)^2 - 2gg' W_\mu^3 B^\mu + g'^2 B_\mu^2 \right] = \frac{1}{8} v^2 \left[gW_\mu^3 - g' B_\mu \right]^2$$

$$+ 0 * \left[g W_\mu^3 + g' B_\mu \right]^2$$

The fields Z_μ A_μ diagonalize the mass matrise.

and are orthogonal.

$$\Rightarrow \quad \frac{1}{2} M_Z^2 Z_\mu^2 + \frac{1}{2} M_A^2 A_\mu^2 .$$

Thus we can identify Z_μ and A_μ as linear combinations of the fields W_μ^3 and B_μ. and Thus in a normalized form ~~we~~ we have

$$A_\mu = \frac{g' W_\mu^3 + g B_\mu}{\sqrt{g'^2 + g^2}} \quad \text{with the eigen value } M_A = 0.$$

$$Z_\mu = \frac{g W_\mu^3 - g' B_\mu}{\sqrt{g'^2 + g^2}} \quad \text{with the eigen value } M_Z = \frac{1}{2} v \sqrt{g'^2 + g^2}$$

in term of θ_ω

$$A_\mu = \cos\theta_\omega B_\mu + \sin\theta_\omega W_\mu^3$$

$$Z_\mu = -\sin\theta_\omega B_\mu + \cos\theta_\omega W_\mu^3$$

and $$\frac{M_\omega}{M_Z} = \frac{\frac{1}{2} v g}{\frac{1}{2} v \sqrt{g'^2 + g^2}} = \frac{g}{\sqrt{g'^2 + g^2}} = \cos\theta_\omega$$

or $$M_\omega = M_Z \cos\theta_\omega .$$

Thus

$$\frac{M_W^2}{M_Z^2} = \frac{\left(\sum_j v_j^2 \left(T_j(T_j+1) - \frac{1}{4} Y_j^2\right)\right) g^2}{\left(\sum_j \frac{1}{2} v_j^2 Y_j^2\right) \left(g^2+g'\right)^2}$$

or

$$\left(\frac{M_W}{M_Z \cos\theta_W}\right)^2 = \frac{\sum_j v_j^2 \left(T_j(T_j+1) - \frac{1}{4} Y_j^2\right)}{\sum_j \frac{1}{2} v_j^2 Y_j^2} \checkmark$$

As a check we can check that This is indeed equal To 1 if only one iso doublet ~~exist~~ exists. in This case there is only one term in the sum.

$$\left(\frac{M_W}{M_Z\cos\theta_W}\right)^2 = \frac{v^2}{v^2} \frac{\left(\frac{1}{2}\left(\frac{1}{2}+1\right) - \frac{1}{4}1\right)}{\frac{1}{2}} = \frac{\frac{3}{4} - \frac{1}{4}}{\frac{1}{2}} = 1$$

as expected.