from the previous lecture ...

requiring local phase invariance $\rightarrow$ the electromagnetic field

Maxwell's equations ($\epsilon_0 = \mu_0 = c = 1$)

$$\vec{\nabla} \cdot \vec{E} = \rho_{em} \tag{1}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{2}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{3}$$

$$\vec{\nabla} \times \vec{B} = \vec{J}_{em} + \frac{\partial \vec{E}}{\partial t} \tag{4}$$

Define $$J^\mu_{em} = (\rho_{em}, \vec{j}_{em})$$

$$\begin{aligned}
\partial_\nu \partial^\nu A^\mu - \partial^\mu(\partial_\nu A^\nu) &= \partial_\nu(\partial^\nu A^\mu - \partial^\mu A^\nu) \\
&= \partial_\nu F^{\nu\mu} = J^\mu_{em}
\end{aligned}$$

where we defined the electromagnetic field strength tensor

$$F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) = -F_{\nu\mu}$$

hence $$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

The elecromagentic field strength tensor and hence Maxwell's equations are invariant under the gauge transformations

$$A^\mu \quad \to \quad A'^\mu = A^\mu + \partial^\mu \chi$$

where χ is a scalar function.

$$\begin{aligned} F^{\mu\nu} \quad \to \quad F'^{\mu\nu} &= \partial^\mu (A^\nu + \partial^\nu \chi) - \partial^\nu (A^\mu + \partial^\mu \partial \chi) \\ &= \partial^\mu A^\nu - \partial^\nu A^\mu + \partial^\mu \partial^\nu \chi - \partial^\nu \partial^\mu \chi \\ &= \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu} \end{aligned}$$

$$\Rightarrow \qquad \text{we can always choose} \quad \partial_\mu A^\mu = 0 \quad \text{(Lorentz gauge)} \tag{5}$$

$$\text{If} \quad \partial_\mu A^\mu = f \neq 0 \quad \to \quad \partial_\mu (A^\mu + \partial^\mu \chi) = 0 \quad \to \quad \partial_\mu \partial^\mu \chi = -f$$

$$\Rightarrow \qquad \text{in free space} \quad (J^\mu = 0) \quad \text{we have} \quad \partial_\nu \partial^\nu A^\mu = 0$$

$\to$ massless KGE for each component of A^μ

$\rightarrow \quad A^\mu$ is the wave function of the photon

$\rightarrow \quad A^\mu$ is a four vector $\rightarrow$ photon has spin 1

Plane–wave solutions :

$$A^\mu = \epsilon^\mu \mathrm{e}^{-i(wt-\vec{k}\cdot\vec{r})}$$

where

$$\begin{aligned} \epsilon^\mu &= \text{polarization four vector} \\ k^\mu &= (w, \vec{k}) \quad \rightarrow \quad \text{wave 4–vector} \end{aligned}$$

From the wave equation for A^μ

$\rightarrow \quad k \cdot k = 0 \quad \Rightarrow \quad w^2 = \vec{k}^2 \quad \Leftrightarrow \quad E^2 = p^2 c^2 \quad (\text{for} \quad m = 0)$

From the Lorentz gauge condition

$$\partial_\mu A^\mu = 0 \quad \Rightarrow \quad \epsilon \cdot k = 0 \quad \Rightarrow \quad \epsilon^0 = \frac{\vec{\epsilon}\cdot\vec{k}}{w}$$

$\Rightarrow \quad \epsilon'^{\mu} = \epsilon^{\mu} + ak^{\mu}$ is equivalent ϵ^{μ} for any $a = \mathrm{costant}$

$\Rightarrow$ choose $\epsilon_0 = 0 \quad \Rightarrow \quad \epsilon_{\mu} k^{\mu} = \vec{\epsilon} \cdot \vec{k} = 0$

$$\Rightarrow \quad \text{for} \quad \vec{k} = (0, 0, k_z) \quad \rightarrow \quad \epsilon_x^{\mu} = (0, 1, 0, 0) \quad , \quad \epsilon_x^{\mu} = (0, 0, 1, 0)$$

$$\text{2 polarization states} \quad \epsilon_{R,L}^{\mu} = \frac{(0, 1, \pm i, 0)}{\sqrt{2}} \quad \leftarrow \quad \text{circular polarization}$$

Electromagentic interactions

We introduce electromagnetic interactions via the minimal substitution in the equations of motion

$$E \rightarrow E - eV \quad , \vec{p} \rightarrow \vec{p} - e\vec{A}$$

where e is the electric charge.

relativistically : $p^\mu \to p^\mu - eA^\mu \quad , \quad \partial^\mu \to \partial^\mu + ieA^\mu$

The KG equation becomes

$$\begin{aligned}
&(\partial_\mu + ieA_\mu)(\partial^\mu + ieA^\mu)\phi + m^2\phi = 0 \\
&(\partial_\mu\partial^\mu + m^2)\phi = -ie[A_\mu\partial^\mu\phi + \partial_\mu(A^\mu\phi)] + e^2A_\mu A^\mu\phi
\end{aligned} \tag{6}$$

We saw that the minimal coupling prescription and the gauge condition $A^\mu \to A^\mu + \partial^\mu\chi$ are the same as the local phase invariance

$$\phi(x) \to \mathrm{e}^{-i\alpha(x)}\phi(x) \to \text{local } U(1) \text{ symmetry}$$

The conserved current is now

$$J^\mu = i(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) - 2eA^\mu(x)\phi^*(x)\phi(x)$$

The second term provides the coupling of the scalar field to the electromagnetic field

The Dirac equation

The Schrödinger equation: $$i\hbar\frac{\partial}{\partial t}\psi = \hat{H}\psi$$

is linear in $i\hbar\frac{\partial}{\partial t} \quad \to \quad$ linear in E

since $P_\mu = i\hbar\partial_\mu \quad \& \quad P^2 = E^2 - \vec{P}^2 = m^2$, the KGE is quadratic in $\frac{\partial}{\partial t}$.

Dirac wanted to find a relativistically covariant equation which is linear in $i\hbar\frac{\partial}{\partial t}$

i.e. linear in energy $\to$ linear in the Hamiltonian

$\to$ linear in the generator of time translations

$\to$ linear in time + relativistically covariant $\Rightarrow$ linear in $\underline{-i\hbar\vec{\nabla}}$

$-i\hbar\vec{\nabla} \quad \to \quad$ the quantum generator of spatial translations relativistically $E^2 = \vec{p}^2 c^2 + m^2 c^4 \quad \Rightarrow \quad E = \pm\sqrt{\vec{p}^2 c^2 + m^2 c^4}$

$$i\hbar \frac{1}{c}\frac{\partial}{\partial t}\psi(\vec{r}, t) = \pm\sqrt{\vec{p}^2 + m^2 c^2}\;\psi(\vec{r}, t)$$

we have to get rid of the $\sqrt{}$