Class test

1) $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dx^2 + dy^2$ ①

a) $g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $g^{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b) $\left.\begin{aligned} x &\Rightarrow x + \varepsilon A(x,y) \\ y &\Rightarrow y + \varepsilon B(x,y) \end{aligned}\right\}$ Transformations

$$dx \Rightarrow dx + \varepsilon\left(\frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy\right)$$

$$dy \Rightarrow dy + \varepsilon\left(\frac{\partial B}{\partial x}dx + \frac{\partial B}{\partial y}dy\right)$$

$$dx^2 \Rightarrow dx^2 + 2\varepsilon\left(\frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy\right)dx + O(\varepsilon^2)$$

$$dy^2 \Rightarrow dy^2 + 2\varepsilon\left(\frac{\partial B}{\partial x}dx + \frac{\partial B}{\partial y}dy\right)dy + O(\varepsilon^2)$$

② $\therefore$ $ds^2 = dx^2 + 2\varepsilon\left(\frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy\right)dx + dy^2 + 2\varepsilon\left(\frac{\partial B}{\partial x}dx + \frac{\partial B}{\partial y}dy\right)dy$.

For ds^2 to be invariant, it must have the same value before and after undergoing the above transformation

③ $\therefore$ ① = ② $\Rightarrow dx^2 + dy^2 = dx^2 + 2\varepsilon\left(\frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy\right)dx + dy^2 + 2\varepsilon\left(\frac{\partial B}{\partial x}dx + \frac{\partial B}{\partial y}dy\right)dy$

$\therefore$ ③ $\Rightarrow dx^2 + dy^2 = dx^2 + 2\varepsilon\frac{\partial A}{\partial x}dx^2 + 2\varepsilon\frac{\partial A}{\partial y}dy\,dx + dy^2 + 2\varepsilon\frac{\partial B}{\partial x}dx\,dy + 2\varepsilon\frac{\partial B}{\partial y}dy^2$

Comparing coefficients in ③ gives use the following,

dx^2 : $1 = 1 + 2\varepsilon\frac{\partial A}{\partial x}$

dy^2 : $1 = 1 + 2\varepsilon\frac{\partial B}{\partial y}$

$dydx: \; 0 = 2\varepsilon \frac{\partial A}{\partial y} + 2\varepsilon \frac{\partial B}{\partial x}$

We can then solve these equations to obtain a value for our parameters A and B.

$\therefore dx^2 : 2\varepsilon \frac{\partial A}{\partial x} = 0 \quad \therefore \frac{\partial A}{\partial x} = 0 \quad$ as $2\varepsilon \neq 0$

$\therefore A = f(y).$

$dy^2 : \frac{\partial B}{\partial y} = 0 \quad \therefore B = g(x)$

$dydx: \; 2\varepsilon \frac{\partial A}{\partial y} = -2\varepsilon \frac{\partial B}{\partial x} \quad \therefore \frac{\partial A}{\partial y} = -\frac{\partial B}{\partial x}$

For $\frac{\partial A}{\partial y} = -\frac{\partial B}{\partial x}$ to be true they both must equal some constant.

$\therefore \frac{\partial A}{\partial y} = -\frac{\partial B}{\partial x} = c_1,$

Using the values for A and B found earlier, $A = f(y)$ and $B = g(x)$, we obtain:

$\frac{\partial A}{\partial y} = f'(y) \quad \& \quad \frac{\partial B}{\partial x} = g'(x)$

$\therefore f'(y) = c_1 \quad \therefore f(y) = c_1 y + c_2$

$-g'(x) = c_1 \quad \therefore g(x) = c_3 - c_1 x$

$\therefore A = c_1 y + c_2, \quad B = c_3 - c_1 x$

Here c_1, c_2 and c_3 are parameters where c_1 is a rotation and c_2 & c_3 are shifts in space.

The transformations then become

$$x \to x + \varepsilon(c_1 y + c_2)$$

$$y \to y + \varepsilon(c_3 - c_1 x)$$

Which both represent rotations and have both been translated by a factor of εc_2 for x and εc_3 for y

2a) $L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu \quad , \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \; F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

The Euler Lagrange equation for A_μ is,

$$\frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial A_{\mu,\nu}}\right) - \frac{\partial L}{\partial A_\mu} = 0.$$

We begin by inputting our ~~expression~~ identity for $F^{\mu\nu}$ into the Lagrangian.

$$\therefore \; L = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - j^\mu A_\mu$$

Which can be re-written as,

$$L = -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\alpha A_\beta - \partial_\beta A_\alpha) - j^\mu A_\mu$$

by using $\eta^{\alpha\mu} A_\alpha \equiv A^\mu$ etc...

We split our Euler Lagrange equation up into two parts,

$$\underbrace{\frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial A_{\mu,\nu}}\right)}_{\text{first part}} \quad \& \quad \underbrace{\frac{\partial L}{\partial A_\mu}}_{\text{Second part.}}$$

We now apply our Lagrangian to the first part of our Euler-Lagrange equation.
We also note that $A_{\mu,\nu} = \partial_\mu A_\nu$

$$\therefore \frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial(\partial_\mu A_\nu)}\right)$$

$$\therefore \frac{\partial}{\partial x^\nu}\left(\frac{\partial\left(-\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(\partial_\mu A_\nu-\partial_\nu A_\mu)(\partial_\alpha A_\beta-\partial_\beta A_\alpha) - j^\mu A_\mu\right)}{\partial(\partial_p A_q)}\right)$$

Where we have changed the dummy indices "μ" and "ν" in the denominator.

expanding the brackets in our numerator we get,

$$\frac{\partial}{\partial x^\nu}\left(\underbrace{\frac{\partial\left(-\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(\partial_\mu A_\nu\partial_\alpha A_\beta - \partial_\mu A_\nu\partial_\beta A_\alpha - \partial_\nu A_\mu\partial_\alpha A_\beta + \partial_\nu A_\mu\partial_\beta A_\alpha) - j^\mu A_\mu\right)}{\partial(\partial_p A_q)}}_{④}\right)$$

using $\dfrac{\partial(\partial_\alpha A_\beta)}{\partial(\partial_\mu A_\nu)} = \delta^\alpha_\mu\delta^\beta_\nu$ we obtain

$$④ \Rightarrow -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}\left(\delta^\mu_p\delta^\nu_q\partial_\alpha A_\beta + \delta^\alpha_p\delta^\beta_q\partial_\mu A_\nu - \delta^\mu_p\delta^\nu_q\partial_\beta A_\alpha - \delta^\beta_p\delta^\alpha_q\partial_\mu A_\nu\right.$$
$$\left.-\delta^\nu_p\delta^\mu_q\partial_\alpha A_\beta - \delta^\alpha_p\delta^\beta_q\partial_\nu A_\mu + \delta^\nu_p\delta^\mu_q\partial_\beta A_\alpha + \delta^\beta_p\delta^\alpha_q\partial_\nu A_\mu\right)$$

$$\Rightarrow -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}\left((\delta^\mu_p\delta^\nu_q - \delta^\nu_p\delta^\mu_q)\partial_\alpha A_\beta + (\delta^\alpha_p\delta^\beta_q - \delta^\beta_p\delta^\alpha_q)\partial_\mu A_\nu\right.$$
$$\left.+(\delta^\nu_p\delta^\mu_q - \delta^\mu_p\delta^\nu_q)\partial_\beta A_\alpha + (\delta^\beta_p\delta^\alpha_q - \delta^\alpha_p\delta^\beta_q)\partial_\nu A_\mu\right)$$

Which further simplifies to

④ $\Rightarrow -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}\Big((\delta^\mu_\rho\delta^\nu_q - \delta^\nu_\rho\delta^\mu_q)(\partial_\alpha A_\beta - \partial_\beta A_\alpha) + (\delta^\beta_\rho\delta^\alpha_q - \delta^\alpha_\rho\delta^\beta_q)\cdot(\partial_\mu A_\nu - \partial_\nu A_\mu)$

Using $\delta^\mu{}_\rho = \delta^\rho_\mu$ and $\eta^{\alpha\mu}\delta^\rho_\mu = \eta^{\alpha\rho}$,

We obtain

④ $\Rightarrow -\frac{1}{4}\Big[(\eta^{\alpha\rho}\eta^{\beta q} - \eta^{\beta\rho}\eta^{\alpha q})(\partial_\alpha A_\beta - \partial_\beta A_\alpha) + (-\eta^{\nu\rho}\eta^{\mu q} + \eta^{\mu\rho}\eta^{\nu q})\cdot(\partial_\mu A_\nu + \partial_\nu A_\mu)\Big]$

④ $\Rightarrow -\frac{1}{4}\Big[\eta^{\alpha\rho}\eta^{\beta q}\partial_\alpha A_\beta - \eta^{\beta\rho}\eta^{\alpha q}\partial_\alpha A_\beta - \eta^{\alpha\rho}\eta^{\beta q}\partial_\beta A_\alpha + \eta^{\beta\rho}\eta^{\alpha q}\partial_\beta A_\alpha$
$+ \eta^{\nu\rho}\eta^{\mu q}\partial_\mu A_\nu - \eta^{\mu\rho}\eta^{\nu q}\partial_\mu A_\nu + \eta^{\nu\rho}\eta^{\mu q}\partial_\nu A_\mu - \eta^{\mu\rho}\eta^{\nu q}\partial_\nu A_\mu\Big]$

④ $\Rightarrow -\frac{1}{4}\Big[\partial^\rho A^q - \partial^q A^\rho - \partial^q A^\rho + \partial^\rho A^q - \partial^q A^\rho + \partial^\rho A^q + \partial^\rho A^q - \partial^q A^\rho\Big]$

$\Rightarrow -\frac{1}{4}\Big[2\partial^\rho A^q - 2\partial^q A^\rho + 2\partial^\rho A^q - 2\partial^q A^\rho\Big]$

$= -(\partial^\rho A^q - \partial^q A^\rho) = \partial^q A^\rho - \partial^\rho A^q = F^{q\rho}$

∴ our first term is $\frac{\partial}{\partial x^\nu}F^{\mu\nu}$

and our second term is trivial, $\frac{\partial L}{\partial A_\mu} = j^\mu$

∴ $\frac{\partial}{\partial x^\nu}F^{\mu\nu} = j^\mu$, $\partial_\mu j^\mu = 0$ as $\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}F^{\mu\nu} = 0$

b) $L' = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu + \frac{1}{2}m^2 A_\mu A^\mu$

∴ $\frac{\partial L}{\partial A_\mu} = -j^\mu + \frac{1}{2}m^2\eta^{\alpha\beta}\partial_\mu(A_\alpha A_\beta)$

∴ $= -j^\nu + \frac{1}{2}m^2\eta^{\alpha\beta}(\delta^\alpha_\mu A_\alpha + \delta^\beta_\mu A_\beta)$

$$= -j^\nu + m^2 A^\mu,$$

applying this to our E-L equation we get

$$-\partial_\mu F^{\mu\nu} = -j^\nu + m^2 A^\mu$$

$$\therefore -\partial_\mu F^{\mu\nu} - m^2 A^\mu = -j^\nu$$

$$\therefore \partial_\mu F^{\mu\nu} + m^2 A^\mu = j^\nu$$

$$\therefore (\partial_\mu \partial^\mu + m^2) A^\mu = j^\nu$$