MATH431 Modern Particle Physics Solutions 5

1. The Lagrangian of the given two-dimensional potential is

$$L \;=\; \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - \frac{1}{4}k(x^2+y^2)^2\,.$$

(ii) The Euler-Lagrange equations are

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} \;&=\; m\ddot{x} + k(x^2+y^2)x \;=\; 0\,,\\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} \;&=\; m\ddot{y} + k(x^2+y^2)y \;=\; 0\,.
\end{aligned}$$

(iii) The Hamiltonian is

$$H \;=\; \frac{1}{2m}(p_x^2+p_y^2) + \frac{1}{4}mw^2(x^2+y^2)^2 \quad \text{with} \quad w=\sqrt{\frac{k}{m}}\,.$$

(iv) In polar coordintes we derive:

$$L \;=\; \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{1}{4}kr^4\,,$$

$$\begin{aligned}
p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r}\,,\\
p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}\,.
\end{aligned}$$

Hence we get

$$\begin{aligned}
H \;&=\; \sum_i p_i\dot{q}_i - L \;=\; m\dot{r}^2 + mr^2\dot{\phi}^2 - \frac{1}{2}m(\dot{r}^2+r^2\dot{\phi}^2) + \frac{1}{4}mw^2r^4\\
&=\; \frac{1}{2}m(\dot{r}^2 + mr^2\dot{\phi}^2) + \frac{1}{4}mw^2r^4\\
&=\; \frac{{p_r}^2}{2m} + \frac{{p_\phi}^2}{2mr^2} + \frac{1}{4}mw^2r^4\,.
\end{aligned}$$

(v) There are two constants of the motion.
Since the Hamiltonian does not depend explicitly on time, the energy is a constant of the motion with $E=H(q_0,p_0)$. Since it does not depend explicitly on ϕ, also p_ϕ is a constant of the motion, corresponding to conservation of the angular momentum w.r.t. the symmetry axis.

2. Two-particle states are defined by

$$
\begin{aligned}
|\mathbf{p}_1, \mathbf{p}_2 >=& a^\dagger(\mathbf{p}_1) a^\dagger(\mathbf{p}_1)|0> \\
|\mathbf{p}_1, \mathbf{p}_2 >=& |\mathbf{p}_2, \mathbf{p}_1 > \quad \text{as} \quad [a(\mathbf{p}_1), a(\mathbf{p}_2)] = 0 \\
< \mathbf{p}_1', \mathbf{p}_2' | \mathbf{p}_1, \mathbf{p}_2 >=& < 0|a(\mathbf{p}_1')a(\mathbf{p}_2')a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)|0> \\
&= < 0|a(\mathbf{p}_1')\{a^\dagger(\mathbf{p}_1)a(\mathbf{p}_2') + (2\pi)^3 2p_1^0 \delta(\mathbf{p}_1 - \mathbf{p}_2')\}a^\dagger(\mathbf{p}_2)|0> \\
&= < 0|a(\mathbf{p}_1')a^\dagger(\mathbf{p}_1)\{a^\dagger(\mathbf{p}_2)a(\mathbf{p}_2') + (2\pi)^3 2p_1^0 \delta(\mathbf{p}_2 - \mathbf{p}_2')\}|0> \\
&\quad + (2\pi)^3 2p_1^0 \delta(\mathbf{p}_1 - \mathbf{p}_2') < 0|a(\mathbf{p}_1')a^\dagger(\mathbf{p}_2)|0> \\
&= (2\pi)^3 2p_2^0 \delta(\mathbf{p}_2 - \mathbf{p}_2') < 0|\{a^\dagger(\mathbf{p}_1)a(\mathbf{p}_1') + (2\pi)^3 2p_1^0 \delta(\mathbf{p}_1 - \mathbf{p}_1')\}|0> \\
&+ (2\pi)^3 2p_1^0 \delta(\mathbf{p}_1 - \mathbf{p}_2') < 0|\{a^\dagger(\mathbf{p}_2)a(\mathbf{p}_1') + (2\pi)^3 2p_2^0 \delta(\mathbf{p}_2 - \mathbf{p}_1')\}|0> \\
&= (2\pi)^6 (2p_1^0)(2p_2^0)\{\delta(\mathbf{p}_1 - \mathbf{p}_1')\delta(\mathbf{p}_2 - \mathbf{p}_2') + \delta(\mathbf{p}_1 - \mathbf{p}_2')\delta(\mathbf{p}_2 - \mathbf{p}_1')\}.
\end{aligned}
$$

3(a).

The electromagnetic field strength tensor,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

$$
\begin{aligned}
L_{e.m.} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\
&= -\frac{1}{4}(A_{\mu,\nu} - A_{\nu,\mu})(A^{\mu,\nu} - A_{\nu,\mu}) \\
&= -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(A_{\beta,\alpha} - A_{\alpha,\beta})(A_{\mu,\nu} - A_{\nu,\mu})
\end{aligned}
$$

The Euler–Lagrange equations of motion are obtained from the Lagrangian

$$\frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial A_{\mu,\nu}}\right) - \frac{\partial L}{\partial A_\mu} = 0$$

$$\frac{\partial L}{\partial A_\mu} = 0 \Rightarrow \frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial A_{\mu,\nu}}\right) = 0$$

$$
\begin{aligned}
\frac{\partial L}{\partial A_{p,q}} =& -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(\delta^p_\alpha \delta^q_\beta - \delta^p_\beta \delta^q_\alpha)(A_{\mu,\nu} - A_{\nu,\mu}) + (A_{\alpha,\beta} - A_{\beta,\alpha})(\delta^p_\mu \delta^q_\nu - \delta^p_\nu \delta^q_\mu) \\
=& -\frac{1}{4}(\eta^{p\mu}\eta^{q\nu} - \eta^{q\mu}\eta^{p\nu})(A_{\mu,\nu} + A_{\nu,\mu}) + (\eta^{\alpha p}\eta^{\beta q} - \eta^{\alpha q}\eta^{\beta p})(A_{\beta,\alpha} - A_{\alpha,\beta}) \\
=& -4\frac{1}{4}(A^{q,p} - A^{p,q}) \\
=& -F^{pq}
\end{aligned}
$$

$$\frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial A_{\mu,\nu}}\right) = -\frac{1}{4}\frac{\partial}{\partial x^\nu}F^{\mu\nu} = 0$$

which are Maxwell's equation in the absence of sources.

$$A_\mu \to \tilde{A}_\mu = A_\mu + \partial_\mu \Lambda$$

$$\begin{aligned}\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu &= \partial_\mu(A_\nu - \partial_\nu \Lambda) - \partial_\nu(A_\mu - \partial_\mu \Lambda) \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - \partial_\mu \partial_\nu \Lambda + \partial_\nu \partial_\mu \Lambda = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}\end{aligned}$$

Therefore $L_{e.m.}$ is also invariant under the transformation.

(b) the Lagrangian for a massive photon without sources is given by

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + m^2 A_\mu A^\mu$$

The first term is invariant under transformation. The second term transforms to

$$\to m^2 \tilde{A}_\mu \tilde{A}^\mu = m^2 A_\mu A^\mu + m^2(\partial_\mu \Lambda \partial^\mu \Lambda - \partial_m u \Lambda A^\mu - A_\mu \partial^\mu \Lambda)$$

therefore the mass term does not vanish under the transformation. Requiring invariance imposes $m^2 = 0$.

4.

$$\begin{aligned}\gamma^{5\dagger} =& (i\gamma^0\gamma^1\gamma^2\gamma^3)^\dagger \\ =& -i\gamma^{3\dagger}\gamma^{2\dagger}\gamma^{1\dagger}\gamma^{0\dagger} \\ =& -i(\gamma^0\gamma^3\gamma^0)(\gamma^0\gamma^2\gamma^0)(\gamma^0\gamma^1\gamma^0)\gamma^0 \\ =& -i\gamma^0\gamma^3\gamma^2\gamma^1 = i\gamma^0\gamma^2\gamma^3\gamma^1 = -i\gamma^0\gamma^2\gamma^1\gamma^3 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5.\end{aligned}$$

For the second part, it's best to do for each μ in turn:

$$\begin{aligned}\gamma^5\gamma^0 =& i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 = -i\gamma^0\gamma^1\gamma^2\gamma^0\gamma^3 = i\gamma^0\gamma^1\gamma^0\gamma^2\gamma^3 \\ =& -i\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^0\gamma^5 \Rightarrow \{\gamma^5, \gamma^0\} = 0, \\ \gamma^5\gamma^1 =& i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1 = -i\gamma^0\gamma^1\gamma^2\gamma^1\gamma^3 = i\gamma^0\gamma^1\gamma^1\gamma^2\gamma^3 \\ =& -i\gamma^1\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^1\gamma^5 \Rightarrow \{\gamma^5, \gamma^1\} = 0.\end{aligned}$$

It is clear that the other two calculations will be similar.

5. N.B. Of course the question was wrong–should have said $(\gamma^0)^2 = 1$, $(\gamma^i)^2 = -1$, $i = 1, 2, 3$.

$$\begin{aligned}\gamma^0\gamma^1\gamma^2 =& -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3 = i\gamma^5\gamma^3 \\ \gamma^0\gamma^1\gamma^3 =& -\gamma^0\gamma^1\gamma^2\gamma^2\gamma^3 = \gamma^0\gamma^1\gamma^2\gamma^3\gamma^2 = -i\gamma^5\gamma^2.\end{aligned}$$

6.

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} 1_4 \Rightarrow \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu} 1_4,$$

where 1_4 is the 4-dimensional identity matrix, usually not written explicitly. Taking the trace, and using $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, $\mathrm{tr} 1_4 = 4$, we get

$$\mathrm{tr}[\gamma_\mu\gamma_\nu] = 4\eta_{\mu\nu}.$$

Now

$$\begin{aligned} \gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma =& - \gamma_\nu\gamma_\mu\gamma_\rho\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma \\ =& \gamma_\nu\gamma_\rho\gamma_\mu\gamma_\sigma - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma \\ =& - \gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu + 2\eta_{\mu\sigma}\gamma_\nu\gamma_\rho - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma \\ \Rightarrow \gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma + \gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu =& 2\eta_{\mu\sigma}\gamma_\nu\gamma_\rho - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma. \end{aligned}$$

Taking the trace and using

$$\mathrm{tr}[\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu] = \mathrm{tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma],$$

together with $\mathrm{tr}[\gamma_\mu\gamma_\nu = 4\eta_{\mu\nu}$, we find

$$\mathrm{tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] = 4[\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}]$$