PAPER CODE NO.
MATH 431

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Introduction to Modern Particle Physics

TIME ALLOWED: Two and half hours

INSTRUCTIONS TO CANDIDATES: In this paper bold-face quantities like $\mathbf{x}$ represent three-dimensional vectors.

Full marks can be obtained for complete answers to FIVE questions. Only the best FIVE answers will be counted.

1. Consider the transformations $q(t) \to q(t)+\delta q(t)$ that leave the Lagrangian invariant up to a total time derivative, *i.e.*

$$L + \delta L = L(q(t) + \delta q(t), \dot{q}(t) + \delta\dot{q}(t), t) = L(q(t), \dot{q}(t), t) + \frac{d}{dt}(\epsilon\Lambda), \qquad (1)$$

where ϵ is an infinitesimal constant and Λ is a calculable function of the coordinates, velocities and possibly of time.

(a) Show that there is an associated conserved charge that takes the form

$$\epsilon Q = \frac{\partial L}{\partial \dot{q}}\delta q - \epsilon\Lambda. \qquad (2)$$

[10 marks]

(b) Consider a Lagrangian $L(q(t), \dot{q}(t))$ that has no explicit time dependence and the transformation

$$q(t) \to q(t+\epsilon) \simeq q(t) + \epsilon\dot{q}(t) \qquad (3)$$

that represents a constant infinitesimal time translation. Show that the transformation (3) leaves the Lagrangian invariant up to an added term which is a total time derivative. Calculate Λ and the conserved charge Q. Give an interpretation of the result.

[10 marks]

2. Consider the infinitesimal line element on a two dimensional surface

$$ds^2 = g_{\mu\nu}d\theta^\mu d\theta^\nu = d\theta^2 + \sin^2\theta d\phi^2$$

(a convenient notation is $\theta^\mu \equiv (\theta, \phi)$ $(\mu = 1, 2)$)

(a) Write the metric $g_{\mu\nu}$ in an explicit matrix form.
Write $g^{\mu\nu}$ in matrix form. [4 marks]

(b) Consider the set of infinitesimal transformations of the form

$$\theta^\mu \to \theta^\mu + \epsilon\zeta^\mu(\theta, \phi) \ ,$$

for which the line element ds^2 is invariant and where ϵ is an infinitesimal constant. Derive the conditions that the functions ζ^1 and ζ^2 must satisfy for ds^2 to remain invariant.

[16 marks]

3. (a) Two-particle states are defined by $|\mathbf{p}_1, \mathbf{p}_2\rangle = a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)|0\rangle$. Show that

$$\langle \mathbf{p}_1', \mathbf{p}_2' | \mathbf{p}_1, \mathbf{p}_2\rangle = (2\pi)^6(2p_1^0)(2p_2^0)\left[\delta(\mathbf{p}_1 - \mathbf{p}_1')\delta(\mathbf{p}_2 - \mathbf{p}_2') + \delta(\mathbf{p}_1 - \mathbf{p}_2')\delta(\mathbf{p}_2 - \mathbf{p}_1')\right] .$$

[6 marks]

(b) Show that the number operator, defined by

$$N := \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{p}}{2p^0}\, a^\dagger(\mathbf{p})a(\mathbf{p}) ,$$

satisfies

$$[N, a^\dagger(\mathbf{p})] = a^\dagger(\mathbf{p}) ,$$

and hence

$$N\,|\mathbf{p}_1 \ldots \mathbf{p}_n\rangle = n\,|\mathbf{p}_1 \ldots \mathbf{p}_n\rangle .$$

[14 marks]

4. (a) Defining $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, show that

$$\begin{aligned} (\gamma^5)^2 &= 1 \\ \gamma^{5\dagger} &= \gamma^5. \end{aligned}$$

[12 marks]

(b) Write each of $\gamma^1\gamma^2\gamma^3$ and $\gamma^0\gamma^2\gamma^3$ as a product $c\gamma^5\gamma^\nu$ for some $\nu = 0, 1, 2, 3$ and some number c. [5 marks]

(c) Show that

$$\mathrm{trace}(\gamma_\mu\gamma_\nu) = 4\eta_{\mu\nu}.$$

[You may assume $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$.] [3 marks]

5. Let A_μ be the electromagnetic vector potential. The electromagnetic field strength tensor is then defined as

$$F_{\mu\nu} \;=\; \partial_\mu A_\nu - \partial_\nu A_\mu \,.$$

(a) Show that the covariant Maxwell equations (in the absence of sources) in four-vector notation can be derived from the electromagnetic Lagrangian given by

$$L_{\text{e.m.}} \;=\; -\frac{1}{4}\, F_{\alpha\beta} F^{\alpha\beta} \,.$$

[10 marks]

(b) Show that $L_{\text{e.m.}}$ is invariant under the transformation $A_\mu \to A_\mu - \partial_\mu \Lambda$, where Λ is a scalar function.

[4 marks]

(c) Show that by imposing a local $U(1)$ symmetry a mass term for the photon is forbidden.

[6 marks]

6.

(a.) Show that orbital angular momentum is a constant of the motion for a free non–relativistic particle. [7 marks]

(b.) Show that orbital angular momentum is not a constant of the motion for a free Dirac particle. [7 marks]

(c.) Show that total angular momentum of a free Dirac particle is a constant of the motion. [6 marks]

7. Consider the simple unitary group $SU(3)$.

(a.) How many diagonal generators of the Lie algebra are there? Write down a representation of the diagonal generators in terms of 3×3 hermitian matrices.

[3 marks]

(b.) What is the dimension of the group? Write down a representation of all generators in terms of 3×3 hermitian matrices.

[3 marks]

(c.) What is the fundamental representation of $SU(3)$? Write down its decomposition in terms of a maximal subgroup.

[3 marks]

(d.) Draw the graphic illustration of the fundamental representation, indicating clearly the eigenvalues of each state under the diagonal generators.

[3 marks]

(e.) Find the product and the decomposition under the maximal subgroup of the fundamental times the anti-fundamental representations of $SU(3)$.

[4 marks]

(f.) Find the product and the decomposition under the maximal subgroup of the fundamental times the fundamental representations of $SU(3)$.

[4 marks]