MATH431 Modern Particle Physics Solutions 7

1.

$$UU^\dagger = 1 \Rightarrow U = \mathrm{e}^{iH} \quad ; \quad U^\dagger = \mathrm{e}^{-iH^\dagger} = \mathrm{e}^{-iH}$$

Hence we must have $H = H^\dagger$. H must be hermitian.

2a. 3 diagonal generators.

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_{15} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

2b. $D = 15$. Three diagonal generators of part $(2a)$ plus:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_9 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_{10} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\lambda_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad , \quad \lambda_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

2c.

$$4 = (3, 1/3) + (1, -1)$$

under the maximal subgroup $SU(3) \times U(1)$

2d.

see separate figure

2e.

$$4 \times \bar{4} = \{(3, 1/3) + (1, -1)\} \times \{(\bar{3}, -1/3) + (1, +1)\} =$$
$$15 + 1 = \{(3, +4/3) + (8, 0) + (1, 0) + (\bar{3}, -4/3)\} + (1, 0)$$

2f.

$$4 \times 4 = \{(3, 1/3) + (1, -1)\} \times \{(3, 1/3) + (1, -1)\} =$$
$$6 + 10 = \{(6, 2/3) + (3, -2/3) + (1, -2)\} + \{(\bar{3}, 2/3) + (3, -2/3)\}$$

3.

To construct a consistent three dimensional theory, we must ensure that the dynamics do not depend on the z–direction. The motion of the particle must be confined in the (x, y) plane. Since the Lorents force is perpendicular to the magnetic field, it follows that the components of the magnetic fields in the (x, y) plane has to vanish. Similarly, the component of the electric field in the z–direction is zero. Hence, we take

$$E_z = B_x = B_y = 0.$$

The remaining components E_x, E_y and B_z can only depend on x and y. Similarly, the velocity and current components in the z–directions are zero, *i.e.* $v_z = j_z = 0$. Maxwell's equations then become

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = \rho \quad \text{from} \quad \vec{\nabla} \cdot \vec{E} = \rho$$

$$\frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y} = -\frac{1}{c}\frac{\partial B_z}{\partial t} \quad \text{from} \quad \vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\begin{aligned} \frac{\partial B_z}{\partial y} =& \frac{j_x}{c} + \frac{1}{c}\frac{\partial E_x}{\partial t} \\ -\frac{\partial B_z}{\partial x} =& \frac{j_y}{c} + \frac{1}{c}\frac{\partial E_y}{\partial t} \end{aligned}$$

$$\text{from} \quad \vec{\nabla} \times \vec{B} = \frac{\vec{j}}{c} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

The remaining Maxwell equation $\vec{\nabla} \cdot \vec{B} = 0$ is trivial because $B_x = B_y = 0$ and $B_z = B_z(x, y)$. The Lorentz force law gives nontrivial equations only for the x and y components:

$$\begin{aligned} \frac{dp_x}{dt} =& q\left(E_x + \frac{v_y}{c}B_z\right), \\ \frac{dp_y}{dt} =& q\left(E_y - \frac{v_x}{c}B_z\right). \end{aligned}$$

3b. In three dimensions we have $A^\mu = \left(\Phi, A^1, A^2\right)$, $A_\mu = (\Phi, -A_1, -A_2)$, and $j^\mu = \left(c\rho, j^1, j^2\right)$. Moreover, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, so

$$
\begin{aligned}
F_{0i} &= \frac{1}{c}\frac{\partial A_i}{\partial t} + \frac{\partial \Phi}{\partial x^i} \equiv -E_i \\
F_{12} &= \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \equiv B_z
\end{aligned}
$$

Thus, the field strength tensor takes the form

$$
F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y \\ E_x & 0 & B_z \\ E_y & -B_z & 0 \end{pmatrix} \qquad F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y \\ -E_x & 0 & B_z \\ -E_y & -B_z & 0 \end{pmatrix}
$$

The above $D = 3$ field strength F can be viewed as the $D = 4$ one with $E_z = B_x = B_y = 0$. The $D = 3$ current j can be viewed as the $D = 4$ current with $j_z = 0$. The three dimensional Maxwell equations are therefore the truncation to $E_z = B_x = B_y = 0$ and $j_z = 0$ of the original four dimensional ones