from the previous lecture ...

Aim: **Extract the algebraic properties of the Lorentz group**

In quantum mechanics operators are unitary,

$$U_{QM}^{-1} = U_{QM}^{\dagger} \quad \Rightarrow \quad U_{QM}^{\dagger} U_{QM} = I$$

$$U = \mathrm{e}^{iO} \rightarrow U^{\dagger} U = \mathrm{e}^{-iO^{\dagger}} \mathrm{e}^{iO} = I$$

exponentiation is a good way to represent unitary operators if O is hermitian

if $AB \neq BA$

$$\mathrm{e}^{A} \mathrm{e}^{B} = \mathrm{e}^{A+B+\frac{1}{2}[A,B]} \tag{1}$$

Suppose we have the exponential representation of a unitary group

$$\mathrm{e}^{i\alpha_a X_a},$$

where X_a are hermitian generators and form a vector space; α_a are infinisetimal numbers; and summation over repeated indices is implied. In general,

$$\mathrm{e}^{i\alpha_a X_A}\mathrm{e}^{i\beta_b X_b} \neq \mathrm{e}^{i(\alpha_a X_a+\beta_b X_b)}$$

but as the elements

$$\mathrm{e}^{i\gamma_a X_a}$$

form a group, we must have

$$\mathrm{e}^{i\alpha_a X_A}\mathrm{e}^{i\beta_b X_b} = \mathrm{e}^{i\delta_a X_a} \tag{2}$$

for some δ_a, where summation over repeated indices is implied.

we expand the exponents in eq. (2) up to quadratic order in α and β

$$
\begin{aligned}
& \left(I + i\alpha_a X_a + \frac{(i\alpha_a X_a)^2}{2}\right)\left(I + i\beta_b X_b + \frac{(i\beta_b X_b)^2}{2}\right) \\
= \quad & I + i\alpha_a X_a + i\beta_b X_b - \frac{(\alpha_a X_a)^2}{2} - \alpha_a X_a \beta_b X_b - \frac{(\beta_b X_b)^2}{2} \\
& (\quad \text{complete the square } of \qquad \alpha_a X_a \beta_b X_b) \\
= \quad & I + i\alpha_a X_a + i\beta_b X_b + \\
& -\frac{(\alpha_a X_a)^2}{2} - \frac{\alpha_a X_a \beta_b X_b}{2} - \frac{\beta_b X_b \alpha_a X_a}{2} - \frac{(\beta_b X_b)^2}{2} \\
& \qquad - \frac{\alpha_a X_a \beta_b X_b}{2} + \frac{\beta_b X_b \alpha_a X_a}{2} \\
= \quad & I + i(\alpha_a X_a + \beta_b X_b) + \frac{(i(\alpha_a X_a + \beta_b X_b))^2}{2} - \frac{[\alpha_a X_a, \beta_b X_b]}{2}
\end{aligned}
$$

We get that

$$e^{i\alpha_a X_A} e^{i\beta_b X_b} = e^{i\alpha_a X_a + i\beta_b X_b - \frac{1}{2}[\alpha_a X_a, \beta_b X_B]} \tag{3}$$

or noting the group property eq. (2) we have

$$\begin{aligned} i\delta_a X_a &= i\alpha_a X_a + i\beta_b X_b - \frac{1}{2}[\alpha_a X_a, \beta_b X_B] \\ &= i\alpha_a X_a + i\beta_b X_b - \frac{1}{2}\alpha_a \beta_b [X_a, X_B] \end{aligned}$$

Hence, we must have

$$[X_a, X_b] = if_{abc} X_c \quad \text{for some } f_{abc}$$

$f_{abc} \;\rightarrow\;$ the structure constants summarising the group multiplication law.

in the case of proper orthochronous Lorentz transformations $\omega_{\mu\nu} = -\omega_{\nu\mu}$.

In four spacetime dimensions that corresponds to 3 boosts $+3$ rotations.

$$\rightarrow \;\; \Lambda = e^{\frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}}$$

where $J^{\mu\nu}$ are generators of the Lorentz algebra.

To first order in $\omega_{\mu\nu}$

$$\Lambda \sim I - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} \tag{4}$$

A representation of the Lorentz group transforms as

$$\phi^i \;\to\; \left[\mathrm{e}^{\frac{i}{2}\omega_{\mu\nu}J_R^{\mu\nu}}\right]^i_j \phi^j \;=\; U(\Lambda)\phi\ , \qquad\qquad j =, 1\ldots, n$$

$J_R^{\mu\nu}$ are the generators in the R–representation of the Lorentz group as $n \times n$ matrices. As $\mu, \nu \;=\; 0, \cdots, 3$ we have 16 $J^{\mu\nu}$ matrices, but on six of those are independent as $J_{\mu\nu} = -J_{\nu\mu}$.

elementary particles transform in representations of the Lorentz group.

scalars $\rightarrow$ Higgs boson

fermions $\rightarrow$ matter

vectors $\rightarrow$ force mediators

The generators of the Lorentz group satisfy the Lie algebra.

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho})$$

which is the Lie algebra of $SO(1,3)$. We define the operators

$$\begin{aligned} K_i &= J_{i0} = -J_{0i} \qquad\qquad i = 1,2,3 \\ J_i &= \frac{1}{2}\sum_{j,k} \epsilon_{ijk} J_{jk} \end{aligned}$$

where $\epsilon_{123} = +1,\ \epsilon_{213} = -1,\ \epsilon_{112} = 0$, etc ... To first order in the infinitesimal parameters the Lorentz rotations and boosts are given by

$$\Lambda = I + i\vec{a}\cdot\vec{K} - i\vec{b}\cdot\vec{J}$$

The second rank tensor $\omega_{\mu\nu}$ is given in terms of the a_i and b_i.

For example, in the case of **rotations:** $A = I + i\vec{\alpha} \cdot \vec{J}$.

Hence, the generators of the Lorentz group are:

$$\begin{aligned} \vec{J} &\quad\rightarrow\quad \text{generators of rotations} \\ \vec{K} &\quad\rightarrow\quad \text{generators of boost} \end{aligned}$$

satisfy the commutation relations

$$[J_i, J_j] = i\epsilon ijk J_k$$

$$[J_i, K_j] = i\epsilon ijk K_k$$

$$[K_i, K_j] = -i\epsilon ijk J_k$$

We **define the combinations:**

$$\vec{J}_+ = \frac{1}{2}\left(\vec{J} + i\vec{K}\right)$$
$$\vec{J}_- = \frac{1}{2}\left(\vec{J} - i\vec{K}\right)$$

Note that the generators $\vec{J}_+$ and $\vec{J}_-$ are not hermitians as,

$$\vec{J}_+^\dagger = \vec{J}_-$$
$$\vec{J}_-^\dagger = \vec{J}_+$$

The commutation relations of $\vec{J}_+$ and $\vec{J}_-$ are

$$\left[J_i^+, J_j^+\right] = \frac{1}{4}\left[J_i + iK_i, J_j + iK_j\right] =$$
$$\frac{1}{4}\left(J_k + iK_k + iK_k + J_k\right) = i\epsilon_{ijk}J_k^+$$

Similarly,

$$\left[J_i^+, J_j^+\right] = i\epsilon_{ijk} J_k^+$$
$$\left[J_i^-, J_j^-\right] = i\epsilon_{ijk} J_k^-$$
$$\left[J_i^+, J_j^-\right] = 0$$

The J_i^+ and J_i^- generate two disjoint generators of an $SU(2)$ algebra, $SU(2) \times SU(2)^\dagger$.

Each representation of the Lorentz group is labelled by the indecis of the two disjoint $SU(2)$ algebras (j_1, j_2).

Each representation has $(2j_1 + 1) \otimes (2j_2 + 1)$ components.

As $\vec{J} = \vec{J}_+ + \vec{J}_-$ spin is given by $j_1 + j_2$.

examples:	(j_1, j_2)	spin	components	
a	(0,0)	0	1	singlet
b	$(\frac{1}{2}, 0)$	$\frac{1}{2}$	2	Weyl spinor
c	$(0, \frac{1}{2})$	$\frac{1}{2}$	2	Weyl spinor
d	$(\frac{1}{2}, 0) + (0, \frac{1}{2})$	$\frac{1}{2}$	4	Dirac spinor
e	$(\frac{1}{2}, \frac{1}{2})$	1,0	4	vector