from the previous lecture ...

The Dirac equation
$$i\hbar \frac{1}{c}\frac{\partial}{\partial t}\Psi(\vec{x},t) = (-i\hbar\vec{\alpha}\cdot\vec{\nabla} + \beta mc)\Psi(\vec{x},t)$$

$$\text{Setting} \quad \hbar = c = 1 \quad \Rightarrow \quad (i\gamma^\mu\partial_\mu - m)\Psi = (i\not{\partial} - m)\Psi = 0$$

spin $\uparrow$ spin $\downarrow$ particles spin $\uparrow$ spin $\downarrow$ anti–particles

Showed: exist an operator $\vec{S}$ such that $\vec{J} = \vec{L} + \vec{S}$ is a constant of the motion,

$$\text{magnetic moment} \quad \mu = \frac{e}{m}\vec{S} = g_e\left(\frac{e}{2m}\right)\vec{S}$$

$g_e = 2$ (experiment $\Rightarrow$ 2.0023193...) (0.0023193...) are QFT corrections.

Dirac density and current

To give a probabilistic interpretation of the Dirac wave function ψ we have to construct a conserved current j^μ.

$$\underline{\text{We have}}: \quad \begin{aligned} (i\gamma^\mu \partial_\mu - m)\psi &= 0 \\ \psi^\dagger(-i\gamma^{\mu\dagger}\overleftarrow{\partial}_\mu - m) &= 0 \end{aligned} \tag{1}$$

$$\begin{aligned} \underline{\text{using}} \quad & \gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \\ \Rightarrow \quad & \psi^\dagger(-i\gamma^0\gamma^\mu\gamma^0\overleftarrow{\partial}_\mu - m) = 0 \qquad / \cdot \gamma^0 \\ \Rightarrow \quad & \psi^\dagger\gamma^0(-i\gamma^\mu\overleftarrow{\partial}_\mu - m) = 0 \end{aligned}$$

We define the Dirac adjoint $\bar{\psi} = \psi^\dagger\gamma^0$.

$$\Rightarrow \quad \bar{\psi}(i\gamma^\mu \overleftarrow{\partial}_\mu + m) = 0 \tag{2}$$

Taking $\quad \bar{\psi} \cdot 1 + 2 \cdot \psi \quad \Longrightarrow \quad i\bar{\psi}\gamma^\mu \vec{\partial}_\mu \psi + i\bar{\psi}\gamma^\mu \overleftarrow{\partial}_\mu \psi = i\partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0$

so $\quad j^\mu = \bar{\psi}\gamma^\mu\psi \quad \Rightarrow \quad \partial_\mu j^\mu = 0 \quad \Rightarrow \quad j^\mu$ is our conserved current.

<u>Then</u> :

$$\begin{aligned} \rho = j^0 &= \bar{\psi}\gamma^0\psi = \psi^\dagger(\gamma^0)^2\psi = \psi^\dagger\psi = \sum_{\alpha=1}^{4} |\psi_\alpha|^2 \geq 0 \\ j^j &= \bar{\psi}\gamma^j\psi = \psi^\dagger\gamma^0\gamma^j\psi = \psi^\dagger\alpha^j\psi \end{aligned}$$

$\rightarrow$ ρ is positive definite but we still get negative energy

the negative energy solutions correspond to antiparticles.

The Dirac field describes a multi–state solution, *i.e.* it is a quantum field.

Solutions of the Dirac equation

$\psi = (\psi_1, \psi_2, \psi_3, \psi_4)$ Each component obeys the Klein–Gordon equation.

$\psi = u(E, \vec{p})\mathrm{e}^{-ip\cdot x} = u(E, \vec{p})\mathrm{e}^{-i(Et-\vec{p}\cdot\vec{x})} \rightarrow$ positive energy plane wave solution

$$(i\gamma^\mu \partial_\mu - m)\psi = (\gamma^\mu p_\mu - m)u = 0, \quad \underline{\text{writing}} \quad u = \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \chi_1 \\ \chi_2 \end{pmatrix}$$

$$\begin{aligned}
(\gamma^0 E - \vec{\gamma}\cdot\vec{p} - m)\begin{pmatrix} \phi \\ \chi \end{pmatrix} &= \left[\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} E - \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} p_j - m \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right] \begin{pmatrix} \phi \\ \chi \end{pmatrix} \\
&= \begin{pmatrix} E - m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E - m \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0
\end{aligned}$$

$$\begin{aligned} \Rightarrow \quad (E-m)\,\phi &= \vec{\sigma}\cdot\vec{p}\;\chi \\ \vec{\sigma}\cdot\vec{p}\;\phi &= (E+m)\,\chi \end{aligned}$$

$$\Rightarrow \quad \chi = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,\phi$$

<u>Recall that</u> $\vec{S} = \frac{1}{2}\vec{\Sigma} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \quad \Rightarrow \quad S_z = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Hence $\phi = N\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for spin up along z–axis

$\phi = N\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for spin down along z–axis

$$\vec{\sigma}\cdot\vec{p}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}p_x + \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}p_y + \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}p_z = \begin{pmatrix}p_z\\p_x + ip_y\end{pmatrix}$$

Similarly : $$\vec{\sigma}\cdot\vec{p}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}p_x - ip_y\\-p_z\end{pmatrix}$$

$$\implies \quad u^{\uparrow} = N\begin{pmatrix}1\\0\\\frac{p_z}{E+m}\\\frac{p_x+ip_y}{E+m}\end{pmatrix} \qquad u^{\downarrow} = N\begin{pmatrix}0\\1\\\frac{p_x-ip_y}{E+m}\\\frac{-p_z}{E+m}\end{pmatrix}$$

Normalisation is calculated from $\rho = \psi^\dagger\psi = u^\dagger u = 2E$

$2E$ is the relativistic particle density per unit volume

This gives $$N^2\left[1 + \frac{p_x^2+p_y^2+p_z^2}{(E+m)^2}\right] = 2E$$

$$\text{Using} \qquad \vec{p}^{\,2} = E^2 - m^2 \quad \text{gives}$$

$$N^2 \left[1 + \frac{(E-m)(E+m)}{(E+m)^2}\right] = 2E$$

$$\Rightarrow \qquad N^2 \left(\frac{2E}{E+m}\right) = 2E \qquad \Rightarrow \qquad N = \sqrt{E+m}$$

For a particle in the rest frame $p^\mu = (m, 0, 0, 0) \quad \Rightarrow \quad \vec{p} = 0$ we get

$$\Longrightarrow \qquad u^\uparrow = N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad\qquad u^\downarrow = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

In the non–relativistic limit $u^\uparrow \ \to$ spin up $\qquad u^\downarrow \ \to$ spin down $\qquad$ fields.

For anti–particles of 4–momentum $(E, \vec{p})$ we need a solution with $p^\mu \to (-E, -\vec{p})$

$$\psi = v(E, \vec{p})\mathrm{e}^{ip_\mu x^\mu} = v(E, \vec{p})\mathrm{e}^{i(Et - \vec{p}\cdot\vec{x})}$$

Dirac Equation :

$$(i\gamma^\mu \partial_\mu - m)\psi = (-\gamma^\mu p_\mu - m)v(E, \vec{p}) =$$

$$\left[\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} E + \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} p_j - m \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right] v(E, \vec{p}) =$$

$$\begin{pmatrix} -E - m & \vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & E - m \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0$$

$$\Rightarrow \qquad \vec{\sigma}\cdot\vec{p}\chi = (E + m)\phi \quad \Rightarrow \quad \phi = \frac{\vec{\sigma}\cdot\vec{p}}{E + m}\chi$$

$$\vec{\sigma}\cdot\vec{p}\phi = (E - m)\chi$$

Like the 4–momentum, spin is reversed

$$\implies \quad v^{\downarrow} = N \begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \qquad\qquad v^{\uparrow} = N \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$