from the previous lecture ...

The Dirac equation

$$i\hbar\frac{1}{c}\frac{\partial}{\partial t}\Psi(\vec{x},t) = (-i\hbar\vec{\alpha}\cdot\vec{\nabla} + \beta mc)\Psi(\vec{x},t)$$

The Dirac field describes a multi–state solution, *i.e.* it is a quantum field.

Solutions of the Dirac equation

$\psi = u(E,\vec{p})\mathrm{e}^{-ip\cdot x} = u(E,\vec{p})\mathrm{e}^{-i(Et-\vec{p}\cdot\vec{x})} \rightarrow$ positive energy plane wave solution

$$\Longrightarrow \quad u^{\uparrow} = N\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \qquad u^{\downarrow} = N\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

$2E \rightarrow$ relativistic particles density per unit volume $\Rightarrow \quad N = \sqrt{E+m}$

For anti–particles of 4–momentum $(E, \vec{p})$, $\qquad p^\mu \to (-E, -\vec{p})$

$$\implies \quad v^{\downarrow} = N \begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \qquad\qquad v^{\uparrow} = N \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

Charge conjugation

Consider the KGE for a charged particle in an electromagnetic field

$$\vec{p} \;\rightarrow\; \vec{p} - e\vec{A} \qquad , \qquad E \;\rightarrow\; E - eV$$

$$(E - eV)^2 = (\vec{P} - e\vec{A})^2 + m^2$$

In quantum mechanics $E \;\rightarrow\; i\hbar\partial_t, \qquad \vec{p} \;\rightarrow\; -i\hbar\vec{\nabla}$

$$\Rightarrow \quad (i\hbar\partial_t - eV)^2\phi(\vec{x}, t) = (-i\hbar\vec{\nabla} - e\vec{A})^2\phi(\vec{x}, t) + m^2\phi(\vec{x}, t)$$

The complexified equation $\qquad (-i\hbar\partial_t - eV)^2\phi^* = (i\hbar\vec{\nabla} - e\vec{A})^2\phi^* + m^2\phi^*$

$$\Rightarrow \qquad (i\hbar\partial_t + eV)^2\phi^* = (-i\hbar\vec{\nabla} + e\vec{A})^2\phi^* + m^2\phi^*$$

Hence if $\phi = \mathrm{e}^{-i(Et - \vec{p}\cdot\vec{x})} = \mathrm{e}^{-iEt + \vec{p}\cdot\vec{x}}$ is a solution of the KGE with $E > 0,\ \vec{p}$.
$\phi^* = \mathrm{e}^{i(Et - \vec{p}\cdot\vec{x})} = \mathrm{e}^{-i(E(-t) + i\vec{p}\cdot\vec{x})}$ is a solution of the KGE
with $E > 0,\ \vec{p} \rightarrow -\vec{p}$ and $e \rightarrow -e,\ m = m_0$.

$$\text{This operation} \quad : \quad \begin{array}{ccc} \phi & \to & \phi^* \\ e & \to & -e \end{array}$$

is called charge conjugation C. The KGE is invariant under charge conjugation. The Dirac equation is also invariant under Charge Conjugation (CC). What is the CC operation that leaves the Dirac equation invariant?

$$\psi \quad \to \psi^C = C\psi^* \quad \to \quad \text{charge conjugation} \quad C$$

Such that ψ^C is a positive energy solution with $e \to -e$.

$$\begin{array}{lll} \text{To find } C \text{ write} & \gamma^\mu(i\partial_\mu - eA_\mu)\psi - m\psi = 0 & \leftarrow \quad \text{Dirac eq.} \\ \to & \gamma^\mu(\partial_\mu + ieA_\mu)\psi + im\psi = 0 & \\ \Rightarrow & \gamma^{\mu *}(\partial_\mu - ieA_\mu)\psi^* - im\psi^* = 0 & \\ & -C\gamma^{\mu *}C^{-1}(\partial_\mu - ieA_\mu)\psi^C + im\psi^C = 0 & \end{array}$$

Hence we need

$$\begin{array}{rcl} C\gamma^{\mu *}C^{-1} & = & -\gamma^{\mu} \\ i.e. \qquad \gamma^{\mu}C & = & -C\gamma^{*\mu} \end{array}$$

Since all γ^{μ} are real, except γ^2 (which is purely imaginary) in our standard representation we can take

$$C = i\gamma^2 = i\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

For free particles we have $v^{\uparrow C} = u^{\uparrow}$ $v^{\downarrow C} = -u^{\downarrow}$

Parity invariance

Similarly to charge conjugation we can also extract the transformation properties of the Dirac wave function under parity transformations

$$\underline{\text{define}}: \qquad \psi(\vec{r},t) \;\rightarrow\; \psi^P(\vec{r},t) = P\psi(-\vec{r},t)$$

$\underline{\text{invariance}}$: we want to find P such that ψ^P is also a solution

$$\begin{aligned}
\underline{\text{Dirac equation}}: \quad &\rightarrow \quad (i\gamma^\mu\partial_\mu - m)\psi(\vec{r},t) = 0 \\
&\rightarrow \quad (\gamma^\mu\partial_\mu + im)\psi(\vec{r},t) = 0 \\
&\rightarrow \quad (\gamma^0\partial_0 + \gamma^j\partial_j + im)\psi(\vec{r},t) = 0 \\
\vec{r} \rightarrow -\vec{r} \quad &\rightarrow \quad (\gamma^0\partial_0 - \gamma^j\partial_j + im)\psi(-\vec{r},t) = 0 \\
\psi \rightarrow \psi^P \quad &\rightarrow \quad (P\gamma^0P^{-1}\partial_0 - P\gamma^jP^{-1}\partial_j + im)\psi^P(\vec{r},t) = 0
\end{aligned}$$

Hence for invariance to hold we need

$$P\gamma^0 P^{-1} = \gamma^0$$
$$P\gamma^j P^{-1} = -\gamma^j$$

$$\Rightarrow \quad P\gamma^0 = \gamma^0 P \qquad , \qquad P\gamma^j = -P\gamma^j$$

This relations are satisfied by

$$P = \gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

particle at rest $\quad \psi = u(m,\vec{0})\mathrm{e}^{-imt} = \sqrt{2m}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\mathrm{e}^{-imt} \;\Rightarrow\; \psi^P = P\psi = +\psi$

$\overline{particle}$ at rest $\quad \psi = v(m,\vec{0})\mathrm{e}^{+imt} = \sqrt{2m}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\mathrm{e}^{+imt} \;\Rightarrow\; \psi^P = P\psi = -\psi$

$\Rightarrow$ particle and anti–particles have opporite intrinsic parity.

For KGE the parity transformation is $\qquad \phi(\vec{r},t) \to \phi^P(\vec{r},t) = \phi(-\vec{r},t)$

Since $\phi(\vec{r},t)$ is a scalar function under LT $\qquad \phi'(\vec{r}',t') = \phi(\vec{r},t)$

which follows from the KGE $$\left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2\right)\phi(\vec{r},t) = 0$$

$$P \rightarrow \quad \left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2\right)\phi'(-\vec{r},t) = 0$$

Hence ϕ and ϕ' are solutions of the same equations.

In the case of the Dirac equation the scalar is $\Phi = \bar{\psi}\psi = \psi^\dagger\gamma^0\psi$

$$\begin{aligned} \text{check :} \quad \Phi(\vec{r},t) &= \psi^\dagger(\vec{r},t)\gamma^0\psi(\vec{r},t) \\ \Phi^P(\vec{r},t) &= \underbrace{\psi^\dagger(-\vec{r},t)\gamma^{0\dagger}}_{\overline{\psi^P}}\ \gamma^0\ \underbrace{\gamma^0\psi(-\vec{r},t)}_{P\psi(-\vec{r},t)} \\ &= \psi^\dagger(-\vec{r},t)\ \gamma^0\ \psi(-\vec{r},t) \ = \ \Phi(-\vec{r},t) \end{aligned}$$

$\rightarrow \quad \bar{\psi}\psi$ transforms as a scalar under parity.

Similarly, j^μ is a true vector.

$$\begin{aligned} j^\mu(\vec{r},t) &= \psi^\dagger\gamma^0\gamma^\mu\psi(\vec{r},t) \\ j^{\mu P}(\vec{r},t) &= \psi^\dagger(-\vec{r},t)\gamma^{0\dagger}\gamma^0\gamma^\mu\gamma^0\psi(-\vec{r},t) \end{aligned}$$

$$\begin{aligned} \text{But}\quad \gamma^{0\dagger}\gamma^0\gamma^\mu\gamma^0 = \gamma^\mu\gamma^0 &= \gamma^0\gamma^\mu \quad \text{for} \quad \mu = 0 \\ &= -\gamma^0\gamma^\mu \quad \text{for} \quad \mu = 1,2,3 \end{aligned}$$

$$\underline{\text{Hence}}: \quad j^{P0}(\vec{r},t) = j^0(-\vec{r},t) \quad , \quad \vec{j}^P(\vec{r},t) = -\vec{j}(-\vec{r},t)$$

As we would expect from a true 4–vector $\quad (t,\vec{x}) \longrightarrow (t,-\vec{x})$
under parity transformation