Modern Particle Physics
Instructor's Manual
Version 1.03

MARK THOMSON

University of Cambridge

Contents

Preface to the instructor's manual

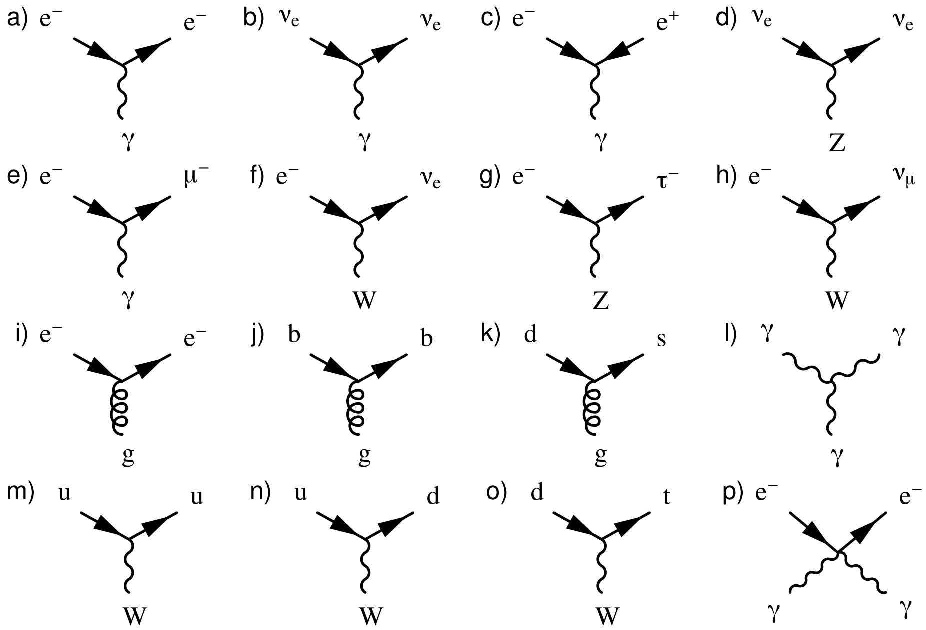
The first version of the Instructor's manual to Modern Particle Physics contains fully-worked solutions to all the problems in Chapters 1–18 of the main text. This document has not been proof-read to the extent of the main text, so I apologies in advance for any errors. Many of the problems have been used in the course that taught for a number of years, so these are battle-hardened. For new questions, introduced to address specific points in the text (particularly in the later chapters), there are a couple issues which have been noted in the solutions.

In some cases there may be more elegant approaches to the problems, the intention was to keep the solutions as straightforward as possible. Comments and suggestions are always welcome.

Mark Thomson, Cambridge, December 14th 2013

1 Introduction

1.1 Feynman diagrams are constructed out of the Standard Model vertices shown in Figure 1.4. Only the weak charged-current ($W^{\pm}$) interaction can change the flavour of the particle at the interaction vertex. Explaining your reasoning, state whether each of the sixteen diagrams below represents a valid Standard Model vertex.



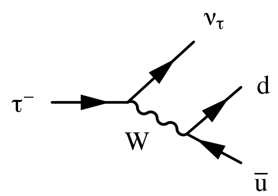
The purpose of this question is to get students to understand the that (with the exception of gauge boson triple and quartic coupling) all Feynman diagrams are built out of the Standard Model three-point vertices of Figure 1.4. Of the sixteen vertices in this question, the only valid Standard Model vertices are: a), d), f), j), n) and o). The other diagrams are forbidden for the following reasons:

- b) The electron neutrino is neutral and therefore does not couple to the gauge boson of the electromagnetic interaction;
- c) This diagram violates both charge conservation and has the effect of turning a particle into an antiparticle (the arrows on the electron lines both point towards the vertex);

e) The electron magnetic interaction does not change flavour, and hence a diagram coupling an electron to a muon is not allowed;
g) The weak neutral current also does not change flavour and hence this diagram is forbidden;
h) The weak charged current does change flavour, but *by definition* only couple together leptons with the corresponding neutrino, hence this diagram which couples together an electron and a muon neutrino is not allowed.
i) The electron does not carry the colour charge of the strong interaction - it is colour neutral - and hence the electron does not participate in the strong interaction;
k) The strong interaction does not change flavour and hence a coupling between a down-quark and a strange-quark is forbidden;
l) In the Standard Model there is no three-photon vertex;
m) Since W bosons are charged, the weak charged current must change flavour;
p) There is no Standard Model vertex couping two fermion lines to two boson lines - all fermion vertices involve a coupling to a single gauge boson.

1.2 Draw the Feynman diagram for $\tau^- \to \pi^- \nu_\tau$ (the π^- is the lightest $d\bar{u}$ meson).

Since the decay involves a change of flavour it can only be a weak charged-current interaction ($W^\pm$):



1.3 Explain why it is *not* possible to construct a valid Feynman diagram using the Standard Model vertices for the following processes:
a) $\mu^- \to e^+e^-e^+$,
b) $\nu_\tau + p \to \mu^- + n$,
c) $\nu_\tau + \underline{p} \to \tau^+ + n$,
d) $\pi^+(u\bar{d}) + \pi^-(d\bar{u}) \to n(udd) + \pi^0(u\bar{u})$.

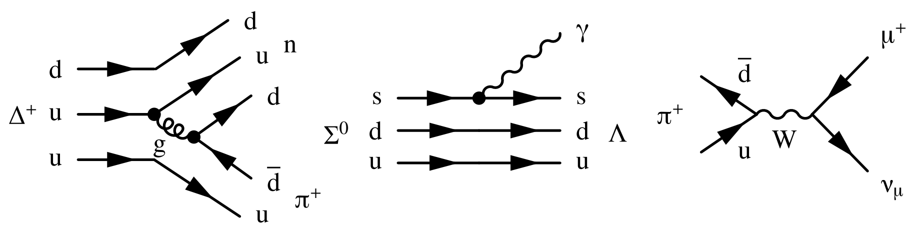
a) The process $\mu^- \to e^+e^-e^+$ would require a change of flavour. There is no problem with producing a particle and its antiparticle (here the e^+e^-) and therefore the required flavour change is $\mu^- \to e^-$ and there is no corresponding Standard Model vertex and the process cannot occur. This used to be referred to as conservation of muon and electron numbers. However, with the discovery of neutrino oscillations this concept is obsolete.

b) Two changes of flavour are required $u \to d$ and $\nu_\tau \to \mu^-$. Whilst the first of these flavour changes can be achieved by a charged-interaction vertex, the second cannot. Leptons only couple to the corresponding weak eigenstate neutrino flavour. In addition, electric charge is not conserved.
c) This process requires a vertex that has the effect $\nu_\tau \to \tau^+$, i.e. turning a particle into an antiparticle. No such vertices exist.
d) Here the net number of particles − antiparticles changes. This can not happen because all Standard Model vertices involving fermions are three point interactions with a single boson, as a consequence the net number of particles − antiparticles in the Universe is constant.

1.4 Draw the Feynman diagrams for the decays:
a) $\Delta^+(uud) \to n(udd)\,\pi^+(u\bar{d})$,
b) $\Sigma^0(uds) \to \Lambda(uds)\,\gamma$,
c) $\pi^+(u\bar{d}) \to \mu^+\nu_\mu$,

and place them in order of increasing lifetime.

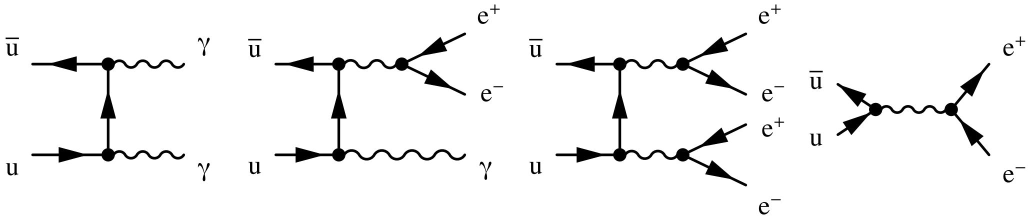
All other things being equal, strong decays will dominate over EM decays, and EM decays will dominate over weak decays. So here the order is a), b), c) with the Feynman diagrams below.



1.5 Treating the π^0 as a $u\bar{u}$ bound state, draw the Feynman diagrams for:
a) $\pi^0 \to \gamma\gamma$,
b) $\pi^0 \to \gamma e^+e^-$,
c) $\pi^0 \to e^+e^-e^+e^-$,
d) $\pi^0 \to e^+e^-$.

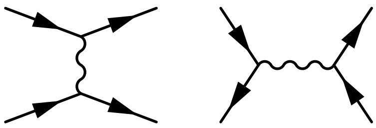
By considering the number of QED vertices present in each decay, *estimate* the relative decay rates taking $\alpha = 1/137$.

The observed branching ratios are $BR(\pi^0 \to \gamma\gamma) = 98.8\%$, $BR(\pi^0 \to \gamma e^+e^-) = 1.2\%$, $BR(\pi^0 \to \gamma e^+e^-e^+e^-) \sim 3 \times 10^{-5}$ and $BR(\pi^0 \to e^+e^-) = 6 \times 10^{-8}$. By counting the number of QED vertices it might be expected that the matrix elements for the processes are: a) $\mathcal{M} \propto e^2$; b) $\mathcal{M} \propto e^3$; c) $\mathcal{M} \propto e^4$; and d) $\mathcal{M} \propto e^2$. Consequently, one would expect the branching ratios to be proportional to a) $|\mathcal{M}|^2 \propto \alpha^2$; b) $|\mathcal{M}|^2 \propto \alpha^3$; c) $|\mathcal{M}|^2 \propto \alpha^4$; and d) $|\mathcal{M}|^2 \propto \alpha^2$. Relative to the dominant $\pi^0 \to \gamma\gamma$ decay modes it might be expected that $BR(\pi^0 \to \gamma e^+e^-) \sim \mathcal{O}(10^{-2})$, and $BR(\pi^0 \to e^+e^-e^+e^-) \sim \mathcal{O}(10^{-4})$, in reasonable agreement with the observed values.



The observed branching ratio to e^+e^- is much smaller than that predicted from simple vertex counting (the contribution from this Feynman diagram is heavily helicity suppressed, see for example 11). This is an important point, simple vertex factor counting only addresses one of the contributions to the matrix element squared, other factors may be just as important.

1.6 Particle interactions fall into two main categories, scattering processes and annihilation processes. as indicated by the Feynman diagrams below.

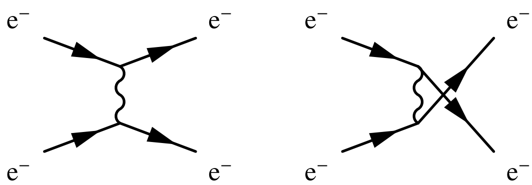


Draw the lowest-order Feynman diagrams for the scattering and/or annihilation processes:

a) $e^-e^- \to e^-e^-$,
b) $e^+e^- \to \mu^+\mu^-$,
c) $e^+e^- \to e^+e^-$,
d) $e^-\nu_e \to e^-\nu_e$,
e) $e^-\overline{\nu}_e \to e^-\overline{\nu}_e$.

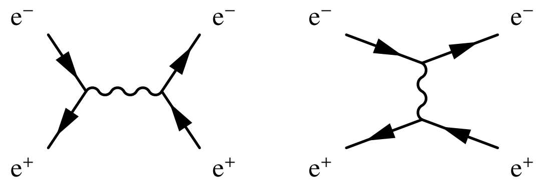
In some cases there may be more than one lowest-order diagram.

a) For this scattering process there are two diagrams, the *u*-channel diagram has to be included since there are identical particles in the final state (the exchanged gauge boson could also be a Z or even the H):

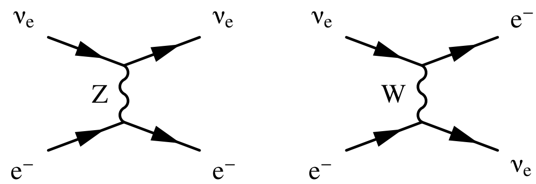


b) Here there is just the *s*-channel annihilation diagram.

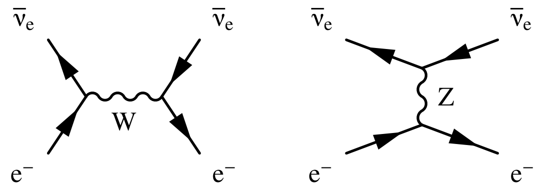
c) Here there are both s-channel and t-channel diagrams:



d) Here there are two t-channel diagrams (involving either an exchanged Z or W):



e) Here there are s- and t-channel diagrams:



1.7 High-energy muons traversing matter lose energy according to

$$-\frac{1}{\rho}\frac{\mathrm{d}E}{\mathrm{d}x} \approx a + bE\,,$$

where a is due to ionisation energy loss and b is due to the bremsstrahlung and e^+e^- pair-production processes. For standard rock, taken to have $A = 22$, $Z = 11$ and $\rho = 2.65\,\mathrm{g\,cm^{-3}}$, the parameters a and b depend only weakly on the muon energy and have values $a \approx 2.5\,\mathrm{MeV\,g^{-1}\,cm^2}$ and $b \approx 3.5\times10^{-6}\,\mathrm{g^{-1}\,cm^2}$.

a) At what muon energy are the ionisation and bremsstrahlung/pair production processes equally important?
b) Approximately how far does a 100 GeV cosmic-ray muon propagate in rock?

a) Ionisation and bremsstrahlung/pair production processes are equally likely (in standard rock) when $a = bE$, i.e. for muons with an energy of $E = 714\,\mathrm{GeV}$.

b) Integrating the energy loss equations gives

$$
\begin{aligned}
-\frac{\mathrm{d}E}{a+bE} &= \rho\,\mathrm{d}x \\
\Rightarrow \qquad -\left[\ln(a+bE)\right]_{E_0}^{0} &= b\rho\,[x]_0^L \\
\Rightarrow \qquad L &= \frac{1}{\rho b}\ln\left[1+\frac{b}{a}E_0\right] \\
&= 141\,\mathrm{m}
\end{aligned}
$$

From this it can be seen that high-energy muons are highly penetrating particles. Muons with $E > 10\,\mathrm{GeV}$ will usually traverse the entire volume of a collider detector.

1.8 Tungsten has a radiation length of $X_0 = 0.35\,\mathrm{cm}$ and a critical energy of $E_c = 7.97\,\mathrm{MeV}$. Roughly what thickness of Tungsten is required to fully contain a 500 GeV electromagnetic shower from an electron?

The number of particles in a shower doubles every radiation length of material traversed. Hence the typical particle energy $\langle E_n \rangle$ in a shower after n radiation lengths is

$$\langle E_n \rangle = E/2^n\,, \tag{1.1}$$

and the shower terminates when this mean energy is equal to the critical energy, hence

$$n \ln 2 = \ln\left(E/E_c\right)\,.$$

For a 500 GeV electromagnetic shower in Tungsten this corresponds to $n = 16$ radiation lengths or 5.6 cm of Tungsten. The above analysis is overly simplistic. In practice, due to fluctuations, the shower extends deeper than the above calculation would suggest, although with ever decreasing numbers of particles.

1.9 The CPLEAR detector (see Section 14.5.2) consisted of: tracking detectors in a magnetic field of 0.44 T; an electromagnetic calorimeter; and Čerenkov detectors with a radiator of refractive index $n = 1.25$ used to distinguish $\pi^\pm$ from $\mathrm{K}^\pm$.

A charged particle travelling perpendicular to the direction of the magnetic field leaves a track with a measured radius of curvature of $R = 4\,\mathrm{m}$. If it is observed to give a Čerenkov signal, is it possible to distinguish between the particle being a pion or kaon? Take $m_\pi \approx 140\,\mathrm{MeV}/c^2$ and $m_\mathrm{K} = 494\,\mathrm{MeV}/c^2$.

The momentum of the particle can be obtained from

$$\mathrm{p}\cos\lambda = 0.3\,\mathrm{B}R$$

giving $\mathrm{p} = 528\,\mathrm{MeV}/c$. Since the particle produced a Čerenkov signal $\beta > 1/n$, i.e.

$\beta > 0.8$. Using $E = \gamma mc^2$ and $p = \gamma m\beta c$, the particle's velocity is given by

$$\beta = \frac{\mathrm{p}c}{E} = \frac{\mathrm{p}}{\sqrt{\mathrm{p}^2c^2 + m^2c^4}}\,.$$

For the two hypothesis considered here, for $\mathrm{p} = 528\,\mathrm{MeV}/c$, $\beta_\pi = 0.97$ and $\beta_\mathrm{K} = 0.73$. Hence the observed particle track could be due to a pion but cannot be a kaon.

1.10 In a fixed-target pp experiment, what proton energy would be required to achieve the same centre-of-mass energy as the LHC, which will ultimately operate at 14 TeV.

The centre-of-mass energy $\sqrt{s}$ is given by

$$s = p^2 = \left(\sum_i E_i\right)^2 - \left(\sum_i \mathbf{p}\right)^2 = (E + m_\mathrm{p})^2 - \mathrm{p}^2\,,$$

where E and p are the energy and momentum of the proton beam. Since $E \gg \mathrm{p}$,

$$s \approx (E + m_\mathrm{p})^2 - E^2 \approx 2E\,m_\mathrm{p}\,.$$

Consequently, to achieve a centre-of-mass energy of 14 TeV,

$$E = \frac{14^2}{2 \times 940 \times 10^{-6}} = 1.05 \times 10^5\,\mathrm{TeV}\,,$$

which is far above the energy achievable by any foreseeable particle accelerator. The reason such high energy is required for a fixed target experiment is that momentum must be conserved, and hence, most of the energy in the final state is associated with the high velocity of the centre-of-mass of the final state system.

1.11 *Note that the question is an updated and clearer version of the original problem.* At the LEP $\mathrm{e}^+\mathrm{e}^-$ collider, which had a circumference of 27 km, the electron and positron beams consisted of four equally spaced bunches in the accelerator. Each bunch corresponded to a beam current of 1.0 mA. The beams collided head-on at the interaction point, where the beam spot had an rms profile of $\sigma_x \approx 250\,\mu$m and $\sigma_y \approx 4\,\mu$m, giving an effective area of $1.0 \times 10^3\,\mu\mathrm{m}^2$. Calculate the instantaneous luminosity and estimate the event rate for the process $\mathrm{e}^+\mathrm{e}^- \to \mathrm{Z}$, which has a cross section of about 40 nb.

The instantaneous luminosity is given by (1.5)

$$\mathcal{L} = f\frac{n_1 n_2}{4\pi\,\sigma_x\sigma_y}\,.$$

Here $\sigma_x\sigma_y = 1.0 \times 10^3\,\mu\mathrm{m}^2 = 1.0 \times 10^{-5}\,\mathrm{cm}^2$. If the number of electrons/positrons in each bunch is n_e and each bunch circulates the ring at a frequency of $f_b = c/(27 \times 10^3) = 11.1\,\mathrm{kHZ}$, the bunch current I is given by

$$I = f_b \times n_\mathrm{e} e\,.$$

A current of 1 mA therefore corresponds to $n_e = 5.6 \times 10^{11}$ per bunch,

$$n_1 = n_2 = 5.6 \times 10^{11} \, .$$

With four bunches per beam, the bunch-crossing frequency of

$$f = 4 \cdot c/27000 = 4 f_b = 44.4 \, \mathrm{kHz} \, .$$

The instantaneous luminosity is

$$\mathcal{L} = 44.4 \times 10^3 \, \frac{(5.6 \times 10^{11})^2}{4\pi \, 1.0 \times 10^{-5}} = 1.1 \times 10^{32} \, \mathrm{cm}^{-2}\mathrm{s}^{-1} \, .$$

Using the values in the original text $\mathcal{L} = 6 \times 10^{30} \, cm^{-2} s^{-1}$.

At the peak of the $e^+e^- \to Z$ resonance, the total cross section is approximately 40 nb, or equivalently $4 \times 10^{-8} \times 10^{-24} \, \mathrm{cm}^2$ and consequently the event rate for a luminosity of $\mathcal{L} = 1 \times 10^{32} \, \mathrm{cm}^{-2}\mathrm{s}^{-1}$, given by $\sigma \mathcal{L}$, is approximately $4 \, \mathrm{s}^{-1}$.

2 Underlying Concepts

2.1 When expressed in natural units the lifetime of the W boson is approximately $\tau \approx 0.5\,\text{GeV}^{-1}$. What is the corresponding value in S.I. units?

To restore the correct dimensions a factor of $\hbar$ needs to be inserted. Hence

$$\tau = \hbar \times 0.5\,\text{GeV}^{-1} = 0.5\,\frac{1.06 \times 10^{-34}}{1.6 \times 10^{-10}} = 3.3 \times 10^{-25}\,\text{s}\,.$$

In natural units, the decay rate

$$\Gamma = \frac{1}{\tau}\,.$$

Dimensional analysis gives

$$[E] = [T]^{-1}\,,$$

and it is necessary to insert a factor of $\hbar$ on the RHS:

$$\Gamma = \frac{\hbar}{\tau}\,.$$

2.2 A cross section is measured to be 1 pb, convert this to natural units.

In natural units, the dimension of area is $[\text{GeV}]^{-2}$ and hence

$$\sigma[\text{pb}] = \sigma[\text{GeV}^{-2}] \times \frac{(\hbar c)^2}{\text{GeV}^2}$$

Here the cross section of 1 pb expressed in natural units is

$$\sigma = 1 \times 10^{-40} \times \frac{(1.6 \times 10^{-10})^2}{(1.06 \times 10^{-34} \cdot 3 \times 10^8)^2} = 2.6 \times 10^{-9}\,\text{GeV}^{-2}\,.$$

Alternatively converting into fm^2 and using the conversion factor $\hbar c = 0.197\,\text{GeV fm}$ gives

$$\sigma = 10^{-10}\,\text{fm}^2 \equiv 10^{-10}/0.197^2\,\text{GeV}^{-2} = 2.6 \times 10^{-9}\,\text{GeV}^{-2}\,.$$

2.3 Show that the process $\gamma \to e^+e^-$ can not occur in the vacuum.

This is most easily demonstrated by considering the reaction in the rest frame of the e^+e^- pair, namely the frame in which the total momentum is zero. In this frame $\mathbf{p} = 0$ and $E = E_{e^-} + E_{e^+} > 2m_e$. Since energy and momentum are conserved this would imply the photon has $\mathbf{p} = 0$ and $E > 2m_e$, which is not consistent with $E = \mathrm{p}$ for the massless photon.

2.4 A particle of mass 3 GeV is travelling in the positive z-direction with momentum 4 GeV, what are its energy and velocity?

Here the key equations are (in natural units):

$$E = \gamma m\,, \quad \mathrm{p} = \gamma m\beta \quad \text{and} \quad E^2 = p^2 + m^2\,.$$

From $E^2 = \mathrm{p}^2 + m^2$ the particle's energy is $E = 5\,\mathrm{GeV}$ and from the above expressions for E and p,

$$\beta = \mathrm{p}/E = 0.8\,.$$

2.5 In the laboratory frame, denoted Σ, a particle travelling in the z-direction has momentum $\mathbf{p} = \mathrm{p}_z\hat{\mathbf{z}}$ and energy E.

a) Use the Lorentz transformation to find expressions for the momentum p'_z and energy E' of the particle in a frame Σ', which is moving in a velocity $\mathbf{v} = +\mathrm{v}\hat{\mathbf{z}}$ relative to Σ, and show that $E^2 - \mathrm{p}_z^2 = (E')^2 - (\mathrm{p}'_z)^2$.

b) For a system of particles, prove that the total four momentum squared,

$$p^\mu p_\mu \equiv \left(\sum_i E_i\right)^2 - \left(\sum_i \mathbf{p}_i\right)^2\,,$$

is invariant under Lorentz transformations.

a) Remembering that $\gamma^2 = 1/(1-\beta^2)$ or equivalently $\gamma^2(1-\beta^2) = 1$, and using the explicit energy-momentum Lorentz transformations

$$E' = \gamma(E - \beta p_z)\,, \quad p'_x = p_x\,, \quad p'_y = p_y \quad \text{and} \quad p'_z = \gamma(p_z - \beta E)\,,$$

then $E' - \mathrm{p}'$ can be written:

$$\begin{aligned} E' - \mathrm{p}' &= \gamma^2(E - \beta p_z)^2 - p_x^2 - p_y^2 - \gamma^2(p_z - \beta E)^2 \\ &= \gamma^2(E^2 - 2\beta E p_z + \beta^2 p_z^2) - p_x^2 - p_y^2 - \gamma^2(p_z^2 - 2\beta E p_z + \beta^2 E^2) \\ &= \gamma^2(1-\beta^2)E^2 - p_x^2 - p_y^2 - \gamma^2(1-\beta^2)p_z^2 \\ &= E^2 - \mathrm{p}^2\,. \end{aligned}$$

b) Here

$$p^\mu p_\mu \equiv \sum_i \left(E_i^2 - \mathrm{p}_i^2\right) + \sum_{i \neq j} \left(E_i E_j - p_{xi} p_{xj} - p_{yi} p_{yj} - p_{zi} p_{zj}\right) .$$

From the first part of the question, to show explicitly that $p^\mu p_\mu$ for a system of particles is Lorentz invariant it is only necessary to show that $E'_i E'_j - p'_{zi} p'_{z_j} = E_i E_j - p_{zi} p_{z_j}$, the proof of which is almost identical to the first part of the question.

2.6 For the decay $a \to 1 + 2$, show that the mass of the particle a can be expressed as

$$m_a^2 = m_1^2 + m_2^2 + 2E_1 E_2 (1 - \beta_1 \beta_2 \cos\theta) ,$$

where β_1 and β_2 are the velocities of the daughter particles ($\beta_i = \mathrm{v}_i/c$) and θ is the angle between them.

Since $m_a^2 = E_a^2 - \mathrm{p}_a^2$ and energy and momentum are conserved in the decay

$$\begin{aligned} m_a^2 &= (E_1 + E_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2 \\ &= E_1^2 + E_2^2 + 2E_1E_2 - \mathrm{p}_1^2 - \mathrm{p}_2^2 - 2\mathbf{p}_1 \cdot \mathbf{p}_2 \\ &= m_1^2 + m_2^2 + 2E_1E_2 - 2\mathrm{p}_1\mathrm{p}_2 \cos\theta \\ &= m_1^2 + m_2^2 + 2E_1E_2(1 - \beta_1\beta_2\cos\theta) , \end{aligned}$$

where the last step follows from $\mathrm{p} = \beta E$.

2.7 In a collider experiment, Λ baryons can be identified from the decay $\Lambda \to \pi^- \mathrm{p}$ that gives rise to a displaced vertex in a tracking detector. In a particular decay, the momenta of the π^+ and p are measured to be 0.75 GeV and 4.25 GeV respectively, and the opening angle between the tracks is 9°. The masses of the pion and proton are 139.6 MeV and 938.3 MeV.

a) Calculate the mass of the Λ baryon.

b) On average, Λ baryons of this energy are observed to decay at a distance of 0.35 m from the point of production. Calculate the lifetime of the Λ.

a) From $E^2 = \mathrm{p}^2 + m^2$ the energies of the two decay products are

$$E_\pi = 0.763\,\mathrm{GeV} \quad \text{and} \quad E_\mathrm{p} = 4.352\,\mathrm{GeV} .$$

The corresponding velocities ($\beta = \mathrm{p}/E$) are

$$\beta_\pi = 0.983 \quad \text{and} \quad \beta_\mathrm{p} = 0.976 .$$

Using the result of the previous question,

$$m_\Lambda^2 = m_1^2 + m_2^2 + 2E_1E_2(1 - \beta_1\beta_2\cos\theta) = 1.244\,\mathrm{GeV}^2 .$$

Hence the mass of the Λ obtained from the measurements give is $m_\Lambda = 1.115\,\mathrm{GeV}$.

b) Accounting for relativistic time dilation the mean distance travelled will be

$$d = \gamma\beta c\tau\,.$$

From the relation $\mathrm{p} = \gamma m\beta$, $\gamma\beta = \mathrm{p}/m$. The energy of the Λ is simply

$$E_\Lambda = E_\pi + E_\mathrm{p} = 5.115\,\mathrm{GeV}\,,$$

from which $\mathrm{p}_\Lambda = 4.99\,\mathrm{GeV}$ and therefore $\gamma\beta = \mathrm{p}/m = 4.47$. From which

$$\tau = 0.35/4.47c = 2.6 \times 10^{-10}\,\mathrm{s}\,.$$

2.8 In the laboratory frame, a proton with total energy E collides with proton at rest. Find the minimum proton energy such that process

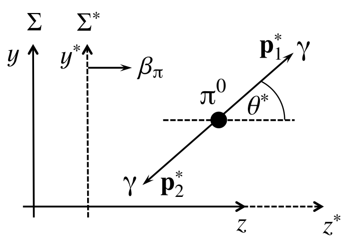
$$\mathrm{p} + \overline{\mathrm{p}} \to \mathrm{p} + \mathrm{p} + \overline{\mathrm{p}} + \overline{\mathrm{p}}$$

is kinematically allowed.

The lowest energy configuration is where all four final-state particles are at rest in the centre-of-mass frame, thus in general, the centre-of-mass energy $\sqrt{s} > 4m_\mathrm{p}$, i.e. $s > 16m_\mathrm{p}^2$. Because $s = E_{TOT}^2 - \mathbf{p}_{TOT}^2$ is Lorentz invariant, writing the proton energy and momentum as E and $\mathbf{p}$, the condition in the laboratory frame is

$$\begin{aligned}
(E + m_\mathrm{p})^2 - (\mathbf{p} + \mathbf{0})^2 &> 16m_\mathrm{p}^2\\
E^2 + 2Em_\mathrm{p} + m_\mathrm{p}^2 - \mathrm{p}^2 &> 16m_\mathrm{p}^2\\
2Em_\mathrm{p} + m_\mathrm{p}^2 &> 16m_\mathrm{p}^2\\
\Rightarrow \qquad E &> 7m_\mathrm{p}\,.
\end{aligned}$$

2.9 Find the maximum opening angle between the photons produced in the decay $\pi^0 \to \gamma\gamma$ if the energy of the neutral pion is 10 GeV, given that $m_{\pi^0} = 135$ MeV.



In the rest frame of the π^0, the four-momenta of the photons are $p_1^* = (E, 0, E\sin\theta^*, E\cos\theta^*)$ and $p_1^* = (E, 0, -E\sin\theta^*, -E\cos\theta^*)$, where $E = m_\pi^0/2$. The four-momenta of the photons in the laboratory frame can be found from the (inverse) Lorentz transformation of Equation 2.6 where $p_x = p_x^* = 0$, $p_y = p_y^* = \pm E\sin\theta^*$ and

$$\begin{aligned}
E_1 &= \gamma E_1^* + \gamma\beta p_{z1}^* = \gamma E(1 + \beta\cos\theta^*) \quad \text{and} \quad p_{z1} = \gamma p_{z1}^* + \gamma\beta E = \gamma E(\cos\theta^* + \beta)\\
E_2 &= \gamma E_2^* + \gamma\beta p_{z2}^* = \gamma E(1 - \beta\cos\theta^*) \quad \text{and} \quad p_{z2} = \gamma p_{z2}^* + \gamma\beta E = \gamma E(-\cos\theta^* + \beta)
\end{aligned}$$

One can either assume that the extreme values occur for the case where $\theta^* = 0$ and $\theta^* = \pi/2$ or one can derive the general expression for the opening angle in the laboratory frame

$$\begin{aligned}\cos\theta = \frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{\mathrm{p}_1\mathrm{p}_2} &= \frac{p_{x1}p_{x2} + p_{y1}p_{y2} + p_{z1}p_{z2}}{E_1E_2} \\ &= \frac{-E^2\sin^2\theta^* + \gamma^2E^2(\beta^2 - \cos^2\theta^*)}{\gamma^2E^2(1-\beta^2\cos^2\theta^*)} \\ &= \frac{-\sin^2\theta^*/\gamma^2 + \beta^2 - \cos^2\theta^*}{1-\beta^2\cos^2\theta^*} \\ &= \frac{\beta^2(1+\sin^2\theta^*) - 1}{1-\beta^2\cos^2\theta^*}\end{aligned}$$

where the extreme values are $\cos\theta = -1$ and $\cos\theta = 2\beta^2 - 1$.

Here, since $\gamma = E_\pi/m_\pi = 74.1$ and therefore using $\beta^2 = 1 - 1/\gamma^2 = 0.99982$, the minimum opening angle is

$$\cos\theta_{\min} = 2\beta^2 - 1 = 0.9963 \quad \text{or} \quad \theta_{\min} = 0.027\,\mathrm{rad} \equiv 1.5^\circ\,.$$

2.10 The maximum of the π^-p cross section, which occurs at $\mathrm{p}_\pi = 300$ MeV, corresponds to the resonant production of the Δ^0 baryon (*i.e.* $\sqrt{s} = m_\Delta$). What is the mass of the Δ?

Taking the protons to be at rest, the squared centre-of-mass energy s is given by

$$\begin{aligned}s &= (E_\pi + m_\mathrm{p})^2 - (\mathrm{p}_\pi^2) \\ &= E_\pi^2 - \mathrm{p}_\pi^2 + 2E_\pi m_\mathrm{p} + m_\mathrm{p}^2 \\ &= m_\mathrm{p}^2 + m_\pi^2 + 2E_\pi m_\mathrm{p} \\ &= m_\mathrm{p}^2 + m_\pi^2 + 2m_\mathrm{p}(\mathrm{p}_\pi^2 + m_\pi^2)^{1/2} \\ &= 1.52\,\mathrm{GeV}\end{aligned}$$

Therefore the mass of the Δ is

$$m_\Delta = \sqrt{s} = 1.23\,\mathrm{GeV}\,.$$

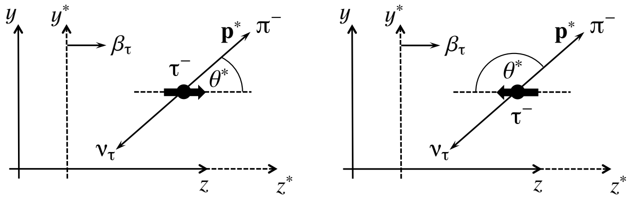
2.11 Tau leptons are produced in the process $e^+e^- \to \tau^+\tau^-$ at a centre-of-mass energy of 91.2 GeV. The angular distribution of the π^- from the decay $\tau^- \to \pi^-\nu_\tau$ is

$$\frac{\mathrm{d}N}{\mathrm{d}(\cos\theta^*)} \propto 1 + \cos\theta^*\,, \tag{2.1}$$

where θ^* is the polar angle of the π^- in the τ-lepton rest frame, relative to the direction defined by the τ spin. Determine the laboratory frame energy distribution of the π^- for the cases where the τ-lepton spin is i) *aligned with* or ii) *opposite to* its direction of flight.

The angular dependence arises from the chiral nature of the weak interaction, which implies that the ν_τ is left-handed. The two cases are indicated below. In the rest frame of the four-momentum of the π^- are respectively given by:

$$p^* = (E^*_\pi, 0, \mathrm{p}^*_\pi \sin\theta^*, \mathrm{p}^*_\pi \cos\theta^*) \quad \text{and} \quad p^* = (E^*_\pi, 0, \mathrm{p}^*_\pi \sin\theta^*, -\mathrm{p}^*_\pi \cos\theta^*)\,.$$



In the laboratory frame, the energies of the charged pions are

$$E_\pm = \gamma E^*_\pi \pm \beta\gamma \mathrm{p}^*_\pi \cos\theta^*\,, \tag{2.2}$$

where E_+ refers to the first case where the spin of the tau lepton is on the same direction as the motion of the tau and β and γ are respectively the velocity and Lorentz factor of the tau-lepton. This implies that the minimum and maximum energies of the decay pions ($\cos\theta^* = \pm 1$) are

$$E_{\min} = \gamma E^*_\pi - \beta\gamma \mathrm{p}^*_\pi \quad \text{and} \quad E_{\max} = \gamma E^*_\pi + \beta\gamma \mathrm{p}^*_\pi\,,$$

which can be used to write

$$\gamma E^*_\pi = \tfrac{1}{2}(E_{\max} + E_{\min}) \quad \text{and} \quad \beta\gamma \mathrm{p}^*_\pi = \tfrac{1}{2}(E_{\max} - E_{\min})\,.$$

For both spin-orientations:

$$\begin{aligned} \frac{\mathrm{d}N}{\mathrm{d}E_\pm} &= \frac{\mathrm{d}N}{\mathrm{d}(\cos\theta^*)} \times \left|\frac{\mathrm{d}E_\pm}{\mathrm{d}(\cos\theta^*)}\right|^{-1} \\ &\propto (1 + \cos\theta^*)/(\beta\gamma \mathrm{p}^*_\pi) \\ &\propto \frac{1 + \cos\theta^*}{E_{\max} - E_{\min}}\,. \end{aligned}$$

This can be written in terms of the measured energy of the pion using (2.2), which when expressed in terms of $E_{\max}$ and $E_{\min}$ gives

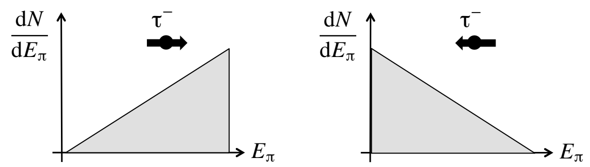
$$1 + \cos\theta^* = \frac{E_{\max} - E_{\min} \pm (2E_\pm - E_{\max} - E_{\min})}{E_{\max} - E_{\min}}\,.$$

Therefore for the two different spin orientations

$$\frac{\mathrm{d}N}{\mathrm{d}E_\pi} = \frac{E_\pi - E_{\min}}{E_{\max} - E_{\min}} \quad \text{and} \quad \frac{\mathrm{d}N}{\mathrm{d}E_\pi} = \frac{E_{\max} - E_\pi}{E_{\max} - E_{\min}}\,,$$

where now $E_\pm$ have simply been written as E_π.

Here the τ energy is simply $m_Z/2$ and therefore $\gamma = m_Z/2m_\tau = 25.65$ and $\beta = 0.9992$. It is straightforward to show that the $E^*_\pi = 0.894\,\text{GeV}$ and $p^*_\pi = 0.883\,\text{GeV}$. Therefore $E_{\text{min}} = 0.3\,\text{GeV}$ and $E_{\text{max}} = 45.5\,\text{GeV}$. Consequently, the energy distributions of the charged pion from $\tau^- \to \pi^-\nu_\tau$ decays (shown below) provides a way of measuring the average polarisation of the tau lepton.



2.12 For the process $1 + 2 \to 3 + 4$, the Mandelstam variables s, t and u are defined as $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$. Show that

$$s + u + t = m_1^2 + m_2^2 + m_3^2 + m_4^2 .$$

This is a fairly straightforward problem if approached correctly:

$$\begin{aligned}
s + u + t &= (p_1 + p_2)^2 + (p_1 - p_3)^2 + (p_1 - p_4)^2 \\
&= p_1^2 + p_2^2 + 2p_1\cdot p_2 + p_1^2 + p_3^2 - 2p_1\cdot p_3 p_1^2 + p_4^2 - 2p_1\cdot p_4 \\
&= 3p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2p_1\cdot(p_2 - p_3 - p_4) \\
&= 3p_1^2 + p_2^2 + p_3^2 + p_4^2 - 2p_1\cdot p_1 \\
&= p_1^2 + p_2^2 + p_3^2 + p_4^2 \\
&= m_1^2 + m_2^2 + m_3^2 + m_4^2 .
\end{aligned}$$

2.13 At the HERA collider, 27.5 GeV electrons were collided head-on with 820 GeV protons. Calculate the centre-of-mass energy.

Both the electron and proton can be treated as ultra-relativistic such that p ≈ E and therefore

$$s = E^2 - \mathbf{p}^2 = (E_p + E_e)^2 - (E_p - E_e) = 90200\,\text{GeV}^2 ,$$

and hence the centre-of-mass energy $\sqrt{s} = 300\,\text{GeV}$.

2.14 Consider the Compton scattering of a photon of momentum $\mathbf{k}$ and energy $E = |\mathbf{k}| = \mathrm{k}$ from an electron *at rest*. Writing the four-momenta of the scattered photon and electron respectively as k' and p', conservation of four-momentum is expressed as $k + p = k' + p'$. Use the relation $p'^2 = m_e^2$ to show that the energy of the scattered photon is given by

$$E' = \frac{E}{1 + (E/m_e)(1 - \cos\theta)} ,$$

where θ is the angle through which the photon is scattered.

Using four-vectors, this is a fairly straightforward problem. Writing $p' = k - k' + p$ and squaring (the four-vectors) gives

$$\begin{aligned} p'^2 &= (k-k')^2 + p^2 + 2p\cdot(k-k') \\ m_e^2 &= k^2 + k'^2 - 2k\cdot k' + m_e^2 + 2m_e(E-E') \\ 2EE'(1-\cos\theta) &= 2m_e(E-E') \\ E'\{E(1-\cos\theta)+m_e\} &= 2m_eE \\ \Rightarrow \qquad E' &= \frac{E}{1+(E/m_e)(1-\cos\theta)} \,. \end{aligned}$$

2.15 Using the commutation relations for position and momentum, prove that

$$\left[\hat{L}_x, \hat{L}_y\right] = i\hat{L}_z \,.$$

Using the commutation relations for the components of angular momenta prove

$$\left[\hat{L}^2, \hat{L}_x\right] = 0 \,,$$

and

$$\hat{L}^2 = \hat{L}_-\hat{L}_+ + \hat{L}_z + \hat{L}_z^2 \,.$$

a) For the first part, starting from the definition of angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$,

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \,, \quad \hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z \quad \text{and} \quad \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \,,$$

the commutator in question can be simplified using $[\hat{z}, \hat{p}_z] = i$ as follows

$$\begin{aligned} \left[\hat{L}_x, \hat{L}_y\right] &= \left[\hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \hat{z}\hat{p}_x - \hat{x}\hat{p}_z\right] \\ &= \left[\hat{y}\hat{p}_z, \hat{z}\hat{p}_x\right] + \left[\hat{z}\hat{p}_y, \hat{x}\hat{p}_z\right] \\ &= \hat{y}\hat{p}_x\left[\hat{p}_z, \hat{z}\right] + \hat{p}_y\hat{x}\left[\hat{z}, \hat{p}_z\right] \\ &= -i\hat{y}\hat{p}_x + i\hat{p}_y\hat{x} \\ &= i(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x) \\ &= i\hat{L}_z \,. \end{aligned}$$

b) To show that $\left[\hat{L}^2, \hat{L}_x\right] = 0$, the angular momentum commutation relations written in the form $\hat{L}_x\hat{L}_y = \hat{L}_y\hat{L}_x + i\hat{L}_z$ can be used:

$$\begin{aligned}\left[\hat{L}^2, \hat{L}_x\right] &= \left[\hat{L}_x\hat{L}_x, \hat{L}_x\right] + \left[\hat{L}_y\hat{L}_y, \hat{L}_x\right] + \left[\hat{L}_z\hat{L}_z, \hat{L}_x\right] \\ &= \hat{L}_y\hat{L}_y\hat{L}_x - \hat{L}_x\hat{L}_y\hat{L}_y + \hat{L}_z\hat{L}_z\hat{L}_x - \hat{L}_x\hat{L}_z\hat{L}_z \\ &= \hat{L}_y(\hat{L}_x\hat{L}_y - i\hat{L}_z) - (\hat{L}_y\hat{L}_x + i\hat{L}_z)\hat{L}_y + \hat{L}_z(\hat{L}_x\hat{L}_z + i\hat{L}_y) - (\hat{L}_z\hat{L}_x - i\hat{L}_y)\hat{L}_z \\ &= -i(\hat{L}_y\hat{L}_z + \hat{L}_z\hat{L}_y) + i(\hat{L}_z\hat{L}_y + \hat{L}_y\hat{L}_z) \\ &= 0\end{aligned}$$

c) First expand $\hat{L}_+\hat{L}_-$

$$\begin{aligned}\hat{L}_-\hat{L}_+ &= (\hat{L}_x - i\hat{L}_y)(\hat{L}_x + i\hat{L}_y) \\ &= \hat{L}_x^2 + \hat{L}_y^2 - i\left[\hat{L}_y, \hat{L}_x\right] \\ &= \hat{L}_x^2 + \hat{L}_y^2 - i(-i\hat{L}_z) \\ &= \hat{L}_x^2 + \hat{L}_y^2 - \hat{L}_z\,.\end{aligned}$$

Then express in terms of $\hat{L}^2$:

$$\begin{aligned}\hat{L}^2 &= \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \\ &= \hat{L}_+\hat{L}_- + \hat{L}_z + \hat{L}_z^2\,.\end{aligned}$$

2.16 Show that the operators $\hat{S}_i = \frac{1}{2}\sigma_i$, where σ_i are the three Pauli spin-matrices,

$$\hat{S}_x = \tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \hat{S}_z = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

satisfy the same algebra as the angular momentum operators, namely

$$\left[\hat{S}_x, \hat{S}_y\right] = i\hat{S}_z\,, \quad \left[\hat{S}_y, \hat{S}_z\right] = i\hat{S}_x \quad \text{and} \quad \left[\hat{S}_z, \hat{S}_x\right] = i\hat{S}_y\,.$$

Find the eigenvalue(s) of the operator $\hat{\mathbf{S}}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$, and deduce that the eigenstates of $\hat{S}_z$ are a suitable representation of a spin-half particle.

a) The commutation relations can be obtained by direct matrix multiplication, e.g.

$$\begin{aligned}\left[\hat{S}_x, \hat{S}_y\right] &= \tfrac{1}{4}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \\ &= \tfrac{1}{4}\left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right] \\ &= i\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= i\hat{S}_z\,.\end{aligned}$$

b) Each of the Pauli spin matrices satisfies $\sigma_k^2 = I$ and therefore

$$\hat{\mathbf{S}}^2 = \tfrac{1}{4}\left(\sigma_x^2 + \sigma_y^2 + \sigma_z^2\right) = \tfrac{3}{4}I\,.$$

Given that the commutation relations are the same as for angular momentum the states in the vector space upon which the $\hat{S}$ act can be described by two quantum numbers $|s, m\rangle$, where the eigenvalue of $\hat{\mathbf{S}}^2$ is $s(s+1)$ and there are $2s+1$ states differing by one unit of m. For the two-component state vector

$$\hat{\mathbf{S}}^2\,|s, m\rangle = \tfrac{3}{4}\,|s, m\rangle \equiv s(s+1)\,|s, m\rangle\ .$$

Hence $s(s+1) = 3/4$ which is satisfied for $s = 1/2$. Hence there are two states in this vector space with $s = 1/2$ and $m_s = \pm 1/2$. Since the operators satisfy the commutation relations of the angular momentum operators, it can be concluded that $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ are a suitable (but not unique) representation of the operators for the algebra of spin-half.

2.17 Find the third-order term in the transition matrix element of Fermi's Golden rule.

The third-order term in the perturbation series can be found be substituting the solution to second-order back into (2.42):

$$i\sum_{k'}\frac{\mathrm{d}c_k}{\mathrm{d}t}\phi_{k'}e^{-iE_{k'}t} \approx \hat{H}'\phi_i e^{-iE_it} - i\sum_{k\neq i}\int\langle k|\hat{H}'|i\rangle e^{i(E_k-E_i)t'}\,\hat{H}'\phi_k e^{-iE_kt'}\,\mathrm{d}t'$$
$$+ (-i)^2\sum_{k\neq i}\sum_{j\neq i\neq k}\int\langle k|\hat{H}'|j\rangle e^{i(E_k-E_j)t''}\,\mathrm{d}t''\int\langle j|\hat{H}'|i\rangle e^{i(E_j-E_i)t'}\,\hat{H}'\phi_k e^{-iE_kt}\,\mathrm{d}t'\,.$$

Multiplying both sides by ϕ_f^* gives

$$i\frac{\mathrm{d}c_f}{\mathrm{d}t}e^{-iE_ft} \approx \langle f|\hat{H}'|i\rangle e^{-iE_it} - i\sum_{k\neq i}\int\langle k|\hat{H}'|i\rangle e^{i(E_k-E_i)t'}\langle f|\hat{H}'|k\rangle e^{-iE_kt'}\,\mathrm{d}t'$$
$$+ (-i)^2\sum_{k\neq i}\sum_{j\neq i\neq k}\int\langle k|\hat{H}'|j\rangle e^{i(E_k-E_j)t''}\,\mathrm{d}t''\int\langle j|\hat{H}'|i\rangle e^{i(E_j-E_i)t'}\langle f|\hat{H}'|k\rangle e^{-iE_kt}\,\mathrm{d}t'\,.$$

Multiplying through by e^{iE_ft} and performing the integrals with respect to t' and t'' leads to

$$i\frac{\mathrm{d}c_f}{\mathrm{d}t} \approx \langle f|\hat{H}'|i\rangle e^{i(E_f-E_i)t} - i\sum_{k\neq i}\langle k|\hat{H}'|i\rangle\frac{e^{i(E_k-E_i)t}}{i(E_k-E_i)}\langle f|\hat{H}'|k\rangle e^{i(E_f-E_k)t}$$
$$+ (-i)^2\sum_{k\neq i}\sum_{j\neq i\neq k}\langle k|\hat{H}'|j\rangle\frac{e^{i(E_k-E_j)t}}{i(E_k-E_j)}\langle j|\hat{H}'|i\rangle\frac{e^{i(E_j-E_i)t}}{i(E_j-E_i)}\langle f|\hat{H}'|k\rangle e^{i(E_f-E_k)t}$$
$$\frac{\mathrm{d}c_f}{\mathrm{d}t} \approx -i\left(\langle f|\hat{H}'|i\rangle + \sum\frac{\langle k|\hat{H}'|i\rangle\langle f|\hat{H}'|k\rangle}{(E_k-E_i)} + \sum\sum\frac{\langle f|\hat{H}'|k\rangle\langle k|\hat{H}'|j\rangle\langle j|\hat{H}'|i\rangle}{(E_k-E_j)(E_j-E_i)}\right)e^{i(E_f-E_i)t}\,,$$

and therefore, to third-order,

$$T_{fi} = \langle f|\hat{H}'|i\rangle + \sum \frac{\langle k|\hat{H}'|i\rangle\langle f|\hat{H}'|k\rangle}{(E_k - E_i)} + \sum\sum \frac{\langle f|\hat{H}'|k\rangle\langle k|\hat{H}'|j\rangle\langle j|\hat{H}'|i\rangle}{(E_k - E_j)(E_j - E_i)} \,.$$

3 Decay Rates and Cross Sections

3.1 Calculate the energy of the μ^- produced in the decay at rest $\pi^- \to \mu^-\overline{\nu}_\mu$. Assume $m_\pi = 140\,\text{MeV}$, $m_\mu = 106\,\text{MeV}$ and take $m_\nu \approx 0$. Momentum is conserved so the

momentum of the muon and neutrino are equal and opposite with magnitude p. Conservation of energy implies that

$$m_\pi = E_\mu + E_\nu$$
$$\Rightarrow \qquad m_\pi - \text{p} = E_\mu$$

where the neutrino mass has been neglected $E_\nu = \text{p}$. Squaring and using $E^2 = m^2 + \text{p}^2$gives

$$m_\pi^2 - 2m_\pi\text{p} + \text{p}^2 = m_\mu^2\text{p}^2$$
$$\Rightarrow \qquad \text{p} = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} = 30\,\text{MeV}\,.$$

Therefore the energy of the muon $E_\mu^2 = m_\mu^2 + \text{p}^2 = 110\,\text{MeV}$.

3.2 For the decay $a \to 1 + 2$, show that the momenta of both daughter particles in the centre-of-mass frame p^* are

$$\text{p}^* = \frac{1}{2m_a}\sqrt{[(m_a^2 - (m_1 + m_2)^2]\,[m_a^2 - (m_1 - m_2)^2]}\,.$$

In the parent particle's rest frame, the momenta of the two daughter particles are

equal and conservation of energy implies $m_a = E_1 + E_2$. Writing this as $m_a - E_2 = E_1$ and squaring gives

$$m_a^2 - 2m_aE_2 + E_2^2 = E_1^2$$
$$m_a^2 - 2m_aE_2 + m_2^2 + \text{p}^{*2} = m_1^2 + \text{p}^{*2}$$
$$\Rightarrow \qquad m_a^2 + (m_2^2 - m_1^2) = 2m_aE_2\,.$$

Squaring again to eliminate E_2 leads to

$$m_a^4 + 2m_a^2(m_2^2 - m_1^2) + (m_2^2 - m_1^2)^2 = 4m_a(m_2^2 + \mathrm{p}^{*2})$$
$$m_a^4 - 2m_a^2(m_2^2 + m_1^2) + (m_1 - m_2)^2(m_1 + m_2)^2 = 4m_a\mathrm{p}^{*2}$$
$$m_a^4 - 2m_a^2\left[(m_1 + m_2)^2 + (m_1 - m_2)^2\right] + (m_1 - m_2)^2(m_1 + m_2)^2 = 4m_a\mathrm{p}^{*2}$$
$$\left[m_a^2 - (m_2 + m_2)^2\right]\left[m_a^2 - (m_1 - m_2)^2\right] = 4m_a\mathrm{p}^{*2}\,,$$

thus showing that

$$\mathrm{p}^* = \frac{1}{2m_a}\sqrt{\left[(m_a^2 - (m_1 + m_2)^2\right]\left[m_a^2 - (m_1 - m_2)^2\right]}\,.$$

3.3 Calculate the branching ratio for the decay $\mathrm{K}^+ \to \pi^+\pi^0$, given the partial decay width $\Gamma(\mathrm{K}^+ \to \pi^+\pi^0) = 1.2 \times 10^{-8}$ eV and the mean kaon lifetime $\tau(\mathrm{K}^+) = 1.2 \times 10^{-8}$ s.

The branching ratio is related to the total decay width Γ by

$$BR(\mathrm{K}^+ \to \pi^+\pi^0) = \frac{\Gamma(\mathrm{K}^+ \to \pi^+\pi^0)}{\Gamma}\,,$$

where (in natural units) $\Gamma = 1/\tau$ or in S.I units (as given here) $\Gamma = \hbar/\tau$. Hence

$$BR(\mathrm{K}^+ \to \pi^+\pi^0) = \frac{\tau}{\hbar} \times \Gamma(\mathrm{K}^+ \to \pi^+\pi^0)$$
$$= \frac{1.2 \times 10^{-8}}{1.06 \times 10^{-34}} \cdot 1.2 \times 10^{-8} \cdot 1.6 \times 10^{-19} = 21\,\%\,.$$

3.4 At a future $\mathrm{e}^+\mathrm{e}^-$ linear collider operating as a Higgs factory at a centre-of-mass energy of $\sqrt{s} = 250$ GeV, the cross section for the process $\mathrm{e}^+\mathrm{e}^- \to \mathrm{HZ}$ is 250 fb. If the collider has an instantaneous luminosity of 2×10^{34} cm^{-2} s^{-1} and is operational for 50 % of the time, how many Higgs bosons will be produced in five years of running? Note: 1 femtobarn $\equiv 10^{-15}$ b.

The total number of events is given by

$$N = \int \sigma\mathcal{L}\,\mathrm{d}t\,,$$

or equivalently, the event rate is $\sigma\mathcal{L}$. Remembering that 1 barn $\equiv 10^{-24}$ cm^2,

$$Rate = \sigma\mathcal{L} = (250 \times 10^{-39}) \times (2 \times 10^{34}) = 0.005\,\mathrm{s}^{-1}$$

Therefore in five years of operation with 50 % lifetime, a total of $0.005 \times 0.5 \times 365.25 \times 86400 = 394000$ $\mathrm{e}^+\mathrm{e}^- \to \mathrm{HZ}$ events would be accumulated.

3.5 The total $\mathrm{e}^+\mathrm{e}^- \to \gamma \to \mu^+\mu^-$ annihilation cross section is $\sigma = 4\pi\alpha^2/3s$, where $\alpha \approx 1/137$. Calculate the cross section at $\sqrt{s} = 50$ GeV, expressing your answer in both natural units and in barns (1 barn = 10^{-28} m^2). Compare this to the total pp cross section at $\sqrt{s} = 50$ GeV which is approximately 40 mb and comment on the result.

In natural units

$$\sigma = \frac{4\pi\alpha^2}{3s} = \frac{4\pi/137^2}{3 \cdot 50^2} = 8.9 \times 10^{-8}\,\mathrm{GeV}^{-2}\,.$$

To convert to natural units, either insert the missing factor of $(\hbar c)^2$ or use the conversion factor $\hbar c = 0.197$ GeV fm and 1 fm^2 = 10^{-30} m^2 = 10^{-2} b, therefore

$$\sigma = 8.9 \times 10^{-8}\,\mathrm{GeV}^{-2} \times 0.197^2 \times 0.01\,\mathrm{b} = 34\,\mathrm{pb}\,.$$

This is a factor of approximately 10^9 smaller than the *strong* interaction pp cross section.

3.6 A 1 GeV muon neutrino is fired at a 1 m thick block of Iron ($^{56}_{28}$Fe) with density $\rho = 7.874 \times 10^3$ kg m^{-3}. If the average neutrino-nucleon interaction cross section is $\sigma = 8 \times 10^{-39}$ cm^2, calculate the (small) probability that the neutrino interacts in the block.

From the definition of cross section the event rate is given by flux × cross section × number of targets, $Rate = \phi\sigma N_t$. The total number of interactions N_{int} is therefore

$$N_{\mathrm{int}} = \sigma N_t \int \phi\,\mathrm{d}t\,.$$

Suppose the ”beam” of one neutrino is confined to an area A, then the integrated flux is just $1/A$. Writing the thickness of the Iron as x, the number of targets traversed by the ”beam” is Axn, where n is the number density of target nuclei. Hence

$$N_{\mathrm{int}} = \sigma \frac{1}{A} nAx = \sigma n x\,.$$

Taking the average mass of a nucleon to be 1.67×10^{-27} kg, here the product of the number density and thickness is

$$nx = \frac{7874}{56 \cdot 1.67 \times 10^{-27}} \times 1\,\mathrm{m} = 8.4 \times 10^{28}\,\mathrm{m}^{-2} \equiv 8.4 \times 10^{24}\,\mathrm{cm}^{-2}\,.$$

Hence the average number of interactions for the single neutrino traversing the block is $8.4 \times 10^{24} \cdot 8 \times 10^{-39} = 7 \times 10^{-10}$. Therefore the interaction probability is less than 10^{-9}.

3.7 For the process $a + b \to 1 + 2$ the Lorentz invariant flux term is

$$F = 4\left[(p_a \cdot p_b)^2 - m_a^2 m_b^2\right]^{\frac{1}{2}}\,.$$

In the non-relativistic limit, $\beta_a \ll 1$ and $\beta_b \ll 1$, show that

$$F \approx 4 m_a m_b\,|\mathbf{v}_a - \mathbf{v}_b|\,,$$

where $\mathbf{v}_a$ and $\mathbf{v}_b$ are the (non-relativistic) velocities of the two particles.

First consider the low energy limit of the four-vector product $p_a{\cdot}p_b = E_aE_b - \mathbf{p}_a \cdot \mathbf{p}_b$ where, (as expected) the non-relativistic limit of the particle energy and momentum are (in natural units)

$$E = \gamma m = m\left(1-\beta^2\right)^{-1/2} \approx m(1+\tfrac{1}{2}\beta^2) = m + \tfrac{1}{2}m\beta^2\,,$$

and to $O(\beta)$:

$$\mathbf{p} = \gamma m\boldsymbol{\beta} = m\left(1-\beta^2\right)^{-1/2}\boldsymbol{\beta} \approx m\boldsymbol{\beta}\,.$$

Hence $p_a{\cdot}p_b = E_aE_b - \mathbf{p}_a \cdot \mathbf{p}_b$ is approximately

$$\begin{aligned}
p_a{\cdot}p_b &\approx m_am_b\left(1+\tfrac{1}{2}\beta_a^2\right)\left(1+\tfrac{1}{2}\beta_b^2\right) - m_am_b\boldsymbol{\beta}_a\cdot\boldsymbol{\beta}_b\\
&= m_am_b\left[1+\tfrac{1}{2}(\beta_a^2+\beta_b^2) - \boldsymbol{\beta}_a\cdot\boldsymbol{\beta}_b + \tfrac{1}{4}\beta_a^2\beta_b^2\right]\\
&= m_am_b\left[1+\tfrac{1}{2}(\boldsymbol{\beta}_a-\boldsymbol{\beta}_b)^2 + \tfrac{1}{4}\beta_a^2\beta_b^2\right]\\
&\approx m_am_b\left[1+\tfrac{1}{2}(\boldsymbol{\beta}_a-\boldsymbol{\beta}_b)^2\right]\,.
\end{aligned}$$

Therefore to $O(\beta^2)$

$$\begin{aligned}
(p_a{\cdot}p_b)^2 &\approx m_a^2m_b^2\left[1+(\boldsymbol{\beta}_a-\boldsymbol{\beta}_b)^2\right]\\
\Rightarrow\qquad F &= 4\left[(p_a{\cdot}p_b)^2 - m_a^2m_b^2\right]^{1/2}\\
&= 4m_am_b|\boldsymbol{\beta}_a-\boldsymbol{\beta}_b|\,,
\end{aligned}$$

or equivalently in S.I. units, $F = 4m_am_b|\mathbf{v}_a-\mathbf{v}_b|$. As expected, in the non-relativistic limit, the flux depends on the relative velocities of the particles.

3.8 The Lorentz invariant flux term for the process $a+b \to 1+2$ in the centre-of-mass frame was shown to be $F = 4\mathrm{p}_i^*\sqrt{s}$, where p_i^* is the momentum of the intial-state particles. Show that the corresponding expression in the frame where b is at rest is

$$F = 4m_b\mathrm{p}_a\,.$$

Independent of rest frame, the Lorentz invariant flux is given by $F = 4\left[(p_a{\cdot}p_b)^2 - m_a^2m_b^2\right]^{1/2}$. Here $p_a = (E_a, 0, 0, \mathrm{p}_a)$ and $p_b = (m_b, 0, 0, 0)$ and therefore

$$\begin{aligned}
F &= 4\left[E_a^2m_b^2 - m_a^2m_b^2\right]^{1/2}\\
&= 4\left[(\mathrm{p}_a^2+m_a^2)m_b^2 - m_a^2m_b^2\right]^{1/2}\\
&= 4\mathrm{p}_am_b\,.
\end{aligned}$$

3.9 Show that the momentum in the centre-of-mass frame of the initial-state particles in a two-body scattering process can be expressed as

$$\mathrm{p}_i^{*2} = \frac{1}{4s}[s-(m_1+m_2)^2][s-(m_1-m_2)^2]\,.$$

In the centre-of-mass frame $\sqrt{s} = E_1^* + E_2^*$ and therefore

$$\begin{aligned}
(\sqrt{s} - E_1^*)^2 &= E_2^{*2} \\
s - 2\sqrt{s}E_1^* + E_1^{*2} &= E_2^{*2} \\
s - 2\sqrt{s}E_1^* + m_1^2 + p_i^{*2} &= m_1^2 + p_i^{*2} \\
\Rightarrow \qquad 2\sqrt{s}E_1^* &= s + (m_1^2 - m_2^2) \\
4s(p_i^{*2} + m_1^2) &= s^2 + 2s(m_1^2 - m_2^2) + (m_1^2 - m_2^2)^2 \\
4sp_i^{*2} &= s^2 - 2s(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2 \\
&= s^2 - 2s(m_1^2 + m_2^2) + (m_1 - m_2)^2(m_1 + m_2)^2 \\
&= \left[s - (m_1 + m_2)^2\right]\left[s - (m_1 - m_2)^2\right] \\
\Rightarrow \qquad p_i^{*2} &= \frac{1}{4s}\left[s - (m_1 + m_2)^2\right]\left[s - (m_1 - m_2)^2\right]
\end{aligned}$$

3.10 Repeat the calculation of Section 3.5.2 for the process $e^-p \to e^-p$ where the mass of the electron is no longer neglected.

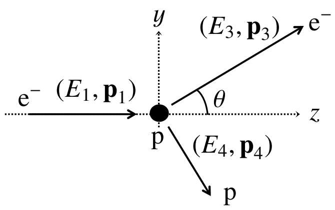
a) First show that

$$\frac{dE_3}{d(\cos\theta)} = \frac{p_1 p_3^2}{p_3(E_1 + m_p) - E_3 p_1 \cos\theta}.$$

b) Then show that

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \cdot \frac{p_3^2}{p_1 m_e} \cdot \frac{1}{p_3(E_1 + m_p) - E_3 p_1 \cos\theta} \cdot |\mathcal{M}_{fi}|^2,$$

where (E_1, p_1) and (E_3, p_3) are the respective energies and momenta of the initial-state and scattered electrons as measured in the laboratory frame.



a) With the electron masses (m) included this is a somewhat tedious calculation. In the laboratory frame where the proton is at rest the four-momenta of the particles (shown above) can be written

$$p_1 = (E_1, 0, 0, p_1), \; p_2 = (m_p, 0, 0, 0), \; p_3 = (E_3, 0, p_3\sin\theta, p_3\cos\theta) \text{ and } p_4 = (E_4, \mathbf{p}_4).$$

Differentiating $E_3^2 = p_3^2 + m_3^2$ with respect to $\cos\theta$ gives

$$2E_3 \frac{dE_3}{d(\cos\theta)} = 2p_3 \frac{dp_3}{d(\cos\theta)}. \tag{3.1}$$

Then equating the expressions for the Mandelstam t variable written in terms of the electron and proton four-momenta gives

$$\begin{aligned} t = (p_1 - p_3)^2 &= (p_2 - p_4)^2 \\ 2m_\mathrm{e}^2 - 2p_1\cdot p_3 &= 2m_\mathrm{p}^2 - 2p_2\cdot p_4 \\ 2m_\mathrm{e}^2 - 2(E_1E_3 - \mathrm{p}_1\mathrm{p}_3\cos\theta) &= 2m_\mathrm{p}^2 - 2m_\mathrm{p}E_4 \\ 2m_\mathrm{e}^2 - 2(E_1E_3 - \mathrm{p}_1\mathrm{p}_3\cos\theta) &= 2m_\mathrm{p}^2 - 2m_\mathrm{p}(E_1 + m_\mathrm{p} - E_3)\,, \end{aligned}$$

which can be differentiated with respect to $\cos\theta$ remembering that E_1 and p_1 are the fixed intial-state electron energy and momentum:

$$\begin{aligned} -E_1\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} + \mathrm{p}_1\frac{\mathrm{dp}_3}{\mathrm{d}(\cos\theta)} + \mathrm{p}_1\mathrm{p}_3 &= m_\mathrm{p}\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} \\ \Rightarrow \qquad (E_1 + m_\mathrm{p})\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} - \mathrm{p}_1\cos\theta\frac{\mathrm{dp}_3}{\mathrm{d}(\cos\theta)} &= \mathrm{p}_1\mathrm{p}_3\,. \end{aligned}$$

Using (3.1) this can be written

$$\begin{aligned} (E_1 + m_\mathrm{p})\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} - \mathrm{p}_1\frac{E_3}{\mathrm{p}_3}\cos\theta\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} &= \mathrm{p}_1\mathrm{p}_3 \\ \Rightarrow \qquad \left(E_1 + m_\mathrm{p} - \frac{\mathrm{p}_1E_3\cos\theta}{\mathrm{p}_3}\right)\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} &= \mathrm{p}_1\mathrm{p}_3 \\ \Rightarrow \qquad \frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} &= \frac{\mathrm{p}_1\mathrm{p}_3^2}{\mathrm{p}_3(E_1 + m_\mathrm{p}) - \mathrm{p}_1E_3\cos\theta}\,, \end{aligned}$$

which is the desired result.

b) From (3.37) of the main text,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{64\pi s\,\mathrm{p}_i^{*2}}|\mathcal{M}_{fi}|^2\,.$$

This can be related to the differential cross section in terms of solid angle using

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}\sigma}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\Omega} = \frac{1}{2\pi}\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)}\frac{\mathrm{d}\sigma}{\mathrm{d}t}\,.$$

But $t = (p_4 - p_2)^2 = 2m_\mathrm{p}^2 - 2m_\mathrm{p}E_4 = 2m_\mathrm{p}^2 - 2m_\mathrm{p}(E_1 + m_\mathrm{p} - E_3)$ and therefore

$$\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)} = 2m_\mathrm{p}\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)}\,.$$

Putting the things together gives

$$\begin{aligned} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{2\pi}2m_\mathrm{p}\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)}\frac{1}{64\pi s\,\mathrm{p}_i^{*2}}|\mathcal{M}_{fi}|^2 \\ &= \frac{1}{64\pi^2}\frac{m_\mathrm{p}\mathrm{p}_1\mathrm{p}_3^2}{sp_i^{*2}}\frac{1}{\mathrm{p}_3(E_1 + m_\mathrm{p}) - \mathrm{p}_1E_3\cos\theta}|\mathcal{M}_{fi}|^2\,. \end{aligned} \tag{3.2}$$

All that remains is to express sp_i^2 in terms of the laboratory frame quantities. The centre-of-mass energy squared is just

$$s = (E_1 + m_\mathrm{p})^2 - \mathrm{p}_1^2 = 2m_\mathrm{p}E_1 + m_\mathrm{p}^2 + m_\mathrm{e}^2\,,$$

and from the previous question

$$\begin{aligned}
sp_i^{*2} &= \tfrac{1}{4}\left[s - (m_\mathrm{e} + m_\mathrm{p})^2\right]\left[s - (m_\mathrm{e} - m_\mathrm{p})^2\right] \\
&= \tfrac{1}{4}\left[s - m_\mathrm{e}^2 - m_\mathrm{p}^2 - 2m_\mathrm{e}m_\mathrm{p}\right]\left[s - m_\mathrm{e}^2 - m_\mathrm{p}^2 + 2m_\mathrm{e}m_\mathrm{p}\right] \\
&= \tfrac{1}{4}\left[2m_\mathrm{p}E_1 - 2m_\mathrm{e}m_\mathrm{p}\right]\left[2m_\mathrm{p}E_1 + 2m_\mathrm{e}m_\mathrm{p}\right] \\
&= m_\mathrm{p}^2E_1^2 - m_\mathrm{e}^2m_\mathrm{p}^2 \\
&= m_\mathrm{p}^2\mathrm{p}_1^2\,.
\end{aligned}$$

Substituting this expression back into (3.2) gives

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\cdot\frac{\mathrm{p}_3^2}{m_\mathrm{p}\mathrm{p}_1}\cdot\frac{1}{\mathrm{p}_3(E_1 + m_\mathrm{p}) - \mathrm{p}_1E_3\cos\theta}\cdot|\mathcal{M}_{fi}|^2\,.$$

4 The Dirac Equation

4.1 Show that

$$[\hat{\mathbf{p}}^2, \hat{\mathbf{r}} \times \hat{\mathbf{p}}] = 0\,,$$

and hence the Hamiltonian of the free particle Schrödinger equation commutes with the angular momentum operator.

Consider the commutator of $\hat{\mathbf{p}}^2$ with the x-component of $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$:

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y\,.$$

Hence

$$\begin{aligned}\left[\hat{\mathbf{p}}^2, \hat{L}_x\right] &= \left[\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2, \hat{y}\hat{p}_z - \hat{z}\hat{p}_y\right] \\ &= \left[\hat{p}_y^2, \hat{y}\hat{p}_z\right] - \left[\hat{p}_z^2, \hat{z}\hat{p}_y\right] \\ &= \left[\hat{p}_y^2, \hat{y}\right]\hat{p}_z - \left[\hat{p}_z^2, \hat{z}\right]\hat{p}_y\,,\end{aligned} \tag{4.1}$$

where the last line follows since, for example, $\hat{p}_z$ commutes with $\hat{y}$. Using

$$\left[y, \hat{p}_y\right] = \hat{y}\hat{p}_y - \hat{p}_y\hat{y} = i\,,$$

the commutators in the previous expression can be simplified, for example

$$\begin{aligned}\left[\hat{p}_y^2, \hat{y}\right] &= \hat{p}_y\hat{p}_y\hat{y} - \hat{y}\hat{p}_y\hat{p}_y \\ &= \hat{p}_y(\hat{y}\hat{p}_y - i) - \hat{y}\hat{p}_y\hat{p}_y \\ &= \hat{p}_y\hat{y}\hat{p}_y - i\hat{p}_y - \hat{y}\hat{p}_y\hat{p}_y \\ &= (\hat{y}_y\hat{p}_y - i)\hat{p}_y - i\hat{p}_y - \hat{y}\hat{p}_y\hat{p}_y \\ &= -2i\hat{p}_y\,.\end{aligned}$$

Similarly $\left[\hat{p}_z^2, \hat{z}\right] = -2i\hat{p}_z$ and therefore (4.1) becomes

$$\begin{aligned}\left[\hat{\mathbf{p}}^2, \hat{L}_x\right] &= \left[\hat{p}_y^2, \hat{y}\right]\hat{p}_z - \left[\hat{p}_z^2, \hat{z}\right]\hat{p}_y \\ &= -2i\hat{p}_y\hat{p}_z + 2i\hat{p}_z\hat{p}_y \\ &= 0\,.\end{aligned}$$

Hence the commentator $\left[\hat{\mathbf{p}}^2, \hat{\mathbf{r}} \times \hat{\mathbf{p}}\right]$ is identically equal to zero. Since the Hamiltonian for a non-relativistic free particle can be written $\hat{H}_D = \hat{\mathbf{p}}^2/2m$, it immediately follows that

$$\left[\hat{H}_D, \hat{\mathbf{L}}\right] = 0\,.$$

Since the Hamiltonian for a non-relativistic free particle commutes with the operator for angular momentum, angular momentum is a conserved quantity for a free Dirac particle.

4.2 Show that u_1 and u_2 are orthogonal, *i.e.* $u_1^\dagger u_2 = 0$.

The two normalised spinors are

$$u_1(p) = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \quad \text{and} \quad u_2(p) = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

and therefore $u_1(p)^\dagger u_2(p)$ is

$$\begin{aligned} u_1(p)^\dagger u_2(p) &= (E+m)\left(1, 0, \tfrac{p_z}{E+m}, \tfrac{p_x-ip_y}{E+m}\right)\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} \\ &= (E+m)\left(0 + 0 + \tfrac{p_z(p_x-ip_y)}{E+m} - \tfrac{p_z(p_x-ip_y)}{E+m}\right) \\ &= 0\,. \end{aligned}$$

Since $u_1(p)^\dagger u_2(p) = 0$, taking the Hermitian conjugate shows that $u_2(p)^\dagger u_1(p) = 0$.

4.3 Verify the statement that the Einstein energy-momentum relationship is recovered if any of the four Dirac spinors of (4.48) are substituted into the Dirac equation written in terms of momentum, $(\gamma^\mu p_\mu - m)u = 0$.

In matrix form $\gamma^\mu p_\mu - m$ is given by

$$\gamma^\mu p_\mu - m = E\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} - p_x\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} - p_y\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} - p_z\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} - mI$$

$$= \begin{pmatrix} E-m & 0 & -p_z & -p_x+ip_y \\ 0 & E-m & -p_x-ip_y & p_z \\ p_z & p_x-ip_y & -(E+m) & 0 \\ p_x+ip_y & -p_z & 0 & -(E+m) \end{pmatrix}.$$

Hence the Dirac equation for u_1, when written out in gory detail, is

$$
\begin{aligned}
(\gamma^\mu p_\mu - m)u &= 0 \\
\begin{pmatrix} E-m & 0 & -p_z & -p_x+ip_y \\ 0 & E-m & -p_x-ip_y & p_z \\ p_z & p_x-ip_y & -(E+m) & 0 \\ p_x+ip_y & -p_z & 0 & -(E+m) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} &= 0 \\
\begin{pmatrix} (E-m)+0-\frac{p_z^2}{E+m}+\frac{-p_x^2-p_y^2}{E+m} \\ 0+0-\frac{p_z}{(p_x+ip_y)}+\frac{p_z}{(p_x+ip_y)} \\ p_z+0-p_z+0 \\ (p_x+ip_y)+0+0-(p_x+ip_y) \end{pmatrix} &= 0 \\
\frac{1}{E+m}\begin{pmatrix} E^2-m^2-p_x^2-p_y^2-p_x^2 \\ 0 \\ 0 \\ 0 \end{pmatrix} &= 0\,,
\end{aligned}
$$

which is only true when $E^2 = \mathrm{p}^2 + m^2$. Of course, this had to be true since the Dirac equation was constructed to be consistent with the Einstein energy-momentum relation.

4.4 For a particle with four-momentum $p^\mu = (E, \mathbf{p})$, the general solution to the free particle Dirac Equation can be written

$$\psi(p) = [au_1(p) + bu_2(p)]e^{i(\mathbf{p}\cdot\mathbf{x}-Et)}\,.$$

Using the explicit forms for u_1 and u_2, show that the four-vector current $j^\mu = (\rho, \mathbf{j})$ is given by

$$j^\mu = 2p^\mu\,.$$

Furthermore, show that the resulting probability density and probability current are consistent with a particle moving with velocity $\beta = \mathrm{p}/E$.

For *arbitrary* spinors ψ and ϕ, with spinor components ψ_i and ϕ_i, matrix multiplication gives, for $\mu = 0$,

$$\overline{\psi}\gamma^0\phi = (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \text{-}1 & 0 \\ 0 & 0 & 0 & \text{-}1 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4\,.$$

Repeating this for $\mu = 1, 2, 3$ gives

$$\begin{aligned}
\overline{\psi}\gamma^0\phi &= \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4 \\
\overline{\psi}\gamma^1\phi &= \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1 \\
\overline{\psi}\gamma^2\phi &= -i(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1) \\
\overline{\psi}\gamma^3\phi &= \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2
\end{aligned}$$

For the free particle spinor $u_1(p)$, the first element of the four-vector is

$$\begin{aligned}
\overline{u}_1\gamma^0 u_1 &= (E+m)\left[1 + \frac{p_z^2}{(E+m)^2} + \frac{(p_x^2+p_y^2)}{(E+m)^2}\right] \\
&= (E+m)\left[1 + \frac{p^2}{(E+m)^2}\right] \\
&= \frac{(E+m)^2 + p^2}{E+m} \\
&= \frac{2E^2 + 2Em}{E+m} = 2E\,.
\end{aligned}$$

Repeating this for the remaining terms in the four-vector current gives

$$\overline{u}_1\gamma^0 u_1 = 2E\,, \quad \overline{u}_1\gamma^1 u_1 = 2p_x\,, \quad \overline{u}_1\gamma^2 u_1 = 2p_y \quad \text{and} \quad \overline{u}_1\gamma^3 u_1 = 2p_z\,,$$

and therefore

$$\overline{u}_1\gamma^\mu u_1 = (2E, 2p_x, 2p_y, 2p_z) \equiv 2p^\mu\,.$$

For u_2, v_1 and v_2 the corresponding expressions are

$$\overline{u}_1\gamma^\mu u_1 = \overline{u}_2\gamma^\mu u_2 = \overline{v}_1\gamma^\mu v_1 = \overline{v}_2\gamma^\mu v_2 = 2p^\mu\,.$$

and the corresponding cross terms all give zero

$$\overline{u}_1\gamma^\mu u_2 = \overline{u}_2\gamma^\mu u_1 = \overline{v}_1\gamma^\mu v_2 = \overline{v}_2\gamma^\mu v_1 = 0\,.$$

For a particle, with $\psi = u(p)e^{ip.x}$, we have

$$\overline{\psi} = \psi^\dagger\gamma^0 = u(p)^\dagger\gamma^0 e^{-ipx} = \overline{u}(p)e^{-ipx}\,,$$

and hence

$$j^\mu = \overline{\psi}\gamma^\mu\psi = \overline{u}\gamma^\mu u\,.$$

For an antiparticle spine the same relation is found; $j^\mu = \overline{v}\gamma^\mu v$.

A particle spinor $u(p)$ can always be expressed as a linear combination of the basis spinors $u_1(p)$, $u_2(p)$:

$$u = \alpha_1 u_1 + \alpha_2 u_2\,, \quad \text{with} \quad |\alpha_1|^2 + |\alpha_2|^2 = 1\,.$$

Hence

$$\overline{u}\gamma^{\mu}u = |\alpha_1|^2\overline{u}_1\gamma^{\mu}u_1 + |\alpha_2|^2\overline{u}_2\gamma^{\mu}u_2 = 2p^{\mu}\,.$$

Thus for *any* free-particle spinor

$$j^{\mu} \equiv \overline{u}\gamma^{\mu}u = 2p^{\mu}\,.$$

The four-vector current $j^{\mu} = (\rho, \mathbf{j})$, leading to the identification

$$\rho = 2E\,, \qquad \mathbf{j} = 2\mathbf{p}\,.$$

This shows that the general spinor $u(p)$ is normalised to $2E$ particles per unit volume and from $E = \gamma m$ and $\mathbf{p} = \gamma m\mathbf{v}$, the corresponding particle flux is $\mathbf{j} = 2E\mathbf{v}$. Hence the particle flux is, as expected,

$$\mathbf{j} = \rho\mathbf{v}\,.$$

4.5 Writing the four-component spinor u_1 in terms of two two-component vectors

$$u = \begin{pmatrix} u_A \\ u_B \end{pmatrix},$$

show that in the non-relativistic limit, where $\beta \equiv \mathrm{v}/c \ll 1$, the components of u_B are smaller than those of u_A by a factor v/c.

Here we are looking for the general order of magnitude of the relative size of the upper and lower components. So for simplicity, consider a particle travelling in the z-direction, which from the definition of u_1 has

$$u_A = N\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad u_B = N\begin{pmatrix} \frac{\mathrm{p}}{E+m} \\ 0 \end{pmatrix}.$$

Here the two lower components of the spinor are smaller than the upper two components by a factor

$$\frac{\mathrm{p}}{E+m} = \frac{\gamma m\beta}{(\gamma+1)m}\,.$$

In the non-relativistic limit where $\gamma \approx 1$ this factor is of order $\beta = v/c$.

4.6 By considering the three cases $\mu = \nu = 0$, $\mu = \nu \neq 0$ and $\mu \neq \nu$ show that

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}\,.$$

For the case where $\mu = \nu = 0$,

$$\gamma^0\gamma^0 + \gamma^0\gamma^0 = 2I = 2g^{00}\,.$$

For the case where $\mu = \nu = k$,

$$\gamma^k\gamma^k + \gamma^k\gamma^k = -2I = 2g^{kk} \text{ for } k = 1, 2, 3\,.$$

For the case where $\mu \neq \nu$, the anti-commutation relation $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ immediately implies that

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0 = 2g^{\mu\nu} \text{ for } \mu \neq \nu\,.$$

In each case $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$.

4.7 By operating on the Dirac equation,

$$(i\gamma^\mu\partial_\mu - m)\psi = 0\,,$$

with $\gamma^\nu\partial_\nu$, prove that the components of ψ satisfy the Klein-Gordon equation,

$$(\partial^\mu\partial_\mu + m^2)\psi = 0\,.$$

Acting on the Dirac equation with $\gamma^\nu\partial_\nu$ and multiplying by $-i$ gives

$$\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu\psi + mi\gamma^\nu\partial_\nu\psi = 0\,.$$

But since ψ satisfies the Dirac equation $i\gamma^\nu\partial_\nu\psi = m\psi$ and thus

$$(\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu + m^2)\psi = 0\,.$$

Since the order of differentiation doesn't matter $\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu$ can be written

$$\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu = \tfrac{1}{2}(\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu)\partial_\nu\partial_\mu = g^{\mu\nu}\,\partial_\nu\partial_\mu\,,$$

and therefore

$$\begin{aligned} &(g^{\mu\nu}\partial_\nu\partial_\mu + m^2)\psi = 0 \\ \Rightarrow \qquad &(\partial^\mu\partial_\mu + m^2)\psi = 0\,, \end{aligned}$$

which is the Klein-Gordon equation.

4.8 Show that

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0\,.$$

Recall the $\gamma^{0\dagger} = \gamma^0$, $\gamma^{k\dagger} = \gamma^k$, $\gamma^0\gamma^0 = I$ and $\gamma^0\gamma^k = -\gamma^k\gamma^0$. For $\mu = 0$,

$$\gamma^{0\dagger} = \gamma^0 = \gamma^0\gamma^0\gamma^0\,.$$

For $\mu = k \neq 0$,

$$\gamma^{k\dagger} = -\gamma^k = -\gamma^0\gamma^0\gamma^k = +\gamma^0\gamma^k\gamma^0\,,$$

and therefore for $\mu = 0, 1, 2, 3$,

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 .$$

4.9 Starting from

$$(\gamma^\mu p_\mu - m)u = 0 ,$$

a) show that the corresponding equation for the adjoint spinor is

$$\overline{u}(\gamma^\mu p_\mu - m) = 0 .$$

b) Hence, without using the explicit form for the u spinors, show that the normalisation condition $u^\dagger u = 2E$ leads to

$$\overline{u}u = 2m ,$$

and that

$$\overline{u}\gamma^\mu u = 2p^\mu .$$

a) Taking the Hermitian conjugate of the Dirac equation $(\gamma^\mu p_\mu - m)u = 0$ (remembering that the p_μ are just real numbers), gives

$$\begin{aligned} 0 &= u^\dagger(\gamma^{\mu\dagger} p_\mu - m) \\ &= u^\dagger\gamma^0\gamma^0(\gamma^{\mu\dagger} p_\mu - m) \\ &= u^\dagger\gamma^0(\gamma^\mu\gamma^0 p_\mu - m\gamma^0) \\ &= \overline{u}(\gamma^\mu p_\mu - m)\gamma^0 , \end{aligned}$$

where one of the intermediate steps used the relation $\gamma^0\gamma^{\mu\dagger} = \gamma^{\mu\dagger}\gamma^0$ for all μ. Finally, pre-multiplying γ^0 by both sides of the above expression leads to

$$\overline{u}(\gamma^\mu p_\mu - m) = 0 .$$

b) Consider the $\overline{u}\gamma^\nu\times$ the Dirac equation and Dirac equation for the adjoint spinor $\times\gamma^\nu u$

$$\overline{u}\gamma^\nu(\gamma^\mu p_\mu - m)u = 0 \quad \text{and} \quad \overline{u}(\gamma^\mu p_\mu - m)\gamma^\nu u = 0$$

and take the sum:

$$\begin{aligned} \overline{u}(\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu)p_\mu u - 2m\overline{u}\gamma^\nu u &= 0 \\ \overline{u}2g^{\mu\nu} p_\mu u &= 2m\overline{u}\gamma^\nu u \\ p^\nu\overline{u}u &= m\overline{u}\gamma^\nu u . \end{aligned}$$

For the case $\nu = 0$ this reduces to

$$\begin{aligned} p^0\overline{u}u &= mu^\dagger\gamma^0\gamma^0 u \\ E\overline{u}u &= mu^\dagger u = 2mE \\ \Rightarrow \qquad \overline{u}u &= 2m . \end{aligned}$$

Substituting this back into the first line:

$$\begin{aligned} p^\nu \overline{u}u &= m\overline{u}\gamma^\nu u \\ p^\nu(2m) &= m\overline{u}\gamma^\nu u \\ \Rightarrow \qquad \overline{u}\gamma^\nu u &= 2p^\nu . \end{aligned}$$

4.10 Demonstrate that the two relations of equation (4.45) are consistent by showing that

$$(\boldsymbol{\sigma}\cdot\mathbf{p})^2 = \mathbf{p}^2 .$$

This can be shown either by writing out the explicit form of $\boldsymbol{\sigma}\cdot\mathbf{p}$ using the Pauli spin matrices or (more elegantly) by using the properties of the matrices, namely $\sigma_k^2 = 1$ and $\sigma_x\sigma_y = -\sigma_y\sigma_x$ from which

$$\begin{aligned} (\boldsymbol{\sigma}\cdot\mathbf{p})^2 &= (p_x\sigma_x + p_y\sigma_y + p_z\sigma_z)(p_x\sigma_x + p_y\sigma_y + p_z\sigma_z) \\ &= p_x^2\sigma_x^2 + p_y^2\sigma_y^2 + p_z^2\sigma_z^2 + \\ &\qquad p_xp_y(\sigma_x\sigma_y + \sigma_y\sigma_x) + p_xp_z(\sigma_x\sigma_z + \sigma_z\sigma_x) + p_yp_z(\sigma_y\sigma_z + \sigma_z\sigma_y) \\ &= p_x^2 + p_y^2 + p_z^2 \\ &= \mathbf{p}^2 . \end{aligned}$$

4.11 Consider the $e^+e^- \to \gamma \to e^+e^-$ annihilation process in the centre-of-mass frame where the energy of the photon is $2E$. Discuss energy and charge conservation for the two cases where: **a)** the negative energy solutions of the Dirac equation are interpreted as negative energy particles propagating backwards in time;

b) the negative energy solutions of the Dirac equation are interpreted as positive energy antiparticles propagating forwards in time.

a) In the first interpretation (left diagram), the intial-state positive e^- of energy $+E$ emits a photon of energy 2E. To conserve energy it is now a negative energy e^- and therefore propagates backwards in time. At the other vertex, the photon interacts with a negative energy e^-, which is propagating backwards in time and scattering results in a positive energy e^-.

b) In the Feynman-Stückelberg interpretation (right diagram), the intial-state positive e^- of energy $+E$ annihilates with a positive energy e^+ to produce a photon of energy 2E. At the second vertex the photon produces an e^+e^- pair. All particles propagate forwards in time.

4.12 Verify that the helicity operator

$$\hat{h} = \frac{\hat{\boldsymbol{\Sigma}}\cdot\hat{\mathbf{p}}}{2p} = \frac{1}{2p}\begin{pmatrix} \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} \end{pmatrix},$$

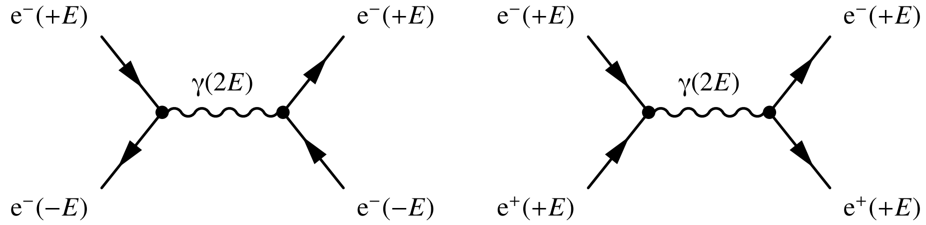


Fig. 4.1 The interpretation of the annihilation process $e^+e^- \to e^+e^-$. In both diagrams (which are not Feynman diagrams) the time axis runs strictly from left to right.

commutes with the Dirac Hamiltonian,

$$\hat{H}_D = \boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m\,.$$

In the Pauli-Dirac representation

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \text{and} \quad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix},$$

and since β contains the identity matrix it is clear that $\left[\hat{h}, =\right] m\beta\hat{h} = 0$ and therefore it is only necessary to consider $\left[\hat{h}, \boldsymbol{\alpha} \cdot \hat{\mathbf{p}}\right]$.

$$\begin{aligned}
\left[\hat{h}, \hat{H}_D\right] &= \frac{1}{2\mathrm{p}}\left[\begin{pmatrix} \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} \end{pmatrix}\begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} \\ \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} & 0 \end{pmatrix} - \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} \\ \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} \end{pmatrix}\right] \\
&= \frac{1}{2\mathrm{p}}\left[\begin{pmatrix} 0 & (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})^2 \\ (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})^2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})^2 \\ (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})^2 & 0 \end{pmatrix}\right] \\
&= 0\,.
\end{aligned}$$

4.13 Show that

$$\hat{P}u_\uparrow(\theta,\phi) = u_\downarrow(\pi-\theta, \pi+\phi)\,,$$

and comment on the result.

In the Dirac-Pauli representation

$$\hat{P} = \gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad u_\uparrow(\theta,\phi) = \sqrt{E+m}\begin{pmatrix} c \\ se^{i\phi} \\ \frac{\mathrm{p}}{E+m}c \\ \frac{\mathrm{p}}{E+m}se^{i\phi} \end{pmatrix}.$$

Therefore

$$\hat{P}u_\uparrow(\theta,\phi) = \sqrt{E+m}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} c \\ se^{i\phi} \\ \frac{\mathrm{p}}{E+m}c \\ \frac{\mathrm{p}}{E+m}se^{i\phi} \end{pmatrix}$$

$$= \sqrt{E+m}\begin{pmatrix} c \\ se^{i\phi} \\ -\frac{\mathrm{p}}{E+m}c \\ -\frac{\mathrm{p}}{E+m}se^{i\phi} \end{pmatrix}.$$

This can be compared to

$$u_\downarrow(\pi-\theta,\pi+\phi) = \sqrt{E+m}\begin{pmatrix} -\sin(\pi/2-\theta/2) \\ \cos(\pi/2-\theta/2)e^{i(\phi+\pi)} \\ \frac{\mathrm{p}}{E+m}\sin(\pi/2-\theta/2) \\ -\frac{\mathrm{p}}{E+m}\cos(\pi/2-\theta/2)e^{i(\phi+\pi)} \end{pmatrix}$$

$$= \sqrt{E+m}\begin{pmatrix} -c \\ -se^{i\phi} \\ \frac{\mathrm{p}}{E+m}c \\ \frac{\mathrm{p}}{E+m}se^{i\phi} \end{pmatrix},$$

and therefore, up to an overall phase factor, $\hat{P}u_\uparrow(\theta,\phi) = u_\downarrow(\pi-\theta,\pi+\phi)$. As expected the action of the parity operator has the effect that $\mathbf{p} \to -\mathbf{p}$ (reversing the direction of the particle), but leaves the orientation of the spin unchanged in space, this transforming a RH particle into a LH particle travelling in the opposite direction.

4.14 Under the combined operation of parity and charge conjugation ($\hat{C}\hat{P}$) spinors transform as

$$\psi \to \psi^c = \hat{C}\hat{P}\psi = i\gamma^2\gamma^0\psi^*.$$

Show that up to an overall complex phase factor

$$\hat{C}\hat{P}u_\uparrow(\theta,\phi) = v_\downarrow(\pi-\theta,\pi+\phi).$$

In the Dirac-Pauli representation

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad u_\uparrow(\theta,\phi) = \sqrt{E+m}\begin{pmatrix} c \\ se^{i\phi} \\ \frac{\mathrm{p}}{E+m}c \\ \frac{\mathrm{p}}{E+m}se^{i\phi} \end{pmatrix}.$$

Hence

$$\hat{C}\hat{P}u_{\uparrow}(\theta,\phi) = i\gamma^2\gamma^0 u_{\uparrow}(\theta,\phi)^* = i\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\sqrt{E+m}\begin{pmatrix} c \\ se^{-i\phi} \\ \frac{\mathrm{p}}{E+m}c \\ \frac{\mathrm{p}}{E+m}se^{-i\phi} \end{pmatrix}$$

$$= i\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}\sqrt{E+m}\begin{pmatrix} c \\ se^{-i\phi} \\ -\frac{\mathrm{p}}{E+m}c \\ -\frac{\mathrm{p}}{E+m}se^{-i\phi} \end{pmatrix}$$

$$= \sqrt{E+m}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} c \\ se^{-i\phi} \\ -\frac{\mathrm{p}}{E+m}c \\ -\frac{\mathrm{p}}{E+m}se^{-i\phi} \end{pmatrix}$$

$$= \sqrt{E+m}\begin{pmatrix} -\frac{\mathrm{p}}{E+m}se^{-i\phi} \\ \frac{\mathrm{p}}{E+m}c \\ -se^{-i\phi} \\ c \end{pmatrix}$$

This can be compared to

$$v_{\downarrow}(\pi-\theta,\phi+\pi) = \sqrt{E+m}\begin{pmatrix} \frac{\mathrm{p}}{E+m}s \\ -\frac{\mathrm{p}}{E+m}ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix} = -e^{i\phi}\sqrt{E+m}\begin{pmatrix} -\frac{\mathrm{p}}{E+m}se^{-i\phi} \\ \frac{\mathrm{p}}{E+m}c \\ -se^{-i\phi} \\ c \end{pmatrix},$$

and thus

$$\hat{C}\hat{P}u_{\uparrow}(\theta,\phi) = -e^{i\phi}v_{\downarrow}(\pi-\theta,\pi+\phi).$$

This overall (unobservable phase) could have been included in the original definition of the $v_{\downarrow}$.

4.15 Starting from the Dirac equation, derive the identity

$$\overline{u}(p')\gamma^{\mu}u(p) = \frac{1}{2m}\overline{u}(p')(p+p')^{\mu}u(p) + \frac{i}{m}\overline{u}(p')\,\Sigma^{\mu\nu}q_{\nu}u(p),$$

where $q = p' - p$ and $\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^{\mu},\gamma^{\nu}]$.

Starting with the Dirac equation for the spinor $u(p)$ and the corresponding equation for the adjoint spinor $\overline{u}(p')$:

$$(\gamma^{\mu}p_{\mu} - m)u(p) = 0 \quad \text{and} \quad \overline{u}(p')(\gamma^{\mu}p'_{\mu} - m) = 0,$$

gives

$$\gamma^\mu p_\mu u(p) = mu(p) \quad \text{and} \quad \overline{u}(p')\gamma^\mu p'_\mu = m\overline{u}(p')\,. \tag{4.2}$$

Now consider the $\overline{u}(p')\,\Sigma^{\mu\nu} q_\nu u(p)$ term, remembering that the p_ν are just numbers and taking care to preserve the order of the γ-matrices:

$$\overline{u}(p')\,\Sigma^{\mu\nu} q_\nu u(p) = \frac{i}{4}\left[\overline{u}(p')\gamma^\mu\gamma^\nu(p'_\nu - p_\nu)u(p) - \overline{u}(p')\gamma^\nu(p'_\nu - p_\nu)\gamma^\mu u(p)\right]\,.$$

Wherever the terms $\overline{u}(p')\gamma^\nu p'_\nu$ and $\gamma^\nu p_\nu u(p)$ appear, the relations of (4.2) can be used to simplify the above expression, hence

$$\begin{aligned}
\overline{u}(p')\,\Sigma^{\mu\nu} q_\nu u(p) &= \frac{i}{4}\left[\overline{u}(p')\gamma^\mu\gamma^\nu p'_\nu u(p) - m\overline{u}(p')\gamma^\mu u(p) - m\overline{u}(p')\gamma^\mu u(p) + \overline{u}(p')\gamma^\nu p_\nu\gamma^\mu u(p)\right] \\
&= \frac{i}{4}\left[-2m\overline{u}(p')\gamma^\mu u(p) + \overline{u}(p')\gamma^\mu\gamma^\nu p'_\nu u(p) + \overline{u}(p')\gamma^\nu p_\nu\gamma^\mu u(p)\right]\,.
\end{aligned}$$

The two terms containing two γ-matrices can be simplified by using $\gamma^\mu\gamma^\nu = 2g^{\mu\nu} - \gamma^\nu\gamma^\mu$ to place them in the order where (4.2) can again be used.

$$\begin{aligned}
\overline{u}(p')\,\Sigma^{\mu\nu} q_\nu u(p) &= \frac{i}{4}\left[-2m\overline{u}(p')\gamma^\mu u(p) + \overline{u}(p')(2g^{\mu\nu} - \gamma^\nu\gamma^\mu)p'_\nu u(p)\right. \\
&\qquad \left.\overline{u}(p')(2g^{\mu\nu} - \gamma^\mu\gamma^\nu)p_\nu u(p)\right] \\
&= \frac{i}{4}\left[-2m\overline{u}(p')\gamma^\mu u(p) + 2\overline{u}(p')p'^\mu u(p) - \overline{u}(p')p'_\nu\gamma^\nu\gamma^\mu u(p)\right. \\
&\qquad \left.+2\overline{u}(p')p^\mu u(p) - \overline{u}(p')\gamma^\mu\gamma^\nu p_\nu u(p)\right] \\
&= \frac{i}{4}\left[-2m\overline{u}(p')\gamma^\mu u(p) + 2\overline{u}(p')(p'^\mu + p^\mu)u(p) - 2m\overline{u}(p')\gamma^\mu u(p)\right] \\
&= \frac{i}{4}\left[-4m\overline{u}(p')\gamma^\mu u(p) + 2\overline{u}(p')(p'^\mu + p^\mu)u(p)\right]
\end{aligned}$$

$$\Rightarrow \qquad i\overline{u}(p')\,\Sigma^{\mu\nu} q_\nu u(p) = m\overline{u}(p')\gamma^\mu u(p) - \frac{1}{2}\overline{u}(p')(p'^\mu + p^\mu)u(p)$$

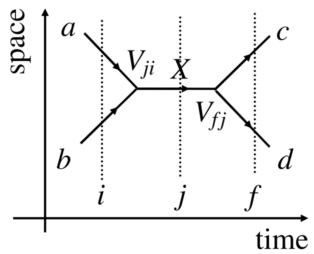
$$\Rightarrow \qquad \overline{u}(p')\gamma^\mu u(p) = \frac{1}{2m}\overline{u}(p')(p'^\mu + p^\mu)u(p) + \frac{i}{m}\overline{u}(p')\,\Sigma^{\mu\nu} q_\nu u(p)\,,$$

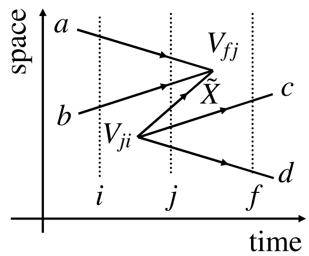
as required.

5 Interaction by Particle Exchange

5.1 Draw the two time-ordered diagrams for the s-channel process shown in Figure 5.5. By repeating the steps of Section 5.1.1, show that the propagator has the same form as obtained for the t-channel process. Hint: one of the time-ordered diagrams is non-intuitive, remember that in second-order perturbation theory the intermediate state does not conserve energy.

The two possible time-orderings are shown below. In the first $a+b$ annihilate into X and then X produces $c+d$. In the second time-ordering, the three particles $c+d+\tilde{X}$ "pop out" of the vacuum and subsequently $a+b+\tilde{X}$ annihilate into the vacuum.





For the first time-ordering, the transition matrix element is given by second-order perturbation theory:

$$T_{fi}^{ab} = \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j} = \frac{\langle c+d|V|X\rangle\langle X|V|a+b\rangle}{(E_a+E_b)-(E_X)}.$$

The non-invariant matrix element V_{ji} is related to the Lorentz invariant (LI) matrix element $\mathcal{M}_{ji}$ by

$$V_{ji} = \mathcal{M}_{ji}\prod_k (2E_k)^{-1/2},$$

where the index k runs over the particles involved at the interaction vertex. For the simplest scalar interaction $\mathcal{M}_{a+b\to X} = \mathcal{M}_{X\to c+d} = g$ and thus

$$T_{fi}^{ab} = \frac{1}{2E_X}\cdot\frac{1}{(2E_a 2E_b 2E_c 2E_d)^{1/2}}\frac{g^2}{(E_a+E_b)-(E_X)},$$

and the corresponding LI matrix element is

$$\mathcal{M}_{fi}^{ab} = \frac{1}{2E_X}\frac{g^2}{(E_a + E_b) - (E_X)}\,.$$

For the second time-ordering all five particles are present in the intermediate state and thus,

$$T_{fi}^{cd} = \frac{\langle c + d + X|V|0\rangle\langle 0|V|a + b + X\rangle}{(E_a + E_b) - (E_a + E_b + E_c + E_d + E_X)}\,,$$

where zero denotes the vacuum state. The corresponding LI matrix element is

$$\mathcal{M}_{fi}^{cd} = -\frac{1}{2E_X}\frac{g^2}{E_c + E_d + E_X} = -\frac{1}{2E_X}\frac{g^2}{(E_a + E_b) + E_X}\,.$$

The total amplitude is the sum of the two time-orderings

$$\begin{aligned}\mathcal{M} &= \frac{g^2}{2E_X}\left[\frac{1}{E_a + E_b - E_X} - \frac{1}{(E_a + E_b) + E_X}\right]\\ &= \frac{g^2}{2E_X}\left[\frac{2E_X}{(E_a + E_b)^2 - E_X^2}\right]\\ &= \frac{g^2}{(E_a + E_b)^2 - E_X^2}\,.\end{aligned}$$

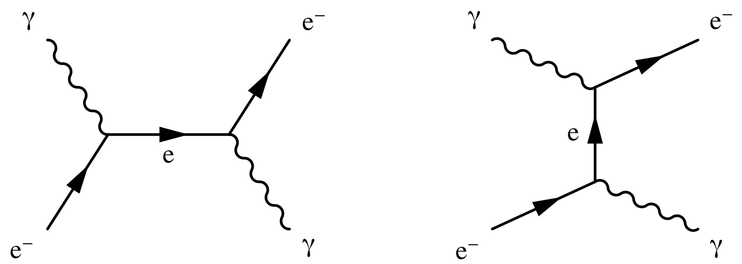
Writing $E_X^2 = (\mathbf{p}_a + \mathbf{p}_b)^2 + m_X^2$ gives

$$\begin{aligned}\mathcal{M} &= \frac{g^2}{(E_a + E_b)^2 - (\mathbf{p}_a + \mathbf{p}_b)^2 - m_X^2}\\ &= \frac{g^2}{q^2 - m_X^2}\,,\end{aligned}$$

where $q^2 = (p_a + p_b)^2$. This is exactly the same form as for the t-channel process considered in the main text, and as before the four-momentum that appears in the expression is obtained from conservation of four-momentum at the interaction vertex.

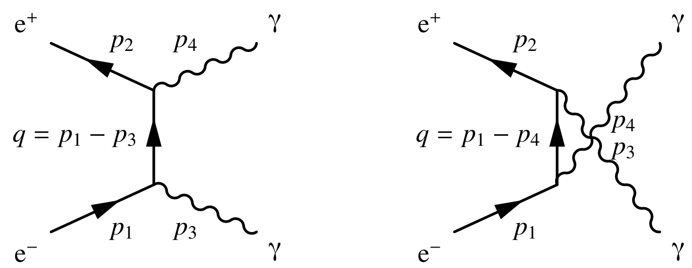
5.2 Draw the *two* lowest-order Feynman diagrams for the Compton scattering process $\gamma e^- \rightarrow \gamma e^-$.

The lowest-order diagrams have just two QED eeγ interaction vertices. Here there is a t-channel and an s-channel diagram. It should be remembered that a Feynman diagram is defined solely by its topology as it already represents all possible time-orderings for a process. In both cases the virtual particle, labelled as e, represents an e^- in one time-ordering and an e^+ in the other.



5.3 Draw the lowest-order t-channel and u-channel Feynman diagrams for $e^+e^- \to \gamma\gamma$ and use the Feynman rules for QED to write down the corresponding matrix elements.

Because the there are identical particles in the final state, both t- and u-channel diagrams can contribute. In the first diagram the virtual photon has four-momentum $q = p_1 - p_3 = p_4 - p_2$, whereas in the second diagram $q = p_1 - p_4 = p_3 - p_2$. Using



the Feynman rules for QED, and remembering to label the vertices with different indices, the matrix elements are

$$-i\mathcal{M}_t = \left[\varepsilon^*_\mu(p_3)ie\gamma^\mu u(p_1)\right] \cdot \left[-\frac{i(\gamma^\rho q_\rho + m_e)}{q^2 - m_e}\right] \cdot \left[\overline{v}(p_2)ie\gamma^\nu \varepsilon^*_\nu(p_4)\right]$$

$$-i\mathcal{M}_u = \left[\varepsilon^*_\mu(p_4)ie\gamma^\mu u(p_1)\right] \cdot \left[-\frac{i(\gamma^\rho q_\rho + m_e)}{q^2 - m_e}\right] \cdot \left[\overline{v}(p_2)ie\gamma^\nu \varepsilon^*_\nu(p_3)\right] .$$

6 Electron-Positron Annihilation

6.1 Using the properties of the γ-matrices of (4.33) and (4.34), and the definition of $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$, show that

$$(\gamma^5)^2 = 1\,, \quad \gamma^{5\dagger} = \gamma^5 \quad \text{and} \quad \gamma^5\gamma^\mu = -\gamma^\mu\gamma^5\,.$$

i) Remembering that $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ for $\mu \neq \nu$ then the γ-matrices can be permuted and then removed using $(\gamma^0)^2 = 1$ and $(\gamma^k)^2 = -1$:

$$\begin{aligned}
(\gamma^5)^2 &= -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 \\
&= +\gamma^0\gamma^1\gamma^2\gamma^0\gamma^3\gamma^1\gamma^2\gamma^3 \\
&= -\gamma^0\gamma^1\gamma^0\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 \\
&= +\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 \\
&= +\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 \\
&= +\gamma^1\gamma^1\gamma^2\gamma^3\gamma^2\gamma^3 \\
&= -\gamma^2\gamma^3\gamma^2\gamma^3 \\
&= +\gamma^2\gamma^2\gamma^3\gamma^3 \\
&= +1\,.
\end{aligned}$$

ii) Similarly

$$\begin{aligned}
\gamma^{5\dagger} &= -i\gamma^3\gamma^2\gamma^1\gamma^0 \\
&= +i\gamma^3\gamma^2\gamma^0\gamma^1 \\
&= -i\gamma^3\gamma^0\gamma^2\gamma^1 \\
&= +i\gamma^0\gamma^3\gamma^2\gamma^1 \\
&= +i\gamma^0\gamma^1\gamma^3\gamma^2 \\
&= -i\gamma^0\gamma^1\gamma^2\gamma^3 \\
&= -\gamma^5\,.
\end{aligned}$$

Of course this can be shown much more quickly be realising that six permutations of the form $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ are required to reorder $\gamma^{5\dagger}$ into γ^5.

iii) Since

$$\gamma^5\gamma^\mu = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu\,,$$

it takes three permutations of the form $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ to place the γ^μ at the front of the expression. It is three, not four, because no permutation is required when γ^μ is next to the corresponding γ-matrix. Thus

$$\gamma^5\gamma^\mu = -i\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^\mu\gamma^5\,.$$

6.2 Show that the chiral projection operators

$$P_R = \tfrac{1}{2}(1+\gamma^5) \quad \text{and} \quad P_L = \tfrac{1}{2}(1-\gamma^5)\,,$$

satisfy

$$P_R + P_L = 1\,, \quad P_R P_R = P_R\,, \quad P_L P_L = P_L \quad \text{and} \quad P_L P_R = 0\,.$$

It is clear that $P_R + P_L = 1$. Using $(\gamma^5)^2 = 1$, it is straightforward to show $P_R P_R = P_R$,

$$\begin{aligned} P_R P_R &= \tfrac{1}{4}(1+\gamma^5)(1+\gamma^5) \\ &= \tfrac{1}{4}(1+2\gamma^5+(\gamma^5)^2) \\ &= \tfrac{1}{4}(2+2\gamma^5) = P_R\,. \end{aligned}$$

The relation $P_L P_L = P_L$ follows in the same way. Finally

$$\begin{aligned} P_R P_L &= \tfrac{1}{4}(1+\gamma^5)(1-\gamma^5) \\ &= \tfrac{1}{4}(1-(\gamma^5)^2) \\ &= 0\,. \end{aligned}$$

6.3 Show that

$$\Lambda^+ = \frac{m+\gamma^\mu p_\mu}{2m} \quad \text{and} \quad \Lambda^- = \frac{m-\gamma^\mu p_\mu}{2m}\,,$$

are also projection operators, and show that they respectively project out particle and antiparticle states, *i.e.*

$$\Lambda^+ u = u\,, \quad \Lambda^- v = v \quad \text{and} \quad \Lambda^+ v = \Lambda^- u = 0\,.$$

i) Firstly it is necessary to show that the $\Lambda^\pm$ matrices are projections operators. Clearly $\Lambda^+ + \Lambda^- = 1$ as required. Secondly, it is necessary to show $\Lambda^+\Lambda^+ = \Lambda^+$, such that the repeated action of a projection operator induces no change. Here (remembering that the p_μ are just numbers):

$$\begin{aligned} \Lambda^+\Lambda^+ &= \tfrac{1}{4m^2}(m+\gamma^\mu p_\mu)(m+\gamma^\nu p_\nu) \\ &= \tfrac{1}{4m^2}(m^2 + m[\gamma^\mu p_\mu + \gamma^\nu p_\nu] + p_\mu p_\nu \gamma^\mu\gamma^\nu)\,. \end{aligned}$$

Because $p_\mu p_\nu \gamma^\mu \gamma^\nu$ is a symmetric tensor it can be written as $\frac{1}{2} p_\mu p_\nu (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu)$ and since $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$

$$\begin{aligned} p_\mu p_\nu \gamma^\mu \gamma^\nu &= \tfrac{1}{2} p_\mu p_\nu (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) \\ &= g^{\mu\nu} p_\mu p_\nu \\ &= p_\mu p^\mu = m^2 \,. \end{aligned}$$

Thus

$$\begin{aligned} \Lambda^+ \Lambda^+ &= \tfrac{1}{4m^2}(m^2 + m[\gamma^\mu p_\mu + \gamma^\nu p_\nu] + p_\mu p_\nu \gamma^\mu \gamma^\nu) \\ &= \tfrac{1}{4m^2}(2m^2 + 2m\gamma^\mu p_\mu) \\ &= \tfrac{1}{2m}(m + \gamma^\mu p_\mu) \\ &= \Lambda^+ \,. \end{aligned}$$

Similarly $\Lambda^+ \Lambda^- = 0$ and $\Lambda^- \Lambda^- = \Lambda^-$.

ii) The free particle u-spinors and v-spinors respectively satisfy

$$(\gamma^\mu p_\mu - m)u = 0 \quad \text{and} \quad (\gamma^\mu p_\mu + m)u = 0 \,,$$

and therefore

$$\gamma^\mu p_\mu u = +mu \quad \text{and} \quad \gamma^\mu p_\mu v = -mv \,.$$

Therefore

$$\begin{aligned} \Lambda^+ u &= \tfrac{1}{2m}(m + \gamma^\mu p_\mu)u \\ &= \tfrac{1}{2m}(m + m)u = u \,. \end{aligned}$$

The other relations follow in the same way.

6.4 Show that the helicity operator can be expressed as

$$\hat{h} = -\frac{1}{2}\frac{\gamma^0 \gamma^5 \boldsymbol{\gamma} \cdot \mathbf{p}}{\mathrm{p}} \,.$$

In the Dirac-Pauli representation, the relevant matrices are

$$\hat{S}_k = \tfrac{1}{2}\hat{\Sigma}_k = \tfrac{1}{2}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \text{and} \quad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

where $k = 1, 2, 3$ and $\hat{S}_k$ are the components of the spin operator *for a Dirac spinor*.

The combination $-\frac{1}{2}\gamma^0\gamma^5\gamma^k$ gives

$$-\tfrac{1}{2}\gamma^0\gamma^5\gamma^k = -\tfrac{1}{2}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$
$$= -\tfrac{1}{2}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$
$$= \tfrac{1}{2}\begin{pmatrix} \sigma_k & 0 \\ \sigma_k & 0 \end{pmatrix}$$
$$= \hat{S}_k\,,$$

and therefore

$$\hat{h} = -\tfrac{1}{2}\frac{\gamma^0\gamma^5\boldsymbol{\gamma}\cdot\mathbf{p}}{\mathrm{p}} = \frac{\hat{\mathbf{S}}\cdot\mathbf{p}}{\mathrm{p}}\,,$$

which gives the projection of the spin of a particle along its direction of motion.

6.5 In general terms, explain why *high-energy* electron-positron colliders must also have high instantaneous luminosities.

Because s-channel QED cross sections decrease as $1/s$, as the centre-of-mass energy increases, higher instantaneous luminosities are required to obtain a reasonable event rate, $Rate = \sigma\mathcal{L}$.

6.6 For a spin-1 system, the eigenstate of the operator $\hat{S}_n = \mathbf{n}\cdot\hat{\mathbf{S}}$ with eigenvalue $+1$ corresponds to the spin being in the direction $\hat{\mathbf{n}}$. Writing this state in terms of the eigenstates of $\hat{S}_z$, *i.e.*

$$|1,+1\rangle_\theta = \alpha|1,-1\rangle + \beta|1,0\rangle + \gamma|1,+1\rangle\,,$$

and taking $\mathbf{n} = (\sin\theta, 0, \cos\theta)$ show that

$$|1,+1\rangle_\theta = \tfrac{1}{2}(1-\cos\theta)\,|1,-1\rangle + \tfrac{1}{\sqrt{2}}\sin\theta\,|1,0\rangle + \tfrac{1}{2}(1+\cos\theta)\,|1,+1\rangle\,.$$

Hint: write $\hat{S}_x$ in terms of the spin ladder operators.

This question is a fairly straightforward but requires care with the algebra. Firstly, it should be noted that $\alpha^2+\beta^2+\gamma^2 = 1$. Secondly, by definition $\hat{S}_n|1,+1\rangle_\theta = +|1,+1\rangle_\theta$, where $\hat{S}_n = \mathbf{n}\cdot\hat{\mathbf{S}}$ and, without loss of generality $\mathbf{n}$ taken to lie in the xz plane.

$$\hat{S}_n = \mathbf{n}\cdot\hat{\mathbf{S}} = \sin\theta\hat{S}_x + \cos\theta\hat{S}_z\,.$$

The operator $\hat{S}_n$ can be written in terms of operators in terms of the $|s,m\rangle$ states using the angular momentum ladder operators,

$$\hat{S}_+ = \hat{S}_x + i\hat{S}_y \quad \text{and} \quad \hat{S}_+ = \hat{S}_x - i\hat{S}_y\,,$$

and therefore $\hat{S}_x = \frac{1}{2}(\hat{S}_+\hat{S}_-)$. Hence

$$\hat{S}_n = \mathbf{n}\cdot\hat{\mathbf{S}} = \tfrac{1}{2}s_\theta[\hat{S}_+ + \hat{S}_-] + c_\theta\hat{S}_z\,,$$

where for compactness the notation $s_\theta = \sin\theta$ and $c_\theta = \cos\theta$ has been used. Since $\hat{S}_n|1,+1\rangle_\theta = +|1,+1\rangle_\theta$, then

$$\left(\tfrac{1}{2}s_\theta[\hat{S}_+ + \hat{S}_-] + c_\theta\hat{S}_z\right)(\alpha|1,-1\rangle + \beta|1,0\rangle + \gamma|1,+1\rangle) = \alpha|1,-1\rangle+\beta|1,0\rangle+\gamma|1,+1\rangle\,.$$

Using

$$\hat{S}_+\,|s,m\rangle = \sqrt{s(s+1)-m(m+1)}\,|s,m+1\rangle$$
$$\hat{S}_-\,|s,m\rangle = \sqrt{s(s+1)-m(m-1)}\,|s,m-1\rangle\ ,$$

$\hat{S}_n\,|1,+1\rangle$ can be expressed in terms of the eigenstates of the $\hat{S}_z$ eigenstates:

$$\begin{aligned}\hat{S}_n\,|1,+1\rangle &= \alpha\left(\sqrt{2}s_\theta\,|1,0\rangle + c_\theta\,|1,-1\rangle\right) + \beta\sqrt{2}s_\theta\,(|1,-1\rangle + |1,+1\rangle) + \gamma(\sqrt{2}s_\theta\,|1,0\rangle + c_\theta\,|1,+1\rangle)\\ &= \left(-\alpha c_\theta + \beta\sqrt{2}s_\theta\right)|1,-1\rangle + \sqrt{2}s_\theta\,(\alpha+\gamma)\,|1,0\rangle + \left(\beta\sqrt{2}s_\theta + \gamma c_\theta\right)|1,+1\rangle\ .\end{aligned}$$

Since this must be equal to $\alpha|1,-1\rangle + \beta|1,0\rangle + \gamma|1,+1\rangle$, equating the individual terms gives the simultaneous equations,

$$\begin{aligned}\beta\sqrt{2}s_\theta &= \alpha(1+c_\theta)\\ \sqrt{2}s_\theta(\alpha+\gamma) &= \beta\\ \beta\sqrt{2}s_\theta &= \gamma(1-c_\theta)\,.\end{aligned}$$

From the first and third equations

$$\alpha(1-c_\theta) = \gamma(1+c_\theta) \quad \text{and} \quad 2\beta^2 = \alpha\gamma\,,$$

which, when combined with $\alpha^2+\beta^2+\gamma^2 = 1$ implies that $\alpha = \frac{1}{2}(1-c_\theta)$, $\gamma = \frac{1}{2}(1+c_\theta)$ and $\beta = \frac{1}{\sqrt{2}}s_\theta$. Hence

$$|1,+1\rangle_\theta = \tfrac{1}{2}(1-\cos\theta)\,|1,-1\rangle + \tfrac{1}{\sqrt{2}}\sin\theta\,|1,0\rangle + \tfrac{1}{2}(1+\cos\theta)\,|1,+1\rangle\,.$$

6.7 Using helicity amplitudes, calculate the differential cross section for $e^-\mu^- \to e^-\mu^-$ scattering in the following steps:

a) From the Feynman rules for QED, show that the lowest-order QED matrix element for $e^-\mu^- \to e^-\mu^-$ is

$$\mathcal{M}_{fi} = -\frac{e^2}{(p_1-p_3)^2}g_{\mu\nu}\,[\overline{u}(p_3)\gamma^\mu u(p_1)]\,[\overline{u}(p_4)\gamma^\nu u(p_2)]\ ,$$

where p_1 and p_3 are the four-momenta of the initial- and final-state e^-, and p_2 and p_4 are the four-momenta of the initial- and final-state μ^-.

b) Working in the centre-of-mass frame, and writing the four-momenta of the initial- and final-state e^- as $p_1^\mu = (E_1, 0, 0, \mathrm{p})$ and $p_3^\mu = (E_1, p\sin\theta, 0, \mathrm{p}\cos\theta)$ respectively, show that the electron currents for the four possible helicity combinations are

$$\begin{aligned}\overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) &= 2(E_1 c, \mathrm{p}s, -i\mathrm{p}s, \mathrm{p}c)\,,\\ \overline{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) &= 2(ms, 0, 0, 0)\,,\\ \overline{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) &= 2(E_1 c, \mathrm{p}s, i\mathrm{p}s, \mathrm{p}c)\,,\\ \overline{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) &= -2(ms, 0, 0, 0)\,,\end{aligned}$$

where m is the electron mass, $s = \sin(\theta/2)$ and $c = \cos(\theta/2)$.

c) Explain why the effect of the parity operator $\hat{P} = \gamma^0$ is

$$\hat{P}u_\uparrow(\mathrm{p}, \theta, \phi) = \hat{P}u_\downarrow(\mathrm{p}, \pi - \theta, \pi + \phi)\,.$$

Hence, or otherwise, show that the muon currents for the four helicity combinations are

$$\begin{aligned}
\overline{u}_\downarrow(p_4)\gamma^\mu u_\downarrow(p_2) &= 2(E_2 c, -\mathrm{p}s, -i\mathrm{p}s, -\mathrm{p}c)\,,\\
\overline{u}_\uparrow(p_4)\gamma^\mu u_\downarrow(p_2) &= 2(Ms, 0, 0, 0)\,,\\
\overline{u}_\uparrow(p_4)\gamma^\mu u_\uparrow(p_2) &= 2(E_2 c, -\mathrm{p}s, i\mathrm{p}s, -\mathrm{p}c)\,,\\
\overline{u}_\downarrow(p_4)\gamma^\mu u_\uparrow(p_2) &= -2(Ms, 0, 0, 0)\,,
\end{aligned}$$

where M is the muon mass.

d) For the relativistic limit where $E \gg M$, show that the matrix element squared for the case where the incoming e^- and incoming μ^- are both left-handed is given by

$$|\mathcal{M}_{LL}|^2 = \frac{4e^4 s^2}{(p_1 - p_3)^4}\,,$$

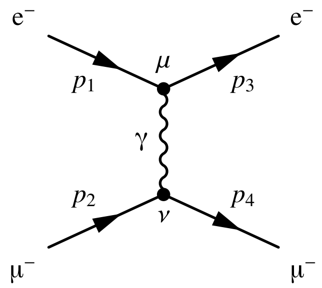
where $s = (p_1 + p_2)^2$. Find the corresponding expressions for $|\mathcal{M}_{RL}|^2$, $|\mathcal{M}_{RR}|^2$ and $|\mathcal{M}_{LR}|^2$.

e) In this relativistic limit, show that the differential cross section for unpolarised $e^-\mu^- \to e^-\mu^-$ scattering in the centre-of-mass frame is

$$\frac{d\sigma}{d\Omega} = \frac{2\alpha^2}{s} \cdot \frac{1 + \frac{1}{4}(1 + \cos\theta)^2}{(1 - \cos\theta)^2}\,.$$

This is a fairly long question, but provides a good introduction to calculating matrix elements using helicity amplitudes.

a) The lowest-order (QED) Feynman diagram for the $e^-\mu^- \to e^-\mu^-$ scattering process is shown below The corresponding matrix element comprises three parts,



the two vertices and the photon propagator. The product of these three parts is equal to $-i\mathcal{M}$. Remembering that the particle attached to the arrow leaving the vertex appears as the adjoint spinor:

$$-i\mathcal{M}_{fi} = [\overline{u}(p_3)(-iQ_e e)\gamma^\mu u(p_1)] \cdot \left[-i\frac{g_{\mu\nu}}{q^2}\right] \cdot \left[\overline{u}(p_4)(-iQ_\mu e)\gamma^\nu u(p_2)\right]\,,$$

where $Q_e = -1$, $Q_\mu = -1$ and $q = p_1 - p_3$, hence

$$\begin{aligned}\mathcal{M}_{fi} &= -\frac{e^2\, g_{\mu\nu}}{(p_1-p_3)^2}\,[\overline{u}(p_3)\gamma^\mu u(p_1)]\,[\overline{u}(p_4)\gamma^\nu u(p_2)]\\ &= -\frac{e^2}{(p_1-p_3)^2}\, j_e\cdot j_\mu\,,\end{aligned}$$

where the electron and muon currents are $j_e^\mu = \overline{u}(p_3)\gamma^\mu u(p_1)$ and $j_\mu^\nu = \overline{u}(p_4)\gamma^\nu u(p_2)$.

b) For a particle of mass m the helicity eigenstate spinors are

$$u_\uparrow = \sqrt{E+m}\begin{pmatrix} c \\ e^{i\phi}s \\ \frac{\mathrm{p}}{E+m}c \\ \frac{\mathrm{p}}{E+m}e^{i\phi}s \end{pmatrix} \quad\text{and}\quad u_\downarrow = \sqrt{E+m}\begin{pmatrix} -s \\ e^{i\phi}c \\ \frac{\mathrm{p}}{E+m}s \\ -\frac{\mathrm{p}}{E+m}e^{i\phi}c \end{pmatrix},$$

where $c = \cos(\theta/2)$ and $s = \sin(\theta/2)$.

For the incoming electron, with $p_1 = (E_1, 0, 0, p)$, with $\theta = 0, \phi = 0$, the two possible spinors are:

$$u_\uparrow(p_1) = \sqrt{E_1+m}\begin{pmatrix} 1 \\ 0 \\ \frac{\mathrm{p}}{E_1+m} \\ 0 \end{pmatrix} \quad\text{and}\quad u_\downarrow(p_1) = \sqrt{E_1+m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{mvp}{E_1+m} \end{pmatrix}.$$

For the outgoing electron, with $p_3 = (E_1, p\sin\theta, 0, p\cos\theta)$, the spinors are:

$$u_\uparrow(p_3) = \sqrt{E_1+m}\begin{pmatrix} c \\ s \\ \frac{\mathrm{p}}{E_1+m}c \\ \frac{\mathrm{p}}{E_1+m}s \end{pmatrix} \quad\text{and}\quad u_\downarrow(p_3) = \sqrt{E_1+m}\begin{pmatrix} -s \\ c \\ \frac{\mathrm{p}}{E_1+m}s \\ -\frac{\mathrm{p}}{E_1+m}c \end{pmatrix}.$$

In the Dirac-Pauli representation, the four vector currents can be calculated from

$$\begin{aligned}\overline{\psi}\gamma^0\phi &= \psi_1\phi_1 + \psi_2\phi_2 + \psi_3\phi_3 + \psi_4\phi_4\\ \overline{\psi}\gamma^1\phi &= \psi_1\phi_4 + \psi_2\phi_3 + \psi_3\phi_2 + \psi_4\phi_1\\ \overline{\psi}\gamma^2\phi &= -i(\psi_1\phi_4 - \psi_2\phi_3 + \psi_3\phi_2 - \psi_4\phi_1)\\ \overline{\psi}\gamma^3\phi &= \psi_1\phi_3 - \psi_2\phi_4 + \psi_3\phi_1 - \psi_4\phi_2\,.\end{aligned}$$

Therefore, for the four possible combined helicity states of the initial- and final-state electrons:

$$\begin{aligned}\overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) &= 2(E_1c, \mathrm{p}s, -i\mathrm{p}s, \mathrm{p}c)\,,\\ \overline{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) &= 2(ms, 0, 0, 0)\,,\\ \overline{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) &= 2(E_1c, \mathrm{p}s, i\mathrm{p}s, \mathrm{p}c)\,,\\ \overline{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) &= -2(ms, 0, 0, 0)\,,\end{aligned}$$

where m is the electron mass.

c) The parity operator reverses the momentum of a particle (a vector quantity) but leaves the spin (an axial-vector quantity) unchanged, and therefore has the effect $\hat{P}u_\uparrow(E, \mathbf{p}) = u_\downarrow(E, -\mathbf{p})$. This can be utilised here to obtain the possible muon currents from the electron currents (once the different masses have been accounted for), since in the centre-of-mass frame the initial- and final-state electron and muon momenta are equal and opposite. Hence

$$\begin{aligned}
\overline{u}_\downarrow(p_4)\gamma^\mu u_\downarrow(p_2) &= \overline{\hat{P}u_\uparrow}(p_3)\gamma^\mu \hat{P}u_\uparrow(p_1) \\
&= (\gamma^0 u_\uparrow(p_3))^\dagger \gamma^0\gamma^\mu\gamma^0 u_\uparrow(p_1) \\
&= u_\uparrow^\dagger(p_3)\gamma^0\gamma^0\gamma^\mu\gamma^0 u_\uparrow(p_1) \\
&= u_\uparrow^\dagger(p_3)\gamma^\mu\gamma^0 u_\uparrow(p_1) \\
&= \begin{cases} u_\uparrow^\dagger(p_3)\gamma^0\gamma^\mu u_\uparrow(p_1) & \text{if } \mu = 0 \\ -u_\uparrow^\dagger(p_3)\gamma^0\gamma^\mu u_\uparrow(p_1) & \text{if } \mu = 1,2,3 \end{cases} \\
&= \begin{cases} \overline{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) & \text{if } \mu = 0 \\ -\overline{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) & \text{if } \mu = 1,2,3 \end{cases} .
\end{aligned}$$

Therefore the muon currents can be obtained from the electron currents by reversing the helicity states and multiplying the space-like components by -1, giving

$$\begin{aligned}
\overline{u}_\downarrow(p_4)\gamma^\mu u_\downarrow(p_2) &= 2(E_2 c, -\mathrm{p}s, -i\mathrm{p}s, -\mathrm{p}c)\,, \\
\overline{u}_\uparrow(p_4)\gamma^\mu u_\downarrow(p_2) &= 2(Ms, 0, 0, 0)\,, \\
\overline{u}_\uparrow(p_4)\gamma^\mu u_\uparrow(p_2) &= 2(E_2 c, -\mathrm{p}s, i\mathrm{p}s, -\mathrm{p}c)\,, \\
\overline{u}_\downarrow(p_4)\gamma^\mu u_\uparrow(p_2) &= -2(Ms, 0, 0, 0)\,,
\end{aligned}$$

where M is the muon mass and E_2 is the energy of the muon in the centre-of-mass frame.

d) For the relativistic limit where terms of order of the electron and muon masses can be neglected, the only non-zero currents are:

$$\begin{aligned}
j^\mu_{\mathrm{e}\downarrow\downarrow} &= \overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = 2E(c, s, -is, c)\,, \\
j^\mu_{\mathrm{e}\uparrow\uparrow} &= \overline{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = 2E(c, s, is, c)\,, \\
j^\nu_{\mu\downarrow\downarrow} &= \overline{u}_\downarrow(p_4)\gamma^\mu u_\downarrow(p_2) = 2E(c, -s, -is, -c)\,, \\
j^\nu_{\mu\uparrow\uparrow} &= \overline{u}_\uparrow(p_4)\gamma^\mu u_\uparrow(p_2) = 2E(c, -s, is, -c)\,.
\end{aligned}$$

When the incoming e^- and incoming μ^- are both left-handed, in this limit, the only non-negligible matrix element is the one where "helicity is conserved" through the vertex:

$$\begin{aligned}\mathcal{M}_{LL} &= -\frac{e^2}{(p_1-p_3)^2}\, j_{\mathrm{e}\downarrow\downarrow}\cdot j_{\mu\downarrow\downarrow}\\ &= -\frac{e^2}{(p_1-p_3)^2} 4E^2\,(c,s,-is,c)\cdot(c,-s,-is,-c)\\ &= -\frac{e^2}{(p_1-p_3)^2} 4E^2\,(c^2+s^2+s^2+c^2) = -\frac{e^2}{(p_1-p_3)^2}8E^2 = -\frac{e^2}{(p_1-p_3)^2}2s\end{aligned}$$

$$\Rightarrow \qquad |\mathcal{M}_{LL}|^2 = \frac{4e^4 s^2}{(p_1-p_3)^4}\,.$$

Similarly,

$$\begin{aligned}\mathcal{M}_{LR} &= -\frac{e^2}{(p_1-p_3)^2}\, j_{\mathrm{e}\downarrow\downarrow}\cdot j_{\mu\uparrow\uparrow}\\ &= -\frac{e^2}{(p_1-p_3)^2} 4E^2\,(c,s,-is,c)\cdot(c,-s,+is,-c)\\ &= -\frac{e^2}{(p_1-p_3)^2} 4E^2\,(c^2+s^2-s^2+c^2) = -\frac{e^2}{(p_1-p_3)^2}8E^2\cos^2(\theta/2)\\ &= -\frac{e^2}{(p_1-p_3)^2} s(1+\cos\theta)\end{aligned}$$

$$\Rightarrow \qquad |\mathcal{M}_{LR}|^2 = \frac{e^4 s^2}{(p_1-p_3)^4}(1+\cos\theta)\,.$$

It is straightforward to show (and can be appreciated from spin arguments) that $|\mathcal{M}_{RR}|^2 = |\mathcal{M}_{LL}|^2$ and $|\mathcal{M}_{RL}|^2 = |\mathcal{M}_{LR}|^2$.

d) In this ultra-relativistic limit, the spin-averaged matrix element squared is

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\left(\mathcal{M}_{RR}|^2 + |\mathcal{M}_{LL}|^2 + |\mathcal{M}_{RL}|^2 = |\mathcal{M}_{LR}|^2\right)\\ &= \frac{2e^2 s^2}{(p_1-p_3)^4}\left[1+\tfrac{1}{4}(1+\cos\theta)\right].\end{aligned}$$

Neglecting the electron mass, $(p_1-p_3)^2 = -2E^2(1-\cos\theta) = -\frac{1}{2}s(1-\cos\theta)$ and the differential cross section (in the centre-of-mass frame) is therefore

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*} &= \frac{1}{64\pi^2 s}\langle|\mathcal{M}|^2\rangle\\ &= \frac{1}{64\pi^2 s}\frac{2e^2 s^2}{E^4(1-\cos\theta)^2}\left[1+\tfrac{1}{4}(1+\cos\theta)\right]\\ &= \frac{2\alpha^2}{s}\frac{\left[1+\tfrac{1}{4}(1+\cos\theta)\right]}{(1-\cos\theta)^2}\,,\end{aligned}$$

where $e^2 = 4\pi\alpha$.

6.8* Using $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$, prove that

$$\gamma^\mu\gamma_\mu = 4\,, \quad \gamma^\mu \not{a}\gamma_\mu = -2\not{a} \quad \text{and} \quad \gamma^\mu \not{a}\not{b}\gamma_\mu = 4a\cdot b\,.$$

i) This question is a good exercise in the manipulation of γ-matrices. The relation $\gamma^\mu\gamma_\mu = 4$ can be shown by writing

$$\begin{aligned}\gamma^\mu\gamma_\mu &= g_{\mu\nu}\gamma^\mu\gamma^\nu \\ &= \tfrac{1}{2}g_{\mu\nu}\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right)\,,\end{aligned}$$

where the second line follows because $g_{\mu\nu}$ is a symmetric tensor. Thus

$$\begin{aligned}\gamma^\mu\gamma_\mu &= \tfrac{1}{2}g_{\mu\nu}\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right) \\ &= g_{\mu\nu}g^{\mu\nu} \\ &= 4\,.\end{aligned}$$

ii) When written out in full (remembering that the components of a are just numbers),

$$\begin{aligned}\gamma^\mu\not{a}\gamma_\mu &= a_\nu\gamma^\mu\gamma^\nu\gamma_\mu \\ &= a_\nu\left(2g^{\mu\nu} - \gamma^\nu\gamma^\mu\right)\gamma_\mu \\ &= a_\nu\left(2\gamma^\nu - \gamma^\nu\gamma^\mu\gamma_\mu\right) \\ &= a_\nu\left(2\gamma^\nu - 4\gamma^\nu\right) \\ &= -2\not{a}\,,\end{aligned}$$

where the result from the first part of the question was used in the second to last step.

iii) When written out in full,

$$\begin{aligned}\gamma^\mu\not{a}\not{b}\gamma_\mu &= a_\nu b_\rho\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu \\ &= a_\nu b_\rho\left(2g^{\mu\nu} - \gamma^\nu\gamma^\mu\right)\gamma^\rho\gamma_\mu \\ &= a_\nu b_\rho\left(2\gamma^\rho\gamma^\nu - \gamma^\nu\gamma^\mu\gamma^\rho\gamma_\mu\right) \\ &= a_\nu b_\rho\left(2\gamma^\rho\gamma^\nu - \gamma^\nu[2g^{\mu\rho} - \gamma^\rho\gamma^\mu]\gamma_\mu\right) \\ &= a_\nu b_\rho\left(2\gamma^\rho\gamma^\nu - 2\gamma^\nu\gamma^\rho + \gamma^\nu\gamma^\rho\gamma^\mu\gamma_\mu\right) \\ &= a_\nu b_\rho\left(2\gamma^\rho\gamma^\nu - 2\gamma^\nu\gamma^\rho + 4\gamma^\nu\gamma^\rho\right) \\ &= 2a_\nu b_\rho\left(\gamma^\rho\gamma^\nu + \gamma^\nu\gamma^\rho\right) \\ &= 4a_\nu b_\rho g^{\rho\nu} \\ &= 4a\cdot b\,;\end{aligned}$$

fun if you like this sort of thing.

6.9* Prove the relation $\left[\overline{\psi}\gamma^\mu\gamma^5\phi\right]^\dagger = \overline{\phi}\gamma^\mu\gamma^5\psi$.

In the main text it was shown that for a vector interaction (e.g. QED and QCD) where the current has the form $j = \overline{\psi}\gamma^\mu\phi$, then $j^\dagger = \overline{\phi}\gamma^\mu\psi$. For the weak interaction one has to consider the axial-vector current $j = \overline{\psi}\gamma^\mu\gamma^5\phi$.

$$\begin{aligned}\left[\overline{\psi}\gamma^\mu\gamma^5\phi\right]^\dagger &= [\psi^\dagger\gamma^0\gamma^\mu\gamma^5\phi]^\dagger \\ &= \phi^\dagger\gamma^{5\dagger}\gamma^{\mu\dagger}\gamma^{0\dagger}\psi \\ &= \phi^\dagger\gamma^5\gamma^{\mu\dagger}\gamma^0\psi \\ &= \phi^\dagger\gamma^{\mu\dagger}\gamma^0\gamma^5\psi \\ &= \phi^\dagger\gamma^0\gamma^\mu\gamma^5\psi \\ &= \overline{\phi}\gamma^\mu\gamma^5\psi\,,\end{aligned}$$

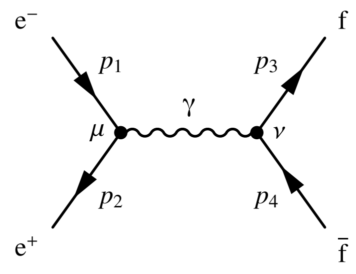
and thus, taking the hermitian conjugate of an axial-vector current retains the form of the current.

6.10* Use the trace formalism to calculate the QED spin-averaged matrix element squared for $e^+e^- \to f\overline{f}$ including the electron mass term.

The QED matrix element for the Feynman diagram shown below is

$$\mathcal{M}_{fi} = \frac{Q_f e^2}{q^2}\,[\overline{v}(p_2)\gamma^\mu u(p_1)]\,g_{\mu\nu}\,[\overline{u}(p_3)\gamma^\nu v(p_4)]\ .$$

Noting the order in which the spinors appear in the matrix element (working back-



wards along the arrows on the fermion lines), the spin-summed matrix element squared is given by

$$\sum_{\text{spins}} |\mathcal{M}_{fi}|^2 = \frac{Q_f^2 e^4}{q^4}\,\mathrm{Tr}\left([\not{p}_2 - m_e]\gamma^\mu[\not{p}_1 + m_e]\gamma^\nu\right)\,\mathrm{Tr}\left([\not{p}_3 + m_f]\gamma_\mu[\not{p}_4 - m_f]\gamma_\nu\right)\ .$$

Since the trace of an odd number of gamma-matrices is zero, the two traces are:

$$\begin{aligned}
\mathrm{Tr}\left([\not{p}_2 - m_\mathrm{e}]\gamma^\mu[\not{p}_1 + m_\mathrm{e}]\gamma^\nu\right) &= \mathrm{Tr}\left(\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu\right) - m_\mathrm{e}^2\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right)\\
&= 4p_1^\mu p_2^\nu - 4g^{\mu\nu}(p_1\cdot p_2) + 4p_1^\nu p_2^\mu - 4m_\mathrm{e}^2 g^{\mu\nu}\,;\\
\mathrm{Tr}\left([\not{p}_3 + m_\mathrm{f}]\gamma_\mu[\not{p}_4 - m_\mathrm{f}]\gamma_\nu\right) &= \mathrm{Tr}\left(\not{p}_3\gamma_\mu\not{p}_4\gamma_\nu\right) - m_\mathrm{f}^2\mathrm{Tr}\left(\gamma_\mu\gamma_\nu\right)\\
&= 4p_{3\mu}p_{4\nu} - 4g_{\mu\nu}(p_3\cdot p_4) + 4p_{3\nu}p_{4\mu} - 4m_\mathrm{f}^2 g_{\mu\nu}\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{\text{spins}}|\mathcal{M}_{fi}|^2 &= \frac{16\,Q_\mathrm{f}^2e^4}{q^4}\left(p_1^\mu p_2^\nu - g^{\mu\nu}(p_1\cdot p_2) + p_1^\nu p_2^\mu - m_\mathrm{e}^2g^{\mu\nu}\right)\times\\
&\qquad\left(p_{3\mu}p_{4\nu} - g_{\mu\nu}(p_3\cdot p_4) + p_{3\nu}p_{4\mu} - m_\mathrm{f}^2 g_{\mu\nu}\right)\\
&= \frac{16\,Q_\mathrm{f}^2e^4}{q^4}\Big[+(p_1\cdot p_4)(p_2\cdot p_3) - (p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_3)(p_2\cdot p_4) - m_\mathrm{f}^2(p_1\cdot p_2)\\
&\qquad - (p_1\cdot p_2)(p_3\cdot p_4) + 4(p_1\cdot p_2)(p_3\cdot p_4) - (p_1\cdot p_2)(p_3\cdot p_4) + 4m_\mathrm{f}^2(p_1\cdot p_2)\\
&\qquad + (p_1\cdot p_3)(p_2\cdot p_4) - (p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) - m_\mathrm{f}^2(p_1\cdot p_2)\\
&\qquad - m_\mathrm{e}^2(p_3\cdot p_4) + 4m_\mathrm{e}^2(p_3\cdot p_4) - m_\mathrm{e}^2(p_3\cdot p_4) + 4m_\mathrm{e}^2m_\mathrm{f}^2\Big]\,,
\end{aligned}$$

where all sixteen terms are shown for completeness. From this expression, it follows that

$$\begin{aligned}
\langle|\mathcal{M}_{fi}|^2\rangle = \frac{8Q_\mathrm{f}^2e^4}{(p_1+p_2)^4}\times\Big[\,&(p_1\cdot p_4)(p_2\cdot p_3) + (p_1\cdot p_3)(p_2\cdot p_4)\\
&+m_\mathrm{e}^2(p_3\cdot p_4) + m_\mathrm{f}^2(p_1\cdot p_1) + 2m_\mathrm{e}^2m_\mathrm{f}^2\Big]\,.
\end{aligned}$$

The same result could have been obtained by applying crossing symmetry,

$$p_1 \to p_1\,,\quad p_2 \to -p_4\,,\quad p_3 \to -p_2\quad\text{and}\quad p_4 \to p_3\,.$$

to the corresponding t-channel matrix element of (6.67) in the main text.

6.11* Neglecting the electron mass term, verify that the matrix element for $\mathrm{e^-f \to e^-f}$ given in (6.67) can be obtained from the matrix element for $\mathrm{e^+e^- \to f\bar{f}}$ given in (6.63) using crossing symmetry with the substitutions

$$p_1 \to p_1\,,\quad p_2 \to -p_3\,,\quad p_3 \to p_4\quad\text{and}\quad p_4 \to -p_2\,.$$

The spin averaged matrix element squared for the s-channel process $\mathrm{e^+e^- \to f\bar{f}}$ is given in (6.63) of the main text:

$$\langle|\mathcal{M}_{fi}|^2\rangle_s = 2\frac{Q_\mathrm{f}^2e^4}{(p_1\cdot p_2)^2}\left[(p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) + m_\mathrm{f}^2(p_1\cdot p_2)\right]\,.$$

Making the replacements

$$p_1 \to p_1\,, \quad p_2 \to -p_3\,, \quad p_3 \to p_4 \quad \text{and} \quad p_4 \to -p_2\,.$$

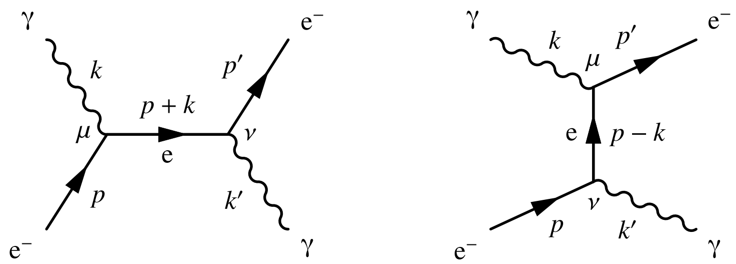
gives the corresponding matrix element for the t-channel scattering process:

$$\begin{aligned}\langle|\mathcal{M}_{fi}|^2\rangle_t &= 2\frac{Q_\mathrm{f}^2 e^4}{(-p_1\cdot p_3)^2}\left[(p_1\cdot p_4)(-p_3\cdot -p_2) + (p_1\cdot -p_2)(-p_3\cdot p_4) + m_\mathrm{f}^2(p_1\cdot -p_3)\right] \\ &= 2\frac{Q_\mathrm{f}^2 e^4}{(p_1\cdot p_3)^2}\left[(p_1\cdot p_4)(p_2\cdot p_3) + (p_1\cdot p_2)(p_3\cdot p_4) - m_\mathrm{f}^2(p_1\cdot p_3)\right]\,.\end{aligned}$$

This can be compared to the matrix element for $\mathrm{e^-f} \to \mathrm{e^-f}$ calculated using the trace formalism, i.e. equation (6.67) of the main text, taking $m_\mathrm{e} \approx 0$.

6.12* Write down the matrix elements, $\mathcal{M}_1$ and $\mathcal{M}_2$, for the two lowest-order Feynman diagrams for the Compton scattering process $\mathrm{e^-\gamma} \to \mathrm{e^-\gamma}$. From first principles, express the spin-averaged matrix element $\langle|\mathcal{M}_1 + \mathcal{M}_2|^2\rangle$ as a trace. You will need the completeness relation for the photon polarisation states (see Appendix D).

The two lowest-order Feynman diagrams for the Compton scattering process $\mathrm{e}^-(p)+\gamma(k) \to \mathrm{e}^- p' + \gamma(k')$ are shown below. In both diagrams the vertex with the incoming photon is labelled μ.



From the QED Feynman rules, the matrix element for the s-channel diagram is given by

$$\begin{aligned}-i\mathcal{M}_s &= \varepsilon_\mu^*(k)\varepsilon_\nu(k')\,\overline{u}(p')\left[ie\gamma^\nu\, i\frac{q^\rho\gamma_\rho + m}{q^2 - m_\mathrm{e}^2}\, ie\gamma^\mu\right]u(p) \\ \mathcal{M}_s &= -e^2\varepsilon_\mu^*(k)\varepsilon_\nu(k')\,\overline{u}(p')\left[\gamma^\nu\,\frac{\not{p} + \not{k} + m}{(p+k)^2 - m_\mathrm{e}^2}\gamma^\mu\right]u(p)\,,\end{aligned}$$

where $q = k + p$ and the slashed notation has been used. Similarly, the matrix element for the second diagram is

$$\mathcal{M}_t = -e^2\varepsilon_\mu^*(k)\varepsilon_\nu(k')\,\overline{u}(p')\left[\gamma^\mu\,\frac{\not{p} - \not{k} + m}{(p-k)^2 - m_\mathrm{e}^2}\gamma^\nu\right]u(p)\,.$$

Therefore the total amplitude, $\mathcal{M} = \mathcal{M}_s + \mathcal{M}_t$, is given by

$$\mathcal{M} = -e^2 \varepsilon^*_\mu(k)\varepsilon_\nu(k') \left\{\bar{u}(p')\left[\gamma^\nu \frac{\not{p} + \not{k} + m}{(p+k)^2 - m_\mathrm{e}^2}\gamma^\mu + \gamma^\mu \frac{\not{p} - \not{k} + m}{(p-k)^2 - m_\mathrm{e}^2}\gamma^\nu\right] u(p)\right\} .$$

The matrix element squared is given by $\mathcal{M}\mathcal{M}^\dagger$ where $\mathcal{M}^\dagger$ is given by

$$\mathcal{M}^\dagger = -e^2 \varepsilon_\rho(k)\varepsilon^*_\sigma(k') \left\{\bar{u}(p')\left[\gamma^\sigma \frac{\not{p} + \not{k} + m}{(p+k)^2 - m_\mathrm{e}^2}\gamma^\rho + \gamma^\rho \frac{\not{p} - \not{k} + m}{(p-k)^2 - m_\mathrm{e}^2}\gamma^\sigma\right] u(p)\right\}^\dagger .$$

Hence $|\mathcal{M}|^2 = \mathcal{M}\mathcal{M}^\dagger$ is given by

$$|\mathcal{M}|^2 = \mathcal{M}\mathcal{M}^\dagger = e^4\, \varepsilon^{\lambda *}_\mu(k)\varepsilon^{\lambda'}_\nu(k')\varepsilon^{\lambda}_\rho(k)\varepsilon^{\lambda' *}_\sigma(k') \Big\{ \quad \Big\}\Big\{ \quad \Big\}^\dagger ,$$

where the photon polarisation states are now made explicit. There are two possible initial-state photon polarisations and two possible spin states for the initial-state electron. Hence the spin-averaged matrix element is give by

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{\lambda=1,2}\sum_{\lambda'=1,2}\sum_{r=1,2}\sum_{r'=1,2} e^4\, \varepsilon^{\lambda *}_\mu(k)\varepsilon^{\lambda'}_\nu(k')\varepsilon^{\lambda}_\rho(k)\varepsilon^{\lambda' *}_\sigma(k') \Big\{ \quad \Big\}\Big\{ \quad \Big\}^\dagger ,$$

where r and r' are the intial- and final-state electron spins. From the completeness relation for photons (see Appendix D),

$$\sum_{\lambda=1,2} \varepsilon^{\lambda *}_\mu(k)\varepsilon^{\lambda}_\rho(k) = -g_{\mu\rho} ,$$

and thus

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{e^4}{4} g_{\mu\rho}g_{\nu\sigma} \sum_{r=1,2}\sum_{r'=1,2} \Big\{ \quad \Big\}\Big\{ \quad \Big\}^\dagger \\
&= \frac{e^4}{4} g_{\mu\rho}g_{\nu\sigma} \sum_{r=1,2}\sum_{r'=1,2} \left\{\bar{u}_{r'}(p')\left[\gamma^\nu \frac{\not{p} + \not{k} + m}{(p+k)^2 - m_\mathrm{e}^2}\gamma^\mu + \gamma^\mu \frac{\not{p} - \not{k} + m}{(p-k)^2 - m_\mathrm{e}^2}\gamma^\nu\right] u_r(p)\right\} \\
&\quad \times \left\{\bar{u}_{r'}(p')\left[\gamma^\sigma \frac{\not{p} + \not{k} + m}{(p+k)^2 - m_\mathrm{e}^2}\gamma^\rho + \gamma^\rho \frac{\not{p} - \not{k} + m}{(p-k)^2 - m_\mathrm{e}^2}\gamma^\sigma\right] u_r(p)\right\}^\dagger \\
&= \frac{e^4}{4} \sum_{r=1,2}\sum_{r'=1,2} \left\{\bar{u}_{r'}(p')\left[\gamma^\nu \frac{\not{p} + \not{k} + m}{(p+k)^2 - m_\mathrm{e}^2}\gamma^\mu + \gamma^\mu \frac{\not{p} - \not{k} + m}{(p-k)^2 - m_\mathrm{e}^2}\gamma^\nu\right] u_r(p)\right\} \\
&\quad \times \left\{\bar{u}_{r'}(p')\left[\gamma_\nu \frac{\not{p} + \not{k} + m}{(p+k)^2 - m_\mathrm{e}^2}\gamma_\mu + \gamma_\mu \frac{\not{p} - \not{k} + m}{(p-k)^2 - m_\mathrm{e}^2}\gamma_\nu\right] u_r(p)\right\}^\dagger \\
&= \frac{e^4}{4} \sum_{r=1,2}\sum_{r'=1,2} \left\{\bar{u}_{r'}(p')\left[\gamma^\nu\,\Gamma_+\gamma^\mu + \gamma^\mu\,\Gamma_-\gamma^\nu\right] u_r(p)\right\} \times \left\{\bar{u}_{r'}(p')\left[\gamma_\nu\Gamma_+\gamma_\mu + \gamma_\mu\,\Gamma_-\gamma_\nu\right] u_r(p)\right\}^\dagger ,
\end{aligned}$$

where

$$\Gamma_\pm = \frac{\not{p} + \not{k} + m}{(p+k)^2 - m_\mathrm{e}^2} .$$

7 Electron-Proton Elastic Scattering

7.1 The derivation of (7.8) used the algebraic relation

$$(\gamma+1)^2(1-\kappa^2)^2 = 4\,,$$

where

$$\kappa = \frac{\beta\gamma}{\gamma+1} \quad \text{and} \quad (1-\beta^2)\gamma^2 = 1\,.$$

Show that this holds.

Using the expression for κ

$$(\gamma+1)(1-\kappa^2) = (\gamma+1) - \frac{\beta^2\gamma^2}{(\gamma+1)}\,,$$

and from the definition of the Lorentz factor,

$$\beta^2\gamma^2 = \gamma^2 - 1 = (\gamma+1)(\gamma-1)\,,$$

and therefore

$$(\gamma+1)(1-\kappa^2) = (\gamma+1) - (\gamma-1) = 2\,.$$

7.2 By considering momentum and energy conservation in e^-p elastic scattering from a proton at rest, find an expression for the fractional energy loss of the scattered electron $(E_1 - E_3)/E_1$ in terms of the scattering angle and the parameter

$$\kappa = \frac{\mathrm{p}}{E_1 + m_e} \equiv \frac{\beta\gamma}{\gamma+1}\,.$$

This question should be ignored - finding a general solution is non-trivial and involves a lot of uninteresting algebra.

The four-momentum of the initial-state proton in the laboratory frame is $p_2 = (m_p, 0, 0, 0)$ and from conservation of four-momentum and the four-momentum of the final-state proton is

$$
\begin{aligned}
p_4 &= (p_1 + p_2 - p_3) \\
\Rightarrow \qquad p_4^2 &= p_1^2 + p_2^2 + p_3^2 + 2p_1\cdot p_2 - 2p_1\cdot p_3 - 2p_2\cdot p_3 \\
\Rightarrow \qquad m_\mathrm{p}^2 &= 2m_\mathrm{e}^2 + m_\mathrm{p}^2 + 2m_\mathrm{p}(E_1 - E_3) - 2(E_1E_3 - \mathrm{p}_1\mathrm{p}_3\cos\theta) \\
\Rightarrow \qquad m_\mathrm{p}(E_1 - E_3) &= (E_1E_3 - \mathrm{p}_1\mathrm{p}_3\cos\theta) - m_\mathrm{e}^2 \\
&= E_1E_3(1 - \beta_1\beta_3\cos\theta) - m_\mathrm{e}^2\,.
\end{aligned}
$$

Assuming the electron is sufficiently relativistic that $\beta_1 \approx \beta_3 \approx 1$, then

$$
\begin{aligned}
m_\mathrm{p}(E_1 - E_3) &\approx E_1E_3(1 - \cos\theta) \\
\Rightarrow \qquad E_3 &= \frac{E_1 m_\mathrm{p}}{m_\mathrm{p} + E_1(1 - \cos\theta)} \\
\Rightarrow \qquad \frac{E_1 - E_3}{E_1} &= \frac{1 - \cos\theta}{m_\mathrm{p}/E_1 + (1 - \cos\theta)}\,.
\end{aligned}
$$

7.3 In an $\mathrm{e}^-\mathrm{p}$ scattering experiment, the incident electron has energy $E_1 = 529.5$ MeV and the scattered electrons are detected at an angle of $\theta = 75°$ relative to the incoming beam.

a) At this angle, almost all of the scattered electrons are measured to have an energy of $E_3 \approx 373$ MeV. What can be concluded from this observation?

b) Find the corresponding value of Q^2.

a) If the process corresponds to the elastic scatter of an electron from the proton one would expect, (7.31),

$$
E_3 = \frac{E_1 m_\mathrm{p}}{m_\mathrm{p} + E_1(1 - \cos\theta)}\,.
$$

For $E_1 = 529.5$ MeV and $\theta = 75°$, elastically scattered electrons would have an energy of 373.3 GeV, consistent with the observed value. Hence the scattering is predominantly elastic in nature.

b) For elastic scattering, Equation (7.32), gives

$$
\begin{aligned}
Q^2 &= \frac{2m_\mathrm{p}E_1^2(1 - \cos\theta)}{m_\mathrm{p} + E_1(1 - \cos\theta)} \\
Q &= 541.3\,\mathrm{MeV}\,.
\end{aligned}
$$

7.4 For a spherically symmetric charge distribution $\rho(\mathrm{r})$, where

$$
\int \rho(\mathrm{r})\,\mathrm{d}^3\mathbf{r} = 1\,,
$$

show that the form factor can be expressed as

$$F(\mathbf{q}^2) = \frac{4\pi}{\mathrm{q}} \int_0^\infty \mathrm{r}\sin(\mathrm{qr})\rho(\mathrm{r})\,\mathrm{dr}\,,$$
$$\simeq 1 - \frac{1}{6}\mathrm{q}^2\langle \mathrm{R}^2\rangle + \ldots\,,$$

where $\langle \mathrm{R}^2\rangle$ is the mean square charge radius. Hence show that

$$\langle \mathrm{R}^2\rangle = -6\left[\frac{\mathrm{d}F(\mathbf{q}^2)}{\mathrm{dq}^2}\right]_{\mathrm{q}^2=0}\,.$$

Starting from the definition of the form factor and measuring the angle θ with respect to the (arbitrary) z axis:

$$\begin{aligned}
F(\mathbf{q}^2) &= \int \rho(\mathbf{r})e^{i\mathbf{q}\cdot\mathbf{r}}\,\mathrm{d}^3\mathbf{r}\\
&= \int\int\int \rho(\mathbf{r}))e^{i\mathrm{qr}\cos\theta}\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{dr}\\
&= 2\pi\int_{\mathrm{r}=0}^\infty \frac{1}{i\mathrm{q}}\left[e^{i\mathrm{qr}\cos\theta}\right]_0^\pi \mathrm{dr}\\
&= 2\pi\int_{\mathrm{r}=0}^\infty \frac{r}{i\mathrm{qr}}\left[e^{-i\mathrm{qr}} - e^{+i\mathrm{qr}}\right] \mathrm{dr}\\
&= \frac{4\pi}{\mathrm{q}}\int_0^\infty \mathrm{r}\sin(\mathrm{qr})\rho(\mathrm{r})\,\mathrm{dr}\,.
\end{aligned}$$

Expanding the sine gives, $\sin\mathrm{qr} \simeq \mathrm{qr} - \frac{1}{3!}(\mathrm{qr})^3 + \ldots$, and therefore

$$\begin{aligned}
F(\mathbf{q}^2) &= \frac{4\pi}{\mathrm{q}}\int_0^\infty \mathrm{r}\sin(\mathrm{qr})\rho(\mathrm{r})\,\mathrm{dr}\\
&= \frac{4\pi}{\mathrm{q}}\int_0^\infty \mathrm{q}\rho(\mathrm{r})\left(\mathrm{qr} - \frac{1}{6}(\mathrm{qr})^3 + \ldots\right)\mathrm{dr}\\
&= 1 - \frac{1}{6}\mathrm{q}^2\langle \mathrm{R}^2\rangle + \ldots\,,
\end{aligned}$$

where the following relations were used

$$\int 4\pi\mathrm{r}^2\rho(\mathrm{r})\,\mathrm{dr} = 1 \quad \text{and} \quad \int 4\pi\mathrm{r}^2\mathrm{r}^2\rho(\mathrm{r})\,\mathrm{dr} = \langle R^2\rangle\,.$$

Differentiating with respect to q^2 and noting that the higher order terms (higher powers of q) vanish at $\mathrm{q}^2 = 0$ gives the desired result of

$$\langle \mathrm{R}^2\rangle = -6\left[\frac{\mathrm{d}F(\mathbf{q}^2)}{\mathrm{dq}^2}\right]_{\mathrm{q}^2=0}\,.$$

7.5 Using the answer to the previous question and the data in Figure 7.8a, estimate the root-mean-squared charge radius of the proton.

From Figure 7.8a the gradient at $Q^2 = 0$ is approximately,

$$\langle \mathrm{R}^2 \rangle = 6/0.4\,\mathrm{GeV}^{-2}$$
$$\langle \mathrm{R}^2 \rangle^{\frac{1}{2}} = 3.9\,\mathrm{GeV}^{-1}\,.$$

Converting into S.I. units using $\hbar c = 0.197$ GeV fm gives a value for the rms charge radius of the proton of approximately 0.8 fm.

7.6 From the slope and intercept of the right plot of Figure 7.7, obtain values for $G_M(0.292\,\mathrm{GeV}^2)$ and $G_E(0.292\,\mathrm{GeV}^2)$.

The form factors can be obtained from

$$m = 2\tau\left[G_M(Q^2)\right]^2 \quad \text{and} \quad c = \frac{\left[G_E(Q^2)\right]^2 + \tau\left[G_M(Q^2)\right]^2}{(1+\tau)}\,,$$

where $\tau = Q^2/4m_\mathrm{p}^2 = 0.083$. Here the intercept and slope are $c \simeq 0.38$ and $c \simeq 0.27$ giving

$$G_M(Q^2 = 0.292\,\mathrm{GeV}^2) = 1.26 \quad \text{and} \quad G_E(Q^2 = 0.292\,\mathrm{GeV}^2) = 0.52\,.$$

7.7 Use the data of Figure 7.7 to estimate $G_E(Q^2)$ at $Q^2 = 0.500\,\mathrm{GeV}^2$.

This slightly fiddly data-analysis question illustrates how the measurements of the electric and magnetic form factors are determined from the raw data. The exact answers obtained will depend on how the interpolation between different data points is performed. The cross section values corresponding to $Q^2 = 500\,\mathrm{MeV}^2$ can be found from Equation (7.32) which can be rearranged to give a quadratic equation in E_1

$$2m_\mathrm{p}(1-\cos\theta)E_1^2 - Q^1(1-\cos\theta)E_1 - m_\mathrm{p}Q^2 = 0\,.$$

Hence for each of the values of θ, shown in the plot, the corresponding value of E_1 for $Q^2 = 500\,\mathrm{MeV}^2$ can be obtained, enabling the cross sections to be read off from the lines, these can then be compared to the expected Mott cross section for a point-like charge. The numbers obtained should be close to those given in the table below.

The plot of the ratio of the measured (interpolated) cross section to $\mathrm{d}\sigma/\mathrm{d}\Omega_0$ plotted against $\tan^2(\theta/2)$ is approximately linear with an intercept of $c = 0.27$ and gradient of $m = 0.28$. The form factors can be obtained from

$$m = 2\tau\left[G_M(Q^2)\right]^2 \quad \text{and} \quad c = \frac{\left[G_E(Q^2)\right]^2 + \tau\left[G_M(Q^2)\right]^2}{(1+\tau)}\,,$$

θ	E_1/MeV	$d\sigma/d\Omega$	$d\sigma/d\Omega_0$	ratio	$\tan^2(\theta/2)$
45°	1066	$\sim 4 \times 10^{-32}$ cm^2/sterad.	1.3×10^{-31} cm^2/sterad.	0.29	0.17
60°	853	$\sim 2 \times 10^{-32}$ cm^2/sterad.	5.9×10^{-32} cm^2/sterad.	0.34	0.33
75°	729	$\sim 1 \times 10^{-32}$ cm^2/sterad.	2.8×10^{-32} cm^2/sterad.	0.36	0.59
90°	650	$\sim 8 \times 10^{-33}$ cm^2/sterad.	1.4×10^{-32} cm^2/sterad.	0.55	0.99
120°	560	$\sim 4 \times 10^{-33}$ cm^2/sterad.	3.8×10^{-33} cm^2/sterad.	1.04	2.99
135°	538	$\sim 3 \times 10^{-33}$ cm^2/sterad.	1.8×10^{-33} cm^2/sterad.	1.65	5.99

where $\tau = Q^2/4m_p^2 = 0.142$. Using these values

$$G_M(Q^2 = 0.5\,\text{GeV}^2) = 0.99 \quad \text{and} \quad G_E(Q^2 = 0.5\,\text{GeV}^2) = 0.41\,,$$

roughly in the expected ratio of 2.79.

7.8 The experimental data of Figure 7.8 can be described by the form factor,

$$G(Q^2) = \frac{G(0)}{(1 + Q^2/Q_0^2)^2}\,,$$

with $Q_0 = 0.71$ GeV. Taking $Q^2 \approx \mathbf{q}^2$, show that this implies that proton has an exponential charge distribution of the form

$$\rho(\mathbf{r}) = \rho_0 e^{-r/a}\,,$$

and find the value of a.

For the exponential charge distribution $\rho(r) = \rho_0 e^{-\lambda r}$:

$$\begin{aligned} G(q^2) &= \frac{4\pi\rho_0}{q}\int_0^\infty r e^{-\lambda r}\sin(qr)\,dr \\ &= \frac{4\pi\rho_0}{q}\frac{1}{2i}\int_0^\infty r\left[e^{-\lambda r + iqr} - e^{-\lambda r - iqr}\right]dr \end{aligned}$$

Integration by parts gives

$$\int_0^\infty r e^{-\alpha r}\,dr = \frac{1}{\alpha^2}$$

for any constant α, so that

$$G(q^2) = \frac{2\pi\rho_0}{iq}\left[\frac{1}{(\lambda - iq)^2} - \frac{1}{(\lambda + iq)^2}\right] = \frac{8\pi\lambda\rho_0}{(\lambda^2 + q^2)^2} = \frac{8\pi\rho_0}{\lambda^3}\frac{1}{(1 + q^2/\lambda^2)^2}\,.$$

Thus the form factor is of the required “dipole” form:

$$G(q^2) = \frac{G(0)}{(1 + |q^2|/0.71)^2}$$

with $G(0) = 8\pi\rho_0/\lambda^3$ and $\lambda = Q_0$. Therefore

$$1/a = \lambda = \sqrt{0.71\,\mathrm{GeV}^2} = 0.84\,\mathrm{GeV}\,.$$

8 Deep Inelastic Scattering

8.1 Use the data in Figure 8.2 to estimate the lifetime of the Δ^+ baryon.

The full-width-half-maximum height of the resonance (the total decay width) is (in natural units) equivalent to the total decay rate, Γ. From the diagram, the Δ^+ peak has a FWHM of

$$\Gamma \approx 1\,\mathrm{GeV}\,.$$

Hence the lifetime is

$$\tau = 1/\Gamma \approx 1\,\mathrm{GeV}^{-1} \equiv \frac{\hbar}{1.6\times 10^{-10}}\,\mathrm{s} = 6.6\times 10^{-25}\,\mathrm{s}\,.$$

8.2 In fixed-target electron-proton *elastic* scattering

$$Q^2 = 2m_\mathrm{p}(E_1 - E_3) = 2m_\mathrm{p}E_1 y \quad \text{and} \quad Q^2 = 4E_1E_3\sin^2(\theta/2)\,.$$

a) Use these relations to show that

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{E_1}{E_3}\frac{m_\mathrm{p}^2}{Q^2}y^2 \quad \text{and hence} \quad \frac{E_3}{E_1}\cos^2\left(\frac{\theta}{2}\right) = 1 - y - \frac{m_\mathrm{p}^2 y^2}{Q^2}\,.$$

b) Assuming azimuthal symmetry and using equations (7.31) and (7.32), show that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} = \left|\frac{\mathrm{d}\Omega}{\mathrm{d}Q^2}\right|\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\pi}{E_3^2}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,.$$

c) Using the results of a) and b) show that the Rosenbluth equation,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4E_1^2\sin^4(\theta/2)}\frac{E_3}{E_1}\left(\frac{G_E^2 + \tau G_M^2}{(1+\tau)}\cos^2\frac{\theta}{2} + 2\tau G_M^2\sin^2\frac{\theta}{2}\right),$$

can be written in the Lorentz invariant form

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[\frac{G_E^2 + \tau G_M^2}{(1+\tau)}\left(1 - y - \frac{m_\mathrm{p}^2 y^2}{Q^2}\right) + \frac{1}{2}y^2 G_M^2\right].$$

a) From second expression given at the start of the question

$$\sin^2(\theta/2) = \frac{E_1}{E_3}\frac{Q^2}{4E_1^2}\,,$$

and from the first expression $E1 = Q^2/2m_p y$ and this,

$$\sin^2(\theta/2) = \frac{E_1}{E_3}\frac{m_p^2}{Q^2}y^2 .$$

Therefore

$$\cos^2(\theta/2) = 1 - \sin^2(\theta/2) = 1 - \frac{E_1}{E_3}\frac{m_p^2}{Q^2}y^2 ,$$

or equivalently

$$\frac{E_1}{E_3}\cos^2(\theta/2) = \frac{E_1}{E_3} - \frac{m_p^2}{Q^2}y^2 = 1 - y - \frac{m_p^2}{Q^2}y^2 ,$$

which is the desired result.

b) Assuming azimuthal symmetry an element of solid angle (integrated in ϕ) is just $\mathrm{d}\Omega = 2\pi \mathrm{d}(\cos\theta)$. Because there is only one independent variable in elastic scattering, the differential cross sections in terms of $\mathrm{d}Q^2$ and $\mathrm{d}\Omega$ are related by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} = \left|\frac{\mathrm{d}\Omega}{\mathrm{d}Q^2}\right|\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = 2\pi\left|\frac{\mathrm{d}(\cos\theta)}{\mathrm{d}Q^2}\right|\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} .$$

Equation (7.32) gives the expression for Q^2 in terms of the $\cos\theta$:

$$Q^2 = \frac{2m_p E_1^2(1-\cos\theta)}{m_p + E_1(1-\cos\theta)} , \tag{8.1}$$

which can be differentiated giving

$$\frac{\mathrm{d}Q^2}{\mathrm{d}(\cos\theta)} = \frac{-2m_p^2E_1^2}{\left[m_p + E_1(1-\cos\theta)\right]^2} . \tag{8.2}$$

and the denominator can be eliminated using (7.32), giving

$$\frac{\mathrm{d}Q^2}{\mathrm{d}(\cos\theta)} = -2E_3^2 . \tag{8.3}$$

Thus

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} = 2\pi\left|\frac{\mathrm{d}(\cos\theta)}{\mathrm{d}Q^2}\right|\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\pi}{E_3^2}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} .$$

c) The Lorenz invariant form of the Rosenbluth equation can be obtained by substituting the results from a) and b) into the original expression:

$$\begin{aligned}
\frac{d\sigma}{dQ^2} &= \frac{\pi}{E_3^2}\frac{d\sigma}{d\Omega} \\
&= \frac{\pi}{E_3^2}\frac{\alpha^2}{4E_1^2\sin^4(\theta/2)}\frac{E_3}{E_1}\left(\frac{G_E^2+\tau G_M^2}{(1+\tau)}\cos^2\frac{\theta}{2}+2\tau G_M^2\sin^2\frac{\theta}{2}\right) \\
&= \frac{\pi}{E_3^2}\frac{16E_1^2E_3^2\alpha^2}{4E_1^2Q^4}\left(\frac{G_E^2+\tau G_M^2}{(1+\tau)}\left(1-y-\frac{m_p^2y^2}{Q^2}\right)+2\tau G_M^2\frac{m_p^2}{Q^2}y^2\right) \\
&= \frac{4\pi\alpha^2}{Q^4}\left[\frac{G_E^2+\tau G_M^2}{(1+\tau)}\left(1-y-\frac{m_p^2y^2}{Q^2}\right)+\frac{1}{2}y^2G_M^2\right].
\end{aligned}$$

8.3 In fixed-target electron-proton inelastic scattering:

a) show that the laboratory frame differential cross section for deep-inelastic scattering is related to the Lorentz invariant differential cross section of equation (8) by

$$\frac{d^2\sigma}{dE_3\,d\Omega} = \frac{E_1E_3}{\pi}\frac{d^2\sigma}{dE_3\,dQ^2} = \frac{E_1E_3}{\pi}\frac{2m_px^2}{Q^2}\frac{d^2\sigma}{dx\,dQ^2},$$

where E_1 and E_3 are the energies of the incoming and outgoing electron.

b) Show that

$$\frac{2m_px^2}{Q^2}\cdot\frac{y^2}{2} = \frac{1}{m_p}\frac{E_3}{E_1}\sin^2\frac{\theta}{2} \quad \text{and} \quad 1-y-\frac{m_p^2x^2y^2}{Q^2} = \frac{E_3}{E_1}\cos^2\frac{\theta}{2}.$$

c) Hence, show that the Lorentz invariant cross section of equation (8) becomes

$$\frac{d^2\sigma}{dE_3\,d\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\left[\frac{F_2}{\nu}\cos^2\frac{\theta}{2}+\frac{2F_1}{m_p}\sin^2\frac{\theta}{2}\right].$$

d) A fixed-target ep scattering experiment consists of an electron beam of maximum energy 20 GeV and a variable angle spectrometer that can detect scattered electrons with energies greater than 2 GeV. Find the range of values of θ over which deep-inelastic scattering events can be studied at $x = 0.2$ and $Q^2 = 2\,\text{GeV}^2$.

a) Changing variables from $d\Omega = 2\pi d(\cos\theta)$ to

$$Q^2 = -q^2 = 2E_1E_3(1-\cos\theta)$$

gives

$$\frac{d^2\sigma}{dE_3\,d\Omega} = \frac{1}{2\pi}\frac{d^2\sigma}{dE_3\,d(\cos\theta)} = \frac{1}{2\pi}\left|\frac{dQ^2}{d(\cos\theta)}\right|\frac{d^2\sigma}{dE_3\,dQ^2} = \frac{1}{2\pi}2E_1E_3\frac{d^2\sigma}{dE_3\,dQ^2}$$

and hence (remembering that E_1 is the fixed initial-state electron energy)

$$\frac{d^2\sigma}{dE_3\,d\Omega} = \frac{E_1E_3}{\pi}\frac{d^2\sigma}{dE_3\,dQ^2} = \frac{E_1E_3}{\pi}\frac{d^2\sigma}{d\nu\,dQ^2}. \tag{8.4}$$

To change variables from ν to x, use

$$x = \frac{Q^2}{2m_\mathrm{p}\nu} \quad \Rightarrow \quad \nu = \frac{Q^2}{2m_\mathrm{p}x}$$

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}Q^2} = \left|\frac{\mathrm{d}\nu}{\mathrm{d}x}\right| \frac{\mathrm{d}^2\sigma}{\mathrm{d}\nu\,\mathrm{d}Q^2}$$

which gives directly

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}\nu\,\mathrm{d}Q^2} = \frac{2m_\mathrm{p}x^2}{Q^2}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}Q^2}$$

and thus

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\,\mathrm{d}\Omega} = \frac{E_1E_3}{\pi}\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\,\mathrm{d}Q^2} = \frac{E_1E_3}{\pi}\frac{2m_\mathrm{p}x^2}{Q^2}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}Q^2}\,.$$

b) Since

$$Q^2 = 4E_1E_3\sin^2(\theta/2)$$

and

$$y = \frac{\nu}{E_1}$$

then

$$\frac{E_3}{E_1}\sin^2(\theta/2) = \frac{Q^2}{4E_1^2} = \frac{Q^2y^2}{4\nu^2}\,.$$

Furthermore, using $\nu = Q^2/2Mx$, one then obtains

$$\frac{2m_\mathrm{p}x^2}{Q^2}\cdot\tfrac{1}{2}y^2 = \frac{1}{m_\mathrm{p}}\frac{E_3}{E_1}\sin^2\frac{\theta}{2}\,. \tag{8.5}$$

Hence

$$1 - y - \frac{m_\mathrm{p}^2x^2y^2}{Q^2} = \frac{E_3}{E_1}\cos^2\frac{\theta}{2}\,. \tag{8.6}$$

c) Starting from Equation (8).

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1 - y - \frac{m_\mathrm{p}^2x^2y^2}{Q^2}\right)\frac{F_2(x,Q^2)}{x} + y^2F_1(x,Q^2)\right],$$

and using the result of part a) gives

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\,\mathrm{d}\Omega} = \frac{E_1E_3}{\pi}\frac{2m_\mathrm{p}x^2}{Q^2}\frac{4\pi\alpha^2}{Q^4}\left[\left(1 - y - \frac{m_\mathrm{p}^2x^2y^2}{Q^2}\right)\frac{F_2(x,Q^2)}{x} + y^2F_1(x,Q^2)\right].$$

Using the results from part c) leads to

$$\begin{aligned}\frac{d^2\sigma}{dE_3\,d\Omega} &= \frac{8E_1E_3m_p x^2\alpha^2}{Q^6}\left[\left(\frac{E_3}{E_1}\cos^2\frac{\theta}{2}\right)\frac{F_2(x,Q^2)}{x} + \left(\frac{Q^2}{m_p^2x^2}\frac{E_3}{E_1}\sin^2\frac{\theta}{2}\right)F_1(x,Q^2)\right]\\ &= \frac{8E_3^2m_p\alpha^2}{Q^4}\left[\left(\frac{x}{Q^2}\cos^2\frac{\theta}{2}\right)F_2(x,Q^2) + \left(\frac{1}{m_p^2}\sin^2\frac{\theta}{2}\right)F_1(x,Q^2)\right].\end{aligned}$$

But $x/Q^2 = 1/(2m_p\nu)$ and $Q^2 = 4E_1E_3\sin^2(\theta/2)$ and therefore

$$\begin{aligned}\frac{d^2\sigma}{dE_3\,d\Omega} &= \frac{m_p\alpha^2}{2E_1^2\sin^4(\theta/2)}\left[\frac{1}{2m_p\nu}\cos^2\frac{\theta}{2}\,\frac{F_2(x,Q^2)}{x} + \frac{1}{m_p^2}\sin^2\frac{\theta}{2}\,F_1(x,Q^2)\right]\\ &= \frac{\alpha^2}{4E_1^2\sin^4(\theta/2)}\left[\frac{1}{\nu}\cos^2\frac{\theta}{2}\,\frac{F_2(x,Q^2)}{x} + \frac{2}{m_p}\sin^2\frac{\theta}{2}\,F_1(x,Q^2)\right],\end{aligned}$$

as required.

d) This is a tricky question, but is informative. It very much follows on from the previous discussion of the measurement of form factors in elastic scattering, except here there is an additional kinematic degree of freedom. Given that we wish to measure the structure functions at $x = 0.2$ and $Q^2 = 2\,\text{GeV}^2$, the electron energies E_1 and E_3 are constrained via

$$E_1 - E_3 = \frac{Q^2}{2Mx} = \frac{2\,\text{GeV}^2}{2\times(0.938\,\text{GeV})\times 0.2} = 5.33\,\text{GeV} \tag{8.7}$$

and

$$E_1E_3 = \frac{Q^2}{4\sin^2\theta/2}. \tag{8.8}$$

The experimental limitations, $E_1 < 20\,\text{GeV}$ and $E_3 > 2\,\text{GeV}$, then lead to constraints on the scattering angle θ. Here it helps to think in terms of graphical solutions of Equations (8.7) and (8.8) on a plot of E_3 versus E_1. Equation (8.7) corresponds to a straight line running at 45°, while Equation (8.8) gives an infinite set of hyperbolae, each hyperbola corresponding to a different possible value of θ.

The minimum possible value of θ corresponds to taking the maximum possible beam energy $E_1 = 20\,\text{GeV}$:

$$\sin^2\theta/2 = \frac{Q^2}{4E_1E_3} = \frac{2}{4\times 20\times(20-5.33)} = 1.70\times 10^{-3}$$

which gives

$$\theta_{\min} = 4.73^\circ.$$

The maximum possible value of θ is determined by the minimum detectable scattered electron energy of $E_3 = 2\,\mathrm{GeV}$:

$$\sin^2\theta/2 = \frac{Q^2}{4E_1E_3} = \frac{2}{4\times(2+5.33)\times 2} = 0.034$$

which gives

$$\theta_{\max} = 21.3^\circ\,.$$

Therefore, the experimental strategy is to choose several values of θ between approximately 5° and 20°, and for each angle, measure the *reduced* cross section,

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\,\mathrm{d}\Omega}\times\frac{4E_1^2\sin^4\theta/2}{\alpha^2\cos^2\theta/2} = \left[\frac{F_2}{\nu}+\frac{2F_1}{m_\mathrm{p}}\tan^2\frac{\theta}{2}\right],$$

and plot this versus $\tan^2\theta/2$. This should give a straight line (since ν is fixed here) with slope $2F_1/m_\mathrm{p}$ and intercept F_2/ν.

Each θ value requires a different beam energy given by solving the quadratic equation

$$E_1(E_1-5.33) = \frac{Q^2}{4\sin^2\theta/2}$$

This gives

$$2E_1 = 5.33+\sqrt{(5.33)^2+\frac{Q^2}{\sin^2\theta/2}}\,,$$

and thus $E_1 = 19.1\,\mathrm{GeV}$ for $\theta = 5^\circ$ and $E_1 = 7.5\,\mathrm{GeV}$ for $\theta = 20^\circ$. Note that $y = (E_1 - E_3)/E_1$ varies between 0.28 and 0.71, giving a significant contribution from F_1 (which is necessary for it to be determined accurately).

8.4 If quarks were spin-0 particles, why would $F_1^{\mathrm{ep}}(x)/F_2^{\mathrm{ep}}(x)$ be zero?

If quarks were spin-0 particles, there would be no magnetic contribution to this QED scattering process. Consequently $F_1^{\mathrm{ep}}(x)$, which is associated with the $\sin^2\theta/2$ angular dependence, would be zero.

8.5 What is the expected value of $\int_0^1 u(x)-\overline{u}(x)\,\mathrm{d}x$ for the proton?

Writing both $u(x)$ and $\overline{u}(x)$ in terms of sea and valance contributions, $u(x) = u_V(x)+u_S(x)$ and $\overline{u}(x) = \overline{u}_S(x)$ and making the very reasonable assumption that the sea contributions for quarks and antiquarks are the same,

$$\int_0^1 u(x)-\overline{u}(x)\,\mathrm{d}x = \int_0^1 u_V(x)\,\mathrm{d}x = 2\,.$$

8.6 Figure 8.1 shows the raw measurements of the structure function $F_2(x)$ in low-energy electron-deuterium scattering. When combined with the measurements of Figure 8.11, it is found that

$$\frac{\int_0^1 F_2^{\mathrm{eD}}(x)\,\mathrm{d}x}{\int_0^1 F_2^{\mathrm{ep}}(x)\,\mathrm{d}x} \simeq 0.84\,.$$

Write down the quark-parton model prediction for this ratio and determine the relative fraction of the momentum of proton carried by down/anti-down quarks compared to that carried by the up/anti-up quarks, $f_{\mathrm{d}}/f_{\mathrm{u}}$.

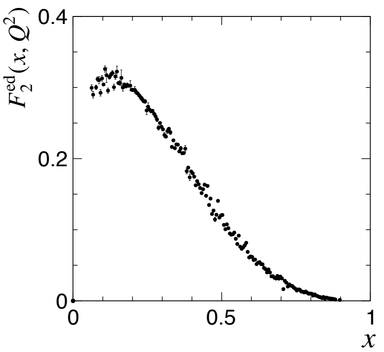


Fig. 8.1 SLAC measurements of $F_2^{\mathrm{eD}}(x, Q^2)$ in for $2 < Q^2/\,\mathrm{GeV}^2 < 30$.

The significance of this questions is that it relates to the actual early measurements of structure functions where the target was usually either liquid Hydrogen or Liquid Deuterium. From (8.27) and (8.28), which are written in terms of the PDFs of the *proton*,

$$F_2^{\mathrm{ep}}(x) = x\left(\frac{4}{9}u(x) + \frac{1}{9}d(x) + \frac{4}{9}\overline{u}(x) + \frac{1}{9}\overline{d}(x)\right),$$

$$F_2^{\mathrm{en}}(x) = x\left(\frac{4}{9}d(x) + \frac{1}{9}u(x) + \frac{4}{9}\overline{d}(x) + \frac{1}{9}\overline{u}(x)\right).$$

Therefore the structure function of the deuteron, with one proton and one neutrino, is given by

$$F_2^{\mathrm{eD}}(x) = \frac{1}{2}\left(F_2^{\mathrm{ep}}(x) + F_2^{\mathrm{en}}(x)\right) = \frac{5}{18}x\left(u(x) + d(x) + \overline{u}(x) + \overline{d}(x)\right).$$

Hence the integrals of the measured F_2 distributions give:

$$\int_0^1 F_2^{\mathrm{ep}}(x)\,\mathrm{d}x = \frac{4}{9}f_{\mathrm{u}} + \frac{1}{9}f_{\mathrm{d}} \quad \text{and} \quad \int_0^1 F_2^{\mathrm{eD}}(x)\,\mathrm{d}x = \frac{5}{18}\left(f_{\mathrm{d}} + f_{\mathrm{u}}\right),$$

where f_u and f_d are defined by

$$f_u = \int_0^1 [xu(x) + x\overline{u}(x)]\,dx \quad \text{and} \quad f_d = \int_0^1 [xd(x) + x\overline{d}(x)]\,dx\,.$$

Hence the ratio

$$\frac{\int_0^1 F_2^{eD}(x)\,dx}{\int_0^1 F_2^{ep}(x)\,dx} = 0.84 = \frac{5}{2}\frac{f_u + f_d}{4f_u + f_d}\,,$$

and thus

$$\frac{f_u + f_d}{4f_u + f_d} \simeq 0.336\,.$$

From this relation is straightforward to show that

$$f_d/f_u \simeq 0.52\,,$$

which is consistent with the result quoted in Chapter 8.

8.7 Including the contribution from strange quarks:
a) Show that $F_2^{ep}(x)$ can be written

$$F_2^{ep}(x) = \frac{4}{9}x\,[u(x) + \overline{u}(x)] + \frac{1}{9}x\left[d(x) + \overline{d}(x) + s(x) + \overline{s}(x)\right]\,,$$

where $s(x)$ and $\overline{s}(x)$ are the strange quark parton distribution functions of the proton.
b) Find the corresponding expression for $F_2^{en}(x)$ and show that

$$\int_0^1 \frac{\left[F_2^{ep}(x) - F_2^{en}(x)\right]}{x}dx \approx \frac{1}{3} + \frac{2}{3}\int_0^1 \left[\overline{u}(x) - \overline{d}(x)\right]dx\,,$$

and interpret the measured value of 0.24 ± 0.03.

a) Since strange quarks have charge $-1/3$, they couple to the photon in the QED deep inelastic scattering process in the same way as down quarks and therefore

$$F_2^{ep}(x) = \frac{4}{9}x\,[u(x) + \overline{u}(x)] + \frac{1}{9}x\left[d(x) + \overline{d}(x) + s(x) + \overline{s}(x)\right]\,,$$

b) Remembering that the PDFs in the above expression refer to the PDFs for the proton,

$$F_2^{en}(x) = \frac{4}{9}x\,[u^n(x) + \overline{u}^n(x)] + \frac{1}{9}x\left[d^n(x) + \overline{d}^n(x) + s^n(x) + \overline{s}^n(x)\right]\,.$$

Making the assumption that $u^p = d^n$, this can be written in terms of the proton PDFs as

$$F_2^{en}(x) = \frac{4}{9}x\left[d(x) + \overline{d}(x)\right] + \frac{1}{9}x\,[u(x) + \overline{u}(x) + s(x) + \overline{s}(x)]\,,$$

where it has been assumed that the strange quark PDFs from the proton and neutron are identical. This is a reasonable assumption since the strange quark content of the nucleons are from the sea.

Using the above expressions

$$\int_0^1 \frac{\left[F_2^{\mathrm{ep}}(x) - F_2^{\mathrm{en}}(x)\right]}{x} \mathrm{d}x = \frac{1}{3}\int_0^1 (u(x) - d(x) + \overline{u}(x) - \overline{d}(x))\,\mathrm{d}x\,.$$

Writing the quark PDFs in terms of valence and sea contributions, and assuming that the quark and antiquark sea contributions (which arise from gluon-splitting) are the same, i.e. $u_S(x) = \overline{u}(x)$ and $d_S(x) = \overline{d}(x)$, then

$$\begin{aligned}\int_0^1 \frac{\left[F_2^{\mathrm{ep}}(x) - F_2^{\mathrm{en}}(x)\right]}{x} \mathrm{d}x &= \frac{1}{3}\int_0^1 (u_V(x) - d_V(x) + u_S(x) - d_S(x) + \overline{u}(x) - \overline{d}(x))\,\mathrm{d}x \\ &= \frac{1}{3}\int_0^1 (u_V(x) - d_V(x) + 2\overline{u}(x) - 2\overline{d}(x))\,\mathrm{d}x \\ &= \frac{1}{3} + \frac{2}{3}\int_0^1 (\overline{u}(x) - \overline{d}(x))\,\mathrm{d}x\,,\end{aligned}$$

where the last step follows from there being two valance up quarks and one valence down quark. The measured value can therefore be interpreted as

$$\int_0^1 (\overline{u}(x) - \overline{d}(x))\,\mathrm{d}x = \frac{3}{2}\,[0.24 - 0.33 \pm 0.03] = -0.14 \pm 0.05\,,$$

demonstrating that there is a deficit of $\overline{\mathrm{u}}$ quarks relative to $\overline{\mathrm{d}}$ quarks in the proton, as can be seen in the global fit to a wide range of data shown in Figure 8.17.

8.8 At the HERA collider, electrons of energy $E_1 = 27.5$ GeV collided with protons of energy $E_2 = 820$ GeV. In deep inelastic scattering events at HERA, show that Bjorken x is given by

$$x = \frac{E_3}{E_2}\left[\frac{1 - \cos\theta}{2 - (E_3/E_1)(1 + \cos\theta)}\right],$$

where θ is the angle through which the electron has scattered and E_3 is the energy of the scattered electron. Estimate x and Q^2 for the event shown in Figure 8.13 assuming that the energy of the scattered electron is 250 GeV.

For $\mathrm{e^+p} \to \mathrm{e^+}X$ DIS at HERA, the masses of the electron and proton can be neglected and thus the four-momenta can be written:

$$p_1 = (E_1, 0, 0, E_1)\,, \quad p_2 = (E_2, 0, 0, -E_2) \quad \text{and} \quad p_3 = (E_3, E_3\sin\theta, 0, E_3\cos\theta)\,.$$

Using these four-momenta

$$q^2 = -2p_1.p_3 = -2E_1E_3(1 - \cos\theta)$$

and

$$p_2.q = p_2.p_1 - p_2.p_3 = 2E_1E_2 - E_2E_3(1 + \cos\theta)\,.$$

Thus, the Bjorken scaling variable x, defined as

$$x \equiv \frac{-q^2}{2p_2.q}\,,$$

can be written

$$x = \frac{E_3}{E_2}\left[\frac{1 - \cos\theta}{2 - (E_3/E_1)(1 + \cos\theta)}\right].$$

For the event shown in the text $\theta \approx 150°$. Thus

$$Q^2 = 2E_1E_3(1 - \cos\theta) = 2 \cdot 27.5 \cdot 250(1 - \cos\theta) \simeq 3 \times 10^4\,\mathrm{GeV}^2\,.$$

Similarly

$$x = \frac{250}{820}\left[\frac{1 - \cos\theta}{2 - (250/27.5)(1 + \cos\theta)}\right] \simeq 0.7\,.$$

Hence the DIS interaction shown is one of the highest Q^2 DIS interactions observed at HERA and has high x.

9 Symmetries and the Quark Model

9.1 By writing down the general term in the Binomial expansion of

$$\left(1 + i\frac{1}{n}\boldsymbol{\alpha}\cdot\hat{\mathbf{G}}\right)^n,$$

show that

$$\hat{U}(\boldsymbol{\alpha}) = \lim_{n\to\infty}\left(1 + i\frac{1}{n}\boldsymbol{\alpha}\cdot\hat{\mathbf{G}}\right)^n = \exp\left(i\boldsymbol{\alpha}\cdot\mathbf{G}\right).$$

For compactness, writing $x = i\boldsymbol{\alpha}\cdot\hat{\mathbf{G}}$, the required expression can be written

$$\left(1+\frac{x}{n}\right)^n = 1 + n\frac{x}{n} + \frac{1}{2!}n(n-1)\left(\frac{x}{n}\right)^2 + \frac{1}{3!}n(n-1)(n-2)\left(\frac{x}{n}\right)^3 + \ldots$$

Taking the limit as $n\to\infty$ terms such as $n(n-1)(n-2)/n^3 \to 1$ and thus

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = e^x,$$

and therefore

$$\hat{U}(\boldsymbol{\alpha}) = \lim_{n\to\infty}\left(1 + i\frac{1}{n}\boldsymbol{\alpha}\cdot\hat{\mathbf{G}}\right)^n = \exp\left(i\boldsymbol{\alpha}\cdot\mathbf{G}\right).$$

9.2 For an infinitesimal rotation about the z-axis through an angle ϵ show that

$$\hat{U} = 1 - i\epsilon\hat{J}_z,$$

where $\hat{J}_z$ is the angular momentum operator $\hat{J}_z = x\hat{p}_y - y\hat{p}_x$.

Here we need to distinguish between active and passive rotations. As written, the question refers to a rotation of the axes through a positive angle ϵ, it could be interpreted as a rotation of all the coordinates in which case the corresponding Unitary operator would be $\left(1 + i\epsilon\hat{J}_z\right)$. For a infinitesimal rotation of the x and y axes

$$x \to x' = x\cos\epsilon + y\sin\epsilon \simeq x - \epsilon y$$
$$y \to y' = y\cos\epsilon - x\sin\epsilon \simeq y + \epsilon x.$$

Under this coordinate transformation, wavefunctions transform as

$$\psi(x,y,z) \to \psi'(x,y,z) = \psi(x+\epsilon y, y-\epsilon x, z)$$
$$= \psi(x,y,z) + y\epsilon\frac{\partial\psi}{\partial x} - x\epsilon\frac{\partial\psi}{\partial y}\,.$$

The partial derivatives are related to the quantum mechanical momentum operators (in Natural Units) by

$$\frac{\partial}{\partial x} = i\hat{p}_x \quad \text{and} \quad \frac{\partial}{\partial y} = i\hat{p}_y\,,$$

and therefore

$$\begin{aligned}\psi(x,y,z) \to \psi'(x,y,z) &= \psi(x,y,z) + y\epsilon\frac{\partial\psi}{\partial x} - x\epsilon\frac{\partial\psi}{\partial y}\\ &= \psi(x,y,z) + i\epsilon(y\hat{p}_x - x\epsilon\hat{p}_y)\psi(x,y,z)\\ &= \left(1 - i\epsilon\hat{J}_z\right)\psi\,.\end{aligned}$$

Thus the corresponding Unitary operator for the transformation is

$$\hat{U} = 1 - i\epsilon\hat{J}_z\,.$$

9.3 By considering the isospin states, show that the rates for the following strong interaction decays occur in the ratios

$$\Gamma(\Delta^- \to \pi^-\mathrm{n}) : \Gamma(\Delta^0 \to \pi^-\mathrm{p}) : \Gamma(\Delta^0 \to \pi^0\mathrm{n}) : \Gamma(\Delta^+ \to \pi^+\mathrm{n}) :$$
$$\Gamma(\Delta^+ \to \pi^0\mathrm{p}) : \Gamma(\Delta^{++} \to \pi^+\mathrm{p}) = 3:1:2:1:2:3\,.$$

This is an interesting application of the use of isospin in strong interactions. We have asserted that SU(2) flavour symmetry is an exact symmetry of the strong interaction. One consequence is that isospin and the third component of isospin is conserved in strong interactions. Furthermore, from the point of view of the strong interaction the Δ^-, Δ^0, Δ^- and Δ^{++} are indistinguishable. The amplitudes for the above decays can be written as

$$\mathcal{M}(\Delta \to \pi\mathrm{N}) \sim \langle \pi\mathrm{N}|\hat{H}_{\mathrm{strong}}|\Delta\rangle\,,$$

which in the case of an exact SU(2) light quark flavour symmetry can be written as

$$\mathcal{M}(\Delta \to \pi\mathrm{N}) \sim A\langle \phi(\pi\mathrm{N})|\phi(\Delta)\rangle\,,$$

where A is a constant and ϕ represents the isospin wavefunctions. Here $\langle \phi(\pi\mathrm{N})|\phi(\Delta)\rangle$ expresses conservation of isospin in the interaction. The question therefore boils down to determining the isospin values for the states involved.

Consider the decay $\Delta^- \to \pi^- n$, which in terms of isospin states corresponds to

$$\phi\left(\tfrac{3}{2},-\tfrac{3}{2}\right) \to \phi\,(1,-1)\phi\left(\tfrac{1}{2},-\tfrac{1}{2}\right).$$

The decay rate will depend on the isospin of the combined $\pi^- n$ system. Since I_3 is an additive quantum number the third component of the combined $\pi^- n$ system is $-3/2$ and this implies that the total isospin must be *at least* 3/2. But since the total isospin lies between $|1-1/2| < I < |1+1/2|$, the isospin of the $\pi^- n$ system is uniquely identified as

$$\phi(\pi^- n) = \phi\,(1,-1)\phi\left(\tfrac{1}{2},-\tfrac{1}{2}\right) = \phi\left(\tfrac{3}{2},-\tfrac{3}{2}\right).$$

Consequently the amplitude for the decay is given by

$$\mathcal{M}(\Delta^- \to \pi^- n) \sim A\langle\phi(\pi^- n)\,|\,\phi(\Delta^-)\rangle = A\left\langle\phi\left(\tfrac{3}{2},-\tfrac{3}{2}\right)|\phi\left(\tfrac{3}{2},-\tfrac{3}{2}\right)\right\rangle = A\,.$$

Now consider the decays of the Δ^0 which the isospin state

$$\phi(\Delta^-) = \phi\left(\tfrac{3}{2},-\tfrac{1}{2}\right).$$

The decomposition of this state into the equivalent πN system (and isospin-1 state combined with an isospin-1/2 state) can be achieved using isospin ladder operators. Starting from the unique assignment

$$\phi\left(\tfrac{3}{2},-\tfrac{3}{2}\right) = \phi\,(1,-1)\phi\left(\tfrac{1}{2},-\tfrac{1}{2}\right),$$

the isospin raising operator gives

$$\begin{aligned}\hat{T}_+\phi\left(\tfrac{3}{2},-\tfrac{3}{2}\right) &= \left(\hat{T}_+\phi\,(1,-1)\right)\phi\left(\tfrac{1}{2},-\tfrac{1}{2}\right) + \phi\,(1,-1)\left(\hat{T}_+\phi\left(\tfrac{1}{2},-\tfrac{1}{2}\right)\right),\\ \phi\left(\tfrac{3}{2},-\tfrac{3}{2}\right) &= \sqrt{2}\phi\,(1,0)\phi\left(\tfrac{1}{2},-\tfrac{1}{2}\right) + \phi\,(1,-1)\phi\left(\tfrac{1}{2},+\tfrac{1}{2}\right),\end{aligned}$$

or equivalently

$$\phi(\Delta^0) = \sqrt{\tfrac{2}{3}}\phi(\pi^0 n) + \sqrt{\tfrac{1}{3}}\phi(\pi^- p)\,.$$

Thus the decay amplitudes of the Δ^0 can be written

$$\mathcal{M}(\Delta^0 \to \pi^0 n) \sim A\left\langle\phi(\pi^0 n)\,|\,\sqrt{\tfrac{2}{3}}\phi(\pi^0 n) + \sqrt{\tfrac{1}{3}}\phi(\pi^- p)\right\rangle = \sqrt{\tfrac{2}{3}}A\,,$$

and

$$\mathcal{M}(\Delta^0 \to \pi^- p) \sim A\left\langle\phi(\pi^- p)\,|\,\sqrt{\tfrac{2}{3}}\phi(\pi^0 n) + \sqrt{\tfrac{1}{3}}\phi(\pi^- p)\right\rangle = \sqrt{\tfrac{1}{3}}A\,.$$

Since decay rates are proportional to the amplitude squared

$$\Gamma(\Delta^- \to \pi^- n) : \Gamma(\Delta^0 \to \pi^- p) : \Gamma(\Delta^0 \to \pi^0 n) = 3:1:2\,.$$

The other ratios follow from the same arguments.

9.4 If quarks and antiquarks were spin-zero particles, what would be the multiplicity of the $L = 0$ multiplet(s). Remember that the overall wavefunction for bosons must be symmetric under particle exchange.

Since the colour quantum numbers of the quarks has nothing to do with spin, the colour singlet states are still

$$\frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b}) \qquad \text{and} \qquad \frac{1}{\sqrt{6}}(rgb - grb + gbr - bgr + brg - rbg)\,.$$

Hence, due to colour confinement, we still expect to see *mesons* containing a quark and an antiquark and *baryons* containing three quarks.

Mesons: Since the flavour quantum numbers of the quarks remain unchanged, the flavour wavefunctions for mesons retain their usual SU(3) form, and we would expect to see the usual flavour nonets. Since there is no spin degree of freedom, the would be a single nonet corresponding to each value of L. The overall parity of a two particle system in a state with orbital angular momentum L is

$$P = P_1.P_2.(-1)^L\,.$$

Spin 0 quarks would be bosons, and would have the same intrinsic parity, thus $P_1 = P_2$. Hence, for a meson formed from spin-0 quarks:

$$P(q\overline{q}) = (-1)^L\,.$$

Consequently one would expect to see nonets (with the total angular momentum equal to L) with

$$J^P = 0^+, 1^-, 2^+, 3^-, \ldots \quad ,$$

which is in contradiction to the observed meson states.

Baryons: for baryons, regarded as being built up from three "identical" spin-1/2 quarks, with appropriate colour, spin and flavour quantum numbers, the overall wavefunction is

$$\psi = \phi_{\text{flavour}}\,\chi_{\text{spin}}\,\xi_{\text{colour}}\,\eta_{\text{space}}\,.$$

For spin-1/2 quarks, i.e. fermions, ψ must be totally antisymmetric under interchange of any pair of quarks within the baryon. For baryons made from spin-0 quarks, the wavefunction would become just

$$\psi = \phi_{\text{flavour}}\,\xi_{\text{colour}}\,\eta_{\text{space}}\,.$$

and the overall wavefunction ψ would be totally *symmetric* under quark interchange since quarks are now bosons. For spin-0 quarks, the colour wavefunction is the usual SU(3) colour singlet, which is totally antisymmetric under interchange of any pair of quarks within the baryon. Hence,

$$\phi(\text{flavour})\,\eta(\text{space}) \quad \text{must now be totally } \textit{antisymmetric}\,.$$

For $L = 0$ baryons, ψ(space) is totally symmetric, so ψ(flavour) must be totally *antisymmetric*. The *only* totally antisymmetric flavour wavefunction which can be constructed out of the three flavours u, d and s is

$$\eta(\text{space}) = \frac{1}{\sqrt{6}}(uds - dus + dsu - sdu + sud - usd)\,.$$

Hence, for spin-0 quarks one would expect only a single $L = 0$ baryon state, with the flavour content uds. This baryon would have parity $P = +1. + 1. + 1.(-1)^0 = +1$ and total spin zero (since $L = 0$ and $S = 0$) giving a

$$J^P = 0^+ \quad \text{singlet}$$

as the lightest baryon multiplet.

9.5 The neutral vector mesons can decay leptonically through a virtual photon, for example by $V(q\bar{q}) \rightarrow \gamma \rightarrow e^+e^-$. The matrix element for this decay is proportional to $\langle\psi|\hat{Q}_q|\psi\rangle$, where ψ is the meson flavour wavefunction and $\hat{Q}_q$ is an operator that is proportional to the quark charge. Neglecting the relatively small differences in phase space, show that

$$\Gamma(\rho^0 \rightarrow e^+e^-) : \Gamma(\omega \rightarrow e^+e^-) : \Gamma(\phi \rightarrow e^+e^-) \approx 9 : 1 : 2\,.$$

The underlying process is the QED annihilation process $q\bar{q} \rightarrow e^+e^-$, where the matrix element can be expressed as

$$\mathcal{M}(q\bar{q} \rightarrow e^+e^-) \sim \langle e^+e^-|\hat{Q}_q|q\bar{q}\rangle = AQ_q\,,$$

where A is assumed to be a constant and Q_q is the charge of the annihilating quark-pair. For the ϕ which is a pure $s\bar{s}$ state, the matrix element

$$\mathcal{M}(\phi \rightarrow e^+e^-) \sim \langle e^+e^-|\hat{Q}_q|s\bar{s}\rangle = AQ_s = -\tfrac{1}{3}A\,.$$

For the ρ^0 with wavefunction $\left|\rho^0,=\right\rangle \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$, the phases of the two components are important and the total amplitude depends on the coherent sum of the contributions from the decays of the $u\bar{u}$ and $d\bar{d}$. Here the matrix element can be written

$$\mathcal{M}(\rho^0 \rightarrow e^+e^-) \sim \langle e^+e^-|\hat{Q}_q|\tfrac{1}{\sqrt{2}}(u\bar{u}-d\bar{d})\rangle = A\tfrac{1}{\sqrt{2}}\,(Q_u - Q_d) = A\tfrac{1}{\sqrt{2}}\left(\tfrac{2}{3} + \tfrac{1}{3}\right) = \tfrac{1}{\sqrt{2}}A\,.$$

Similarly,

$$\mathcal{M}(\omega \rightarrow e^+e^-) \sim \langle e^+e^-|\hat{Q}_q|\tfrac{1}{\sqrt{2}}(u\bar{u}+d\bar{d})\rangle = A\tfrac{1}{\sqrt{2}}\,(Q_u - Q_d) = A\tfrac{1}{\sqrt{2}}\left(\tfrac{2}{3} - \tfrac{1}{3}\right) = \tfrac{1}{3\sqrt{2}}A\,.$$

Neglecting the relatively small differences in phase space, $\Gamma \propto \mathcal{M}^2$ and thus

$$\Gamma(\rho^0 \rightarrow e^+e^-) : \Gamma(\omega \rightarrow e^+e^-) : \Gamma(\phi \rightarrow e^+e^-) \approx 9 : 1 : 2\,.$$

9.6 Using the meson mass formulae of (9.37) and (9.38), obtain predictions for the masses of the $\pi^\pm$, π^0, η, η', ρ^0, $\rho^\pm$, ω and ϕ. Compare the values obtained to the experimental values listed in Table 9.1.

This is a fairly straightforward question, using the meson mass formula

$$m(\mathrm{q}_1\overline{\mathrm{q}}_2) = m_1 + m_2 + \frac{A}{m_1 m_2}\langle \mathbf{S}_1 \cdot \mathbf{S}_2 \rangle\,, \tag{9.1}$$

the masses of the pseudoscalar and vector mesons are given respectively by (9.37) and (9.38):

$$\text{Pseudoscalar mesons (s=0):} \quad m_P = m_1 + m_2 - \frac{3A}{4m_1 m_2}\,,$$
$$\text{Vector mesons (s=1):} \quad m_V = m_1 + m_2 + \frac{A}{4m_1 m_2}\,,$$

where the constants can be taken to be

$$m_\mathrm{d} = m_\mathrm{u} = 0.307\,\mathrm{GeV}\,, \quad m_\mathrm{s} = 0.490\,\mathrm{GeV} \quad \text{and} \quad A = 0.06\,\mathrm{GeV}^3\,.$$

The only complication is how to handle states composed of more than one quark type, for example the η with

$$|\eta\rangle = \tfrac{1}{\sqrt{6}}(\mathrm{u}\overline{\mathrm{u}} + \mathrm{d}\overline{\mathrm{d}} - 2\mathrm{s}\overline{\mathrm{s}})\,.$$

As is often the case, it helps to think of this in terms of a meson mass operator for a $\mathrm{q}\overline{\mathrm{q}}$ system, such that the expectation value of the mass operator is the mass of the bound $\mathrm{q}\overline{\mathrm{q}}$ system

$$\langle \mathrm{q}_1\overline{\mathrm{q}}_2|\hat{M}_{\mathrm{q}\overline{\mathrm{q}}}|\mathrm{q}_1\overline{\mathrm{q}}_2\rangle = m(\mathrm{q}_1\overline{\mathrm{q}}_2)\,,$$

where $m(\mathrm{q}_1\overline{\mathrm{q}}_2)$ is given by (9.1). Stating the problem in this manner, one can write the mass of the η as

$$\begin{aligned} m_\eta &= \langle \eta|\hat{M}_{\mathrm{q}\overline{\mathrm{q}}}|\eta\rangle \\ &= \frac{1}{6}\langle(\mathrm{u}\overline{\mathrm{u}} + \mathrm{d}\overline{\mathrm{d}} - 2\mathrm{s}\overline{\mathrm{s}})|\,\hat{M}_{\mathrm{q}\overline{\mathrm{q}}}\,|(\mathrm{u}\overline{\mathrm{u}} + \mathrm{d}\overline{\mathrm{d}} - 2\mathrm{s}\overline{\mathrm{s}})\rangle \\ &= \frac{1}{6}\left[m(\mathrm{u},\overline{\mathrm{u}}) + m(\mathrm{d},\overline{\mathrm{d}}) + 4m(\mathrm{s},\overline{\mathrm{s}})\right]\,. \end{aligned}$$

Using the meson mass formula for the $S = 0$ pseudoscalar η:

$$m_\eta = \frac{1}{6}\left[2m_\mathrm{u} + 2m_\mathrm{d} + 8m_\mathrm{s} - \frac{3A}{4m_\mathrm{u}^2} - \frac{3A}{4m_\mathrm{d}^2} - 4\frac{3A}{4m_\mathrm{s}^2}\right] = 573\,\mathrm{MeV}\,,$$

in reasonable agreement with the observed mass of 548 MeV.

The meson mass formulae works well for all of the mesons in the question, with the exception of the η', where the prediction of approximately 350 MeV is very different from the measured mass of 958 MeV. However it should be noted that the η' is a flavour singlet state and in principle it could mix with flavourless purely gluonic bound states and given the special nature of the η', it is not surprising that the simple mass formula does not work.

9.7 Compare the experimentally measured values of the masses of the $J^P = \frac{3}{2}^+$ baryons, given in Table 9.2, with the predictions of (9.41). You will need to consider the combined spin of *any* two quarks in a spin-$\frac{3}{2}$ baryon state.

Any two of quarks in a spin-$\frac{3}{2}$ baryon, can be in a combined spin-0 or spin-1 qq state. If they are in a spin-0 state, the additional third quark will always result in a spin-$\frac{1}{2}$ qqq state. Hence, *any* two quarks in a spin-$\frac{3}{2}$ baryon must be in a spin-1 state. Consequently, for any two quarks

$$\begin{aligned}
& \mathbf{S} = \mathbf{S}_i + \mathbf{S}_j \\
\Rightarrow \quad & 2\mathbf{S}_i \cdot \mathbf{S}_j = \mathbf{S}^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2 \\
\Rightarrow \quad & 2\langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle = s(s+1) - s_1(s_1+1) - s_2(s_2) \\
& = 2 - \tfrac{3}{4} - \tfrac{3}{4} \\
\Rightarrow \quad & \langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle = \frac{1}{4}\,.
\end{aligned}$$

Thus the baryon mass formula of (9.41), when applied to the spin-$\frac{3}{2}$ decuplet becomes

$$m(\mathrm{q_1q_2q_3}) = m_1 + m_2 + m_3 + \frac{A}{4}\left(\frac{1}{m_1m_2} + \frac{1}{m_1m_3} + \frac{1}{m_2m_3}\right).$$

Taking the mass of the u and d quarks to be equal, the predicted masses of the decuplet baryons depend only on the number of strange quarks:

$$\begin{aligned}
\Delta \quad &: \quad m = 3m_\mathrm{u} + \frac{A}{4}\frac{3}{m_\mathrm{u}^2} \\
\Sigma^* \quad &: \quad m = 2m_\mathrm{u} + m_\mathrm{s} + \frac{A}{4}\left(\frac{1}{m_\mathrm{u}^2} + \frac{2}{m_\mathrm{u}m_\mathrm{s}}\right) \\
\Xi^* \quad &: \quad m = m_\mathrm{u} + 2m_\mathrm{s} + \frac{A}{4}\left(\frac{1}{m_\mathrm{s}^2} + \frac{2}{m_\mathrm{u}m_\mathrm{s}}\right) \\
\Omega \quad &: \quad m = 3m_\mathrm{s} + \frac{A}{4}\frac{3}{m_\mathrm{s}^2}
\end{aligned}$$

Using

$$m_\mathrm{d} = m_\mathrm{u} = 0.365\,\mathrm{GeV}\,, \quad m_\mathrm{s} = 0.540\,\mathrm{GeV} \quad \text{and} \quad A' = 0.026\,\mathrm{GeV}^3\,,$$

the predicted masses are: $m_\Delta = 1.241$ GeV, $m_{\Sigma^*} = 1.385$ GeV, $m_{\Xi^*} = 1.533$ GeV and $m_\Omega = 1.687$ GeV, which are in good agreement with the measured values.

9.8 **a)** Obtain the wavefunction for the Σ^0 and therefore find the wavefunction for the Λ.

b) Using (9.41), obtain predictions for the masses of the Σ^0 and the Λ baryons and compare these to the measured values.

a) This is a difficult question that requires some insight. If the SU(3) flavour symmetry were exact, the Λ(uds) and Σ^0(uds) baryons would have the same mass – they don't. The situation is similar to the that of the neutral mesons, where the quark flavour wavefunctions for the π^0 and η can be obtained from the operation of the ladder operators on the six states around the "edges" of the octet. The physical states are linear combinations of these states. How treat this ambiguity is not *a priori* obvious. Following the discussion of the light meson states, one expects that the u and d quarks in the uds baryon wavefunction obey an exact SU(2) flavour symmetry.

Starting from the wavefunction for the Σ^-(dds), which has the same form as that of the proton:

$$|\Sigma^-\uparrow\rangle \propto 2\mathrm{d}\!\uparrow\mathrm{d}\!\uparrow\mathrm{s}\!\downarrow - \mathrm{d}\!\uparrow\mathrm{d}\!\downarrow\mathrm{s}\!\uparrow - \mathrm{d}\!\downarrow\mathrm{d}\!\uparrow\mathrm{s}\!\uparrow + \text{ cyclic combinatorics}\,.$$

Note that in this wavefunction the d quarks appear in symmetric spin states. This must be the case since the flavour wavefunction of the dd is always symmetric and for the overall wavefunction to be antisymmetric under the interchange of the two d quarks, they must be in a symmetric spin state (since the colour wavefunction is anti-symmetric). Following the (almost) exact SU(2) light quark flavour symmetry, the wavefunction for the Σ^0 is expected to have the form:

$$|\Sigma^0\uparrow\rangle \propto 2\mathrm{d}\!\uparrow\mathrm{u}\!\uparrow\mathrm{s}\!\downarrow - \mathrm{d}\!\uparrow\mathrm{u}\!\downarrow\mathrm{s}\!\uparrow - \mathrm{u}\!\downarrow\mathrm{d}\!\uparrow\mathrm{s}\!\uparrow + \text{ cyclic combinatorics}\,.$$

The Λ wavefunction must be orthogonal to this state and satisfy the overall requirements of the approximate SU(3) flavour symmetry. We could construct this state by applying ladder operators to the states around the edge of the baryon octet. Alternatively, since in the Σ^0 state identified above, the u and d quarks are in flavour and spin symmetric states ($s_{\mathrm{ud}} = 1$) and orthogonal state can be written down by placing the u and d quarks in flavour and spin anti-symmetric states ($s_{\mathrm{ud}} = 0$), in this case:

$$|\Lambda\uparrow\rangle \propto \mathrm{d}\!\uparrow\mathrm{u}\!\downarrow\mathrm{s}\!\uparrow - \mathrm{u}\!\downarrow\mathrm{d}\!\uparrow\mathrm{s}\!\uparrow + \text{ cyclic combinatorics}$$

b) Following the above arguments the total spin of the ud system assume that the ud quarks in the Λ and Σ^0 are either in a spin-0 or spin-1 state, $s_{\mathrm{ud}} = 0$ or $s_{\mathrm{ud}} = 1$.

The total spin of the three quark system can be written:

$$\begin{aligned}
\mathbf{S} &= \mathbf{S}_{\mathrm{u}} + \mathbf{S}_{\mathrm{d}} + \mathbf{S}_{\mathrm{s}} \\
\mathbf{S}^2 &= \mathbf{S}_{\mathrm{u}}^2 + \mathbf{S}_{\mathrm{d}}^2 + \mathbf{S}_{\mathrm{s}}^2 + 2\mathbf{S}_{\mathrm{u}} \cdot \mathbf{S}_{\mathrm{d}} + 2\mathbf{S}_{\mathrm{s}} \cdot (\mathbf{S}_{\mathrm{u}} + \mathbf{S}_{\mathrm{d}}) \\
\langle \mathbf{S}^2 \rangle &= \langle \mathbf{S}_{\mathrm{u}}^2 \rangle + \langle \mathbf{S}_{\mathrm{d}}^2 \rangle + \langle \mathbf{S}_{\mathrm{s}}^2 \rangle + 2\langle \mathbf{S}_{\mathrm{u}} \cdot \mathbf{S}_{\mathrm{d}} \rangle + 2\langle \mathbf{S}_{\mathrm{s}} \cdot (\mathbf{S}_{\mathrm{u}} + \mathbf{S}_{\mathrm{d}}) \rangle \\
s(s+1) &= 3s_{\mathrm{q}}(s_{\mathrm{q}}+1) + 2\langle \mathbf{S}_{\mathrm{u}} \cdot \mathbf{S}_{\mathrm{d}} \rangle + 2\langle \mathbf{S}_{\mathrm{s}} \cdot (\mathbf{S}_{\mathrm{u}} + \mathbf{S}_{\mathrm{d}}) \rangle \,.
\end{aligned}$$

Here the total angular momentum (”spin”) of the baryon is 1/2 and the above equation can be rearranged to give an expression for the term involving the s-quark spins in terms of the total angular momentum of the ud system.

$$2\langle \mathbf{S}_{\mathrm{s}} \cdot (\mathbf{S}_{\mathrm{u}} + \mathbf{S}_{\mathrm{d}}) \rangle = -\tfrac{3}{2} - 2\langle \mathbf{S}_{\mathrm{u}} \cdot \mathbf{S}_{\mathrm{d}} \rangle \mathbf{??} \tag{9.2}$$

If the total spin of the ud system is $\mathbf{S}_{\mathrm{ud}}$, then

$$\begin{aligned}
\mathbf{S}_{\mathrm{ud}} &= \mathbf{S}_{\mathrm{u}} + \mathbf{S}_{\mathrm{d}} \\
\langle \mathbf{S}_{\mathrm{ud}}^2 \rangle &= \langle \mathbf{S}_{\mathrm{u}}^2 \rangle + \langle \mathbf{S}_{\mathrm{d}}^2 \rangle + 2\langle \mathbf{S}_{\mathrm{u}} \cdot \mathbf{S}_{\mathrm{d}} \rangle \\
s_{\mathrm{ud}}(s_{\mathrm{ud}}+1) &= 3/2 + 2\langle \mathbf{S}_{\mathrm{u}} \cdot \mathbf{S}_{\mathrm{d}} \rangle \,.
\end{aligned}$$

Substituting this back into (**??**) gives

$$\begin{aligned}
2\langle \mathbf{S}_{\mathrm{s}} \cdot (\mathbf{S}_{\mathrm{u}} + \mathbf{S}_{\mathrm{d}}) \rangle &= -\tfrac{3}{2} + \tfrac{3}{2} - s_{\mathrm{ud}}(s_{\mathrm{ud}}+1) \\
\langle \mathbf{S}_{\mathrm{s}} \cdot (\mathbf{S}_{\mathrm{u}} + \mathbf{S}_{\mathrm{d}}) \rangle &= -\tfrac{1}{2} s_{\mathrm{ud}}(s_{\mathrm{ud}}+1) \,.
\end{aligned}$$

Now, neglecting the differences in up and down quark masses, the baryon mass formula for the uds system can be written:

$$\begin{aligned}
m(\mathrm{uds}) &= 2m_{\mathrm{u}} + m_{\mathrm{s}} + \frac{A'}{m_{\mathrm{u}}^2}\langle \mathbf{S}_{\mathrm{u}} \cdot \mathbf{S}_{\mathrm{d}} \rangle + \frac{A'}{m_{\mathrm{u}} m_{\mathrm{s}}}\langle \mathbf{S}_{\mathrm{s}} \cdot (\mathbf{S}_{\mathrm{u}} + \mathbf{S}_{\mathrm{d}}) \rangle \\
&= 2m_{\mathrm{u}} + m_{\mathrm{s}} + \frac{A'}{m_{\mathrm{u}}^2}\left(\tfrac{1}{2} s_{\mathrm{ud}}(s_{\mathrm{ud}}+1) - \tfrac{3}{4}\right) - \frac{A'}{2m_{\mathrm{u}} m_{\mathrm{s}}} s_{\mathrm{ud}}(s_{\mathrm{ud}}+1) \,.
\end{aligned}$$

Putting in the appropriate values for the quark masses and A' gives the numerical result:

$$m(\mathrm{uds})/\mathrm{GeV} = 1.124 + 0.032 s_{\mathrm{ud}}(s_{\mathrm{ud}}+1) \,.$$

For the case when the ud quarks are in a anti-symmetric/symmetric spin state ($s_{\mathrm{ud}} = 0/s_{\mathrm{ud}} = 1$) the resulting masses are predicted to be

$$\begin{aligned}
s_{\mathrm{ud}} = 0 \quad &: \quad m(\Lambda) = 1.124\,\mathrm{GeV} \\
s_{\mathrm{ud}} = 1 \quad &: \quad m(\Sigma^0) = 1.187\,\mathrm{GeV} \,,
\end{aligned}$$

In reasonable agreement with the observed values of $m(\Lambda) = 1.116\,\mathrm{GeV}$ and $m(\Sigma^0) = 1.193\,\mathrm{GeV}$.

9.9 Show that the quark model predictions for the magnetic moments of the Σ^+, Σ^- and Ω^- baryons are

$$\mu(\Sigma^+) = \tfrac{1}{3}\,(4\mu_{\rm u} - \mu_{\rm s})\ , \quad \mu(\Sigma^-) = \tfrac{1}{3}\,(4\mu_{\rm d} - \mu_{\rm s}) \quad \text{and} \quad \mu(\Omega^-) = 3\mu_{\rm s}\,.$$

What values of the quark constituent masses are required to give the best agreement with the measured values of

$$\mu(\Sigma^+) = (2.46 \pm 0.01)\mu_N\,, \quad \mu(\Sigma^-) = (-1.16 \pm 0.03)\mu_N \quad \text{and} \quad \mu(\Omega^-) = (-2.02 \pm 0.06)\mu_N\,.$$

Since the Σ^+(uus) and Σ^-(dds) are part of the same multiplet as the proton (uud), the wavefunctions follow from that of the proton with the respective replacements d $\to$ s and d $\to$ s/u $\to$ d:

$$|\Sigma^-\uparrow\rangle \propto 2\mathrm{d}\!\uparrow\mathrm{d}\!\uparrow\mathrm{s}\!\downarrow - \mathrm{d}\!\uparrow\mathrm{d}\!\downarrow\mathrm{s}\!\uparrow - \mathrm{d}\!\downarrow\mathrm{d}\!\uparrow\mathrm{s}\!\uparrow + \text{ cyclic combinatorics}\,.$$

and

$$|\Sigma^+\uparrow\rangle \propto 2\mathrm{u}\!\uparrow\mathrm{u}\!\uparrow\mathrm{s}\!\downarrow - \mathrm{u}\!\uparrow\mathrm{u}\!\downarrow\mathrm{s}\!\uparrow - \mathrm{u}\!\downarrow\mathrm{u}\!\uparrow\mathrm{s}\!\uparrow + \text{ cyclic combinatorics}\,.$$

The derivations of the magnetic moments then follow exactly that for the proton in the main text. Since

$$\mu_{\rm p} = \tfrac{4}{3}\mu_{\rm u} - \tfrac{1}{3}\mu_{\rm d}\,,$$

it follows that

$$\mu(\Sigma^-) = \tfrac{4}{3}\mu_{\rm d} - \tfrac{1}{3}\mu_{\rm s} \quad \text{and} \quad \mu(\Sigma^-) = \tfrac{4}{3}\mu_{\rm u} - \tfrac{1}{3}\mu_{\rm s}\,.$$

The Ω^-sss is part of the spin-$\frac{3}{2}$ baryon decuplet. The wavefunction for the extreme spin state ($m_s = +3/2$) is simply:

$$|\Omega^-\uparrow\rangle = \mathrm{s}\!\uparrow\mathrm{s}\!\uparrow\mathrm{s}\!\uparrow,$$

and the magnetic moment is clearly just the sum of the magnetic moments of the individual quarks:

$$\mu(\Omega^-) = 3\mu_{\rm s}\,.$$

The simplest way to interpret the data is to use the measurement of the $\mu(\Omega^-)$ to determine the magnetic moment of the s quark:

$$3\mu_{\rm s} = (-2.02 \pm 0.06)\mu_N \quad \Rightarrow \quad \mu_{\rm s} = (-0.673 \pm 0.02)\mu_N\,,$$

and then to use the other two measurements to determine $\mu_{\rm u}$ and $\mu_{\rm d}$:

$$\tfrac{4}{3}\mu_{\rm d} - \tfrac{1}{3}\mu_{\rm s} = \mu(\Sigma^-) = \tfrac{4}{3}\mu_{\rm d} + 0.224\mu_N = (-1.16 \pm 0.03)\mu_N \quad \Rightarrow \quad \mu_{\rm d} = (-1.04 \pm 0.02)\mu_N\,,$$

and

$$\tfrac{4}{3}\mu_{\rm u} - \tfrac{1}{3}\mu_{\rm s} = \mu(\Sigma^+) = \tfrac{4}{3}\mu_{\rm d} + 0.224\mu_N = (2.46 \pm 0.01)\mu_N \quad \Rightarrow \quad \mu_{\rm u} = (+1.68 \pm 0.01)\mu_N\,,$$

All that remains is to determine the best values of the (effective) quark masses

$$\mu_{\mathrm{u}} = \langle \mathrm{u}\uparrow|\hat{\mu}_z|\mathrm{u}\uparrow\rangle = \left(+\tfrac{2}{3}\right)\frac{e}{2m_{\mathrm{u}}} = +\frac{2m_{\mathrm{p}}}{3m_{\mathrm{u}}}\mu_N\,,$$
$$\mu_{\mathrm{d}} = \langle \mathrm{d}\uparrow|\hat{\mu}_z|\mathrm{d}\uparrow\rangle = \left(-\tfrac{1}{3}\right)\frac{e}{2m_{\mathrm{d}}} = -\frac{m_{\mathrm{p}}}{3m_{\mathrm{d}}}\mu_N\,,$$
$$\mu_{\mathrm{s}} = \langle \mathrm{s}\uparrow|\hat{\mu}_z|\mathrm{s}\uparrow\rangle = \left(-\tfrac{1}{3}\right)\frac{e}{2m_{\mathrm{s}}} = -\frac{m_{\mathrm{p}}}{3m_{\mathrm{s}}}\mu_N\,.$$

Using the above measurements, one obtains:

$$\mu_{\mathrm{u}} = (+1.68 \pm 0.01)\mu_N = +\frac{2m_{\mathrm{p}}}{3m_{\mathrm{u}}}\mu_N \quad\Rightarrow\quad m_{\mathrm{u}} = 0.39m_{\mathrm{p}} \simeq 370\,\mathrm{MeV}\,,$$
$$\mu_{\mathrm{d}} = (-1.04 \pm 0.02)\mu_N = -\frac{m_{\mathrm{p}}}{3m_{\mathrm{d}}}\mu_N \quad\Rightarrow\quad m_{\mathrm{d}} = 0.32m_{\mathrm{p}} \simeq 300\,\mathrm{MeV}\,,$$
$$\mu_{\mathrm{s}} = (-0.673 \pm 0.02)\mu_N = -\frac{m_{\mathrm{p}}}{3m_{\mathrm{s}}}\mu_N \quad\Rightarrow\quad m_{\mathrm{s}} = 0.50m_{\mathrm{p}} \simeq 465\,\mathrm{MeV}\,.$$

9.10 If the colour did not exist, baryon wavefunctions would be constructed from

$$\psi = \phi_{\text{flavour}}\chi_{\text{spin}}\,\eta_{\text{space}}\,.$$

Taking $L = 0$ and using the flavour and spin wavefunctions derived in the text:

a) Show that it is still possible to construct a wavefunction for a spin-up proton for which $\phi_{\text{flavour}}\chi_{\text{spin}}$ is totally antisymmetric.

b) Predict the baryon multiplet structure for this model.

c) For this colourless model, show that μ_{p} is negative and that the ratio of the neutron and proton magnetic moments would be

$$\frac{\mu_{\mathrm{n}}}{\mu_{\mathrm{p}}} = -2\,.$$

a) If the colour did not exist, baryon wavefunctions would be constructed from

$$\psi = \phi_{\text{flavour}}\chi_{\text{spin}}\,\eta_{\text{space}}\,.$$

For the $L = 0$ baryons, the spatial wavefunction is symmetric and the requirement that the overall wavefunction is anti-symmetric implies that the combination of $\phi_{\text{flavour}} \times \chi_{\text{spin}}$ must be anti-symmetric under the interchange of any two quarks. The linear combination

$$\psi = \alpha\phi_S\chi_A + \beta\phi_A\chi_S$$

is clearly anti-symmetric under the interchange of quarks $1 \leftrightarrow 2$ and for the right choice of α and β is anti-symmetric under the interchange of any two quarks. Using the forms of the flavour and spin wavefunctions given in the main text,

$$\psi = \frac{\alpha}{\sqrt{12}}(2u\!\uparrow u\!\downarrow d\!\uparrow - 2u\!\downarrow u\!\uparrow d\!\uparrow - u\!\uparrow d\!\downarrow u\!\uparrow + u\!\downarrow d\!\uparrow u\!\uparrow - d\!\uparrow u\!\downarrow u\!\uparrow + d\!\downarrow u\!\uparrow u\!\uparrow) +$$
$$\frac{\beta}{\sqrt{12}}(2u\!\uparrow d\!\uparrow u\!\downarrow - 2d\!\uparrow u\!\uparrow u\!\downarrow - u\!\uparrow d\!\downarrow u\!\uparrow + d\!\uparrow u\!\downarrow u\!\uparrow - u\!\downarrow d\!\uparrow u\!\uparrow + d\!\downarrow u\!\uparrow u\!\uparrow) .$$

The relation between the coefficients α and β can be found by considering the transformation properties under interchange of quarks $2 \leftrightarrow 3$ (or $1 \leftrightarrow 3$):

$$\psi' = \psi_{2\leftrightarrow 3}$$
$$= \frac{\alpha}{\sqrt{12}}(2u\!\uparrow d\!\uparrow u\!\downarrow - 2u\!\downarrow d\!\uparrow u\!\uparrow - u\!\uparrow u\!\uparrow d\!\downarrow + u\!\downarrow u\!\uparrow d\!\uparrow - d\!\uparrow u\!\uparrow u\!\downarrow + d\!\downarrow u\!\uparrow u\!\uparrow) +$$
$$\frac{\beta}{\sqrt{12}}(2u\!\uparrow u\!\downarrow d\!\uparrow - 2d\!\uparrow u\!\downarrow u\!\uparrow - u\!\uparrow u\!\uparrow d\!\downarrow + d\!\uparrow u\!\uparrow u\!\downarrow - u\!\downarrow u\!\uparrow d\!\uparrow + d\!\downarrow u\!\uparrow u\!\uparrow) .$$

The requirement of overall anti-symmetry implies that $\psi_{2\leftrightarrow 3} = -\psi$. For example consider the uud parts of the above wavefunctions, the requirement of antisymmetry implies:

$$2\alpha u\!\uparrow u\!\downarrow d\!\uparrow - 2\alpha u\!\downarrow u\!\uparrow d\!\uparrow =$$
$$\alpha u\!\uparrow u\!\uparrow d\!\downarrow - \alpha u\!\downarrow u\!\uparrow d\!\uparrow - 2\beta u\!\uparrow u\!\downarrow d\!\uparrow + \beta u\!\uparrow u\!\uparrow d\!\downarrow + \beta u\!\downarrow u\!\uparrow d\!\uparrow ,$$

which is satisfied for $\beta = -\alpha$ and hence

$$\psi = \tfrac{1}{\sqrt{2}}(\phi_S\chi_A - \phi_A\chi_S) .$$

With a little straightforward algebra it can then be shown that

$$\psi = \tfrac{1}{\sqrt{6}}(u\!\uparrow u\!\downarrow d\!\uparrow - u\!\downarrow u\!\uparrow d\!\uparrow + u\!\downarrow d\!\uparrow u\!\uparrow - d\!\uparrow u\!\downarrow u\!\uparrow - u\!\uparrow d\!\uparrow u\!\downarrow + d\!\uparrow u\!\uparrow u\!\downarrow) . \quad (9.3)$$

b) Within the assumed SU(3) flavour symmetry, just as before, the combinations of mixed symmetry flavour and spin states will give rise to a spin-$\frac{1}{2}$ octet. In addition, it is is possible to construct a state where $\phi_{\text{flavour}} \times \chi_{\text{spin}}$ is anti-symmetric by combining the totally anti-symmetric flavour singlet with a symmetric spin-$\frac{3}{2}$ state. Hence, in this model there would be an octet of spin-$\frac{1}{2}$ baryons and a singlet spin-$\frac{3}{2}$ uds state.

c) In this model the wavefunctions of the ”spin-up” proton and neutron are

$$|p\!\uparrow\rangle = \tfrac{1}{\sqrt{6}}(u\!\uparrow u\!\downarrow d\!\uparrow - u\!\downarrow u\!\uparrow d\!\uparrow + u\!\downarrow d\!\uparrow u\!\uparrow - d\!\uparrow u\!\downarrow u\!\uparrow - u\!\uparrow d\!\uparrow u\!\downarrow + d\!\uparrow u\!\uparrow u\!\downarrow) ,$$

and

$$|n\!\uparrow\rangle = \tfrac{1}{\sqrt{6}}(d\!\uparrow d\!\downarrow u\!\uparrow - d\!\downarrow d\!\uparrow u\!\uparrow + d\!\downarrow u\!\uparrow d\!\uparrow - u\!\uparrow d\!\downarrow d\!\uparrow - d\!\uparrow u\!\uparrow d\!\downarrow + u\!\uparrow d\!\uparrow d\!\downarrow) .$$

Since, for each term, the spins of the two like quarks are always opposite, they do not contribute to the overall magnetic moment, which is determined solely by the magnetic moment of the other quark. Hence, taking $m_{\mathrm{u}} \sim m_{\mathrm{d}}$,

$$\frac{\mu_{\mathrm{n}}}{\mu_{\mathrm{p}}} = \frac{\mu_{\mathrm{u}}}{\mu_{\mathrm{d}}} = -2\,.$$

This colourless model, therefore, does not predict the observed ratio of magnetic moments of the proton and neutron.

10 Quantum Chromodynamics (QCD)

10.1 By considering the symmetry of the wavefunction, explain why the existence of the Ω^-(sss) $L = 0$ baryon provides evidence for a degree of freedom in addition to space × spin × flavour.

In the absence of colour, the overall wavefunction has the following degrees of freedom:

$$\psi = \phi_{\text{flavour}}\chi_{\text{spin}}\,\eta_{\text{space}}\,.$$

The overall wavefunction must be anti-symmetric under the interchange of any two quarks (since they are fermions). For the a state with zero orbital angular momentum ($\ell = 0$), the spatial wavefunction is symmetric. The flavour wavefunction sss is clearly symmetric under the interchange of any two quarks. Therefore, the required overall anti-symmetric wavefunction would imply a totally anti-symmetric spin wavefunction, however, there is no totally anti-symmetic spin wavefunction for the combination of three spin-half particles ($2 \otimes 2 \otimes 2 = 4 \oplus 2 \oplus 2$). Hence, without an additional degree of freedom, in this case colour, the Ω^- would not exist.

10.2 From the expression for the running of α_S with $N_f = 3$, determine the value of q^2 at which α_S appears to become infinite. Comment on this result.

The strong coupling constant runs according to

$$\alpha_S(q^2) = \frac{\alpha_S(\mu^2)}{1 + B\,\alpha_S(\mu^2)\ln\left(\dfrac{q^2}{\mu^2}\right)},$$

where $B = (11N_c - 2N_f)/(12\pi)$. For $N_c = 3$ colours and $N_f = 3$ (valid at low values of $|q^2|$),

$$B = \frac{27}{12\pi}\,.$$

The strong coupling constant appears to become infinite when

$$\alpha_S(\mu^2)\ln\left(\frac{q^2_\infty}{\mu^2}\right) = -\frac{12\pi}{27}\,.$$

At $|q| = 10\,\mathrm{GeV}$, $\alpha_S \simeq 0.18$, and therefore α_S becomes infinite when

$$\begin{aligned} 0.18\ln\left(\frac{q_\infty^2}{100}\right) &= -\frac{12\pi}{27} \\ \Rightarrow \quad \frac{q_\infty^2}{100} &= 4.3\times 10^{-4} \\ \Rightarrow \quad q_\infty &\approx 200\,\mathrm{MeV}\,. \end{aligned}$$

At this scale, known as Λ_{QCD}, QCD is clearly non-perturbative.

10.3 Find the overall “colour factor” for qq → qq if QCD corresponded to a SU(2) colour symmetry.

If QCD corresponded to an SU(2) rather than SU(3) ”colour” symmetry, the corresponding vertex factor would be

$$-\tfrac{1}{2}ig_S\lambda^a_{ji}\gamma^\mu \to -\tfrac{1}{2}ig_S\sigma^a_{ji}\gamma^\mu\,,$$

where the colour indices i, j=1,2 and σ^a with $a = 1,2,3$ are the three Pauli spin matrices corresponding to the three ”gluons” in this model.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Following the arguments for SU(3) QQD, the colour factor for a particular $\mathrm{q}_i\mathrm{q}_k \to \mathrm{q}_j\mathrm{q}_l$ scattering process is

$$C(ik \to jl) = \frac{1}{4}\sum_{a=1}^{3}\lambda^a_{ji}\lambda^a_{kl}\,.$$

Denoting the colours as r and b, it is straightforward to obtain the following results

$$\begin{aligned} C(rr \to rr) = C(bb \to bb) &= \tfrac{1}{4}(1)(1) = \tfrac{1}{4}\,, \\ C(rb \to rb) = C(br \to br) &= \tfrac{1}{4}(1)(-1) = -\tfrac{1}{4}\,, \\ C(rb \to br) = C(br \to rb) &= \tfrac{1}{4}\left[(1)(1) + (i)(-i)\right] = \tfrac{1}{2}\,, \end{aligned}$$

and all other combinations are zero. Hence the spin-averaged colour factor (averaging over the four possible initial colour combinations of the two initial-state quarks) is

$$\begin{aligned} \langle |C|^2\rangle &= \frac{1}{4}\sum_{i,j,k,l=1}^{2}|C(ij \to kl)|^2 \\ &= \frac{1}{4}\left[2\times\left(\frac{1}{4}\right)^2 + 2\times\left(\frac{1}{4}\right)^2 + 2\times\left(\frac{1}{2}\right)^2\right] \\ &= \frac{3}{16}\,. \end{aligned}$$

10.4 Calculate the non-relativistic QCD potential between quarks q_1 and q_2 in a $q_1q_2q_3$ baryon with colour wavefunction

$$\psi = \frac{1}{\sqrt{6}}(rgb - grb + gbr - bgr + brg - rbg)\,.$$

The NRQCD potential between two quarks can be expressed as

$$V_{qq}(\mathbf{r}) = +C\frac{\alpha_S}{\mathrm{r}}\,,$$

where C is the appropriate colour factor. Consequently, the expectation value of the NRQCD potential between quarks 1 and 2 an a baryon with colour wavefunction ψ

$$\begin{aligned}
\langle V_{qq}^{12}\rangle &= \langle\psi|V_{qq}|\psi\rangle \\
&= \tfrac{1}{6}\langle(rg - gr + gb - bg + br - rb)|V_{qq}|(rg - gr + gb - bg + br - rb)\rangle \\
&= +\frac{\alpha_S}{6\mathrm{r}}\left[6C(rg \to rg) - 6C(rg \to rg)\right] \\
&= +\frac{\alpha_S}{6\mathrm{r}}\left[6\left(-\tfrac{1}{6}\right) - 6\left(+\tfrac{1}{2}\right)\right],
\end{aligned}$$

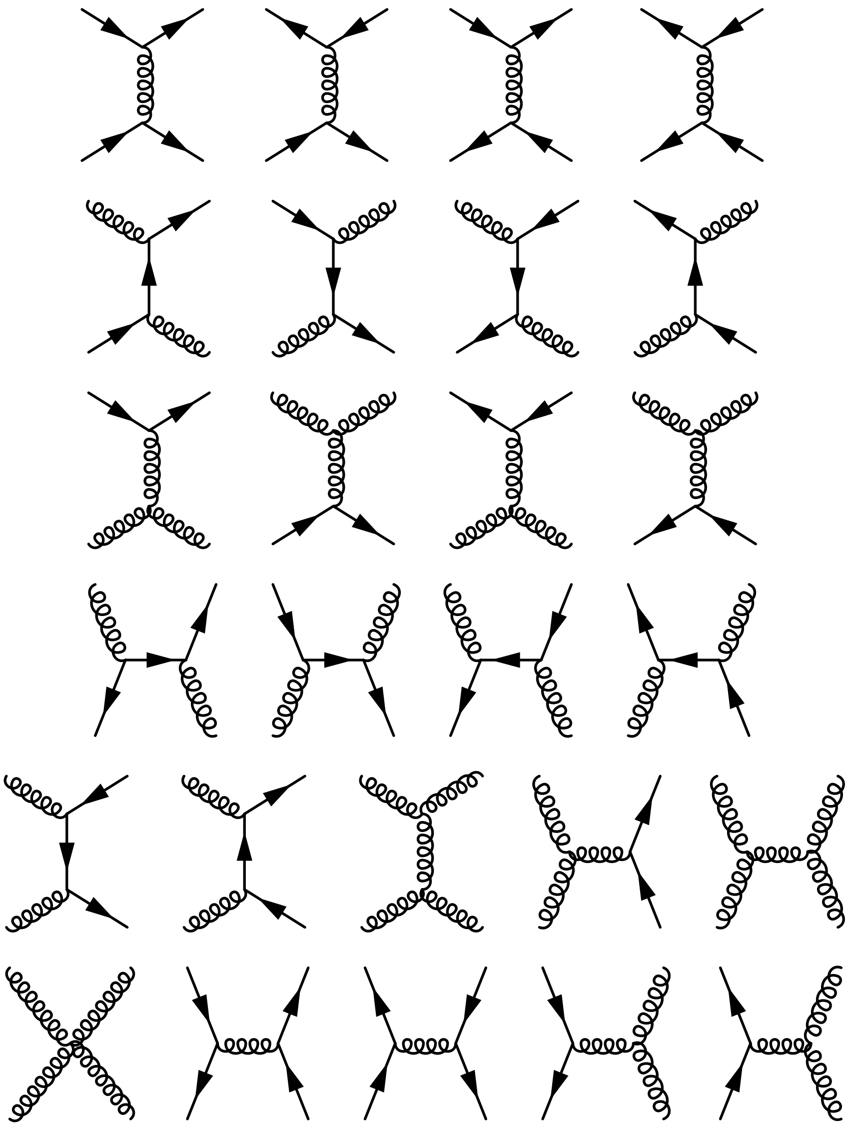
and thus

$$\langle V_{qq}^{12}\rangle = -\frac{2\alpha_S}{3\mathrm{r}}\,.$$

Hence, in the non-relativistic limit, the QCD potential between any two quarks in a baryon is attractive.

10.5 Draw the lowest-order QCD Feynman diagrams for the process $\mathrm{p\overline{p}} \to 2\,\mathrm{jets} + X$, where X represents the remnants of the colliding hadrons.

There are diagrams involving: i) the scattering of quarks and antiquarks, ii) the scattering of a quark/antiquark and a gluon and iii) the scattering of gluons, where the anti-quarks/quarks can either be from the valance or sea content of the proton and antiproton.



10.6 The observed events in the process $\mathrm{pp} \to 2\,\mathrm{jets}$ at the LHC can be described in terms of the jet p_T and the jet rapidities y_3 and y_4.

a) Assuming that the jets are massless, $E^2 = p_\mathrm{T}^2 + p_z^2$, show that the four-momenta of the final-state jets can be written as

$$p_3 = (p_\mathrm{T}\cosh y_3, +p_\mathrm{T}\sin\phi, +p_\mathrm{T}\cos\phi, p_\mathrm{T}\sinh y_3)\,,$$
$$p_4 = (p_\mathrm{T}\cosh y_4, -p_\mathrm{T}\sin\phi, -p_\mathrm{T}\cos\phi, p_\mathrm{T}\sinh y_4)\,.$$

b) By writing the four-momenta of the colliding partons in a pp collision as

$$p_1 = \frac{\sqrt{s}}{2}(x_1, 0, 0, x_1) \quad \text{and} \quad p_2 = \frac{\sqrt{s}}{2}(x_2, 0, 0, -x_1)\,,$$

show that conservation of energy and momentum implies

$$x_1 = \frac{p_\mathrm{T}}{\sqrt{s}}(e^{+y_3} + e^{+y_4}) \quad \text{and} \quad x_2 = \frac{p_\mathrm{T}}{\sqrt{s}}(e^{-y_3} + e^{-y_4})\,.$$

c) Hence show that

$$Q^2 = p_\mathrm{T}^2(1 + e^{y_4 - y_3})\,.$$

a) From the definition of rapidity

$$2y = \ln\left(\frac{E + p_z}{E - p_z}\right) \quad \Rightarrow \quad e^{2y} = \frac{E + p_z}{E - p_z}\,,$$

which can be rearranged to give

$$E(e^{2y} - 1) = p_z(e^{2y} + 1)$$
$$\Rightarrow \quad \frac{E}{p_z} = \frac{e^{2y} + 1}{e^{2y} - 1} = \frac{e^{y} + e^{-y}}{e^{y} - e^{-y}} = \frac{\cosh y}{\sinh y}\,.$$

In the massless limit, the jet energy is the sum of the squares of the transverse and longitudinal momentum components $E^2 = p_\mathrm{T}^2 + p_z^2$ and thus,

$$E^2 - p^2 = p_\mathrm{T}^2$$
$$E^2\left(1 - \frac{p_z^2}{E^2}\right) = p_\mathrm{T}^2$$
$$\Rightarrow \quad E^2\left(1 - \frac{\sinh^2 y}{\cosh^2 y}\right) = p_\mathrm{T}^2$$
$$\Rightarrow \quad E^2 = p_\mathrm{T}^2\cosh^2 y\,,$$

from which it follows that $p_z^2 = E^2 - p_\mathrm{T}^2 = p_\mathrm{T}^2\sinh^2 y$. Therefore the jet four-momentum can be written in terms of y, p_T and the azimuthal angle,

$$p = (p_\mathrm{T}\cosh y, p_\mathrm{T}\sin\phi, p_\mathrm{T}\cos\phi, p_\mathrm{T}\sinh y)\,,$$

and since the two jets here are produced back-to-back in the transverse plane:

$$
\begin{aligned}
p_3 &= (p_\mathrm{T}\cosh y_3, +p_\mathrm{T}\sin\phi, +p_\mathrm{T}\cos\phi, p_\mathrm{T}\sinh y_3)\,,\\
p_4 &= (p_\mathrm{T}\cosh y_4, -p_\mathrm{T}\sin\phi, -p_\mathrm{T}\cos\phi, p_\mathrm{T}\sinh y_4)\,.
\end{aligned}
$$

First recall that

$$\cosh y + \sinh y = \tfrac{1}{2}\,(e^y + e^{-y}) + \tfrac{1}{2}\,(e^y - e^{-y}) = e^y \tag{10.1}$$

$$\cosh y - \sinh y = \tfrac{1}{2}\,(e^y + e^{-y}) - \tfrac{1}{2}\,(e^y - e^{-y}) = e^{-y}\,. \tag{10.2}$$

Here conservation of energy and momentum imply

$$
\begin{aligned}
&\frac{\sqrt{s}}{2}(x_1 + x_2) = p_\mathrm{T}(\cosh y_3 + \cosh y_4)\\
\text{and}\quad &\frac{\sqrt{s}}{2}(x_1 - x_2) = p_\mathrm{T}(\sinh y_3 + \sinh y_4)\,.
\end{aligned}
$$

Taking the sum and difference of these two relations and using the identities of (10.1) and (10.2) gives

$$
\begin{aligned}
\sqrt{s}x_1 &= p_\mathrm{T}(\cosh y_3 + \cosh y_4 + \sinh y_3 + \sinh y_4) = p_\mathrm{T}(e^{y_3} + e^{y_4})\\
\text{and}\quad \sqrt{s}x_2 &= p_\mathrm{T}(\cosh y_3 + \cosh y_4 - \sinh y_3 - \sinh y_4) = p_\mathrm{T}(e^{-y_3} + e^{-y_4})\,,
\end{aligned}
$$

and, as required,

$$x_1 = \frac{p_\mathrm{T}}{\sqrt{s}}(e^{+y_3} + e^{+y_4}) \quad \text{and} \quad x_2 = \frac{p_\mathrm{T}}{\sqrt{s}}(e^{-y_3} + e^{-y_4})\,.$$

In the massless limit ($p_1^2 = p_3^2 = 0$), the four-momentum transfer

$$
\begin{aligned}
Q^2 &= -q^2 = -(p_1 - p_3)^2 = 2p_1\cdot p_3\\
&= 2\frac{\sqrt{s}}{2}(x_1, 0, 0, x_1)\cdot(p_\mathrm{T}\cosh y_3, +p_\mathrm{T}\sin\phi, +p_\mathrm{T}\cos\phi, p_\mathrm{T}\sinh y_3)\\
&= p_\mathrm{T}\sqrt{s}x_1(\cosh y_3 - \sinh y_3)\\
&= p_\mathrm{T}\sqrt{s}x_1 e^{-y_3}\,.
\end{aligned}
$$

Using the result for x_1 from part b) above,

$$
\begin{aligned}
Q^2 &= p_\mathrm{T}\sqrt{s}x_1 e^{-y_3}\\
&= p_\mathrm{T}^2(e^{y_3} + e^{y_4})e^{-y_3}\\
&= p_\mathrm{T}^2(1 + e^{y_4 - y_3})\,.
\end{aligned}
$$

It is worth recalling that rapidly *differences* are invariant under boosts along the z-direction, and therefore the result for (the Lorentz invariant) quantity Q^2 is invariant under boosts along the z-direction (as it must be).

10.7 Using the results of the previous question show that the Jacobian

$$\frac{\partial(y_3, y_4, p_T^2)}{\partial(x_1, x_2, q^2)} = \frac{1}{x_1 x_2}.$$

The transformation from $\{y_3, y_4, Q^2\}$ to $\{x_1, x_2, p_T\}$ is defined by the relations

$$\begin{aligned} x_1 &= \frac{p_T}{\sqrt{s}}(e^{+y_3} + e^{+y_4}), \\ x_2 &= \frac{p_T}{\sqrt{s}}(e^{-y_3} + e^{-y_4}), \\ Q^2 &= p_T^2(1 + e^{y_4 - y_3}), \end{aligned}$$

and the differential elements are related

$$\mathrm{d}y_3\mathrm{d}y_3\mathrm{d}p_T^2 = \left|\frac{\partial(y_3, y_4, p_T^2)}{\partial(x_1, x_2, Q^2)}\right| \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}Q^2 \ .$$

Since we already have $\{x_1, x_2, p_T\}$ expressed in terms of $\{y_3, y_4, Q^2\}$, it is simpler to calculate

$$\frac{\partial(x_1, x_2, Q^2)}{\partial(y_3, y_4, p_T^2)} \quad \text{and use} \quad \frac{\partial(y_3, y_4, p_T^2)}{\partial(x_1, x_2, Q^2)} = \left(\frac{\partial(x_1, x_2, Q^2)}{\partial(y_3, y_4, p_T^2)}\right)^{-1} .$$

The actual calculation requires care but is fairly straightforward. Here, the Jacobian (it terms of p_T rather than p_T^2) is

$$\begin{aligned} J = \frac{\partial(x_1, x_2, Q^2)}{\partial(y_3, y_4, p_T)} &= \begin{vmatrix} \frac{\partial x_1}{\partial y_3} & \frac{\partial x_1}{\partial y_4} & \frac{\partial x_1}{\partial p_T} \\ \frac{\partial x_2}{\partial y_4} & \frac{\partial x_2}{\partial y_4} & \frac{\partial x_2}{\partial p_T} \\ \frac{\partial Q^2}{\partial y_3} & \frac{\partial Q^2}{\partial y_4} & \frac{\partial Q^2}{\partial p_T} \end{vmatrix} \\ &= \begin{vmatrix} \frac{p_T}{\sqrt{s}} e^{y_3} & \frac{p_T}{\sqrt{s}} e^{y_4} & \frac{1}{\sqrt{s}}(e^{y_3} + e^{y_4}) \\ -\frac{p_T}{\sqrt{s}} e^{-y_3} & -\frac{p_T}{\sqrt{s}} e^{-y_4} & \frac{1}{\sqrt{s}}(e^{-y_3} + e^{-y_4}) \\ -p_T^2 e^{y_4 - y_3} & p_T^2 e^{y_4 - y_3} & 2p_T(1 + e^{y_4 - y_3}) \end{vmatrix} \\ &= \frac{p_T^3}{s} \begin{vmatrix} e^{y_3} & e^{y_4} & e^{y_3} + e^{y_4} \\ -e^{-y_3} & -e^{-y_4} & e^{-y_3} + e^{-y_4} \\ -e^{y_4 - y_3} & e^{y_4 - y_3} & 2(1 + e^{y_4 - y_3}) \end{vmatrix} \end{aligned}$$

Multiplying out the terms in the determinant leads to

$$
\begin{aligned}
J = \frac{p_{\mathrm{T}}^3}{s} \Big\{ & e^{y_3} \left[-2(e^{-y_4} + e^{-y_3}) - e^{y_4-y_3}(e^{-y_3} + e^{-y_4})\right] \\
& - e^{y_4} \left[-e^{-y_3}(1 + e^{y_4-y_3}) + e^{y_4-y_3}(e^{-y_3} + e^{-y_4})\right] \\
& + (e^{y_3} + e^{y_4}) \left[-e^{y_4-2y_3} - e^{-y_3}\right] \Big\} \\
= -\frac{p_{\mathrm{T}}^3}{s} & \left[2e^{y_3-y_4} + 2 + e^{y_4-y_3} + 1 - 2e^{y_4-y_3} - 2e^{2y_4-2y_3} + e^{2y_4-2y_3} \right. \\
& \left. + e^{y_4-y_3} + e^{y_4-y_3} + 1 + e^{2y_4-2y_3} + e^{y_4-y_3}\right] \\
= -\frac{p_{\mathrm{T}}^3}{s} & [2 + e^{y_4-y_3} + e^{y_3-y_4} \\
= -2p_{\mathrm{T}} & x_1 x_2 \, .
\end{aligned}
$$

The next step is to transform from $Q^2 \to q^2 = -Q^2$ and from $p_{\mathrm{T}} \to p_{\mathrm{T}}^2$ giving

$$
\begin{aligned}
\frac{\partial(x_1, x_2, q^2)}{\partial(y_3, y_4, p_{\mathrm{T}}^2)} &= -\frac{1}{2p_{\mathrm{T}}} \frac{\partial(x_1, x_2, Q^2)}{\partial(y_3, y_4, p_{\mathrm{T}}^2)} \\
&= x_1 x_2 \, ,
\end{aligned}
$$

and from the reciprocity properties of the Jacobian J, the require result is obtained:

$$
\frac{\partial(y_3, y_4, p_{\mathrm{T}}^2)}{\partial(x_1, x_2, q^2)} = \frac{1}{x_1 x_2} \, .
$$

10.8 The total cross section for the Drell-Yan process $\mathrm{p\overline{p}} \to \mu^+\mu^- X$ was shown to be

$$
\sigma_{\mathrm{DY}} = \frac{4\pi\alpha^2}{81s} \int_0^1 \int_0^1 \frac{1}{x_1 x_2} \left[4u(x_1)u(x_2) + 4\overline{u}(x_1)\overline{u}(x_2) + d(x_1)d(x_2) + \overline{d}(x_1)\overline{d}(x_2)\right] \mathrm{d}x_1 \mathrm{d}x_2 \, .
$$

a) Express this cross section in terms of the valence quark PDFs and a single PDF for the sea contribution, where $S(x) = \overline{u}(x) = \overline{d}(x)$.

b) Obtain the corresponding expression for $\mathrm{pp} \to \mu^+\mu^- X$.

c) Sketch the region in the x_1-x_2 plane corresponding $s_{\mathrm{q\overline{q}}} > s/4$. Comment on the expected ratio of the Drell-Yan cross sections in pp and $\mathrm{p\overline{p}}$ collisions (at the same centre-of-mass energy) for the two cases: i) $\hat{s} \ll s$ and ii) $\hat{s} > s/4$, where $\hat{s}$ is the centre-of-mass energy of the colliding partons.

a) Writing $u(x) = u_V(x) + S(x)$ and $d(x) = d_V(x) + S(x)$,

$$\begin{aligned}
d^2\sigma_{DY}^{p\bar{p}} &= \frac{4\pi\alpha^2}{81 s x_1 x_2}\{4[u_V(x_1) + S(x_1)][u_V(x_2) + S(x_2)] + 4S(x_1)S(x_2) + \\
&\qquad [d_V(x_1) + S(x_1)][d_V(x_2) + S(x_2)] + S(x_1)S(x_2)\}dx_1 dx_2 \\
&= \frac{4\pi\alpha^2}{81 s x_1 x_2}\{4u_V(x_1)u_V(x_2) + 4u_V(x_1)S(x_2) + 4S(x_1)u_V(x_2) + 10S(x_1)S(x_2) + \\
&\qquad d_V(x_1)d_V(x_2) + d_V(x_1)S(x_2) + S(x_1)d_V(x_2)\}dx_1 dx_2
\end{aligned}$$

If we also assume that $u_V(x) = 2d_V(x)$ then

$$\begin{aligned}
d^2\sigma_{DY}^{p\bar{p}} &= \frac{4\pi\alpha^2}{81 s x_1 x_2}\{17 d_V(x_1)d_V(x_2) + \\
&\qquad 9d_V(x_1)S(x_2) + 9S(x_1)d_V(x_2) + 10S(x_1)S(x_2)\}dx_1 dx_2 .
\end{aligned}$$

b) For pp collisions, the Drell-Yan cross section is

$$\begin{aligned}
d^2\sigma_{DY}^{pp} &= \frac{4\pi\alpha^2}{81 s x_1 x_2}\{4u(x_1)\bar{u}(x_2) + 4\bar{u}(x_1)u(x_2) + \\
&\qquad d(x_1)\bar{d}(x_2) + \bar{d}(x_1)d(x_2)\}dx_1 dx_2
\end{aligned}$$

This can be expressed in terms of sea and valance quark pdfs:

$$\begin{aligned}
d^2\sigma_{DY}^{pp} &= \frac{4\pi\alpha^2}{81 s x_1 x_2}\{4u_V(x_1)S(x_2) + 4S(x_1)u_V(x_2) + \\
&\qquad d_V(x_1)S(x_2) + S(x_1)d_V(x_2) + 10S(x_1)S(x_2)\}dx_1 dx_2
\end{aligned}$$

If we again assume that $u_V(x) = 2d_V(x)$ then

$$d^2\sigma_{DY}^{pp} = \frac{4\pi\alpha^2}{81 s x_1 x_2}\{9d_V(x_1)S(x_2) + 9S(x_1)d_V(x_2) + 10S(x_1)S(x_2)\}dx_1 dx_2$$

c) Since $\hat{s} = x_1 x_2 s$, lines of constant $\hat{s}$ define hyperbolae in the $\{x_1, x_2\}$ plane. For $\hat{s} \ll s$ both x_1 and x_2 will usually be small, and in this region the Drell-Yen cross section will be dominated the sea quarks and, from the above results, $d^2\sigma_{DY}^{p\bar{p}} \sim \sigma_{DY}^{pp}$. Consequently the cross section for the Drell-Yan production of low-mass $\mu^+\mu^-$-pairs will be approximately the same for pp and $p\bar{p}$ collisions. In contrasts for $\hat{s} > s/4$, both x_1 and x_2 will be greater than 0.5 and the valance quark contributions will dominate over the sea. In this case $d^2\sigma_{DY}^{p\bar{p}} \gg \sigma_{DY}^{pp}$, and the cross section for the production of high-mass $\mu^+\mu^-$-pairs will be much greater for $p\bar{p}$ collisions.

10.9 Drell-Yan production of $\mu^+\mu^-$-pairs with an invariant mass Q^2 has been studied in $\pi^\pm$ interactions with Carbon (which has equal numbers of protons and neutrons). Explain why the ratio

$$\frac{\sigma(\pi^+ C \to \mu^+\mu^- X)}{\sigma(\pi^- C \to \mu^+\mu^- X)}$$

tends to unity for small Q^2 and tends to $\frac{1}{4}$ as Q^2 approaches s.

The PDFs for the $\pi^+(\mathrm{u\overline{d}})$ can be written in terms of valance and sea quark distributions:

$$\begin{aligned}
u^{\pi^+}(x) &= u_V^{\pi^+}(x) + S^{\pi^+}(x) \equiv u_V^{\pi}(x) + S^{\pi}(x)\\
\overline{d}^{\pi^+}(x) &= \overline{d}_V^{\pi^+}(x) + S^{\pi^+}(x) \equiv \overline{d}_V^{\pi}(x) + S^{\pi}(x)\\
d^{\pi^+}(x) &= S^{\pi^+}(x) \equiv S^{\pi}(x)\\
\overline{u}^{\pi^+}(x) &= S^{\pi^+}(x) \equiv S^{\pi}(x)\,,
\end{aligned}$$

where the symbols with a superscript π implicitly refer to the PDFs for the π^+. Assuming isospin symmetry, e.g. the down-quark PDF in the $\pi^-(\mathrm{d\overline{u}})$ will be identical to the up-quark PDF in the π^+, the PDFs for the π^- are

$$\begin{aligned}
u^{\pi^-}(x) &= S^{\pi^-}(x) \equiv S^{\pi}(x)\\
\overline{d}^{\pi^-}(x) &= S^{\pi^-}(x) \equiv S^{\pi}(x)\\
d^{\pi^-}(x) &= d_V^{\pi^-}(x) + S^{\pi^-}(x) \equiv u_V^{\pi}(x) + S^{\pi}(x)\\
\overline{u}^{\pi^-}(x) &= \overline{u}_V^{\pi^-}(x) + S^{\pi^-}(x) \equiv \overline{d}_V^{\pi}(x) + S^{\pi}(x)\,.
\end{aligned}$$

For low Q^2 Drell-Yan production annihilation of sea quarks will dominate and therefore

$$\frac{\sigma(\pi^+\mathrm{C} \to \mu^+\mu^- X)}{\sigma(\pi^-\mathrm{C} \to \mu^+\mu^- X)} \to 1 \quad \text{as} \quad Q^2 \to 0\,.$$

As $Q^2 \to s$, both x_1 and x_2 tend to unity and the annihilation of valance quarks will dominate. Since $\overline{d}^{\pi^+}(x) = \overline{u}^{\pi^-}(x)$ and carbon contains an equal number of valance up- and down-quarks

$$\frac{\sigma(\pi^+\mathrm{C} \to \mu^+\mu^- X)}{\sigma(\pi^-\mathrm{C} \to \mu^+\mu^- X)} \to \frac{Q_\mathrm{d}^2\overline{d}^{\pi^+}(1)d^\mathrm{C}(1)}{Q_\mathrm{u}^2\overline{u}^{\pi^-}(1)u^\mathrm{C}(1)} \to \frac{1}{4} \quad \text{as} = Q^2 \to s\,.$$

11 The Weak Interaction

11.1 Explain why the strong decay $\rho^0 \to \pi^-\pi^+$ is observed, but the strong decay $\rho^0 \to \pi^0\pi^0$ is not.

Hint: you will need to consider conservation of angular momentum, parity and the symmetry of the $\pi^0\pi^0$ wavefunction.

The spins and parities of the particles involved are:

$$\pi^-\,,\ \pi^0\,,\ \pi^+\ :\ J^P = 0^-$$
$$\rho^0\ :\ J^P = 1^-\,.$$

Conservation of angular momentum implies that the two pions in the final state must be produced with one unit of orbital angular momentum, $\ell = 1$. Conservation of parity in this *strong* decay requires that

$$P_\rho^0 = P_\pi \cdot P_\pi \cdot (-1)^\ell$$
$$(-1) = (-1)\cdot(-1)\cdot(-1) = -1\ \checkmark$$

and therefore angular momentum and parity are conserved in the decay, explaining why $\rho^0 \to \pi^-\pi^+$ is observed. In the decay $\rho^0 \to \pi^0\pi^0$, the final state consists of two identical boson that must be produced in an overall symmetric wavefunction, which is not the case, since the orbital angular momentum $\ell = 1$ stare is anti-symmetric under the interchange of the two π^0s.

11.2 When π^- mesons are stopped in a deuterium target they can form a bound $(\pi^- - \mathrm{D})$ state with zero orbital angular momentum, $\ell = 0$. The bound state decays by the strong interaction

$$\pi^-\mathrm{D} \to \mathrm{nn}\,.$$

By considering the possible spin and orbital angular momentum states of the nn system, and the required symmetry of the wavefunction, show that the pion has negative intrinsic parity.

Note: the deuteron has $J^P = 1^+$ and the pion is a spin-0 particle.

This is an example of how the parity of particles can be inferred from their decays. Writing the spin (total angular momentum) and parity of the pion as $J_\pi^P = 0^{P_\pi}$ and the total orbital angular momentum in the final state as L, conservation of parity in the strong interaction implies that parity of the initial state is equal to the parity of the final state:

$$\begin{aligned} P_\pi \cdot P_\mathrm{D} \cdot (-1)^{\ell}_{i} &= P_\mathrm{n} \cdot P_\mathrm{n} \cdot (-1)^{L} \\ P_\pi \cdot (+1) \cdot (+1) &= P_\mathrm{n}^2 (-1)^{L} \\ P_\pi &= (-1)^{L}\,. \end{aligned}$$

Conservation of angular momentum implies that

$$\begin{aligned} \mathbf{J}_\mathrm{D} + \mathbf{J}_\pi + \boldsymbol{\ell} &= \mathbf{L} + \mathbf{S}_\mathrm{nn} \\ \mathbf{1} &= \mathbf{L} + \mathbf{S}_\mathrm{nn} \end{aligned} \qquad (11.1)$$

where $\mathbf{S}_\mathrm{nn} = \mathbf{0}$ or $\mathbf{1}$, is the total spin of the neutron-neutron system. Since the final state consists of identical fermions the overall wavefunction of the neutron-neutron system must be anti-symmetric:

$$\psi_\mathrm{space} \times \psi_\mathrm{spin} : \text{anti-symmetric}\,.$$

If the neutrons are produced in the anti-symmetric spin-0 final state, this implies that L must be even, i.e. $L = 0, 2, 4, ..$, however in this case, it is impossible to satisfy (11.1). Consequently in can be inferred that $S_\mathrm{nn} = 1$. In this case, the spin part of the wavefunction is symmetric and thus the spatial part must be anti-symmetric: L is odd. If $L = 1$, then the condition for conservation of angular momentum (11.1) becomes:

$$\mathbf{1} = \mathbf{1} + \mathbf{1}\,.$$

Remembering that total angular momentum sums as a vector and therefore $\mathbf{1} + \mathbf{1} = \mathbf{0}$, $\mathbf{1}$ or $\mathbf{2}$, and therefore it is possible to satisfy both the overall symmetry requirement on the nn system and conservation of angular momentum with (and ono;y with) $L = 1$. Thus the intrinsic parity of the pion is negative since $P_\pi = (-1)^L$.

11.3 Classify the following quantities as either scalars (S), pseudoscalars (P), vectors (V) or axial-vectors (A):

a) mechanical power, $P = \mathbf{F} \cdot \mathbf{v}$;
b) force, $\mathbf{F}$;
c) torque, $\mathbf{G} = \mathbf{r} \times \mathbf{F}$;
d) vorticity, $\boldsymbol{\Omega} = \boldsymbol{\nabla} \times \mathbf{v}$;
e) magnetic flux, $\phi = \int \mathbf{B} \cdot \mathrm{d}\mathbf{S}$;
f) divergence of the electric field strength, $\boldsymbol{\nabla} \cdot \mathbf{E}$.

Classify the following quantities as either scalars (S), pseudoscalars (P), vectors (V) or axial-vectors (A):

a) $P = \mathbf{F} \cdot \mathbf{v}$: **scalar** - scalar product of two vectors ;
b) $\mathbf{F}$: **vector**;

c) $\mathbf{G} = \mathbf{r} \times \mathbf{F}$: **axial-vector** - cross product of two vectors ;

d) $\mathbf{\Omega} = \mathbf{\nabla} \times \mathbf{v}$: **axial-vector** - cross product of two vectors (even if one is a vector operator) ;

e) magnetic flux, $\phi = \int \mathbf{B} \cdot \mathrm{d}\mathbf{S}$: **pseudo-scalar** - scalar product an axial vector ($\mathbf{B}$) with a vector ;

f) divergence of the electric field strength, $\mathbf{\nabla} \cdot \mathbf{E}$: **scalar** - scalar product of two vectors .

11.4 In the annihilation process $e^+e^- \to q\bar{q}$, the QED vector interaction leads to non-zero matrix elements only for the chiral combinations $LR \to LR$, $LR \to RL$, $RL \to RL$, $RL \to LR$. What are the corresponding allowed chiral combinations for S, P and $S - P$ interactions.

First consider a scalar interaction at the e^+e^- vertex for the LR chiral combination $\bar{v}_L u_R$:

$$\begin{aligned}\bar{v}_L u_R &= v_L^\dagger \gamma^0 u_R \\ &= (P_R v_L)^\dagger \gamma^0 P_R u_R\,,\end{aligned}$$

where the appropriate chiral project operators have been inserted (remembering that for antiparticles, $v_L = P_R v$). Hence

$$\begin{aligned}\bar{v}_L u_R &= v_L^\dagger P_R^\dagger \gamma^0 P_R u_R \\ &= v_L^\dagger \tfrac{1}{2}(1+\gamma^5)^\dagger \gamma^0 \tfrac{1}{2}(1+\gamma^5) u_R \\ &= \tfrac{1}{4} v_L^\dagger (1+\gamma^5)\gamma^0(1+\gamma^5) u_R && (\text{using } \gamma^5 = \gamma^{5\dagger}) \\ &= \tfrac{1}{4} v_L^\dagger \gamma^0 (1-\gamma^5)(1+\gamma^5) u_R && (\text{using } \gamma^5\gamma^0 = \gamma^0\gamma^5) \\ &= 0 && (\text{since } (\gamma^5)^2 = 1)\end{aligned}$$

In the same way it is straightforward to show that the RL combination yields zero, but the LL and RR are non-zero. Hence for a scalar interaction, the chiral combinations that contribute to the annihilation process are $LL \to LL$, $LL \to RR$, $RR \to LL$ and $RR \to RR$.

For a pseudoscalar interaction of the form $\bar{v}_L \gamma^5 u_R$:

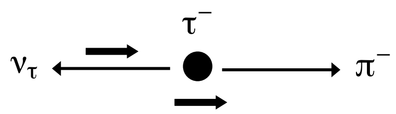
$$\begin{aligned}\bar{v}_L \gamma^5 u_R &= v_L^\dagger P_R^\dagger \gamma^0 \gamma^5 P_R u_R \\ &= v_L^\dagger \tfrac{1}{2}(1+\gamma^5)^\dagger \gamma^0 \gamma^5 \tfrac{1}{2}(1+\gamma^5) u_R \\ &= \tfrac{1}{4} v_L^\dagger (1+\gamma^5)\gamma^0\gamma^5(1+\gamma^5) u_R && (\text{using } \gamma^5 = \gamma^{5\dagger}) \\ &= \tfrac{1}{4} v_L^\dagger \gamma^0 (1-\gamma^5)\gamma^5(1+\gamma^5) u_R && (\text{using } \gamma^5\gamma^0 = \gamma^0\gamma^5) \\ &= \tfrac{1}{4} v_L^\dagger \gamma^0 \gamma^5 (1-\gamma^5)(1+\gamma^5) u_R \\ &= 0\,, && (\text{since } (\gamma^5)^2 = 1)\end{aligned}$$

and thus for a pseudoscalar interaction, the chiral combinations that contribute to the annihilation process are $LL \to LL$, $LL \to RR$, $RR \to LL$ and $RR \to RR$.

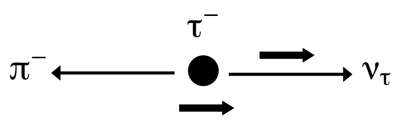
For an $S - P$ interaction, the e^+e^- vertex has a factor $\overline{v}(1 - \gamma^5)u$, and the $1 - \gamma^5$ factor projects out LH chiral particle states and therefore the only chiral combination in the initial state is LL. The $q\overline{q}$ vertex factor is $\overline{u}(1 - \gamma^5)v$. Here the $1 - \gamma^5$ factor projects out RH chiral antiparticle states. Hence, for an $S - P$ interaction the annihilation process, the only non-zero contribution to the amplitude comes from $RR \to LL$.

11.5 Consider the decay at rest $\tau^- \to \pi^-\nu_\tau$, where the spin of the tau is in the positive z-direction and the ν_τ and π^- travel in the $\pm\, z$-directions. Sketch the allowed spin configurations assuming that the form of the weak charged-current interaction is i) $V - A$ and ii) $V + A$.

i) Here the $V - A$ for of the interaction projects out LH particle states and RH antiparticle states. Hence in the decay $\tau^- \to \pi^-\nu_\tau$, the neutrino is produced in a LH chiral state. Since the neutrino is almost massless, it is highly relativistic and the chiral and helicity states are the same. Hence the neutrino must be produced in a LH helicity state and the allowed spin combination is:



ii) Here the $V + A$ for of the interaction projects out RH particle states and LH antiparticle states. Hence in the decay $\tau^- \to \pi^-\nu_\tau$, the neutrino would now be produced in a RH chiral state:



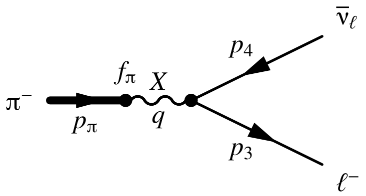
11.6 Repeat the pion decay calculation for a pure scalar interaction and show that the predicted ratio of decay rates is

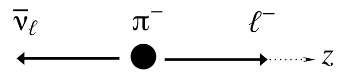
$$\frac{\Gamma(\pi^- \to e^-\overline{\nu}_e)}{\Gamma(\pi^- \to \mu^-\overline{\nu}_\mu)} \approx 5.5\,.$$

Consider the $\pi^- \to \ell^-\overline{\nu}_\ell$ decay in its rest frame, where the direction of the charged lepton defines the z-axis, as shown below. In this case, the four-momenta of the π^-, ℓ^- and $\overline{\nu}_\ell$ are respectively,

$$p_\pi = (m_\pi, 0, 0, 0)\,, \quad p_\ell = p_3 = (E_\ell, 0, 0, \mathrm{p}) \quad \text{and} \quad p_{\overline{\nu}} = p_4 = (\mathrm{p}, 0, 0, -\mathrm{p})\,,$$

where p is the magnitude of the momentum of both the charged lepton and antineutrino in the centre-of-mass frame. Here it is assumed that the interaction is mediated by a massive scalar particle X.





The scalar quantity associated with the $\ell^-\overline{\nu}_\ell$ vertex, with coupling g_Z, is

$$j_\ell = g_X \overline{u}(p_3) v(p_4)\,.$$

Because the pion is a bound $q\overline{q}$ state, the corresponding hadronic "current" cannot be expressed in terms of free particle Dirac spinors. For the overall amplitude to be a Lorentz invariant quantity (i.e. a scalar), the hadronic vertex has to be described by a scalar quantity (just as was the case of the leptonic vertex). The only scalar quantity available is the mass of the pion, hence, the most general expression for the pion current is obtained by replacing $\overline{v}u$ with $f_\pi m_\pi$, where f_π is a constant associated with the decay. The matrix element for the decay $\pi^- \to \ell^-\overline{\nu}_\ell$ therefore can be written as

$$\begin{aligned}\mathcal{M}_{fi} &= [g_X f_\pi m_\pi] \times \left[\frac{1}{m_X^2}\right] \times [g_X \overline{u}(p_3) v(p_4)] \\ &= G_X f_\pi m_\pi \overline{u}(p_3) v(p_4)\,,\end{aligned}$$

where the propagator has been approximated by a Fermi-like contact interaction (assuming $q^2 = m_\pi^2 \ll m_X^2$) and G_X is analogous to the Fermi constant.

In the Standard Model, there are only LH neutrinos and RH antineutrinos, and we will assume this is the case here (this assumption doesn't affect the final result). Thus, the antineutrino is produced in a RH chiral (and helicity) state. To conserve angular momentum, the lepton must also be right-handed. Thus

$$\mathcal{M}_{fi} = G_X f_\pi m_\pi u_\uparrow^\dagger(p_3) \gamma^0 v_\uparrow(p_4)\,,$$

with

$$u_\uparrow(p_3) = \sqrt{E_\ell + m_\ell}\begin{pmatrix}1\\0\\\frac{\mathrm{p}}{E_\ell+m_\ell}\\0\end{pmatrix} \quad\text{and}\quad v_\uparrow(p_4) = \sqrt{\mathrm{p}}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}.$$

Hence

$$\begin{aligned}\mathcal{M}_{fi} &= G_X f_\pi m_\pi u_\uparrow^\dagger(p_3)\gamma^0 v_\uparrow(p_4)\\ &= G_X f_\pi m_\pi \sqrt{E_\ell + m_\ell}\,\sqrt{\mathrm{p}}\left(1 + \frac{\mathrm{p}}{E_\ell + m_\ell}\right)\\ \mathcal{M}_{fi}^2 &= G_X^2 f_\pi^2 m_\pi^2 (E_\ell + m_\ell)\mathrm{p}\left(1 + \frac{\mathrm{p}}{E_\ell + m_\ell}\right)^2 .\end{aligned}$$

The above expression can be can be simplified using the expressions for E_ℓ and p given in (11.19) of the main text, such that

$$\begin{aligned}\mathcal{M}_{fi}^2 &= G_X^2 f_\pi^2 m_\pi^2 (E_\ell + m_\ell)\mathrm{p}\left(1 + \frac{\mathrm{p}}{E_\ell + m_\ell}\right)^2\\ &= G_X^2 f_\pi^2 m_\pi^2 \mathrm{p}\left(\frac{m_\pi + m_\ell}{2m_\pi}\right)^2 \frac{4m_\pi^2}{(m_\pi + m_\ell)^2}\\ &= G_X^2 f_\pi^2 m_\pi^2 \mathrm{p} .\end{aligned}$$

Since the pion is a spin-0 particle, there is no need to average over the initial-state spins and matrix element squared is given by

$$\langle|\mathcal{M}_{fi}|^2\rangle \equiv |\mathcal{M}_{fi}|^2 = G_X^2 f_\pi^2 m_\pi^2 \mathrm{p} .$$

Finally, the decay rate can be determined from the expression for the two-body decay rate given by (3.49), where the integral over solid angle introduces a factor of 4π as there is no angular dependence in $\langle|\mathcal{M}_{fi}|^2\rangle$. Hence

$$\Gamma = \frac{4\pi}{32\pi^2 m_\pi^2}\mathrm{p}\,\langle|\mathcal{M}_{fi}|^2\rangle = \frac{G_X^2 f_\pi^2}{8\pi}\mathrm{p}^2 .$$

Therefore, to lowest order, the predicted ratio of the $\pi^- \to e^-\overline{\nu}_e$ to $\pi^- \to \mu^-\overline{\nu}_\mu$ decay rates is

$$\frac{\Gamma(\pi^- \to e^-\overline{\nu}_e)}{\Gamma(\pi^- \to \mu^-\overline{\nu}_\mu)} = \frac{p_e^2}{p_\mu^2} = \left[\frac{(m_\pi^2 - m_e^2)}{(m_\pi^2 - m_\mu^2)}\right]^2 = 5.49 .$$

11.7 Predict the ratio of the $K^- \to e^-\overline{\nu}_e$ and $K^- \to \mu^-\overline{\nu}_\mu$ weak interaction decay rates and compare your answer to the measured value of

$$\frac{\Gamma(K^- \to e^-\overline{\nu}_e)}{\Gamma(K^- \to \mu^-\overline{\nu}_\mu)} = (2.488 \pm 0.012) \times 10^{-5} .$$

From the result derived in the text for pion decay, the predicted ratio of the two leptonic decays of the charged kaon is

$$\frac{\Gamma(K^- \to e^-\overline{\nu}_e)}{\Gamma(K^- \to \mu^-\overline{\nu}_\mu)} = \left[\frac{m_e(m_K^2 - m_e^2)}{m_\mu(m_K^2 - m_\mu^2)}\right]^2 = 2.55 \times 10^{-5} ,$$

which is in reasonable agreement with the observed ratio; given the calculation is only to lowest-order, perfect agreement is not expected. Note that the decay to electrons is even more suppressed relative to $\pi^- \to e^-\overline{\nu}_e$. This is because the final-state electron is more relativistic and, therefore, the helicity states correspond more closely to pure chiral states.

11.8 Charged kaons have several weak interaction decay modes, the largest of which are

$$K^+(u\overline{s}) \to \mu^+\nu_\mu\,, \quad K^+ \to \pi^+\pi^0 \quad \text{and} \quad K^+ \to \pi^+\pi^+\pi^-\,.$$

a) Draw the Feynman diagrams for these three weak decays.

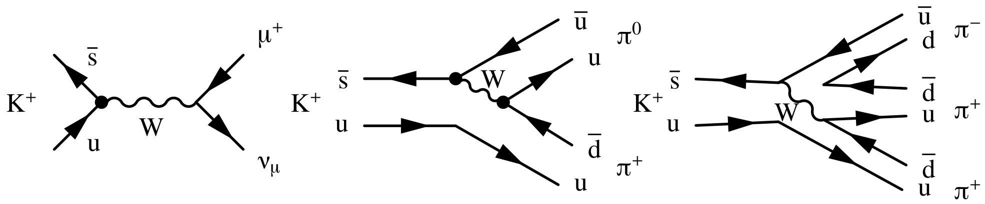
b) Using the measured branching ratio

$$Br(K^+ \to \mu^+\nu_\mu) = 63.55 \pm 0.11\,\%\,,$$

estimate the lifetime of the charged kaon.

Note: charged pions decay almost 100 % of the time by the weak interaction $\pi^+ \to \mu^+\nu_\mu$ and have a lifetime of $(2.6033 \pm 0.0005) \times 10^{-8}$ s.

a) The lowest-order quark-level Feynman diagrams are:



In the final diagram the $d\overline{d}$ pair are produced from a gluon in the hadronisation process.

b) From the derivation in the main text

$$1/\tau_{\pi^-} = \Gamma_\pi^+ = \Gamma(\pi^+ \to \mu^+\nu_\mu) = \frac{G_F^2}{8\pi m_\pi^3} f_\pi^2 \left[m_\ell(m_\pi^2 - m_\ell^2)\right]^2\,,$$

and the total K^+ decay width is related to the partial decay width $\Gamma(K^+ \to \mu^-\overline{\nu}_\mu)$ by

$$BR(K^+ \to \mu^+\nu_\mu) = \frac{\Gamma(K^+ \to \mu^+\nu_\mu)}{\Gamma_{K^+}}$$

Hence

$$\frac{\tau_{K^+}}{\tau_{\pi^+}} = \frac{\Gamma_{\pi^+}}{\Gamma_{K^+}}$$
$$= \frac{\Gamma(\pi^+ \to \mu^+\nu_\mu)}{\Gamma(K^+ \to \mu^+\nu_\mu)} \times BR(K^+ \to \mu^+\nu_\mu)$$
$$= \frac{m_K^3}{m_\pi^3} \frac{f_\pi^2 \left[(m_\pi^2 - m_\mu^2)\right]^2}{f_K^2 \left[(m_K^2 - m_\mu^2)\right]^2} .$$

Making the brave assumption that $f_K \approx f_\pi$ (both are pseudo scalar mesons) and putting in the numbers

$$\tau_{K^+} \approx 0.05\tau_{\pi^+}$$
$$= 1.3 \times 10^{-9}\,\mathrm{s}$$

This is a factor 10 shorter than the measured value of $\tau_{K^+} = 1.2\times10^{-8}$ s because the K^+ decay rate is suppressed by a factor of $\tan^2\theta_C = 0.053$ (see Chapter 14) relative to the π^+ decay rate; in charged kaon decay the weak decay vertex is $\bar{s} \to \bar{u}$, whereas for pion decay it is $\bar{d} \to \bar{u}$.

11.9 From the prediction of (11.25) and the above measured value of the charged pion lifetime, obtain a value for f_π.

From the derivation in the main text

$$\Gamma(\pi^+ \to \mu^+\nu_\mu) = \frac{G_F^2}{8\pi m_\pi^3} f_\pi^2 \left[m_\mu(m_\pi^2 - m_\mu^2)\right]^2 .$$

Assuming the π^+ decays almost entirely by $\pi^+ \to \mu^+\nu_\mu$, the decay rate is given by:

$$\Gamma(\pi^+ \to \mu^+\nu_\mu) \approx \Gamma_{\pi^+} = \frac{\hbar}{\tau_{\pi^+}} = 2.54 \times 10^{-17}\,\mathrm{GeV} .$$

Putting in the numerical values,

$$f_\pi \approx 0.135\,\mathrm{GeV} .$$

It should be noted that $f_\pi \sim m_\pi$.

11.10 Calculate the partial decay width for the decay $\tau^- \to \pi^-\nu_\tau$ in the following steps:

a) Draw the Feynman diagram and show that the corresponding matrix element is

$$\mathcal{M} \approx \sqrt{2} G_F f_\pi \bar{u}(p_\nu)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)u(p_\tau)g_{\mu\nu}p_\pi^\nu .$$

b) Taking the τ^- spin to be in the z-direction and the four-momentum of the neutrino to be

$$p_\nu = \mathrm{p}^*(1, \sin\theta, 0, \cos\theta) ,$$

show that the leptonic current is

$$j^{\mu} = \sqrt{2m_{\tau}p^{*}}\,(-s, -c, -ic, s)\ ,$$

where $s = \sin\left(\frac{\theta}{2}\right)$ and $c = \cos\left(\frac{\theta}{2}\right)$. Note that for this configuration, the spinor for the τ^- can be taken to be u_1 for a particle at rest.

c) Write down the four-momentum of the π^- and show that

$$|\mathcal{M}|^2 = 4G_F^2 f_\pi^2 m_\tau^3 p^* \sin^2\left(\frac{\theta}{2}\right)\ .$$

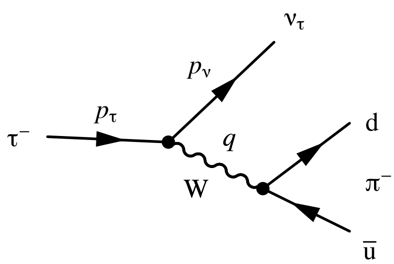
d) Hence show that

$$\Gamma(\tau^- \to \pi^-\nu_\tau) = \frac{G_F^2 f_\pi^2}{16\pi} m_\tau^3 \left(\frac{m_\tau^2 - m_\pi^2}{m_\tau^2}\right)^2\ .$$

e) Using the value of f_π obtained in the previous problem, find a numerical value for $\Gamma(\tau^- \to \pi^-\nu_\tau)$.

f) Given that the lifetime of the τ-lepton is measured to be $\tau_\tau = 2.906 \times 10^{-13}$ s, find an approximate value for the $\tau^- \to \pi^-\nu_\tau$ branching ratio.

a) The Feynman diagram for the $\tau^- \to \pi^-\nu_\tau$ decay is Following the arguments for



π decay, the interaction at the pion weak vertex is described by $f_\pi p_\pi^\nu$ and therefore the matrix element for $\tau^- \to \pi^-\nu_\tau$ can be written as

$$\mathcal{M} = -\left[\frac{g_W}{\sqrt{2}}\overline{u}(p_\nu)\tfrac{1}{2}\gamma^\mu(1-\gamma^5)u(p_\tau)\right] \times \left[\frac{g^{\mu\nu}}{q^2 - m_W^2}\right] \times \left[\frac{g_W}{\sqrt{2}}\tfrac{1}{2}f_\pi p_\pi^\nu\right].$$

Making the good approximation that $q^2 \ll m_W^2$ this becomes

$$\begin{aligned}\mathcal{M} &= \frac{g_W^2}{4m_W^2} f_\pi \overline{u}(p_\nu)\tfrac{1}{2}\gamma^\mu(1-\gamma^5)u(p_\tau)\, g^{\mu\nu}\, p_\pi^\nu \\ &= \sqrt{2}G_F\, f_\pi\, \overline{u}(p_\nu)\tfrac{1}{2}\gamma^\mu(1-\gamma^5)u(p_\tau)\, g^{\mu\nu} . p_\pi^\nu\end{aligned}$$

b) For the tau-lepton, with spin in the $+z$-direction, the appropriate spinor is u_1 for a particle at rest and the factor $\frac{1}{2}(1-\gamma^5)$ projects out the left-handed chiral state of the neutrino, which is equivalent to the left-handed helicity state. Hence, the leptonic current associated with the above matrix element is

$$\begin{aligned} j^\mu &= \overline{u}(p_\nu)\tfrac{1}{2}\gamma^\mu(1-\gamma^5)u(p_\tau) \\ &= u_\downarrow^\dagger(p_\nu)\gamma^0\gamma^\mu u_\uparrow(p_\tau) \\ &= \sqrt{p^*}\,(-s,c,s,-c)\,\gamma^0\gamma^\mu\,\sqrt{2m_\tau}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \end{aligned}$$

Using the explicit forms for the γ-matrices, it is straightforward to show

$$j^\mu = \sqrt{2p^*m_\tau}\,(-s,-c,-ic,s)\,,$$

where $s = \sin(\theta/2)$ and $c = \cos(\theta/2)$.

c) The four-momentum of the pion is

$$p_\pi^\nu = (E_\pi, -p^*\sin\theta, 0, -p^*\cos\theta)\,,$$

and

$$\begin{aligned} \mathcal{M} &= \sqrt{2}G_\mathrm{F}\sqrt{2m_\tau p^*}f_\pi \times (j\cdot p) \\ &= 2G_\mathrm{F}\sqrt{m_\tau p^*}f_\pi\left(-E_\pi\sin\tfrac{\theta}{2} - p^*\sin\theta\cos\tfrac{\theta}{2} + p^*\cos\theta\sin\tfrac{\theta}{2}\right) \\ &= 2G_\mathrm{F}\sqrt{m_\tau p^*}f_\pi\left(-E_\pi\sin\tfrac{\theta}{2} - p^*\sin\tfrac{\theta}{2}\right) \\ &= 2G_\mathrm{F}\sqrt{m_\tau p^*}f_\pi m_\tau\sin\tfrac{\theta}{2}\,, \end{aligned}$$

where the last step follows from conservation of energy $m_\tau = E_\pi + E_\nu = E_\pi + p^*$, which gives the desired result:

$$|\mathcal{M}|^2 = 4G_\mathrm{F}^2 f_\pi^2 m_\tau^3 \mathrm{p}^* \sin^2\tfrac{\theta}{2}\,.$$

d) The general expression for the decay rate for $a \to 1+2$ is

$$\Gamma = \frac{p^*}{32\pi^2 m_a^2}\int |\mathcal{M}|^2\,\mathrm{d}\Omega\,.$$

Note there is no need to average over the initial spin states, it is perfectly reasonable to consider the decay rate from a particular spin configuration. Here The general expression for the decay rate for $a \to 1 + 2$ is

$$\begin{aligned}
\Gamma &= \frac{p^*}{32\pi^2 m_\tau^2} \times 4G_F^2 f_\pi^2 m_\tau^3 p^* \times \int_{-1}^{+1} \int_{\phi=0}^{2\pi} \sin^2 \tfrac{\theta}{2} \mathrm{d}(\cos\theta)\,\mathrm{d}\phi \\
&= \frac{p^{*2} G_F^2 f_\pi^2 m_\tau}{8\pi^2} \times 2\pi \int_{-1}^{+1} \tfrac{1}{2}(1 - \cos\theta)\,\mathrm{d}(\cos\theta) \\
&= \frac{p^{*2} G_F^2 f_\pi^2 m_\tau}{8\pi} \times \int_{-1}^{+1} (1 - x)\,\mathrm{d}x \\
&= \frac{p^{*2} G_F^2 f_\pi^2 m_\tau}{4\pi}\,.
\end{aligned}$$

It is straightforward to show that

$$p^* = \frac{m_\tau^2 - m_\pi^2}{2m_\tau}\,,$$

and therefore

$$\begin{aligned}
\Gamma(\tau^- \to \pi^- \nu_\tau) &= \frac{G_F^2 f_\pi^2}{16\pi} m_\tau \left(\frac{m_\tau^2 - m_\pi^2}{m_\tau}\right)^2 \\
&= \frac{G_F^2 f_\pi^2}{16\pi} m_\tau^3 \left(\frac{m_\tau^2 - m_\pi^2}{m_\tau^2}\right)^2\,.
\end{aligned}$$

e) Taking $f_\pi = m_\pi$ the expression for $\Gamma(\tau^- \to \pi^- \nu_\tau)$ gives:

$$\Gamma(\tau^- \to \pi^- \nu_\tau) = 2.93 \times 10^{-13}\,\mathrm{GeV}\,.$$

f) From the observed value of $\tau_\tau = 2.906 \times 10^{-13}$, the total decay width of the tau lepton is:

$$\Gamma_\tau = \frac{\hbar}{\tau_\tau} = 2.28 \times 10^{-12}\,\mathrm{GeV}\,.$$

Using this and the calculated value of $\Gamma(\tau^- \to \pi^- \nu_\tau) = 2.93 \times 10^{-13}$ GeV,

$$BR(\tau^- \to \pi^- \nu_\tau) = \frac{\Gamma(\tau^- \to \pi^- \nu_\tau)}{\Gamma_\tau} = 11.9\,\%\,,$$

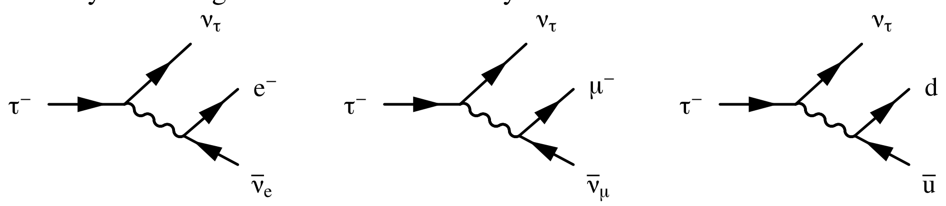
which is in fair agreement with the measured value of $10.83 \pm 0.06\,\%$.

12 The Weak Interactions of Leptons

12.1 Explain why the tau lepton branching ratios are observed to be approximately

$$Br(\tau^- \to e^-\nu_\tau\overline{\nu}_e) \;:\; Br(\tau^- \to e^-\nu_\tau\overline{\nu}_\mu) \;:\; Br(\tau^- \to \nu_\tau + \text{hadrons}) \approx 1 \;:\; 1 \;:\; 3\,.$$

The Feynman diagrams for the main decay modes are shown below: In the case



of the hadronic decays the $d\overline{u}$ system can form a π^- with $J^P = 0^-$, a ρ^- with $J^P = 1^-$, or system of light mesons produced through an intermediate mesonic state or through the hadronisation of the $d\overline{u}$ system. Assuming a universal strength for the weak interaction vertex (i.e. putting aside the discussion of the CKM matrix for now), the amplitudes for the three decays shown would be expected to be roughly equal. Thus, remembering that the diagram with quarks is repeated for the three separate colour combinations $r\overline{r}$, $b\overline{b}$ and $g\overline{g}$ one would expect

$$Br(\tau^- \to e^-\nu_\tau\overline{\nu}_e) \;:\; Br(\tau^- \to e^-\nu_\tau\overline{\nu}_\mu) \;:\; Br(\tau^- \to \nu_\tau + \text{hadrons}) \approx 1 \;:\; 1 \;:\; 3\,.$$

In reality, the branching ratios are:

$$\begin{aligned} &Br(\tau^- \to e^-\nu_\tau\overline{\nu}_e) \;:\; Br(\tau^- \to e^-\nu_\tau\overline{\nu}_\mu) \;:\; Br(\tau^- \to \nu_\tau + \text{hadrons}) \\ &\qquad\approx 17.83\,\% \;:\; 17.41\,\% \;:\; 62.65\,\%\,, \end{aligned}$$

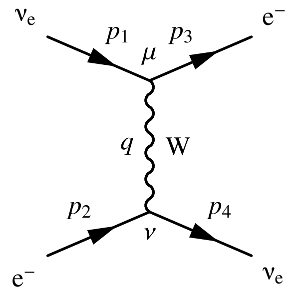
with the additional 2 % being due to $\tau^- \to e^-\nu_\tau\overline{\nu}_e\gamma$ and $\tau^- \to \mu^-\nu_\tau\overline{\nu}_\mu\gamma$. The BR to muons is slightly lower than that to electrons, due to mass of the muon reducing the available phase space. Furthermore, the branching ratio to hadrons is slightly higher than the naive prediction do to enhancements from QCD corrections (for example the diagram with an additional gluon in the final state).

12.2 Assuming that the process $\nu_e e^- \to e^-\nu_e$ only occurs by the weak charged-current interaction (*i.e.* ignoring the Z-exchange neutral-current process), show that

$$\sigma_{CC}^{\nu_e e^-} \approx \frac{2m_e E_\nu G_F^2}{\pi}\,,$$

where E_ν is neutrino energy in the laboratory frame in which the struck e^- is at rest.

The Feynman diagram for the CC $\nu_e e^- \to e^- \nu_e$ process is shown below. Ne-



glecting the q^2 dependence of the propagator, the corresponding matrix element is

$$-i\mathcal{M}_{fi} = \left[-i\frac{g_W}{\sqrt{2}}\overline{u}(p_3)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)u(p_1)\right]\frac{ig_{\mu\nu}}{m_W^2}\left[-i\frac{g_W}{\sqrt{2}}\overline{u}(p_4)\gamma^\nu \tfrac{1}{2}(1-\gamma^5)u(p_2)\right]$$

$$\mathcal{M}_{fi} = \frac{g_W^2}{2m_W^2}g_{\mu\nu}\left[\overline{u}(p_3)\gamma^\mu\tfrac{1}{2}(1-\gamma^5)u(p_1)\right]\left[\overline{u}(p_4)\gamma^\nu\tfrac{1}{2}(1-\gamma^5)u(p_2)\right]$$

$$= \frac{g_W^2}{2m_W^2}g_{\mu\nu}\left[\overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1)\right]\left[\overline{u}_\downarrow(p_4)\gamma^\nu u_\downarrow(p_2)\right] ,$$

with the spinors,

$$u_\downarrow(p_1) = \sqrt{E}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix},\ u_\downarrow(p_2) = \sqrt{E}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix},\ u_\downarrow(p_3) = \sqrt{E}\begin{pmatrix}-s\\c\\s\\-c\end{pmatrix}\ u_\downarrow(p_4) = \sqrt{E}\begin{pmatrix}-c\\-s\\c\\s\end{pmatrix},$$

where $c = \cos\frac{\theta^*}{2}$ and $s = \sin\frac{\theta^*}{2}$. If this were the only Feynman diagram contributing to the process $\nu_e e^- \to \nu_e e^-$, the following the derivation of Chapter 12.2.1, one would obtain

$$\sigma_{CC}(\nu_e e^- \to \nu_e e^-) = \frac{G_F^2 s}{\pi} .$$

It should be noted that this neglects the NC Z-exchange diagram and that $\mathcal{M} \to \mathcal{M}_{CC} + \mathcal{M}_{NC}$, which has the effect to reduce the $\nu_e e^- \to \nu_e e^-$ cross section through negative interference. Carrying on regardless, the centre-of-mass energy is

$$s = (p_1 + p_2)^2 = (E_\nu + m_e)^2 - E_\nu^2 = 2m_e E_\nu + m_e^2 \approx 2m_e E_\nu ,$$

and therefore

$$\sigma_{CC}(\nu_e e^- \to \nu_e e^-) = \frac{2m_e E_\nu G_F^2}{\pi} .$$

12.3 Using the above result, estimate the probability that a 10 MeV Solar ν_e will

undergo a charged-current weak interaction with an electron in the Earth if it travels along a trajectory passing through the centre of the Earth.
Take the Earth to be a sphere of radius 6400 km and uniform density $\rho = 5520\,\mathrm{kg\,m^{-3}}$.

From the previous question the "CC cross section" for a 10 MeV electron neutrino is given by

$$\sigma_{CC}(\nu_e e^- \to \nu_e e^-) = \frac{2m_e E_\nu G_F^2}{\pi} = 1.3 \times 10^{-48}\,\mathrm{cm}^2\,.$$

A neutrino traversing the Earth passes through 12800 km of rock. Consider a cylinder of this length and area 1 cm^2. The number of nucleons contained in this cylinder will be:

$$n_N = \frac{\rho V}{m_N} = \frac{5520 \cdot 1 \times 10^{-4} \cdot 12.8 \times 10^6}{6.67 \times 10^{-27}} \simeq 1.1 \times 10^{33}\,.$$

Assuming half the nucleons are protons (for which there will be an equal number of electrons), the number of electrons is therefore

$$n_e \simeq 5 \times 10^{32}\,\mathrm{cm}^{-2}\,.$$

Hence the probability of an interaction is

$$P = \sigma_{CC}(\nu_e e^- \to \nu_e e^-)[\mathrm{cm}^2] \times n_e[\mathrm{cm}^{-2}] < 10^{-15}\,.$$

12.4 By equating the powers of y in (12.34) with those in the parton model prediction of (12.20), show that the structure functions can be expressed as

$$F_2^{\nu p} = 2xF_1^{\nu p} = 2x\left[d(x) + \overline{u}(x)\right] \quad \text{and} \quad xF_3^{\nu p} = 2x\left[d(x) - \overline{u}(x)\right]\,.$$

Equating (12.34) and (12.20) gives

$$\begin{aligned}
\frac{G_F^2}{\pi} sx\left[d(x) + (1-y)^2\overline{u}(x)\right] &= \frac{G_F^2}{2\pi} s\left[(1-y)F_2^{\nu p} + xy^2F_1^{\nu p} + xy\left(1 - \frac{y}{2}\right)F_3^{\nu p}\right] \\
2xd(x) + 2x(1-y)^2\overline{u}(x) &= (1-y)F_2^{\nu p} + xy^2F_1^{\nu p} + xy\left(1 - \frac{y}{2}\right)F_3^{\nu p} \\
2x(d(x) + \overline{u}(x)) - 4xy\overline{u}(x) + 2xy^2\overline{u}(x) &= F_2^{\nu p} + y(xF_3^{\nu p} - F_2^{\nu p}) + y^2(xF_1^{\nu p})
\end{aligned}$$

Equating powers of y^0 gives:

$$F_2^{\nu p} = 2x(d(x) + \overline{u}(x))\,.$$

Equating powers of y^1 gives:

$$\begin{aligned}
4x\overline{u}(x) &= F_2^{\nu p} - xF_3^{\nu p} \\
&= 2x(d(x) + \overline{u}(x)) - xF_3^{\nu p} \\
\Rightarrow \quad F_3^{\nu p} &= 2(d(x) - \overline{u}(x))\,.
\end{aligned}$$

Finally, equating powers of y^2 gives:

$$\begin{aligned} F_1^{\nu p} &= \tfrac{1}{2}F_3^{\nu p} + 2\overline{u}(x) \\ &= d(x) - \overline{u}(x) + 2\overline{u}(x) \\ &= d(x) + \overline{u}(x)\,, \end{aligned}$$

and therefore

$$F_2^{\nu p} = 2xF_1^{\nu p} = 2x\left[d(x) + \overline{u}(x)\right] \quad \text{and} \quad xF_3^{\nu p} = 2x\left[d(x) - \overline{u}(x)\right]\,.$$

12.5 In the quark-parton model, show that $F_2^{eN} = \frac{1}{2}(Q_u^2 + Q_d^2)F_2^{\nu N}$.
Hence show that the measured value of

$$F_2^{eN}/F_2^{\nu N} = 0.29 \pm 0.02\,,$$

is consistent with the up and down quarks having respective charges of $+2/3$ and $-1/3$.

From the previous question,

$$F_2^{\nu p} = 2x\left[d(x) + \overline{u}(x)\right]\,.$$

Similarly, assuming isospin symmetry (and remembering that PDFs without superscripts refer to the proton):

$$F_2^{\nu n} = 2x\left[d^n(x) + \overline{u}^n(x)\right] = 2x\left[u(x) + \overline{d}(x)\right]\,.$$

Hence

$$F_2^{\nu N} = \tfrac{1}{2}(F_2^{\nu p} + F_2^{\nu n}) = x\left[u(x) + d(x) + \overline{u}(x) + \overline{d}(x)\right]\,.$$

From the discussion of deep inelastic electron-nucleon scattering in Chapter 8, the corresponding expressions for F_2, given by (8.27) and (8.28), are

$$\begin{aligned} F_2^{ep}(x) &= x\left(Q_u^2 u(x) + Q_d^2 d(x) + Q_u^2 \overline{u}(x) + Q_d^2 \overline{d}(x)\right)\,, \\ F_2^{en}(x) &= x\left(Q_u^2 d(x) + Q_d^2 u(x) + Q_u^2 \overline{d}(x) + Q_d^2 \overline{u}(x)\right)\,, \end{aligned}$$

and thus

$$\begin{aligned} F_2^{eN} &= \tfrac{1}{2}\left[F_2^{ep}(x) + F_2^{en}(x)\right] \\ &= \tfrac{1}{2}(Q_u^2 + Q_d^2)x\left[u(x) + d(x) + \overline{u}(x) + \overline{d}(x)\right] \\ &= \tfrac{1}{2}(Q_u^2 + Q_d^2)F_2^{\nu N} \end{aligned}$$

For $Q_u = +2/3$ and $Q_d = -1/3$ the parton model predicts:

$$F_2^{eN}/F_2^{\nu N} = \tfrac{1}{2}(Q_u^2 + Q_d^2) = \tfrac{5}{18} = 0.278,$$

consistent with the measured value of 0.29 ± 0.02.

12.6 Including the contributions from strange quarks, the neutrino-nucleon scattering structure functions can be expressed as

$$F_2^{\nu p} = 2x[d(x) + s(x) + \overline{u}(x)] \quad \text{and} \quad F_2^{\nu n} = 2x[u(x) + \overline{d}(x) + \overline{s}(x)]\,,$$

where $s(x)$ and $\overline{s}(x)$ are respectively the strange and anti-strange quark PDFs of the nucleon. Assuming $s(x) = \overline{s}(x)$, obtain an expression for $xs(x)$ in terms of the structure functions for neutrino-nucleon and electron-nucleon scattering

$$F_2^{\nu N} = \tfrac{1}{2}\left(F_2^{\nu p}(x) + F_2^{\nu n}(x)\right) \quad \text{and} \quad F_2^{eN} = \tfrac{1}{2}\left(F_2^{ep}(x) + F_2^{en}(x)\right).$$

Including the strange quark contribution

$$\begin{aligned} F_2^{\nu N} &= \tfrac{1}{2}\left(F_2^{\nu p}(x) + F_2^{\nu n}(x)\right) \\ &= x\left[u(x) + d(x) + \overline{u}(x) + \overline{d}(x)\right] + x\left[s(x) + \overline{s}(x)\right]. \end{aligned} \tag{12.1}$$

Including the strange quark contribution (which is assumed to be the same for the proton and neutron)

$$\begin{aligned} F_2^{ep}(x) &= x\left(Q_u^2 u(x) + Q_d^2 d(x) + Q_s^2 s(x) + Q_u^2\overline{u}(x) + Q_d^2\overline{d}(x) + Q_s^2\overline{s}(x)\right), \\ F_2^{en}(x) &= x\left(Q_u^2 d(x) + Q_d^2 u(x) + Q_s^2 s(x) Q_u^2\overline{d}(x) + Q_d^2\overline{u}(x) + Q_s^2\overline{s}(x)\right), \end{aligned}$$

and therefore

$$\begin{aligned} F_2^{eN} &= \tfrac{1}{2}\left(F_2^{ep}(x) + F_2^{en}(x)\right) \\ &= \tfrac{1}{2}x(Q_u^2 + Q_d^2)\left[u(x) + d(x) + \overline{u}(x) + \overline{d}(x)\right] + xQ_s^2(s(x) + \overline{s}(x)) \\ &= \tfrac{5}{18}x\left[u(x) + d(x) + \overline{u}(x) + \overline{d}(x)\right] + \tfrac{1}{9}x(s(x) + \overline{s}(x)). \end{aligned}$$

Combining with the corresponding result for $F_2^{\nu N}$ given in (12.1), leads to

$$\begin{aligned} \tfrac{5}{18}F_2^{\nu N} - F_2^{eN} &= \left[\tfrac{5}{18} - \tfrac{1}{9}\right]x(s(x) + \overline{s}(x)) \\ &= \tfrac{3}{18}x(s(x) + \overline{s}(x)). \end{aligned}$$

Assuming $s(x) = \overline{s}(x)$,

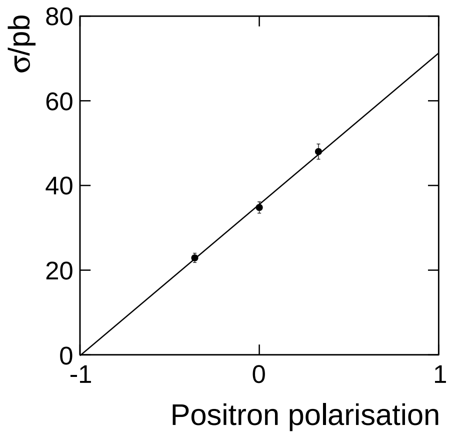
$$\begin{aligned} \tfrac{5}{18}F_2^{\nu N} - F_2^{eN} &= \tfrac{6}{18}xs(x) \\ \Rightarrow \quad xs(x) &= \tfrac{5}{6}F_2^{\nu N} - 3F_2^{eN}. \end{aligned}$$

12.7 The H1 and ZEUS experiments at HERA measured the cross sections for the charged-current processes $e^-p \to \nu_e X$ and $e^+p \to \overline{\nu}_e X$ for different degrees of electron longitudinal polarisation. For example, the ZEUS measurements of the total $e^+p \to \overline{\nu}_e X$ cross section at $\sqrt{s} = 318\,\text{GeV}$ and $Q^2 > 200\,\text{GeV}^2$ for positron polarisations of $P_e =$ -36 %, 0 % and +33 % are:

$$\sigma(-0.36) = 22.9 \pm 1.1\,\text{pb}, \quad \sigma(0) = 34.8 \pm 1.34\,\text{pb} \quad \text{and} \quad \sigma(+0.33) = 48.0 \pm 1.8\,\text{pb},$$

see Abramowicz *et al.* (2010) and references therein. Plot these data and predict the corresponding cross section for $P_e = -1.0$, *i.e.* when the positrons are all left-handed. What does this tell you about the nature of the weak charged-current interaction?

The data are plotted in the figure below, along with a linear fit (χ^2-minimization). The linear fit has $\chi^2 = 0.48$ for one degree of freedom, and therefore the data are consistent with the hypothesis that he cross section depends linearly on the degree of positron polarisation. The fit results indicate that the cross section is expected to be zero for $P(e^+) = -1$ when the positrons are all left-handed. Consequently the data support the hypothesis that the weak charged current only couples to RH antiparticles and thus has the form $V - A$. Add the weak charged current been of the form $V + A$ a negative slope with intercept at $P(e^+) = +1$ would have been observed.



13 Neutrinos and Neutrino Oscillations

13.1 By writing $p_1 = \beta E_1$ and $p_2 = \beta E_2$, and assuming $\beta_1 = \beta_2 = \beta$, show that equation (13.13) reduces to (13.12), *i.e.*

$$\Delta\phi_{12} = (E_1 - E_2)\left[T - \left(\frac{E_1 + E_2}{p_1 + p_2}\right)L\right] + \left(\frac{m_1^2 - m_2^2}{p_1 + p_2}\right)L \approx \frac{m_1^2 - m_2^2}{2p}L\,,$$

where $p = p_1 \approx p_2$ and it is assumed that $p_1 \gg m_1$ and $p_2 \gg m_2$.

Assuming the mass eigenstates propagate with equal velocity, $\beta_1 = \beta_2 = \beta$, and $T = L/\beta$, the expression for the phase difference of equation (13.13) can be written

$$\begin{aligned}
\Delta\phi_{12} &= (E_1 - E_2)\left[T - \left(\frac{E_1 + E_2}{p_1 + p_2}\right)L\right] + \left(\frac{m_1^2 - m_2^2}{p_1 + p_2}\right)L \\
&= (E_1 - E_2)\left[\frac{L}{\beta} - \left(\frac{E_1 + E_2}{\beta E_1 + \beta E_2}\right)L\right] + \left(\frac{m_1^2 - m_2^2}{p_1 + p_2}\right)L \\
&= \left(\frac{m_1^2 - m_2^2}{p_1 + p_2}\right)L \\
&\approx \frac{m_1^2 - m_2^2}{2p}L\,.
\end{aligned}$$

13.2 Show that when L is given in km and Δm^2 is given in eV^2, the two-flavour oscillation probability expressed in natural units becomes

$$\sin^2(2\theta)\sin^2\left(\frac{\Delta m^2[\mathrm{GeV}^2]L[\mathrm{GeV}^{-1}]}{4E_\nu[\mathrm{GeV}]}\right) \;\rightarrow\; \sin^2(2\theta)\sin^2\left(1.27\frac{\Delta m^2[\mathrm{eV}^2]L[\mathrm{km}]}{E_\nu[\mathrm{GeV}]}\right).$$

First note that

$$\Delta m^2[\mathrm{GeV}^2] = 10^{-18}\Delta m^2[\mathrm{eV}^2] \quad \text{and} \quad L[\mathrm{m}] = 10^3 L[\mathrm{km}]\,.$$

To convert from natural units $L[\mathrm{GeV}^{-1}]$ to SI units $L[\mathrm{m}]$ the expression in brackets needs to be multiplied by the factor

$$\frac{\mathrm{GeV}}{\hbar c}\,,$$

Hence

$$\sin^2(2\theta)\sin^2\left(\frac{\Delta m^2[\mathrm{GeV}^2]L[\mathrm{GeV}^{-1}]}{4E_\nu[\mathrm{GeV}]}\right) \to \sin^2(2\theta)\times$$
$$\sin^2\left(\frac{10^3\cdot 10^{-18}\cdot 1.6\times 10^{-10}}{1.06\times 10^{-34}\cdot 3\times 10^8}\frac{\Delta m^2[\mathrm{eV}^2]L[\mathrm{km}]}{4E_\nu[\mathrm{GeV}]}\right)$$
$$\to \sin^2(2\theta)\times\sin^2\left(1.27\frac{\Delta m^2[\mathrm{eV}^2]L[\mathrm{km}]}{E_\nu[\mathrm{GeV}]}\right).$$

13.3 From equation (13.24) and the unitarity relation of (13.18), show that

$$\begin{aligned}P(\nu_e\to\nu_e) = 1 &+ 2|U_{e1}|^2|U_{e2}|^2\,\Re e\{[e^{-i(\phi_1-\phi_2)}-1]\}\\ &+ 2|U_{e1}|^2|U_{e3}|^2\,\Re e\{[e^{-i(\phi_1-\phi_3)}-1]\}\\ &+ 2|U_{e2}|^2|U_{e3}|^2\,\Re e\{[e^{-i(\phi_2-\phi_3)}-1]\}.\end{aligned}$$

The expression for $P(\nu_e\to\nu_e)$ can be obtained from equation (13.24) by making the replacing in the sub-scripts $\mu\to e$:

$$\begin{aligned}P(\nu_e\to\nu_e) =&|U^*_{e1}U_{e1}|^2+|U^*_{e2}U_{e2}|^2+|U^*_{e3}U_{e3}|^2+2\,\Re e\{U^*_{e1}U_{e1}U_{e2}U^*_{e2}e^{-i(\phi_1-\phi_2)}\}\\ &+2\,\Re e\{U^*_{e1}U_{e1}U_{e3}U^*_{e3}e^{-i(\phi_1-\phi_3)}\}+2\,\Re e\{U^*_{e2}U_{e2}U_{e3}U^*_{e3}e^{-i(\phi_2-\phi_3)}\}.\end{aligned}$$

Applying the identity

$$|z_1+z_2+z_3|^2 \equiv |z_1|^2+|z_2|^2+|z_3|^2+2\,\Re e\{z_1z_2^*+z_1z_3^*+z_2z_3^*\},$$

to the unitarity relation of (13.18),

$$U_{e1}U^*_{e1}+U_{e2}U^*_{e2}+U_{e3}U^*_{e3}=1,$$

namely $|U_{e1}U^*_{e1}+U_{e2}U^*_{e2}+U_{e3}U^*_{e3}|^2=1^2$ implies that

$$\begin{aligned}|U_{e1}U^*_{e1}|^2+|U_{e2}U^*_{e2}|^2+|U_{e3}U^*_{e3}|^2 = 1 &- 2\,\Re e\{U_{e1}U^*_{e1}U^*_{e2}U_{e2}\}\\ &-2\,\Re e\{U_{e1}U^*_{e1}U^*_{e3}U_{e3}\}-2\,\Re e\{U_{e2}U^*_{e2}U^*_{e3}U_{e3}\}.\end{aligned}$$

Substituting this into the above expression for $P(\nu_e\to\nu_e)$ gives

$$\begin{aligned}P(\nu_e\to\nu_e) &= 1+2\,\Re e\{U^*_{e1}U_{e1}U_{e2}U^*_{e2}\left[e^{-i(\phi_1-\phi_2)}-1\right]\}+\ldots\\ &= 1+2|U_{e1}|^2|U_{e2}|^2 2\,\Re e\{[e^{-i(\phi_1-\phi_2)}-1]\}+\ldots,\end{aligned}$$

as required.

13.4 Derive equation (13.30) in the following three steps:

a) By writing the oscillation probability $P(\nu_e\to\nu_\mu)$ as

$$P(\nu_e\to\nu_\mu)=2\sum_{i<j}\Re e\left\{U^*_{ei}U_{\mu i}U_{ej}U^*_{\mu j}\left[e^{i(\phi_j-\phi_i)}-1\right]\right\},$$

and writing $\Delta_{ij} = (\phi_i - \phi_j)/2$, show that

$$P(\nu_e \to \nu_\mu) = -4\sum_{i<j} \Re\mathfrak{e}\{U_{ei}^* U_{\mu i} U_{ej} U_{\mu j}^*\} \sin^2 \Delta_{ij} + 2\sum_{i<j} \Im\mathfrak{m}\{U_{ei}^* U_{\mu i} U_{ej} U_{\mu j}^*\} \sin 2\Delta_{ij}\,.$$

b) Defining $-J \equiv \Im\mathfrak{m}\{U_{e1}^* U_{\mu 1} U_{e3} U_{\mu 3}^*\}$, use the unitarity of the PMNS matrix to show that

$$\Im\mathfrak{m}\{U_{e1}^* U_{\mu 1} U_{e3} U_{\mu 3}^*\} = -\Im\mathfrak{m}\{U_{e2}^* U_{\mu 2} U_{e3} U_{\mu 3}^*\} = -\Im\mathfrak{m}\{U_{e1}^* U_{\mu 1} U_{e2} U_{\mu 2}^*\} = -J\,.$$

c) Hence, using the identity

$$\sin A + \sin B - \sin(A+B) = 4 \sin\left(\frac{A}{2}\right) \sin\left(\frac{B}{2}\right) \sin\left(\frac{A+B}{2}\right),$$

show that

$$P(\nu_e \to \nu_\mu) = -4\sum_{i<j} \Re\mathfrak{e}\{U_{ei}^* U_{\mu i} U_{ej} U_{\mu j}^*\} \sin^2 \Delta_{ij} + 8J \sin\Delta_{12} \sin\Delta_{13} \sin\Delta_{23}\,.$$

d) Hence show that

$$P(\nu_e \to \nu_\mu) - P(\overline{\nu}_e \to \overline{\nu}_\mu) = 16\, \Im\mathfrak{m}\{U_{e1}^* U_{\mu 1} U_{e2} U_{\mu 2}^*\} \sin\Delta_{12} \sin\Delta_{13} \sin\Delta_{23}\,.$$

e) Finally, using the current knowledge of the PMNS matrix determine the maximum possible value of $P(\nu_e \to \nu_\mu) - P(\overline{\nu}_e \to \overline{\nu}_\mu)$.

This is a fairly long and technical derivation, but the result is important as it predicts the scale of CP violation for neutrinos propagating in the vacuum.
a) First expand the exponential into cosine and sine terms

$$\begin{aligned}
P(\nu_e \to \nu_\mu) &= 2\sum_{i<j} \Re\mathfrak{e}\left\{U_{ei}^* U_{\mu i} U_{ej} U_{\mu j}^* \left[e^{i(\phi_j-\phi_i)} - 1\right]\right\} \\
&= 2\sum_{i<j} \Re\mathfrak{e}\left\{U_{ei}^* U_{\mu i} U_{ej} U_{\mu j}^* \left[\cos(\phi_j-\phi_i) - 1 + i\sin(\phi_j-\phi_i)\right]\right\} \\
&= 2\sum_{i<j} \Re\mathfrak{e}\left\{U_{ei}^* U_{\mu i} U_{ej} U_{\mu j}^* \left[\cos(\phi_j-\phi_i) - 1\right]\right\} \\
&\quad - 2\sum_{i<j} \Im\mathfrak{m}\left\{U_{ei}^* U_{\mu i} U_{ej} U_{\mu j}^* \left[\sin(\phi_j-\phi_i)\right]\right\} \\
&= -4\sum_{i<j} \Re\mathfrak{e}\left\{U_{ei}^* U_{\mu i} U_{ej} U_{\mu j}^* \left[\sin^2 \frac{(\phi_j-\phi_i)}{2}\right]\right\} \\
&\quad + 2\sum_{i<j} \Im\mathfrak{m}\left\{U_{ei}^* U_{\mu i} U_{ej} U_{\mu j}^* \left[\sin(\phi_i-\phi_j)\right]\right\} \\
&= -4\sum_{i<j} \Re\mathfrak{e}\{U_{ei}^* U_{\mu i} U_{ej} U_{\mu j}^*\} \sin^2 \Delta_{ij} + 2\sum_{i<j} \Im\mathfrak{m}\{U_{ei}^* U_{\mu i} U_{ej} U_{\mu j}^*\} \sin 2\Delta_{ij}\,,
\end{aligned} \tag{13.1}$$

as required.

b) The relevant Unitarity relations are:

$$U_{e1}^* U_{\mu 1} + U_{e2}^* U_{\mu 2} + U_{e3}^* U_{\mu 3} = 0\,,$$
$$\text{and} \quad U_{e1} U_{\mu 1}^* + U_{e2} U_{\mu 2}^* + U_{e3} U_{\mu 3}^* = 0\,.$$

Using these relations

$$\mathfrak{Im}\{U_{e1}^* U_{\mu 1} U_{e3} U_{\mu 3}^*\} = \mathfrak{Im}\{U_{e1}^* U_{\mu 1}(-U_{e1} U_{\mu 1}^* - U_{e2} U_{\mu 2}^*)\} = -\,\mathfrak{Im}\{U_{e1}^* U_{\mu 1} U_{e2} U_{\mu 2}^*\}$$
$$\mathfrak{Im}\{U_{e2}^* U_{\mu 2} U_{e3} U_{\mu 3}^*\} = \mathfrak{Im}\{(-U_{e1}^* U_{\mu 1} - U_{e3}^* U_{\mu 3}) U_{e3} U_{\mu 3}^*\} = -\,\mathfrak{Im}\{U_{e1}^* U_{\mu 1} U_{e3} U_{\mu 3}^*\}$$

Defining $-J \equiv \mathfrak{Im}\{U_{e1}^* U_{\mu 1} U_{e3} U_{\mu 3}^*\}$, these relations can be written as

$$\mathfrak{Im}\{U_{e1}^* U_{\mu 1} U_{e3} U_{\mu 3}^*\} = -\,\mathfrak{Im}\{U_{e2}^* U_{\mu 2} U_{e3} U_{\mu 3}^*\} = -\,\mathfrak{Im}\{U_{e1}^* U_{\mu 1} U_{e2} U_{\mu 2}^*\} = -J\,.$$

c) Using J defined above, the second term in (13.1) can be written:

$$\begin{aligned}
\mathcal{P} &= 2\sum_{i<j} \mathfrak{Im}\{U_{ei}^* U_{\mu i} U_{ej} U_{\mu j}^*\} \sin 2\Delta_{ij} \\
&= 2\,\mathfrak{Im}\{U_{e1}^* U_{\mu 1} U_{e2}^* U_{\mu 2}\} \sin 2\Delta_{12} + 2\,\mathfrak{Im}\{U_{e1}^* U_{\mu 1} U_{e3}^* U_{\mu 3}\} \sin 2\Delta_{13} \\
&\quad + 2\,\mathfrak{Im}\{U_{e2}^* U_{\mu 2} U_{e3}^* U_{\mu 3}\} \sin 2\Delta_{23} \\
&= 2J(\sin 2\Delta_{12} - \sin 2\Delta_{13} + \sin 2\Delta_{23})\,.
\end{aligned}$$

But $\Delta_{13} = \Delta_{12} + \Delta_{23}$, thus

$$\mathcal{P} = 2J(\sin 2\Delta_{12} + \sin 2\Delta_{23} - \sin 2(\Delta_{12} + \Delta_{23}))\,,$$

which has the form $\sin A + \sin B - \sin(A+B)$ with $A = \Delta_{12}$ and $B = \Delta_{23}$. Hence, using the identity,

$$\sin A + \sin B - \sin(A+B) = 4 \sin\left(\frac{A}{2}\right) \sin\left(\frac{B}{2}\right) \sin\left(\frac{A+B}{2}\right)\,,$$

this can be rewritten as

$$\mathcal{P} = 8J \sin \Delta_{12} \sin \Delta_{13} \sin \Delta_{23}\,,$$

and finally

$$P(\nu_e \to \nu_\mu) = -4\sum_{i<j} \mathfrak{Re}\{U_{ei}^* U_{\mu i} U_{ej} U_{\mu j}^*\} \sin^2 \Delta_{ij} + 8J \sin \Delta_{12} \sin \Delta_{13} \sin \Delta_{23}\,.$$

d) The corresponding expression for $P(\overline{\nu}_e \to \overline{\nu}_\mu)$ is obtained by interchanging $U_{\alpha i} \leftrightarrow U_{\alpha i}^*$ and therefore:

$$P(\overline{\nu}_e \to \overline{\nu}_\mu) = -4\sum_{i<j} \mathfrak{Re}\{U_{ei} U_{\mu i}^* U_{ej}^* U_{\mu j}\} \sin^2 \Delta_{ij} + 8\overline{J} \sin \Delta_{12} \sin \Delta_{13} \sin \Delta_{23}\,,$$

where $-J \equiv \Im\mathfrak{m}\{U_{e1}^* U_{\mu 1} U_{e3} U_{\mu 3}^*\}$ and $-\overline{J} = \Im\mathfrak{m}\{U_{e1} U_{\mu 1}^* U_{e3}^* U_{\mu 3}\}$. Since $\Im\mathfrak{m}\{z\} = -\Im\mathfrak{m}\{z^*\}$, $\overline{J} = -J$. Similarly, since $\Re\mathfrak{e}\{z\{= \Re\mathfrak{e}\{z^*\}$,

$$\Re\mathfrak{e}\{U_{ei} U_{\mu i}^* U_{ej}^* U_{\mu j}\} = \Re\mathfrak{e}\{U_{ei}^* U_{\mu i} U_{ej} U_{\mu j}^*\}\,.$$

Hence

$$\begin{aligned} P(\nu_e \to \nu_\mu) - P(\overline{\nu}_e \to \overline{\nu}_\mu) &= 16J \sin\Delta_{12} \sin\Delta_{13} \sin\Delta_{23} \\ &= 16\,\Im\mathfrak{m}\{U_{e1}^* U_{\mu 1} U_{e2} U_{\mu 2}^*\} \sin\Delta_{12} \sin\Delta_{13} \sin\Delta_{23}\,. \end{aligned}$$

e) Finally, J can be expressed in terms of the elements of the PMNS matrix using,

$$\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}.$$

Since U_{e1} and U_{e2} are real

$$\begin{aligned} J &= U_{e1} U_{e2}\,\Im\mathfrak{m}\{U_{\mu 1} U_{\mu 2}^*\} \\ &= c_{12}c_{13}^2 s_{12}\,\Im\mathfrak{m}\left\{\left[-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta}\right]\left[c_{12}c_{23} - s_{12}s_{23}s_{13}e^{-i\delta}\right]\right\} \\ &= c_{12}c_{13}^2 s_{12}\left[-s_{12}c_{23}s_{12}s_{23}s_{13}\sin\delta - c_{12}c_{23}c_{12}s_{23}s_{13}\sin\delta\right] \\ &= -c_{12}c_{13}^2 s_{12}c_{23}s_{23}s_{13}\sin\delta \\ &= -\tfrac{1}{8}\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{13}\sin 2\theta_{23}\sin\delta\,. \end{aligned}$$

Hence the difference in the $P(\nu_e \to \nu_\mu) - P(\overline{\nu}_e \to \overline{\nu}_\mu)$ oscillation probability is:

$$\Delta P = 2\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{13}\sin 2\theta_{23}\sin\delta\sin\Delta_{12}\sin\Delta_{13}\sin\Delta_{23}\,.$$

Using $\theta_{12} \approx 35°$, $\theta_{23} \approx 45°$ and $\theta_{13} \approx 10°$,

$$\Delta P = 0.63\sin\delta\sin\Delta_{12}\sin\Delta_{13}\sin\Delta_{23}\,.$$

Since $\Delta_{13} \approx \Delta_{23}$, this can be approximated as

$$\Delta P = 0.63\sin\delta\sin\Delta_{12}\sin\Delta_{23}\,.$$

The maximum difference occurs for $|\sin\delta| = 1$, in which case

$$\Delta P = 0.63\sin\Delta_{12}\sin\Delta_{23}\,.$$

For a terrestrial experiment the issue is that the $\sin\Delta_{12}$ term results in oscillations over very large distances. Consider a beam neutrino experiment, similar to MINOS with a peak beam energy of 3 GeV. The first oscillation maximum from the Δ_{23} term will occur at

$$\Delta_{23} = \frac{\pi}{2} \quad \Rightarrow \quad L = 1500\,\text{km}\,.$$

But at this distance

$$
\begin{aligned}
\frac{\Delta_{12}}{\Delta_{23}} &= \frac{\Delta m^2_{12}}{\Delta m^2_{32}} \approx 0.033 \\
\Rightarrow \quad \Delta_{12} &= 0.033\frac{\pi}{2} = 0.05 \\
\Rightarrow \quad \sin\Delta_{12} &\approx 0.05\,.
\end{aligned}
$$

Consequently, at this baseline, $P(\nu_e \to \nu_\mu) - P(\overline{\nu}_e \to \overline{\nu}_\mu) \approx 0.03$.

It should be noted that the above treatment uses the vacuum oscillation formula and neglects "matter effects".

13.5 The general unitary transformation between mass and weak eigenstates for two flavours can be written as

$$
\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta\exp(i\delta_1) & \sin\theta\exp\left(i\left[\frac{\delta_1+\delta_2}{2} - \delta\right]\right) \\ -\sin\theta\exp\left(i\left[\frac{\delta_1+\delta_2}{2} + \delta\right]\right) & \cos\theta\exp(i\delta_2) \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}.
$$

a) Show that the matrix in the above expression is indeed unitary.

b) Show that the three complex phases δ_1, δ_2 and δ can be eliminated from the above expression by the transformation

$$
\ell_\alpha \to \ell_\alpha e^{i(\theta_e+\theta'_\alpha)}, \quad \nu_k \to \nu_k e^{i(\theta_e+\theta'_k)} \quad \text{and} \quad U_{\alpha k} \to U_{\alpha k} e^{i(\theta'_\alpha-\theta'_k)},
$$

without changing the physical form of the two-flavour weak charged current

$$
-i\frac{g_W}{\sqrt{2}}\,(\overline{e},\,\overline{\mu})\,\gamma^\mu\tfrac{1}{2}(1-\gamma^5)\begin{pmatrix} U_{e1} & U_{e2} \\ U_{\mu 1} & U_{\mu 2} \end{pmatrix}\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}.
$$

a) A Unitary matrix satisfies $UU^\dagger = 1$. Here

$$
\begin{aligned}
U &= \begin{pmatrix} \cos\theta\exp(i\delta_1) & \sin\theta\exp\left(i\left[\frac{\delta_1+\delta_2}{2} - \delta\right]\right) \\ -\sin\theta\exp\left(i\left[\frac{\delta_1+\delta_2}{2} + \delta\right]\right) & \cos\theta\exp(i\delta_2) \end{pmatrix}, \\
\Rightarrow \quad U^\dagger &= \begin{pmatrix} \cos\theta\exp(-i\delta_1) & -\sin\theta\exp\left(-i\left[\frac{\delta_1+\delta_2}{2} + \delta\right]\right) \\ \sin\theta\exp\left(-i\left[\frac{\delta_1+\delta_2}{2} - \delta\right]\right) & \cos\theta\exp(-i\delta_2) \end{pmatrix},
\end{aligned}
$$

and thus

$$
\begin{aligned}
UU^\dagger &= \begin{pmatrix} \cos^2\theta + \sin^2\theta & (\sin\theta\cos\theta - \sin\theta\cos\theta)e^{\frac{i\delta_1}{2}}e^{\frac{i\delta_2}{2}}e^{-i\delta} \\ (\sin\theta\cos\theta - \sin\theta\cos\theta)e^{\frac{i\delta_1}{2}}e^{\frac{-i\delta_2}{2}}e^{i\delta} & \cos^2\theta + \sin^2\theta \end{pmatrix}, \\
&= 1\,.
\end{aligned}
$$

Therefore U, which depends on four independent real parameters, is one possible representation of a general 2×2 unitary matrix.

b) Under the above redefinition of the phases of the fermion fields:

$$U \to \begin{pmatrix} \cos\theta e^{i(\delta_1+\theta'_e-\theta'_1)} & \sin\theta e^{i\left(\frac{\delta_1+\delta_2}{2}-\delta+\theta'_e-\theta'_2\right)} \\ -\sin\theta e^{i\left(\frac{\delta_1+\delta_2}{2}+\delta+\theta'_\mu-\theta'_1\right)} & \cos\theta e^{i(\delta_2+\theta'_\mu-\theta'_2)} \end{pmatrix}.$$

All complex phases can be eliminated if the following four conditions are satisfied

$$\theta'_1 - \theta'_e = \delta_1 \quad \text{and} \quad \theta'_2 - \theta'_e = \frac{\delta_1+\delta_2}{2} - \delta\,, \tag{13.2}$$

$$\theta'_1 - \theta'_\mu = \frac{\delta_1+\delta_2}{2} + \delta \quad \text{and} \quad \theta'_2 - \theta'_\mu = \delta_2\,. \tag{13.3}$$

Choosing $\theta'_e = 0$, which is equivalent to writing all the phases relative to the phase of the electron, then the two equations of (13.2) become

$$\theta'_1 = \delta_1 \quad \text{and} \quad \theta'_2 = \frac{\delta_1+\delta_2}{2} - \delta\,.$$

The two equations of (13.3) then give a consistent solution for θ'_μ:

$$\theta'_\mu = \theta'_1 - \frac{\delta_1+\delta_2}{2} - \delta = \delta_1 - \frac{\delta_1+\delta_2}{2} - \delta = \frac{\delta_1-\delta_2}{2} - \delta$$

$$\theta'_\mu = \theta'_2 - \delta_2 = \frac{\delta_1+\delta_2}{2} - \delta - \delta_2 = \frac{\delta_1-\delta_2}{2} - \delta\,.$$

Therefore, by redefining the phases of the fields using:

$$\begin{aligned} \theta'_e &= \phi\,, \\ \theta'_\mu &= \phi + \frac{\delta_1-\delta_2}{2} - \delta\,, \\ \theta'_1 &= \phi + \delta_1\,, \\ \theta'_2 &= \phi + \frac{\delta_1+\delta_2}{2} - \delta\,, \end{aligned}$$

all complex phases can be removed from the 2×2 analogue of the PMNS matrix, where the ϕ fixes the overall phase of (for example) the electron field. Therefore, for two generations, all complex phases in the analogue of the PMNS matrix can be absorbed into the definitions of the phases of the fields, without any physical consequences: the weak interaction vertices $\overline{\nu}_k\frac{1}{2}\gamma^\mu(1-\gamma^5)\ell_\alpha$ are identical and, consequently, the two flavour phenomenology for neutrino oscillations is the same.

13.6 The derivations of (13.37) and (13.38) used the trigonometric relations

$$1 - \tfrac{1}{2}\sin^2(2\theta_{13}) = \cos^4(\theta_{13}) + \sin^4(\theta_{13})\,,$$

and

$$4\sin^2\theta_{23}\cos^2\theta_{13}(1-\sin^2\theta_{23}\cos^2\theta_{13}) = (\sin^2 2\theta_{23}\cos^4\theta_{13} + \sin^2 2\theta_{13}\sin^2\theta_{23})\,.$$

Convince yourself these relations hold.

In both cases the double angle formula $\sin 2\theta = 2\sin\theta\cos\theta$ is used. The first identify follows from

$$\begin{aligned}
1 - \tfrac{1}{2}\sin^2 2\theta_{13} &= 1 - \tfrac{1}{2}4\sin^2\theta_{13}\cos^2\theta_{13} \\
&= \cos^2\theta_{13} + \sin^2\theta_{13} - 2\sin^2\theta_{13}\cos^2\theta_{13} \\
&= \cos^2\theta_{13} - \sin^2\theta_{13}\cos^2\theta_{13} + \sin^2\theta_{13} - \sin^2\theta_{13}\cos^2\theta_{13} \\
&= \cos^2\theta_{13}(1-\sin^2\theta_{13}) + \sin^2\theta_{13}(1-\cos^2\theta_{13}) \\
&= \cos^4\theta_{13} + \sin^4\theta_{13}\,.
\end{aligned}$$

For the second identify, it is easiest to start from

$$\begin{aligned}
\sin^2 2\theta_{23}\cos^4\theta_{13} + \sin^2 2\theta_{13}\sin^2\theta_{23} &= 4\sin^2\theta_{23}\cos^2\theta_{23}\cos^4\theta_{13} + 4\sin^2\theta_{13}\cos^2\theta_{13}\sin^2\theta_{23} \\
&= 4\sin^2\theta_{23}\cos^2\theta_{13}\left[\cos^2\theta_{23}\cos^2\theta_{13} + \sin^2\theta_{13}\right] \\
&= 4\sin^2\theta_{23}\cos^2\theta_{13}\left[(1-\sin^2\theta_{23})\cos^2\theta_{13} + \sin^2\theta_{13}\right] \\
&= 4\sin^2\theta_{23}\cos^2\theta_{13}\left[1-\sin^2\theta_{23}\cos^2\theta_{13}\right]\,.
\end{aligned}$$

13.7 Use the data of Figure 13.20 to obtain estimates of $\sin^2(2\theta_{12})$ and $|\Delta m^2_{21}|$.

In Figure 13.20 the distance L_0 in $L_0/E_{\overline{\nu}_e}$, is the average distance to many reactors weighted by expected flux. The variety of actual distances, smears out the calculated form of the oscillation probability, with the smearing becoming more notable at small values of E, or equivalently large values of $L_0/E_{\overline{\nu}_e}$. The first oscillation minimum occurs at $L/E < 30\,\text{km}$ but is not clearly resolved. The second oscillation minimum is clearly defined at

$$L_0/E_{\overline{\nu}_e} \approx 50\,\text{km}\,\text{MeV}^{-1} = 50000\,\text{km}\,\text{GeV}^{-1}\,.$$

In S.I. units, the survival probability is

$$P(\overline{\nu}_e \to \overline{\nu}_e) = \cos^4\theta_{13}\left[1 - \sin^2(2\theta_{12})\sin^2\left(1.27\frac{\Delta m^2_{21}[\text{eV}^2]\,L[\text{km}]}{E[\text{GeV}]}\right)\right],$$

and the second oscillation minimum occurs at

$$\begin{aligned}
1.27\frac{\Delta m^2_{21}[\text{eV}^2]\,L[\text{km}]}{E[\text{GeV}]} &= \frac{3\pi}{2} \\
\Rightarrow \quad \Delta m^2_{21}[\text{eV}^2] &= \frac{3\pi}{2\cdot 1.27\cdot 50000} \\
&= 7.4\times 10^{-5}\,\text{eV}^2\,.
\end{aligned}$$

Determining the angle θ_{12} requires care. The amplitude of the oscillations (estimated from the difference between the amplitudes of the first oscillation maximum

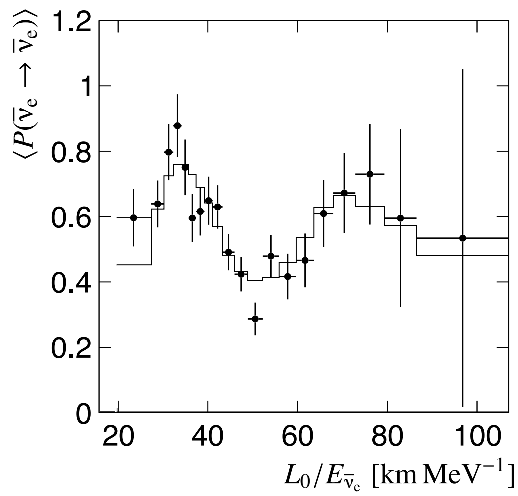


Fig. 13.1 KamLAND data showing the measured mean survival probability as a function of the measured neutrino energy divided by the flux-weighted mean distance to the reactors, L_0. The histogram shows the expected distribution for the oscillation parameters that best describe the data.

and the well-resolved second oscillation minimum) is about 0.4. Without experimental effects this would be equal to $\cos^4\theta_{13}\sin^2 2\theta_{12}$. However, the sharpness of the oscillation structure is smeared out due to the reactors being at a variety of distances from the experiment. The effect of this smearing can be estimated. According to the survival probability formula, the peak at $L/E = \pi$ should correspond to a survival probability of $\cos^4\theta_{13} \simeq 0.95$. The measured survival probability is about 0.75, due to the smearing out of the peak due to the ranges of L to the different reactors. If the range of L values had been very large, then the peak would have been smeared out to an average value of 0.95/2 = 0.475. Hence the effect of the smearing has been to reduce the peak from the expected value of 0.95 (which is 0.475 above the totally smeared case) to 0.75 (which is 0.275 above the totally smeared case). The net effect of the smearing in L is to reduce the height of the oscillation maximum by a factor 0.275/0.475 = 0.58, giving the naive estimate of

$$\cos^4\theta_{13}\sin^2 2\theta_{12} \approx \frac{0.4}{0.58}$$
$$\Rightarrow \sin^2 2\theta_{12} \approx \frac{0.4}{0.58\cdot 0.95} = 0.73\,.$$

A more accurate estimate could be obtained with a more sophisticated treatment of the data.

13.8 Use the data of Figure 13.22 to obtain estimates of $\sin^2(2\theta_{23})$ and $|\Delta m^2_{32}|$.

The interpretation of the MINOS data is relatively straightforward as the distance

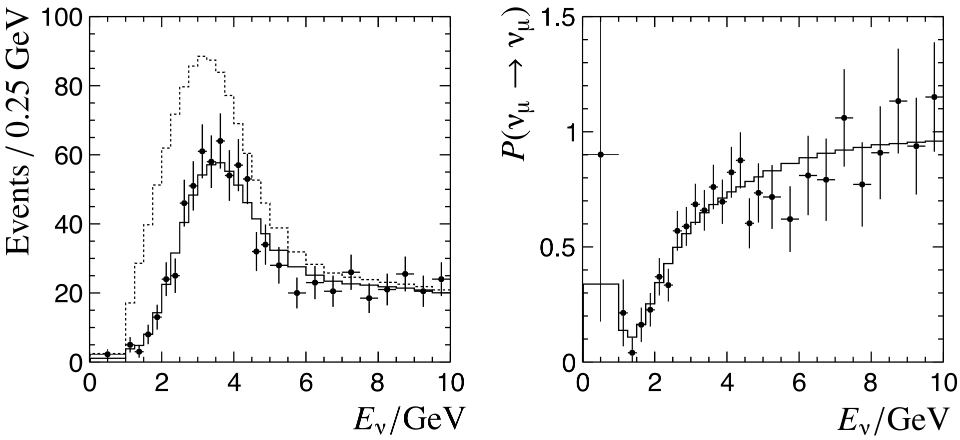


Fig. 13.2 The MINOS far detector energy spectrum compared to the unoscillated prediction and the oscillation probability as measured from the ratio of the far detector data to the unoscillated prediction.

from the source of the beam to the far detector is fixed, $L = 735$ km and the energy of the neutrino is relatively well measured. From the figure, the first oscillation minimum occurs at $E_\nu \simeq 1.4$ GeV (the second oscillation minimum at lower energies is not observed due to the nearly negligible flux).

$$\begin{aligned} 1.27\frac{\Delta m_{32}^2[\mathrm{eV}^2]\, L[\mathrm{km}]}{E[\mathrm{GeV}]} &= \frac{\pi}{2} \\ \Rightarrow \quad \Delta m_{32}^2[\mathrm{eV}^2] &= \frac{\pi}{2}\frac{1.4}{1.27 \cdot 735} \\ &= 2.3 \times 10^{-3}\,\mathrm{eV}^2\,. \end{aligned}$$

The measured the survival probability at the oscillation minimum is $P_{meas.} \approx 0.1$, but it without additional knowledge of the experimental resolution, which will contribute to the observed depth of the minimum, all that can be stated for certain is:

$$P(\nu_\mu \to \nu_\mu) < 0.1 \quad \Rightarrow \quad \sin^2(2\theta_{23}) > 0.9\,.$$

13.9 The T2K experiment uses an off-axis ν_μ beam produced from $\pi^+ \to \mu^+\nu_\mu$ decays. Consider the case where the pion has velocity β along the z-direction in the laboratory frame and a neutrino with energy E^* is produced at an angle θ^* with respect to the z'-axis in the π^+ rest frame.

a) Show that the neutrino energy in the pion rest frame is $\mathrm{p}^* = (m_\pi^2 - m_\mu^2)/2m_\pi$.

b) Using a Lorentz transformation, show that the energy E and angle of production θ of the neutrino in the laboratory frame are

$$E = \gamma E^*(1 + \beta\cos\theta^*) \quad \text{and} \quad E\cos\theta = \gamma E^*(\cos\theta^* + \beta)\,,$$

where $\gamma = E_\pi/m_\pi$.

c) Using the expressions for E^* and θ^* in terms of E and θ, show that

$$\gamma^2(1-\beta\cos\theta)(1+\beta\cos\theta^*) = 1\,.$$

d) In the limit $\theta \ll 1$, show that

$$E \approx 0.43E_\pi \frac{1}{1+\beta\gamma^2\theta^2}\,,$$

and therefore on-axis ($\theta = 0$) the neutrino energy spectrum follows that of the pions.

e) Assuming that the pions have a flat energy spectrum in the range $1-5$ GeV, sketch the form of the resulting neutrino energy spectrum at the T2K far detector (Super-Kamiokande), which is off-axis at $\theta = 2.5°$. Given that the Super-Kamiokande detector is 295 km from the beam, explain why this angle was chosen.

a) From conservation of energy

$$\begin{aligned} E_\mu &= m_\pi - E_\nu = m_\pi - \mathrm{p}^* \\ \Rightarrow \quad E_\mu^2 &= m_\mu^2 + p^* = m_\pi^2 - 2m_\pi \mathrm{p}^* + (\mathrm{p}^*)^2 \\ \Rightarrow \quad \mathrm{p}^* &= \frac{m_\pi^2 - m_\mu^2}{2m_\pi}\,. \end{aligned}$$

b) In the pion rest frame (Σ') the four-momentum of the neutrino is:

$$p'_\nu = (E^*, 0, \mathrm{p}^*\sin\theta^*, \mathrm{p}^*, \cos\theta^*) = E^*(1, 0, \sin\theta^*, \cos\theta^*)\,,$$

where the neutrino mass has been neglected: $E^* \simeq \mathrm{p}^*$. The laboratory frame four-momentum is

$$\begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & +\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ +\gamma\beta & 0 & 0 & \gamma \end{pmatrix} E^* \begin{pmatrix} 1 \\ 0 \\ \sin\theta^* \\ \cos\theta^* \end{pmatrix} = E^* \begin{pmatrix} \gamma + \gamma\beta\cos\theta^* \\ 0 \\ \sin\theta^* \\ \gamma\beta + \gamma\cos\theta^* \end{pmatrix}.$$

Hence the laboratory from neutrino energy is

$$E = \gamma E^*(1+\beta\cos\theta^*)\,,$$

where γ is determined by the boost of the pion, $\gamma = E_\pi/m_\pi$. In the laboratory frame (where for the almost massless neutrino $\mathrm{p} \simeq E$), the polar angle with respect to the z-axis is given by

$$\begin{aligned} \cos\theta &= \frac{p_z}{\mathrm{p}} = \frac{p_z}{E} = \frac{\gamma E^*(\beta + \cos\theta^*)}{E} \\ \Rightarrow \quad E\cos\theta &= \gamma E^*(\beta + \gamma\cos\theta^*)\,. \end{aligned}$$

c) By flipping the sign of β, the Lorentz transformation from the laboratory frame to the pion rest frame gives

$$E^* = \gamma E(1-\beta\cos\theta) \quad \text{and} \quad E^*\cos\theta^* = \gamma E(\cos\theta - \beta)\,,$$

where it should be remembered that $\gamma = E_\pi/m_\pi$. Taken the product of this expression for E^* and the expression for E in part b) for the question,

$$\begin{aligned} E^*E &= \gamma E(1-\beta\cos\theta)\gamma E^*(1+\beta\cos\theta^*) \\ \Rightarrow \quad 1 &= \gamma^2(1-\beta\cos\theta)(1+\beta\cos\theta^*)\,. \end{aligned}$$

d) The lab. frame energy of the neutrino produced from pions of laboratory frame energy E_π and a decay angle θ^* in the pion rest frame is given in by the expression in part a):

$$\begin{aligned} E_\nu &= \gamma E^*(1+\beta\cos\theta^*) \\ &= \frac{E_\pi}{m_\pi}\gamma E^*(1+\beta\cos\theta^*) \\ &= \frac{E_\pi}{m_\pi}\frac{m_\pi^2 - m_\mu^2}{2m_\pi}(1+\beta\cos\theta^*) \\ &= \tfrac{1}{2}E_\pi\frac{m_\pi^2 - m_\mu^2}{m_\pi^2}(1+\beta\cos\theta^*)\,. \end{aligned}$$

Using the result from part c) this can be expressed in terms of the laboratory frame angle:

$$\begin{aligned} E_\nu &= \tfrac{1}{2}E_\pi\frac{m_\pi^2 - m_\mu^2}{m_\pi^2}(1+\beta\cos\theta^*) \\ &= \tfrac{1}{2}E_\pi\frac{m_\pi^2 - m_\mu^2}{m_\pi^2}\frac{1}{\gamma^2(1-\beta\cos\theta)}\,. \end{aligned} \tag{13.4}$$

In the small angle limit, the denominator can be approximated by

$$\begin{aligned} \gamma^2(1-\beta\cos\theta) &\approx \gamma^2\left[1-\beta\left(1-\frac{\theta^2}{2}\right)\right] \\ &= \gamma^2(1-\beta) + \frac{\gamma^2\theta^2\beta}{2}\,. \end{aligned} \tag{13.5}$$

Assuming the $E_\nu \gg m_\pi$ such that $\gamma \gg 1$,

$$\beta = \left(1-\frac{1}{\gamma^2}\right)^{\frac{1}{2}} \approx 1 - \frac{1}{2\gamma^2}\,,$$

and hence (13.5) becomes

$$\begin{aligned}\gamma^2(1-\beta\cos\theta) &\approx \gamma^2(1-\beta) + \frac{\gamma^2\theta^2\beta}{2} \\ &\approx \tfrac{1}{2} + \frac{\gamma^2\theta^2\beta}{2} \\ &= \tfrac{1}{2}(1+\gamma^2\theta^2\beta)\,.\end{aligned}$$

Inserting this expression in (13.4) leads to

$$\begin{aligned}E_\nu &= E_\pi \frac{m_\pi^2 - m_\mu^2}{m_\pi^2}\frac{1}{1+\beta\gamma^2\theta^2} \\ &= 0.43 E_\pi \frac{1}{1+\beta\gamma^2\theta^2}\,.\end{aligned}$$

Hence, along the beam-axis ($\theta \approx 0$) the neutrino energy distribution follows that of the pions creating the beam.

e) The neutrino energies for a set of pion beam energies are tabulated below for $\theta = 0°$ and $\theta = 2.5°$: The effect of going away from the beam axis is to produce

E_π	E_ν at $\theta = 0°$	E_ν at $\theta = 2.5°$
1.0 GeV	0.43 GeV	0.39 GeV
1.5 GeV	0.65 GeV	0.53 GeV
2.0 GeV	0.86 GeV	0.62 GeV
2.5 GeV	1.08 GeV	0.67 GeV
3.0 GeV	1.29 GeV	0.68 GeV
3.5 GeV	1.50 GeV	0.68 GeV
4.0 GeV	1.72 GeV	0.67 GeV
4.5 GeV	1.93 GeV	0.65 GeV
5.0 GeV	2.15 GeV	0.62 GeV

a "narrow-band" beam, where most the neutrino energy depends only very weakly on the energy of the decaying pion producing the neutrino. This is in contrast to an on-axis "wide-band" beam where the neutrino energy is proportional to the the decaying pion energy. The oscillation probability (corresponding to Δm_{32}) depends on

$$P \propto \sin^2\left(1.27\frac{\Delta m_{32}^2[\mathrm{eV}^2]L[\mathrm{km}]}{E_\nu[\mathrm{GeV}]}\right)\,.$$

For $\Delta m_{32} = 2.3\times10^{-3}\,\mathrm{eV}^2$ and $L = 295\,\mathrm{km}$, the first oscillation maximum occurs at

$$\begin{aligned} 1.27\frac{\Delta m_{32}^2[\mathrm{eV}^2]L[\mathrm{km}]}{E_\nu[\mathrm{GeV}]} &= \frac{\pi}{2} \\ \Rightarrow \quad E_\nu &= \frac{2.54}{\pi}\Delta m_{32}^2[\mathrm{eV}^2]L[\mathrm{km}]\,\mathrm{GeV} \\ &= 0.55\,\mathrm{GeV}\,. \end{aligned}$$

Hence the off-axis angle was chosen such that the peak of narrow-band neutrino beam energy is close to the first oscillation maximum.

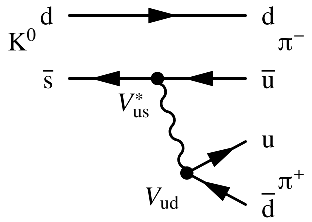
14 CP Violation and Weak Hadronic Interactions

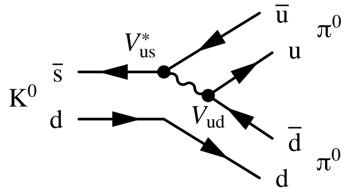
14.1 Draw the lowest-order Feynman diagrams for the decays

$$\mathrm{K}^0 \to \pi^+\pi^-\,, \quad \mathrm{K}^0 \to \pi^0\pi^0\,, \quad \overline{\mathrm{K}}^0 \to \pi^+\pi^- \quad \text{and} \quad \overline{\mathrm{K}}^0 \to \pi^0\pi^0\,,$$

and state how the corresponding matrix elements depend on the Cabibbo angle θ_c.

The two lowest-order Feynman diagrams for the $\mathrm{K}^0 \to \pi^+\pi^-$ and $\mathrm{K}^0 \to \pi^0\pi^0$ decays are:

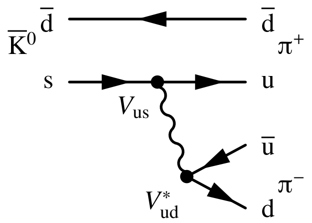


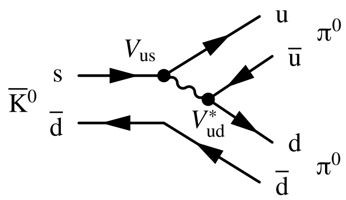


In the two flavour approximation, the matrix element for both diagrams is proportional to

$$\mathcal{M} \propto |V_{us}|\,|V_{ud}| \approx \sin\theta_C \cos\theta_C\,.$$

The two corresponding lowest-order Feynman diagrams for the $\overline{\mathrm{K}}^0 \to \pi^+\pi^-$ and $\overline{\mathrm{K}}^0 \to \pi^0\pi^0$ decays are: Again, for both diagrams:





$$\mathcal{M} \propto |V_{us}|\,|V_{ud}| \approx \sin\theta_C \cos\theta_C\,.$$

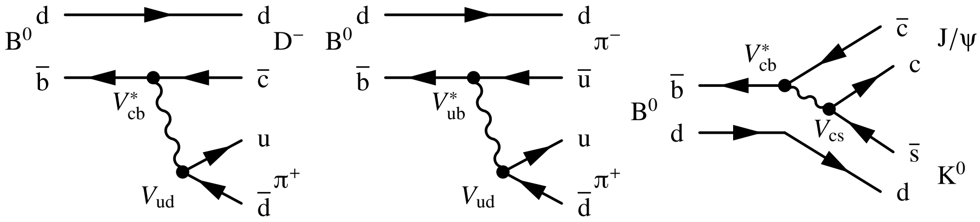
14.2 Draw the lowest-order Feynman diagrams for the decays

$$\mathrm{B}^0 \to \mathrm{D}^-\pi^+\,, \quad \mathrm{B}^0 \to \pi^+\pi^- \quad \text{and} \quad \mathrm{B}^0 \to \mathrm{J}/\psi\,\mathrm{K}^0\,,$$

and place them in order of decreasing decay rate.

The flavour content of the above mesons is $B^0(d\bar{b})$, $D^-(d\bar{c})$, $J/\psi(c\bar{c})$ and $K^0(d\bar{s})$.

The lowest-order Feynman diagrams for the given decays are:



From consideration of the CKM elements alone the matrix elements scale as

$$\mathcal{M}(B^0 \to D^- \pi^+) : \mathcal{M}(B^0 \to \pi^+ \pi^-) : \mathcal{M}(B^0 \to J/\psi K^0) \\ = |V_{cb}||V_{ud}| : |V_{ub}||V_{ud}| : |V_{cb}||V_{cs}| .$$

To first order the CKM matrix elements depend on the number of generations changed at the vertex. Thus the matrix elements for the decays $B^0 \to D^- \pi^+$ and $\mathcal{M}(B^0 \to J/\psi K^0)$, which both have one vertex at which there is no change of generation and one vertex where there is one change of generation $\bar{b} \to \bar{c}$, will be larger than that for $\mathcal{M}(B^0 \to \pi^+ \pi^-)$, which has one vertex at which there are two changes of generation $\bar{b} \to \bar{u}$. Being more quantitative, from consideration of the CKM matrix alone,

$$\begin{aligned} Br(B^0 \to D^- \pi^+) &: Br(B^0 \to \pi^+ \pi^-) : Br(B^0 \to J/\psi K^0) \\ &= |V_{cb}|^2|V_{ud}|^2 : |V_{ub}|^2|V_{ud}|^2 : |V_{cb}|^2|V_{cs}|^2 \\ &= 1.6 \times 10^{-3} : 1.5 \times 10^{-5} : 1.6 \times 10^{-3} . \end{aligned}$$

From consideration of the CKM matrix alone, one would expect the $Br(B^0 \to \pi^+ \pi^-)$ to be about 100 times smaller than for the other two decays.

$$Br^{obs}(B^0 \to D^- \pi^+) : Br^{obs}(B^0 \to \pi^+ \pi^-) : Br^{obs}(B^0 \to J/\psi K^0) \\ = 2.7 \times 10^{-3} : 5.1 \times 10^{-6} : 8.7 \times 10^{-4} ,$$

in reasonable agreement with the dependence from CKM matrix elements alone. A more sophisticated estimate of the *BR*s would account for the differences in phase space for the decays, where

$$\Gamma \propto p^* \propto \left\{\left[m_B^2 - (m_1 + m_2)^2\right]\left[m_B^2 - (m_1 - m_2)^2\right]\right\}^{\frac{1}{2}} .$$

Taking account the different phase space factors and scaling to the observed *BR* for

$(B^0 \to D^-\pi^+$:

$$
\begin{aligned}
Br(B^0 \to D^-\pi^+) : Br(B^0 \to \pi^+\pi^-) &: Br(B^0 \to J/\psi\,K^0) \\
&= |V_{cb}|^2|V_{ud}|^2 p^* : |V_{ub}|^2|V_{ud}|^2 p^* : |V_{cb}|^2|V_{cs}|^2 p^* \\
&= 2.7 \times 10^{-3} : 2.9 \times 10^{-5} : 2.0 \times 10^{-3}\,.
\end{aligned}
$$

The remaining differences, compared to the observed values, can be attributed to other differences in the matrix element, which includes factors accounting for the formation of the final-state mesons.

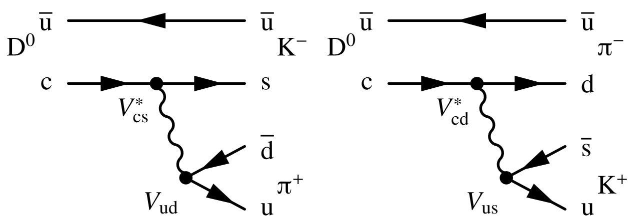
14.3 Draw the lowest-order Feynman diagrams for the weak decays

$$D^0(c\bar{u}) \to K^-(s\bar{u}) + \pi^+(u\bar{d}) \quad \text{and} \quad D^0(c\bar{u}) \to K^+(u\bar{s}) + \pi^-(d\bar{u})\,,$$

and explain the observation that

$$\frac{\Gamma(D^0 \to K^+\pi^-)}{\Gamma(D^0 \to K^-\pi^+)} \approx 4 \times 10^{-3}\,.$$

The lowest-order Feynman diagrams for the given decays are:



On the basis of the CKM matrix alone, one would expect

$$\frac{\Gamma(D^0 \to K^+\pi^-)}{\Gamma(D^0 \to K^-\pi^+)} \approx \frac{|V_{cd}|^2|V_{us}|^2}{|V_{ud}|^2|V_{cs}|^2} = \frac{0.225^2 \cdot 0.225^2}{0.974^2 \cdot 0.973^2} = 3 \times 10^{-3}\,,$$

explaining most of the difference in the observed decay rates.

14.4 A hypothetical $\overline{T}^0(t\bar{u})$ meson decays by the weak charged-current decay chain,

$$\overline{T}^0 \to W\pi \to (X\pi)\,\pi \to (Y\pi)\,\pi\,\pi \to (Z\pi)\,\pi\,\pi\,\pi\,.$$

Suggest the most likely identification of the W, X, Y and Z mesons and state why this decay chain would be preferred over the direct decay $\overline{T}^0 \to Z\,\pi$.

In each case the decay of the quark paired with the $\bar{u}$ decays according to the largest CKM matrix element, in this case

$$t \to b \to c \to s \to u\,,$$

and mesons can be identified as

$$W = B^-(b\bar{u})\,, \quad X = \overline{D}^0(c\bar{u})\,, \quad Y = K^-(s\bar{u}) \quad \text{and} \quad Z = \pi^0(u\bar{u})\,.$$

The direct decay to $\pi^0\pi^0$ final state would involve a $t \to d$ decay with the the CKM element $|V_{td}|^2 \sim 10^{-4}$.

14.5 For the cases of two, three and four generations, state:

a) the number of free parameters in the corresponding $n \times n$ unitary matrix relating the quark flavour and weak states;

b) how many of these parameters are real and how many are complex phases;

c) how many of the complex phases can be absorbed into the definitions of phases of the fermions without any physical consequences;

d) whether CP violation can be accommodated in quark mixing.

a) An $n \times n$ matrix has n^2 complex elements and therefore has $2n^2$ free parameters. The condition $U^\dagger U = 1$ gives $n \times n$ equations that the elements must satisfy, leaving $n \times n$ free parameters:

$$N_2^{\text{free}} = 4\,, \quad N_3^{\text{free}} = 9 \quad \text{and} \quad N_4^{\text{free}} = 16\,.$$

b) The real parameters correspond to rotations around different axes. In the case of a 2×2 matrix, there is just one rotation angle θ_{12}, in the case of a 2×2 matrix, there are three rotation angles θ_{12}, θ_{13} and θ_{23}. For a 4×4 matrix, there are six rotation angles θ_{12}, θ_{13}, θ_{14}, θ_{23}, θ_{24} and θ_{34}. Thus

$$N_2^{\text{real}} = 1 \;:\; N_2^{\text{phase}} = 3\,, \quad N_3^{\text{real}} = 3 \;:\; N_3^{\text{phase}} = 6 \quad \text{and} \quad N_4^{\text{real}} = 6 \;:\; N_4^{\text{phase}} = 10\,.$$

c) An $n \times n$ matrix expresses the copings between $2n$ fermions. Hence there it is possible to define $2n$ phases for the individual fermions, but there is always one overall phase that can be defined without any physical consequences so all phases can be defined with respect to one of the fermions. Hence $2n - 1$ phases of the $n \times n$ unitary matrix can be absorbed into the definitions of the phases of the fermions, leaving

$$N_2^{\text{real}} = 1 \;:\; N_2^{\text{phase}} = 0\,, \quad N_3^{\text{real}} = 3 \;:\; N_3^{\text{phase}} = 1 \quad \text{and} \quad N_4^{\text{real}} = 6 \;:\; N_4^{\text{phase}} = 3\,.$$

d) CP violation arises from at least one complex phase in the mixing matrix, and therefore CP violation can arise in quark mixing for three or more generations, but not for two generations.

14.6 Draw the lowest-order Feynman diagrams for the strong interaction processes

$$\overline{\mathrm{p}}\mathrm{p} \to \mathrm{K}^-\pi^+\mathrm{K}^0 \quad \text{and} \quad \overline{\mathrm{p}}\mathrm{p} \to \mathrm{K}^+\pi^-\overline{\mathrm{K}}^0 .$$

The quark content of the particles involved are:

$$\overline{\mathrm{p}}(\overline{\mathrm{u}}\overline{\mathrm{u}}\overline{\mathrm{d}})\,\mathrm{p}(\mathrm{uud}) \to \mathrm{K}^-(\mathrm{s}\overline{\mathrm{u}})\,\pi^+(\mathrm{u}\overline{\mathrm{d}})\,\mathrm{K}^0(\mathrm{d}\overline{\mathrm{s}}) \quad : \quad \overline{\mathrm{u}}\overline{\mathrm{u}}\overline{\mathrm{d}}\mathrm{uud} \to \mathrm{s}\overline{\mathrm{u}}\mathrm{u}\overline{\mathrm{d}}\mathrm{d}\overline{\mathrm{s}}\,,$$

and

$$\overline{\mathrm{p}}(\overline{\mathrm{u}}\overline{\mathrm{u}}\overline{\mathrm{d}})\,\mathrm{p}(\mathrm{uud}) \to \mathrm{K}^+(\mathrm{u}\overline{\mathrm{s}})\,\pi^-(\mathrm{d}\overline{\mathrm{u}})\,\overline{\mathrm{K}}^0(\mathrm{s}\overline{\mathrm{d}}) \quad : \quad \overline{\mathrm{u}}\overline{\mathrm{u}}\overline{\mathrm{d}}\mathrm{uud} \to \mathrm{s}\overline{\mathrm{u}}\mathrm{u}\overline{\mathrm{d}}\mathrm{d}\overline{\mathrm{s}}\,.$$

In both cases the flavour change is $\mathrm{u}\overline{\mathrm{u}} \to \mathrm{s}\overline{\mathrm{s}}$ and the (rather messy) Feynman diagrams involve the annihilation of a $\mathrm{u}\overline{\mathrm{u}} \to \mathrm{g} \to \mathrm{s}\overline{\mathrm{s}}$ and subsequent arrangement of the final-state quarks into the mesons given.

14.7 In the neutral kaon system, time-reversal violation can be expressed in terms of the asymmetry

$$A_T = \frac{\Gamma(\overline{\mathrm{K}}^0 \to \mathrm{K}^0) - \Gamma(\mathrm{K}^0 \to \overline{\mathrm{K}}^0)}{\Gamma(\overline{\mathrm{K}}^0 \to \mathrm{K}^0) + \Gamma(\mathrm{K}^0 \to \overline{\mathrm{K}}^0)}\,.$$

Show that this is equivalent to

$$A_T = \frac{\Gamma(\overline{\mathrm{K}}^0_{t=0} \to \pi^-\mathrm{e}^+\nu_\mathrm{e}) - \Gamma(\mathrm{K}^0_{t=0} \to \pi^+\mathrm{e}^-\overline{\nu}_\mathrm{e})}{\Gamma(\overline{\mathrm{K}}^0_{t=0} \to \pi^-\mathrm{e}^+\nu_\mathrm{e}) + \Gamma(\mathrm{K}^0_{t=0} \to \pi^+\mathrm{e}^-\overline{\nu}_\mathrm{e})}\,,$$

and therefore

$$A_T \approx 4|\varepsilon|\cos\phi\,.$$

To measure the time reversal symmetry it is necessary to tag the decays of the flavour states $\mathrm{K}^0(\mathrm{d}\overline{\mathrm{s}})$ and $\overline{\mathrm{K}}^0(\mathrm{s}\overline{\mathrm{d}})$ through their weak leptonic decays, $\mathrm{s} \to \mathrm{u}\ell^-\overline{\nu}_\ell$ and $\overline{\mathrm{s}} \to \overline{\mathrm{u}}\ell^+\nu_\ell$. For example, the rate of $\overline{\mathrm{K}}^0 \to \mathrm{K}^0$ is measured from the ℓ^+ decay rate from tagged $\overline{\mathrm{K}}^0$ production. Consequently,

$$A_T = \frac{\Gamma(\overline{\mathrm{K}}^0 \to \mathrm{K}^0) - \Gamma(\mathrm{K}^0 \to \overline{\mathrm{K}}^0)}{\Gamma(\overline{\mathrm{K}}^0 \to \mathrm{K}^0) + \Gamma(\mathrm{K}^0 \to \overline{\mathrm{K}}^0)} = \frac{\Gamma(\overline{\mathrm{K}}^0_{t=0} \to \pi^-\mathrm{e}^+\nu_\mathrm{e}) - \Gamma(\mathrm{K}^0_{t=0} \to \pi^+\mathrm{e}^-\overline{\nu}_\mathrm{e})}{\Gamma(\overline{\mathrm{K}}^0_{t=0} \to \pi^-\mathrm{e}^+\nu_\mathrm{e}) + \Gamma(\mathrm{K}^0_{t=0} \to \pi^+\mathrm{e}^-\overline{\nu}_\mathrm{e})}\,.$$

The mass eigenstates expressed in terms of the flavour eigenstates are

$$|\mathrm{K}_S\rangle = \frac{1}{\sqrt{2(1+|\varepsilon|^2)}}\left[(1+\varepsilon)|\mathrm{K}^0\rangle + (1-\varepsilon)|\overline{\mathrm{K}}^0\rangle\right]\,,$$

$$|\mathrm{K}_L\rangle = \frac{1}{\sqrt{2(1+|\varepsilon|^2)}}\left[(1+\varepsilon)|\mathrm{K}^0\rangle - (1-\varepsilon)|\overline{\mathrm{K}}^0\rangle\right]\,,$$

and therefore

$$
\begin{aligned}
|\mathrm{K}^0\rangle &= \sqrt{\frac{(1+|\varepsilon|^2)}{2}}\frac{1}{1+\varepsilon}\left[|\mathrm{K}_S\rangle + |\mathrm{K}_L\rangle\right] , \\
|\overline{\mathrm{K}}^0\rangle &= \sqrt{\frac{(1+|\varepsilon|^2)}{2}}\frac{1}{1-\varepsilon}\left[|\mathrm{K}_S\rangle - |\mathrm{K}_L\rangle\right] .
\end{aligned}
$$

Hence the neutral kaon state initially produced as a K^0 evolves as

$$
\begin{aligned}
|\mathrm{K}^0_{t=0}(t)\rangle &= \sqrt{\frac{(1+|\varepsilon|^2)}{2}}\frac{1}{1+\varepsilon}\left[\theta_S(t)|\mathrm{K}_S\rangle + \theta_L(t)|\mathrm{K}_L\rangle\right] \\
&= \sqrt{\frac{(1+|\varepsilon|^2)}{2}}\frac{1}{1+\varepsilon}\times\frac{1}{\sqrt{2(1+|\varepsilon|^2)}}\times \\
&\quad \Big[\theta_S(t)(1+\varepsilon)|\mathrm{K}^0\rangle + \theta_S(t)(1-\varepsilon)|\overline{\mathrm{K}}^0\rangle + \theta_L(t)(1+\varepsilon)|\mathrm{K}^0\rangle - \theta_L(t)(1-\varepsilon)|\overline{\mathrm{K}}^0\rangle\Big] \\
&= \tfrac{1}{2}\left(\theta_S(t)+\theta_L(t)\right)|\mathrm{K}^0\rangle + \tfrac{1}{2}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)\left(\theta_S(t)-\theta_L(t)\right)|\overline{\mathrm{K}}^0\rangle .
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\Gamma(\mathrm{K}^0_{t=0}\to\mathrm{K}^0) &= \tfrac{1}{4}|\theta_S+\theta_L|^2 , \\
\Gamma(\mathrm{K}^0_{t=0}\to\overline{\mathrm{K}}^0) &= \tfrac{1}{4}\left|\frac{1-\varepsilon}{1+\varepsilon}\right|^2|\theta_S-\theta_L|^2 .
\end{aligned}
$$

Since ε is small

$$
\begin{aligned}
\left|\frac{1-\varepsilon}{1+\varepsilon}\right|^2 &= \frac{(1-\varepsilon)(1-\varepsilon^*)}{(1+\varepsilon)(1+\varepsilon^*)} \\
&\approx \frac{1-2\,\Re\mathrm{e}\,\{\varepsilon\}}{1+2\,\Re\mathrm{e}\,\{\varepsilon\}} \\
&\approx 1-4\,\Re\mathrm{e}\,\{\epsilon\} \\
&= 1-4|\varepsilon|\cos\phi ,
\end{aligned}
$$

and thus

$$
\begin{aligned}
\Gamma(\mathrm{K}^0_{t=0}\to\mathrm{K}^0) &= \tfrac{1}{4}|\theta_S+\theta_L|^2 , \\
\Gamma(\mathrm{K}^0_{t=0}\to\overline{\mathrm{K}}^0) &= \tfrac{1}{4}\left[1-4|\varepsilon|\cos\phi\right]|\theta_S-\theta_L|^2 .
\end{aligned}
$$

Working through the same algebra for an initial $\overline{\mathrm{K}}^0$ gives

$$
\begin{aligned}
\Gamma(\overline{\mathrm{K}}^0_{t=0}\to\overline{\mathrm{K}}^0) &= \tfrac{1}{4}|\theta_S+\theta_L|^2 , \\
\Gamma(\overline{\mathrm{K}}^0_{t=0}\to\mathrm{K}^0) &= \tfrac{1}{4}\left|\frac{1+\varepsilon}{1-\varepsilon}\right|^2|\theta_S-\theta_L|^2 \\
&= \tfrac{1}{4}\left[1+4|\varepsilon|\cos\phi\right]|\theta_S-\theta_L|^2 .
\end{aligned}
$$

Using the rates derived above

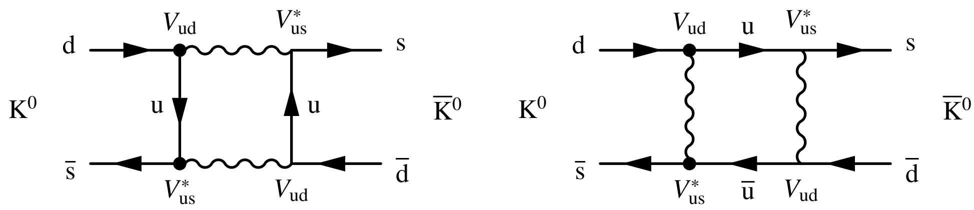
$$\begin{aligned} A_T &= \frac{\Gamma(\overline{\mathrm{K}}^0 \to \mathrm{K}^0) - \Gamma(\mathrm{K}^0 \to \overline{\mathrm{K}}^0)}{\Gamma(\overline{\mathrm{K}}^0 \to \mathrm{K}^0) + \Gamma(\mathrm{K}^0 \to \overline{\mathrm{K}}^0)} \\ &= \frac{8|\varepsilon|\cos\phi\,|\theta_S - \theta_L|^2}{2|\theta_S - \theta_L|^2} \\ &= 4|\varepsilon|\cos\phi\,, \end{aligned}$$

and it is worth noting that the time reversal asymmetry is independent of time.

14.8 The K_S–K_L mass difference can be expressed as

$$\Delta m = m(\mathrm{K}_L) - m(\mathrm{K}_S) \approx \sum_{\mathrm{q,q'}} \frac{G_\mathrm{F}^2}{3\pi^2} f_\mathrm{K}^2 m_\mathrm{K} |V_\mathrm{qd} V_\mathrm{qs}^* V_\mathrm{q'd} V_\mathrm{q's}^*|\, m_\mathrm{q} m_\mathrm{q'}\,,$$

where q and q′ are the quark flavours appearing in the box diagram. Using the values for the CKM matrix elements given in (14.8), obtain expressions for the relative contributions to Δm arising from the different combinations of quarks in the box diagrams.



The relative importance of the contributions from the u, c and t quarks in the box diagram depends on $V_\mathrm{qd}V_\mathrm{qs}^* m_\mathrm{q}$. Taking $m_\mathrm{u} \approx 0.3\,GeV$, $m_\mathrm{c} \approx 1.5\,\mathrm{GeV}$ and $m_\mathrm{t} \approx 175\,\mathrm{GeV}$ the relative contributions are in the ratio:

$$V_\mathrm{ud}V_\mathrm{us}m_\mathrm{u} :: V_\mathrm{cd}V_\mathrm{cs}m_\mathrm{c} : V_\mathrm{td}V_\mathrm{ts}m_\mathrm{t} = 0.07\,\mathrm{GeV} : 0.33\,\mathrm{GeV} : 0.06\,\mathrm{GeV}\,,$$

and, therefore, the largest contribution to the mass difference comes from the box diagrams with two charm quarks. Taking $f_\mathrm{K} \sim 0.1\,\mathrm{GeV}$, the numerically value for the contribution from the diagrams involving the charm quarks alone is

$$\Delta m = m(\mathrm{K}_L) - m(\mathrm{K}_S) = \frac{G_\mathrm{F}^2}{3\pi^2} f_\mathrm{K}^2 m_\mathrm{K} |V_\mathrm{cd}|^2 |V_\mathrm{cs}|^2 m_\mathrm{c}^2 = 2.3 \times 10^{-15}\,\mathrm{GeV}\,,$$

which is not too far from the measured value of $\Delta m = 3.5 \times 10^{-15}\,\mathrm{GeV}$.

14.9 Indirect CP violation in the neutral kaon system is expressed in terms of $\varepsilon = |\varepsilon| e^{i\phi}$. Writing

$$\xi = \frac{1-\varepsilon}{1+\varepsilon} \approx 1 - 2\varepsilon = \left(\frac{M_{12}^* - \frac{i}{2}\Gamma_{12}^*}{M_{12} - \frac{i}{2}\Gamma_{12}} \right)^{\frac{1}{2}}\,,$$

show that

$$\varepsilon \approx \frac{1}{2} \times \left(\frac{(\Im\mathfrak{m}\,\{M_{12}\} - \frac{i}{2}\,\Im\mathfrak{m}\,\{\Gamma_{12}\})}{M_{12} - \frac{i}{2}\Gamma_{12}} \right)^{\frac{1}{2}} \approx \frac{\Im\mathfrak{m}\,\{M_{12}\} - i\,\Im\mathfrak{m}\,\{\Gamma_{12}/2\}}{\Delta m - i\Delta\Gamma/2} .$$

Using the knowledge that $\phi \approx 45^\circ$ and the measurements of Δm and $\Delta\Gamma$, deduce that $\Im\mathfrak{m}\,\{M_{12}\} \gg \Im\mathfrak{m}\,\{\Gamma_{12}\}$ and therefore

$$|\varepsilon| \sim \frac{1}{\sqrt{2}} \frac{\Im\mathfrak{m}\,\{M_{12}\}}{\Delta m} .$$

The first part of the problem is more obvious if one starts from the required solution. Recalling that the differences in mass and decay width of the physical kaon mass eigenstates can be expressed as

$$\Delta m - \tfrac{i}{2}\Delta\Gamma = \lambda_+ - \lambda_- = \left[(M_{12} - \tfrac{i}{2}\Gamma_{12})(M_{12}^* - \tfrac{i}{2}\Gamma_{12}^*)\right]^{\frac{1}{2}}$$

then

$$\begin{aligned}
\frac{\Im\mathfrak{m}\,\{M_{12}\} - i\,\Im\mathfrak{m}\,\{\Gamma_{12}/2\}}{\Delta m - i\Delta\Gamma/2} &= \frac{\Im\mathfrak{m}\,\{M_{12}\} - \frac{i}{2}\,\Im\mathfrak{m}\,\{\Gamma_{12}\}}{\left[M_{12} - \frac{i}{2}\Gamma_{12}\right]^{1/2}\left[M_{12}^* - \frac{i}{2}\Gamma_{12}^*\right]^{1/2}} \\
&= \frac{1}{2} \frac{M_{12} - M_{12}^* - \frac{i}{2}(\Gamma_{12} - \Gamma_{12}^*)}{\left[M_{12} - \frac{i}{2}\Gamma_{12}\right]^{1/2}\left[M_{12}^* - \frac{i}{2}\Gamma_{12}^*\right]^{1/2}} \\
&= \frac{1}{2} \frac{(M_{12} - \frac{i}{2}\Gamma_{12}) - (M_{12}^* - \frac{i}{2}\Gamma_{12}^*)}{\left[M_{12} - \frac{i}{2}\Gamma_{12}\right]^{1/2}\left[M_{12}^* - \frac{i}{2}\Gamma_{12}^*\right]^{1/2}} \\
&= \frac{1}{2}\left[\left(\frac{M_{12} - \frac{i}{2}\Gamma_{12}}{M_{12}^* - \frac{i}{2}\Gamma_{12}^*}\right)^{\frac{1}{2}} - \left(\frac{M_{12}^* - \frac{i}{2}\Gamma_{12}^*}{M_{12} - \frac{i}{2}\Gamma_{12}}\right)^{\frac{1}{2}}\right] \\
&\approx \frac{1}{2}\left[(1 - 2\varepsilon)^{-1/2} - (1 - 2\varepsilon)^{+1/2}\right] \\
&= \frac{1}{2}\left[(1 + \varepsilon) - (1 - \varepsilon)\right] \\
&= \varepsilon .
\end{aligned}$$

The second part of the question uses the measured properties of the neutral kaon system to extract information about the effective Hamiltonian. The measured value of Δm is

$$\Delta m = (3.483 \pm 0.006) \times 10^{-15}\,\text{GeV} .$$

From the measured K_S and K_L lifetimes:

$$\begin{aligned}\Delta\Gamma &= \Gamma_S - \Gamma_L \\ &= \frac{\hbar}{\tau_S} - \frac{\hbar}{\tau_L} \\ &= \frac{\hbar}{8.95\times10^{-11}} - \frac{\hbar}{5.12\times10^{-8}} \\ &= (7.36\times10^{-15} - 1.29\times10^{-17})\,\text{GeV} \\ &= 7.34\times10^{-15}\,\text{GeV} \\ \Rightarrow \quad \Delta\Gamma/2 &= 3.67\times10^{-15}\,.\end{aligned}$$

From these experimental measurements it is found that $\Delta m \sim \Delta\Gamma/2$. The angle ϕ, defined by $\varepsilon = |\varepsilon|e^{i\phi}$, was measured by CPLEAR

$$\phi = \arg\varepsilon = (43.19 \pm 0.73)^\circ\,.$$

From the expression for ε derived previously,

$$\varepsilon = \frac{\Im\mathfrak{m}\{M_{12}\} - i\,\Im\mathfrak{m}\{\Gamma_{12}/2\}}{\Delta m - i\Delta\Gamma/2} = \frac{(\Im\mathfrak{m}\{M_{12}\} - i\,\Im\mathfrak{m}\{\Gamma_{12}/2\})(\Delta m + i\Delta\Gamma/2)}{\Delta m^2 + \Delta\Gamma^2/4}\,.$$

But since $\Delta m \sim \Delta\Gamma/2$,

$$\varepsilon \approx \frac{\Im\mathfrak{m}\{M_{12}\} - i\,\Im\mathfrak{m}\{\Gamma_{12}/2\}}{\Delta m - i\Delta\Gamma/2} = \frac{(\Im\mathfrak{m}\{M_{12}\} - i\,\Im\mathfrak{m}\{\Gamma_{12}/2\})\Delta m e^{i\pi/4}}{\Delta m^2 + \Delta\Gamma^2/4}\,,$$

and given $\arg\varepsilon \approx \pi/4$, this implies that $\Im\mathfrak{m}\{M_{12}\} \gg \Im\mathfrak{m}\{\Gamma_{12}\}$, and consequently

$$\begin{aligned}\varepsilon &\approx \frac{\Im\mathfrak{m}\{M_{12}\}}{\Delta m - i\Delta\Gamma/2} \\ &\approx \frac{\Im\mathfrak{m}\{M_{12}\}}{\Delta m(1-i)} \\ \Rightarrow \quad |\varepsilon| &= \frac{1}{\sqrt{2}}\frac{\Im\mathfrak{m}\{M_{12}\}}{\Delta m}\,.\end{aligned}$$

as required. Thus the measured value of $|\varepsilon| \sim 2.3\times10^{-3}$ implies that

$$\Im\mathfrak{m}\{M_{12}\} \approx 1\times10^{-17}\,\text{GeV}\,.$$

14.10 Using (14.53) and the explicit form of Wolfenstein parametrisation of the CKM matrix, show that

$$|\varepsilon| \propto \eta(1 - \rho + \text{constant})\,.$$

Equation (14.53) gives the relation

$$|\varepsilon| \propto \mathcal{A}_{\text{ut}}\,\Im\mathfrak{m}\,(V_{\text{ud}}V_{\text{us}}^*V_{\text{td}}V_{\text{ts}}^*) + \mathcal{A}_{\text{ct}}\,\Im\mathfrak{m}\,(V_{\text{cd}}V_{\text{cs}}^*V_{\text{td}}V_{\text{ts}}^*) + \mathcal{A}_{\text{tt}}\,\Im\mathfrak{m}\,(V_{\text{td}}V_{\text{ts}}^*V_{\text{td}}V_{\text{ts}}^*)\,,$$

where the $\mathcal{A}$ are real constants. In the Wolfenstein parameterisation of the CKM matrix:

$$\begin{aligned}
V_{\rm ud}V_{\rm us}^*V_{\rm td}V_{\rm ts}^* &= (1-\lambda^2/2)\cdot\lambda\cdot A\lambda^3(1-\rho-i\eta)\cdot(-A\lambda^2)\\
&= -A^2\lambda^6(1-\lambda^2/2)(1-\rho-i\eta)\\
V_{\rm cd}V_{\rm cs}^*V_{\rm td}V_{\rm ts}^* &= (-\lambda)\cdot(1-\lambda^2/2)\cdot A\lambda^3(1-\rho-i\eta)\cdot(-A\lambda^2)\\
&= A^2\lambda^6(1-\lambda^2/2)(1-\rho-i\eta)\\
V_{\rm td}V_{\rm ts}^*V_{\rm td}V_{\rm ts}^* &= \left[A\lambda^3(1-\rho-i\eta)\cdot(-A\lambda^2)\right]^2\\
&= A^4\lambda^{10}\left[(1-\rho)^2-\eta^2-2i\eta(1-\rho)\right]\,.
\end{aligned}$$

Hence

$$\begin{aligned}
|\varepsilon| \propto\ & \mathcal{A}_{\rm ut}\,\Im\mathfrak{m}\,(V_{\rm ud}V_{\rm us}^*V_{\rm td}V_{\rm ts}^*) + \mathcal{A}_{\rm ct}\,\Im\mathfrak{m}\,(V_{\rm cd}V_{\rm cs}^*V_{\rm td}V_{\rm ts}^*) + \mathcal{A}_{\rm tt}\,\Im\mathfrak{m}\,(V_{\rm td}V_{\rm ts}^*V_{\rm td}V_{\rm ts}^*)\\
&= a\eta + b\eta + c\eta(1-\rho)\\
&= a\eta(1-\rho+b+c)\,,
\end{aligned}$$

where a, b and c are just real constants. Hence a measurement of $|\varepsilon|$ gives

$$\begin{aligned}
& |\varepsilon| = a\eta(1-\rho+b+c)\\
\Rightarrow\quad & \eta(1-\rho+\text{constant}) = \text{constant}\,,
\end{aligned}$$

which is the equation of a hyperbola in the (ρ,η) plane.

14.11 Show that the B^0–$\overline{B}^0$ mass difference is dominated by the exchange of two top quarks in the box diagram.

From question 8, the mass difference of the B_H and B_L mass eigenstates can be expressed as

$$\Delta m = m(B_H) - m(B_L) \approx \sum_{\rm q,q'} \frac{G_{\rm F}^2}{3\pi^2} f_{\rm B}^2 m_{\rm B} |V_{\rm qd}V_{\rm qb}^*V_{\rm q'd}V_{\rm q'b}^*|\, m_{\rm q}m_{\rm q'}\,,$$

where q and q′ are the quark flavours appearing in the box diagram. Taking $m_{\rm u} \approx 0.3\,GeV$, $m_{\rm c} \approx 1.5\,\text{GeV}$ and $m_{\rm t} \approx 175\,\text{GeV}$ the relative contributions are in the ratio:

$$V_{\rm ud}V_{\rm ub}m_{\rm u} :: V_{\rm cd}V_{\rm cb}m_{\rm c} : V_{\rm td}V_{\rm tb}m_{\rm t} = 0.001\,\text{GeV} : 0.014\,\text{GeV} : 1.57\,\text{GeV}\,,$$

and the contributions from the top quarks in the box diagrams clearly dominate.

14.12 Calculate the velocities of the B-mesons produced in the decay at rest of the $\Upsilon(4S) \to B^0\overline{B}^0$.

The masses of the particles are

$$m(\Upsilon(4S)) = 10.579\,\text{GeV} \quad \text{and} \quad m(\text{B}^0) = m(\overline{\text{B}}^0) = 5.279\,\text{GeV}\,,$$

and for the decay at rest, conservation of energy gives

$$E(\text{B}^0)^* = E(\overline{\text{B}}^0)^* = m(\Upsilon(4S))/2 = 5.289\,\text{GeV}\,.$$

The momenta of the daughter particles are

$$\text{p}^* = (E^{*2} - m^2)^{1/2} = 0.333\,\text{GeV}\,,$$

and thus the velocities are

$$\beta^* = \frac{\text{p}^*}{E^*} \equiv \frac{\gamma m\beta}{\gamma m} = 0.063\,.$$

14.13 Given the lifetimes of the neutral B-mesons are $\tau = 1.53$ ps, calculate the mean distance they travel when produced at the KEKB collider in collisions of 8 GeV electrons and 3.5 GeV positrons.

The electron and positron energies are chosen such that $\sqrt{s} = m(\Upsilon_{4S})$ and here the $\Upsilon(4S)$ is produced with momentum 4.5 GeVin the laboratory frame, corresponding to velocity of

$$\beta_\Upsilon = \frac{\text{p}_\Upsilon}{E_\Upsilon} = \frac{4.5}{11.5} = 0.39 \quad \Rightarrow \quad \gamma_\Upsilon = 1.086\,.$$

In the laboratory frame, the energy of the boosted decay B^0 is given by

$$E' = \gamma_\Upsilon E^* + \gamma_\Upsilon\beta_\Upsilon \text{p}^* \cos\theta^*\,,$$

where θ^* is the polar angle of the B^0 relative to the z-axis along the direction of the boost and p^* and E^* are the rest frame decay momentum and energy, calculated in the previous question. Since $p^* = \beta^* E^* = 0.063$, the energy of the decay B^0 in the laboratory frame

$$E' = \gamma_\Upsilon E^*(1 + \beta_\Upsilon\beta^* \cos\theta^*) = \gamma_\Upsilon E^*(1 + 0.024\cos\theta^*)\,,$$

does not depend strongly on the decay angle. Taking into account time dilation, the mean distance the B^0 will travel is

$$d = \gamma\tau\beta c = \frac{\text{p}'}{m}\tau c\,.$$

Taking $E' \approx \gamma_\Upsilon E^* = 5.74$ GeV, the momentum of the B^0 is approximately $\text{p}' = 2.23$, and therefore $\beta\gamma = p'/m = 0.43$. The mean distance the B^0 travels in the laboratory frame is:

$$d = \frac{\text{p}'}{m}\tau c = 197\,\mu\text{m}\,.$$

14.14 From the measured values

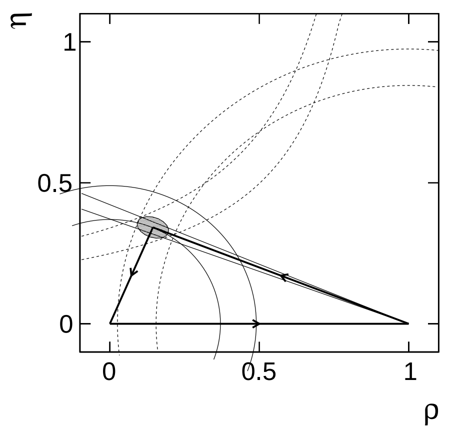
$$|V_{ud}| = 0.97425 \pm 0.00022 \quad \text{and} \quad |V_{ub}| = (4.15 \pm 0.49) \times 10^{-3}\,,$$
$$|V_{cd}| = 0.230 \pm 0.011 \quad \text{and} \quad |V_{cb}| = 0.041 \pm 0.001\,,$$

calculate the length of the corresponding side of the unitarity triangle in Figure 14.25 and its uncertainty. By sketching this constraint and that from the measured value of β, obtain approximate constraints on the values of ρ and η.

The length of the shortest side of the unitarity triangle shown Figure 14.25 is

$$x = \left|\frac{V_{ub}^{*}\,|V_{ud}|}{|V_{cd}|\,|V_{cb}^{*}|}\right| = \frac{|V_{ub}|\,|V_{ud}|}{|V_{cd}|\,|V_{cb}|} = 0.43 \pm 0.06\,,$$

where the fractional uncertainty is given by the sum in quadrature of the fractional errors on the CKM matrix elements involved. As can be seen from the figure below, due to the relatively large uncertainty, the length of this side of this unitarity triangle does not significantly constrain the values of ρ and η. For example, at one standard the side length is consistent with roughly the range $-0.2 < \rho < 0.4$.

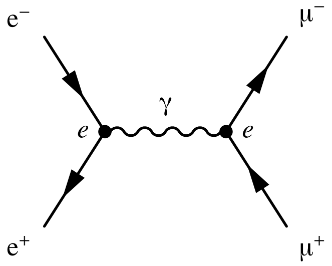


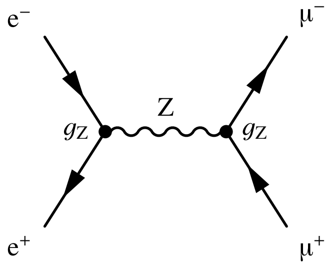
15 Electroweak Unification

15.1 Draw all possible lowest-order Feynman diagrams for the processes:

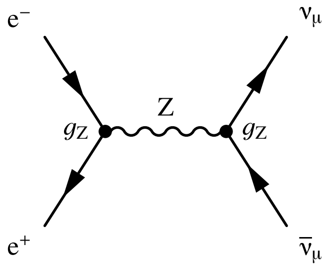
$$e^+e^- \to \mu^+\mu^-\,, \quad e^+e^- \to \nu_\mu\overline{\nu}_\mu\,, \quad \nu_\mu e^- \to \nu_\mu e^- \quad \text{and} \quad \overline{\nu}_e e^- \to \overline{\nu}_e e^-\,.$$

In all cases the Higgs exchange diagrams will give negligible contributions and are ignored. The two possible lowest-order diagrams for $e^+e^- \to \mu^+\mu^-$ are

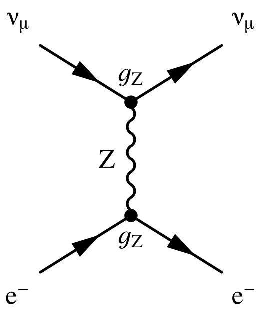




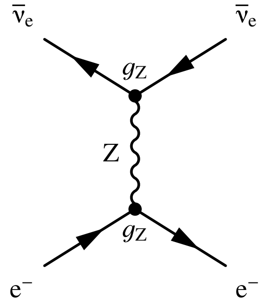
Since neutrinos are neutral, there is no QED diagram for $e^+e^- \to \nu_\mu\overline{\nu}_\mu$ but a the Z-exchange diagram is still present.

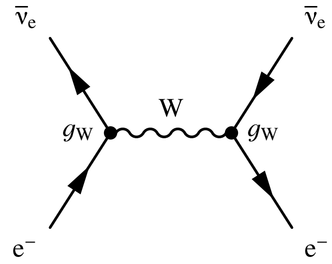


For $\nu_\mu e^- \to \nu_\mu e^-$ only the neutral current weak interaction contributes at lowest order.



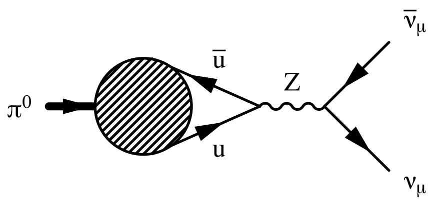
Finally, for $\overline{\nu}_e e^- \to \overline{\nu}_e e^-$ there are weak charged-current and weak neutral current diagrams.





15.2 Draw the lowest-order Feynman diagram for the decay $\pi^0 \to \nu_\mu \overline{\nu}_\mu$ and explain why this decay is effectively forbidden.

The lowest-order Feynman diagram is show below. The general form of the neutral current vertex, $\gamma^\mu(c_V - c_A\gamma^5)$, reduces to a $V - A$ form for the coupling at $Z\nu_\mu\overline{\nu}_\mu$ vertex, and thus neutrino is produced in a LH chiral state and anti-neutrino is produced in a RH chiral state. Because neutrinos are almost massless ($E \gg m$) the chiral states effectively correspond to helicity states and thus the decay would result in a $J = 1$ final state, violating conservation of angular momentum.

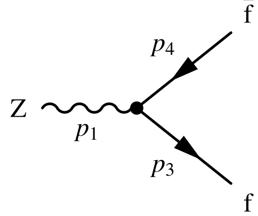


15.3 Starting from the matrix element, work through the calculation of the $Z \to f\overline{f}$ partial decay rate, expressing the answer in terms of the vector and axial-vector couplings of Z. Taking $\sin^2\theta_W = 0.2315$, show that

$$R_\mu = \frac{\Gamma(Z \to \mu^+\mu^-)}{\Gamma(Z \to \text{hadrons})} \approx \frac{1}{20}\,.$$

The matrix element for the $Z \to f\overline{f}$ decay, shown below is

$$\mathcal{M}_{fi} = g_Z\,\epsilon_\mu^\lambda(p_1)\,\overline{u}(p_3)\gamma^\mu\tfrac{1}{2}(c_V - c_A\gamma^5)v(p_4)\,,$$



or equivalently

$$\mathcal{M}_{fi} = g_Z\,\epsilon_\mu^\lambda(p_1)\,\overline{u}(p_3)\gamma^\mu\left[c_L\tfrac{1}{2}(1-\gamma^5) + c_R\tfrac{1}{2}(1+\gamma^5)\right]v(p_4)\,,$$

where $c_V = c_L + c_R$ and $c_A = c_L - c_R$. Written in this form it should be clear that only two chiral combinations give non-zero matrix elements, and in the limit where the final-state fermions are ultra-relativistic, only two helicity combinations give non-zero matrix elements:

$$\mathcal{M}_{LR} = g_Z\,c_L\,\epsilon_\mu^\lambda(p_1)\,\overline{u}_\downarrow(p_3)\gamma^\mu v_\uparrow(p_4) \quad \text{and} \quad \mathcal{M}_{RL} = g_Z\,c_R\,\epsilon_\mu^\lambda(p_1)\,\overline{u}_\uparrow(p_3)\gamma^\mu v_\downarrow(p_4)\,.$$

The leptonic currents are given by (6.17) and (6.16) with $E = m_Z/2$,

$$j^\mu_{LR} = m_Z(0, -\cos\theta, -i, \sin\theta) \quad \text{and} \quad j^\mu_{RL} = m_Z(0, -\cos\theta, +i, \sin\theta)\,.$$

Without loss of generality, we are free to choose the polarisation state of the Z. Here take the Z to be at rest and to be longitudinally polarised, such that

$$\epsilon_\mu = \epsilon_L^\mu = (0,0,0,1)\,.$$

In this case, the matrix elements reduce to

$$\mathcal{M}_{LR} = -g_Z\,c_L\,j^3_{LR} = -g_Z\,c_L m_Z\sin\theta \quad \text{and} \quad \mathcal{M}_{RL} = -g_Z\,c_R\,j^3_{RL} = -g_Z\,c_R m_Z\sin\theta\,.$$

The total decay rate is determined by the summed matrix element squared (no need to average since we have chosen a particular initial state polarisation)

$$\left\langle|\mathcal{M}|^2\right\rangle = |\mathcal{M}_{LR}|^2 + |\mathcal{M}_{RL}|^2 = g_Z^2 m_Z^2(c_L^2 + c_R^2)\sin^2\theta\,.$$

Substituting this into the decay rate formula of (3.49),

$$\begin{aligned}
\Gamma(\mathrm{Z}\to \mathrm{f\bar{f}}) &= \frac{\mathrm{p}^*}{32\pi^2 m_Z^2}\int\langle|\mathcal{M}|^2\rangle\,\mathrm{d}\Omega^* \\
&= \frac{2\pi\mathrm{p}^*}{32\pi^2 m_Z^2}g_Z^2 m_Z^2(c_L^2+c_R^2)\int_{-1}^{+1}\sin^2\theta\,\mathrm{d}(\cos\theta) \\
&= \frac{\mathrm{p}^*}{16\pi^2}g_Z^2(c_L^2+c_R^2)\int_{-1}^{+1}(1-x^2)\,\mathrm{d}(x) \\
&= \frac{\mathrm{p}^*}{24\pi}g_Z^2(c_L^2+c_R^2)\,,
\end{aligned}$$

where p^* is the momentum of the final state fermion in the centre-of-mass frame. If the masses of the final-state particles are neglected, $\mathrm{p}^* = m_Z/2$, and therefore the $\mathrm{Z}\to\mathrm{f\bar{f}}$ decay rate is given by

$$\Gamma(\mathrm{Z}\to\mathrm{f\bar{f}}) = \frac{g_Z^2 m_Z}{24\pi}(c_L^2+c_R^2) = \frac{g_Z^2 m_Z}{48\pi}(c_V^2+c_A^2)\,.$$

The partial decay widths therefore depend on the sum of the squares of the vector and axial-vector couplings of the Z to the fermions. Taking into account the three colours and that the Z cannot decay to top quarks, the ratio

$$\begin{aligned} R_\mu &= \frac{\Gamma(\mathrm{Z} \to \mu^+\mu^-)}{\Gamma(\mathrm{Z} \to \text{hadrons})} \\ &= \frac{\Gamma_{\mathrm{Z}\to\mu^+\mu^-}}{9\Gamma(\mathrm{Z} \to \mathrm{d\bar{d}}) + 6\Gamma(\mathrm{Z} \to \mathrm{u\bar{u}})} \,. \end{aligned}$$

The individual partial decay widths are proportional to:

$$\mu \,:\, c_V^2 + c_A^2 = 0.2516\,, \quad \mathrm{d} \,:\, c_V^2 + c_A^2 = 0.3725 \quad \text{and} \quad \mathrm{u} \,:\, c_V^2 + c_A^2 = 0.2861\,,$$

and therefore

$$\begin{aligned} R_\mu &= \frac{\Gamma(\mathrm{Z} \to \mu^+\mu^-)}{\Gamma(\mathrm{Z} \to \text{hadrons})} \\ &= \frac{0.2516}{9 \cdot 0.3725 + 6 \cdot 0.2861} = 0.496 \approx \frac{1}{20} \,. \end{aligned}$$

15.4 Consider the purely neutral-current (NC) process $\nu_\mu \mathrm{e}^- \to \nu_\mu \mathrm{e}^-$.

a) Show that in the limit where the electron mass can be neglected, the spin-averaged matrix element for $\nu_\mu \mathrm{e}^- \to \nu_\mu \mathrm{e}^-$ can be written

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{2}\left(\left|\mathcal{M}_{LL}^{\mathrm{NC}}\right|^2 + \left|\mathcal{M}_{LR}^{\mathrm{NC}}\right|^2 \right),$$

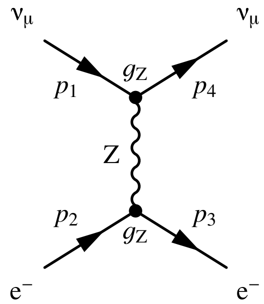
where

$$\mathcal{M}_{LL}^{\mathrm{NC}} = 2c_L^{(\nu)} c_L^{(\mathrm{e})} \frac{g_\mathrm{Z}^2 s}{m_\mathrm{Z}^2} \quad \text{and} \quad \mathcal{M}_{LR}^{\mathrm{NC}} = 2c_L^{(\nu)} c_R^{(\mathrm{e})} \frac{g_\mathrm{Z}^2 s}{m_\mathrm{Z}^2} \tfrac{1}{2}(1 + \cos\theta^*)\,,$$

and θ^* is the angle between the directions of the incoming and scattered neutrino in the centre-of-mass frame.

b) Hence find an expression for the $\nu_\mu \mathrm{e}^-$ neutral-current cross section in terms of the laboratory-frame neutrino energy.

a) The lowest-order Feynman diagram for the NC process $\nu_\mu \mathrm{e}^- \to \nu_\mu \mathrm{e}^-$ is shown below.



In this t-channel process, $q^2 \ll m_\mathrm{Z}^2$ and the Z propagator can be approximated by (a Fermi-like contact interaction),

$$\frac{-ig_{\mu\nu}}{q^2 - m_Z^2} \to \frac{ig_{\mu\nu}}{m_Z^2}\,,$$

and the matrix element is

$$-i\mathcal{M}_{fi} = \left[-ig_Z\overline{u}(p_3)\gamma^\mu \tfrac{1}{2}(c_V^\nu - c_A^\nu\gamma^5)u(p_1)\right]\frac{ig_{\mu\nu}}{m_Z^2}\left[-ig_Z\overline{u}(p_4)\gamma^\nu\tfrac{1}{2}(c_V^e - c_A^e\gamma^5)u(p_2)\right]$$

$$\begin{aligned}
\mathcal{M}_{fi} &= \frac{g_Z^2}{m_Z^2}g_{\mu\nu}\left[\overline{u}(p_3)\gamma^\mu\tfrac{1}{2}(c_V^\nu - c_A^\nu\gamma^5)u(p_1)\right]\left[\overline{u}(p_4)\gamma^\nu\tfrac{1}{2}(c_V^e - c_A^e\gamma^5)u(p_2)\right]\\
&= \frac{g_Z^2}{2m_Z^2}g_{\mu\nu}\left[\overline{u}(p_3)\gamma^\mu\tfrac{1}{2}(1-\gamma^5)u(p_1)\right]\left[\overline{u}(p_4)\gamma^\nu\left\{c_L^e\tfrac{1}{2}(1-\gamma^5) + c_R^e\tfrac{1}{2}(1+\gamma^5)\right\}u(p_2)\right]\\
&= \frac{g_Z^2}{m_Z^2}g_{\mu\nu}\left[c_L^\nu\overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1)\right]\left[\overline{u}(p_4)\gamma^\nu\left\{c_L^e\tfrac{1}{2}(1-\gamma^5) + c_R^e\tfrac{1}{2}(1+\gamma^5)\right\}u(p_2)\right]\,,
\end{aligned}$$

where it should be noted that because $c_V^\nu = c_A^\nu = \frac{1}{2}$, only left-handed neutrinos couple to the Z. Consequently, in the ultra-relativistic limit, there are two non-zero matrix elements:

$$\begin{aligned}
\mathcal{M}_{LL} &= \frac{g_Z^2}{m_Z^2}g_{\mu\nu}c_L^\nu c_L^e\left[\overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1)\right]\left[\overline{u}_\downarrow(p_4)\gamma^\nu u_\downarrow(p_2)\right]\,,\\
\mathcal{M}_{LR} &= \frac{g_Z^2}{m_Z^2}g_{\mu\nu}c_L^\nu c_R^e\left[\overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1)\right]\left[\overline{u}_\uparrow(p_4)\gamma^\nu u_\uparrow(p_2)\right]\,.
\end{aligned}$$

The matrix elements are most easily evaluated in the centre-of-mass frame. Taking the initial neutrino direction to define the z-axis and θ^* to be the polar angle of the final-state neutrino, then the spherical polar angles of the four particles, as indicated in Figure 12.4, are

$$(\theta_1,\phi_1) = (0,0)\,,\ \ (\theta_2,\phi_2) = (\pi,\pi)\,,\ \ (\theta_3,\phi_3) = (\theta^*,0)\ \text{ and }\ (\theta_4,\phi_4) = (\pi-\theta^*,\pi)\,.$$

The corresponding LH spinors are given by (4.65),

$$u_\downarrow(p_1) = \sqrt{E}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix},\ u_\downarrow(p_3) = \sqrt{E}\begin{pmatrix}-s\\c\\s\\-c\end{pmatrix}\ u_\downarrow(p_2) = \sqrt{E}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix},\ u_\downarrow(p_4) = \sqrt{E}\begin{pmatrix}-c\\-s\\c\\s\end{pmatrix},$$

$$u_\uparrow(p_2) = \sqrt{E}\begin{pmatrix}0\\-1\\0\\-1\end{pmatrix},\ u_\uparrow(p_4) = \sqrt{E}\begin{pmatrix}s\\-c\\s\\-c\end{pmatrix},$$

where $c = \cos\frac{\theta^*}{2}$, $s = \sin\frac{\theta^*}{2}$ and E is the energy of each of the four particles in the centre-of-mass frame. The corresponding currents are:

$$
\begin{aligned}
j_\nu^\mu &= \overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = 2E(c, s, -is, c)\,,\\
j_{eL}^\nu &= \overline{u}_\downarrow(p_4)\gamma^\nu u_\downarrow(p_2) = 2E(c, -s, -is, -c)\,,\\
j_{eR}^\nu &= \overline{u}_\uparrow(p_4)\gamma^\nu u_\uparrow(p_2) = 2E(c, -s, +is, -c)\,,
\end{aligned}
$$

and hence

$$
\begin{aligned}
\mathrm{J}_\nu \cdot j_{eL}^\nu &= 4E^2(c^2 + s^2 + s^2 + c^2) = 8E^2 = 2s\\
\mathrm{J}_\nu \cdot j_{eR}^\nu &= 4E^2(c^2 + s^2 - s^2 + s^2) = 8E^2c^2 = 2s\tfrac{1}{2}(1 + \cos\theta^{)}\,.
\end{aligned}
$$

Finally, putting all the pieces together:

$$
\mathcal{M}_{LL}^{\mathrm{NC}} = 2c_L^\nu c_L^e \frac{g_Z^2 s}{m_Z^2} \quad \text{and} \quad \mathcal{M}_{LR}^{\mathrm{NC}} = 2c_L^\nu c_R^e \frac{g_Z^2 s}{m_Z^2}\tfrac{1}{2}(1 + \cos\theta^*)\,,
$$

where θ^* is the angle between the directions of the incoming and scattered neutrino in the centre-of-mass frame.

b) The spin-averaged matrix element squared (averaging over the two spin states of the electron since the neutrino is left-handed) for the NC scattering process is

$$
\begin{aligned}
\langle|\mathcal{M}_{fi}|^2\rangle &= \frac{1}{2}\frac{g_Z^4 s^2}{m_Z^4}\left[4(c_L^\nu)^2(c_L^e)^2 + 4(c_L^\nu)^2(c_R^e)^2\tfrac{1}{4}(1 + \cos\theta^*)^2\right]\\
&= \frac{1}{2}\frac{g_Z^4 s^2}{m_Z^4}\left[(c_L^e)^2 + (c_R^e)^2\tfrac{1}{4}(1 + \cos\theta^*)^2\right].
\end{aligned}
$$

The differential cross section can be obtained from the expression

$$
\frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2 s}\langle|\mathcal{M}_{fi}|^2\rangle = \frac{s}{128\pi^2}\frac{g_Z^4}{m_Z^4}\left[(c_L^e)^2 + (c_R^e)^2\tfrac{1}{4}(1 + \cos\theta^*)^2\right].
$$

Using $G_F = \sqrt{2}g_W^2/8m_W^2 = \sqrt{2}g_Z^2/8m_Z^2$, this can be written as

$$
\frac{d\sigma}{d\Omega^*} = \frac{s}{4\pi^2}G_F^2\left[(c_L^e)^2 + (c_R^e)^2\tfrac{1}{4}(1 + \cos\theta^*)^2\right].
$$

Finally, writing $s = 2m_e E_\nu$ and integrating

$$
\begin{aligned}
\sigma &= \frac{m_e E_\nu G_F^2}{\pi}\int_{-1}^{+1}\left[(c_L^e)^2 + (c_R^e)^2\tfrac{1}{4}(1 + 2x + x^2)\right]dx\\
&= \frac{2m_e E_\nu G_F^2}{\pi}\left[(c_L^e)^2 + \tfrac{1}{3}(c_R^e)^2\right]\\
&= \frac{2m_e E_\nu G_F^2}{\pi}\left[(-0.27)^2 + \tfrac{1}{3}(0.23)^2\right]\\
&\approx \frac{2m_e E_\nu G_F^2}{\pi}\times 0.09\,.
\end{aligned}
$$

15.5 The two lowest-order Feynman diagrams for $\nu_e e^- \to \nu_e e^-$ are shown in Figure 13.5. Because both diagrams produce the same final state, the amplitudes have to be added before the matrix element is squared. The matrix element for the charged-current (CC) process is

$$\mathcal{M}_{LL}^{\mathrm{CC}} = \frac{g_{\mathrm{W}}^2 s}{m_{\mathrm{W}}^2}\,.$$

a) In the limit where the lepton masses and the q^2 term in the W-boson propagator can be neglected, write down expressions for spin-averaged matrix elements for the processes

$$\nu_\mu e^- \to \nu_\mu e^-\,, \quad \nu_e e^- \to \nu_e e^- \quad \text{and} \quad \nu_\mu e^- \to \nu_e \mu^-\,.$$

b) Using the relation $g_Z/m_Z = g_W/m_W$, show that

$$\sigma(\nu_\mu e^- \to \nu_\mu e^-) : \sigma(\nu_e e^- \to \nu_e e^-) : \sigma(\nu_\mu e^- \to \nu_e \mu^-) = c_L^2 + \tfrac{1}{3}c_R^2 : (1 + c_L)^2 + \tfrac{1}{3}c_R^2 : 1\,,$$

where c_L and c_R refer to the couplings of the left- and right-handed charged leptons to the Z.

c) Find numerical values for these ratios of NC+CC : NC : CC cross sections and comment on the sign of the interference between the NC and CC diagrams.

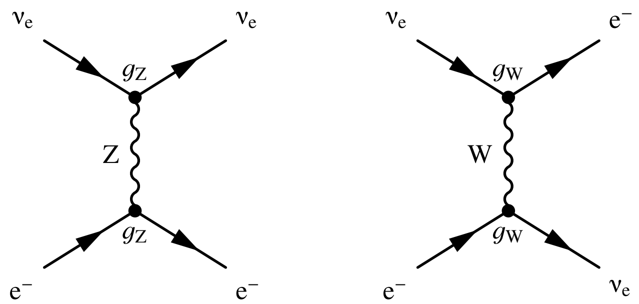
a) From the previous question, the spin-averaged matrix element for $\nu_\mu e^- \to \nu_\mu e^-$

$$\begin{aligned}\langle|\mathcal{M}|^2_{\mathrm{NC}}\rangle &= \frac{1}{2}\left[(\mathcal{M}_{LL}^{\mathrm{NC}})^2 + (\mathcal{M}_{LR}^{\mathrm{NC}})^2\right]\\ &= \frac{1}{2}\frac{g_{\mathrm{W}}^4 s^2}{m_{\mathrm{W}}^4}\left[c_L^2 + c_R^2\tfrac{1}{4}(1+\cos\theta^*)^2\right].\end{aligned}$$

where the relation $g_Z/m_Z = g_W/m_W$ has been used. The spin-averaged matrix element for the pure CC weak interaction $\nu_\mu e^- \to \nu_e \mu^-$ is

$$\begin{aligned}\langle|\mathcal{M}|^2_{\mathrm{CC}}\rangle &= \frac{1}{2}(\mathcal{M}_{LL}^{\mathrm{CC}})^2\\ &= \frac{1}{2}\frac{g_{\mathrm{W}}^4 s^2}{m_{\mathrm{W}}^4}\,.\end{aligned}$$

In the process $\sigma(\nu_e e^- \to \nu_e e^-)$, both charged-current and neutral-current diagrams contribute and can interfere.



Consequently the spin-averaged matrix element for this mixed NC and CC weak interaction is

$$\begin{aligned}\langle|\mathcal{M}|^2_{\mathrm{NC+CC}}\rangle &= \frac{1}{2}\left[(\mathcal{M}^{\mathrm{CC}}_{LL} + \mathcal{M}^{\mathrm{NC}}_{LL})^2 + (\mathcal{M}^{\mathrm{NC}}_{LR})^2\right] \\ &= \frac{1}{2}\frac{g_{\mathrm{W}}^4 s^2}{m_{\mathrm{W}}^4}\left[(1+c_L)^2 + \tfrac{1}{4}c_R^2(1+\cos\theta^*)^2\right],\end{aligned}$$

where the relation $g_{\mathrm{Z}}/m_{\mathrm{Z}} = g_{\mathrm{W}}/m_{\mathrm{W}}$ has been used again.

b) When the spin-averaged matrix elements are integrate over solid angle, the integrals over the angular distributions of the LL and RR contributions give respectively

$$\int_{-1}^{+1} \mathrm{d}\cos\theta^* = 2 \quad \text{and} \quad \int_{-1}^{+1} \tfrac{1}{4}(1+\cos\theta)^2\,\mathrm{d}\cos\theta^* = \tfrac{2}{3}\,,$$

and from part a) it immediately follows that

$$\sigma(\nu_\mu \mathrm{e}^- \to \nu_\mu \mathrm{e}^-) : \sigma(\nu_\mathrm{e} \mathrm{e}^- \to \nu_\mathrm{e} \mathrm{e}^-) : \sigma(\nu_\mu \mathrm{e}^- \to \nu_\mathrm{e} \mu^-) = c_L^2 + \tfrac{1}{3}c_R^2 : (1+c_L)^2 + \tfrac{1}{3}c_R^2 : 1\,.$$

c) Putting in the couplings of the left- and right-handed leptons to the Z, $c_L = -0.27$ and $c_R = 0.23$ gives

$$\begin{aligned}\sigma(\nu_\mu \mathrm{e}^- \to \nu_\mu \mathrm{e}^-) : \sigma(\nu_\mathrm{e} \mathrm{e}^- \to \nu_\mathrm{e} \mathrm{e}^-) : \sigma(\nu_\mu \mathrm{e}^- \to \nu_\mathrm{e} \mu^-) &= c_L^2 + \tfrac{1}{3}c_R^2 \;:\; (1+c_L)^2 + \tfrac{1}{3}c_R^2 \;:\; 1 \\ &= 0.09 \;:\; 0.55 \;:\; 1\,.\end{aligned}$$

The pure neutral-current scattering process has a cross section that is about an order of magnitude smaller than the pure charged-current process. The effect of the NC diagram in the process $\nu_\mathrm{e}\mathrm{e}^- \to \nu_\mathrm{e}\mathrm{e}^-$ is to reduce the cross section compared to having just the CC diagram; because $c_L = -0.27$, the neutral current diagram interferes negatively with the charged current diagram.

16 Tests of the Standard Model

16.1 After correcting for QED effects, including initial-state radiation, the measured $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to$ hadrons cross sections at the peak of the Z resonance give

$$\sigma^0(e^+e^- \to Z \to \mu^+\mu^-) = 1.9993\,\text{nb} \quad \text{and} \quad \sigma^0(e^+e^- \to Z \to \text{hadrons}) = 41.476\,\text{nb}\,.$$

a) Assuming lepton universality, determine $\Gamma_{\ell\ell}$ and Γ_{hadrons}.

b) Hence, using the measured value of $\Gamma_Z = 2.4952 \pm 0.0023\,\text{GeV}$ and the theoretical value of $\Gamma_{\nu\nu}$ given by equation (15.41), obtain an estimate of the number of light neutrino flavours.

a) The cross section at $\sqrt{s} = m_Z$ is given by (16.17):

$$\sigma^0_{\text{ff}} = \frac{12\pi}{m_Z^2}\frac{\Gamma_{\text{ee}}\Gamma_{\text{ff}}}{\Gamma_Z^2}\,,$$

which can be inverted to give

$$\Gamma_{\text{ee}}\Gamma_{\text{ff}} = \frac{\sigma^0_{\text{ff}}\Gamma_Z^2 m_Z^2}{12\pi}\,.$$

Assuming lepton universality, whereby $\Gamma_{\mu\mu} = \Gamma_{\ell\ell}$,

$$\Gamma^2_{\text{ee}} = \frac{\sigma^0_{\mu\mu}\Gamma_Z^2 m_Z^2}{12\pi}\,.$$

Converting the measured peak cross section of $\sigma^0(e^+e^- \to Z \to \mu^+\mu^-) = 1.9993\,\text{nb}$ into natural units gives

$$\begin{aligned}\sigma^0(e^+e^- \to Z \to \mu^+\mu^-) &= 1.9993 \times 10^{-37}\,\text{m}^2 \cdot (\hbar c)^{-2}\\ &= 1.9993 \times 10^{-37}\,\text{m}^2 \frac{1}{(0.197\,\text{GeV} \times 10^{-15}\,\text{m})^2}\\ &= 5.152 \times 10^{-6}\,\text{GeV}^{-2}\,.\end{aligned}$$

Using $m_Z = 91.1875\,\text{GeV}$

$$\begin{aligned}
\Gamma_{ee}^2 &= \frac{\sigma_{\mu\mu}^0 \Gamma_Z^2 m_Z^2}{12\pi} \\
&= \frac{5.152\times10^{-6}\cdot 91.1875^2}{12\pi}\Gamma_Z^2 \\
&= 1.136\times10^{-3}\Gamma_Z^2 \\
\Rightarrow\quad \Gamma_{ee} &= 0.03371\Gamma_Z\,.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\Gamma_{ee}\Gamma_{\text{hadrons}} &= \frac{\sigma_{had}^0 \Gamma_Z^2 m_Z^2}{12\pi} \\
&= \frac{106.88\times10^{-6}\cdot 91.1875^2}{12\pi}\Gamma_Z^2 \\
&= 2.357\times10^{-2}\Gamma_Z^2 \\
\Rightarrow\quad 0.03371\Gamma_Z\Gamma_{\text{hadrons}} &= 2.357\times10^{-2}\Gamma_Z^2 \\
\Rightarrow\quad \Gamma_{\text{hadrons}} &= 0.6992\,\Gamma_Z
\end{aligned}$$

b) The total width of the Z is given by:

$$\Gamma_Z = 3\Gamma_{\ell\ell} + \Gamma_{\text{hadrons}} + N_\nu\Gamma_{\nu\nu}\,.$$

From the results of part a):

$$\begin{aligned}
N_\nu\Gamma_{\nu\nu} &= \Gamma_Z - 3\Gamma_{\ell\ell} - \Gamma_{\text{hadrons}} \\
&= 0.1997\,\Gamma_Z \\
&= 498\,\text{MeV}\,.
\end{aligned}$$

In Chapter 15, the partial decay width for $Z \to \nu_e\overline{\nu}_e$ was calculated to be 167 MeV, and the above measurements therefore imply

$$N_\nu = \frac{498}{167} = 2.98\,,$$

consistent with the claim that there are three light neutrino generations.

16.2 Show that the $e^+e^- \to Z \to \mu^+\mu^-$ differential cross section can be written as

$$\frac{d\sigma}{d\Omega} \propto (1+\cos^2\theta) + \tfrac{8}{3}A_{\text{FB}}\cos\theta\,.$$

From equation (16.25) the $e^+e^- \to Z \to f\overline{f}$ differential cross section can be written as

$$\frac{d\sigma}{d\Omega} = \kappa\left[a(1+\cos^2\theta) + 2b\cos\theta\right]\,,$$

where a and b are constants related to the couplings to the Z, and κ is a normalisation factor. Writing $\cos\theta = x$, then $d\Omega = 2\pi\, d(\cos\theta) = 2\pi dx$ and the number of events produced in the forward and backwards hemispheres can be written:

$$\begin{aligned} N_F &= 2\pi\kappa \int_0^1 a(1+x^2) + 2bx\, dx \\ &= 2\pi\left[\tfrac{4}{3}a + b\right], \\ N_B &= 2\pi\kappa \int_{-1}^0 a(1+x^2) + 2bx\, dx \\ &= 2\pi\left[\tfrac{4}{3}a - b\right]. \end{aligned}$$

Therefore the toward-backward asymmetry is

$$A_{\mathrm{FB}} = \frac{N_F - N_B}{N_F + N_B} = \frac{3b}{4a} \Rightarrow b = \frac{4aA_{\mathrm{FB}}}{3}.$$

Substituting this back into the original equation gives:

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \kappa\left[a(1+\cos^2\theta) + \tfrac{8}{3}aA_{\mathrm{FB}}\cos\theta\right] \\ &\propto (1+\cos^2\theta) + \tfrac{8}{3}A_{\mathrm{FB}}\cos\theta. \end{aligned}$$

16.3 From the measurement of the muon asymmetry parameter,

$$\mathcal{A}_\mu = 0.1456 \pm 0.0091,$$

determine the corresponding value of $\sin^2\theta_W$.

The muon asymmetry parameter is related to the couplings of the Z to muons by

$$\begin{aligned} \mathcal{A}_\mu = \frac{(c_L^\mu)^2 - (c_R^\mu)^2}{(c_L^\mu)^2 + (c_R^\mu)^2} &\equiv \frac{2c_V^\mu c_A^\mu}{(c_V^\mu)^2 + (c_A^\mu)^2} \\ &= \frac{2c_V^\mu/c_A^\mu}{(c_V^\mu/c_A^\mu)^2 + 1} \\ &= \frac{2x}{x^2+1}, \end{aligned}$$

where $x = c_V^\mu/c_A^\mu$. Hence the measured value gives the quadratic equation

$$\begin{aligned} (0.1456 \pm 0.0091) &= \frac{2x}{x^2+1} \\ \Rightarrow \quad x^2 - (13.74 \pm 0.85)x + 1 &= 0, \\ \Rightarrow \quad x = 0.0732 \pm 0.0046 \quad &\text{or} \quad x = 13.7 \pm 0.8. \end{aligned}$$

In the Standard Model

$$x = \frac{c_V^\mu}{c_A^\mu} = 1 - 4\sin^2\theta_W\,,$$

and therefore the measurement can be interpreted as

$$\begin{aligned} 1 - 4\sin^2\theta_W &= 0.0732 \pm 0.0046 \\ 4\sin^2\theta_W &= 0.9268 \pm 0.0046 \\ \sin^2\theta_W &= 0.2317 \pm 0.0012\,. \end{aligned}$$

16.4 The e^+e^- Stanford Linear Collider (SLC), operated at $\sqrt{s} = m_Z$ with left- and right-handed longitudinally polarised beams. This enabled the $e^+e^- \to Z \to f\bar{f}$ cross section to be measured separately for left-handed and right-handed electrons.

Assuming that the electron beam is 100 % polarised and that the positron beam is unpolarised, show that the left-right asymmetry A_{LR} is given by

$$A_{LR} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} = \frac{(c_L^e)^2 - (c_R^e)^2}{(c_L^e)^2 + (c_R^e)^2} = \mathcal{A}_e\,,$$

where σ_L and σ_R are respectively the measured cross sections at the Z resonance for LH and RH electron beams.

This is a relatively straight forward question. The matrix-elements for the different helicity combinations in the process $e^+e^- \to Z \to \mu^+\mu^-$ are given by equations (16.9)- (16.12)

$$\begin{aligned} |\mathcal{M}_{RL\to RL}|^2 &= |P_Z(s)|^2\, g_Z^4 s^2 (c_R^e)^2 (c_R^\mu)^2 (1+\cos\theta)^2\,, \\ |\mathcal{M}_{RL\to LR}|^2 &= |P_Z(s)|^2\, g_Z^4 s^2 (c_R^e)^2 (c_L^\mu)^2 (1-\cos\theta)^2\,, \\ |\mathcal{M}_{LR\to RL}|^2 &= |P_Z(s)|^2\, g_Z^4 s^2 (c_L^e)^2 (c_R^\mu)^2 (1-\cos\theta)^2\,, \\ |\mathcal{M}_{LR\to LR}|^2 &= |P_Z(s)|^2\, g_Z^4 s^2 (c_L^e)^2 (c_L^\mu)^2 (1+\cos\theta)^2\,, \end{aligned}$$

where $|P_Z(s)|^2 = 1/[(s - m_Z^2)^2 + m_Z^2\Gamma_Z^2]$ and $RL \to LR$ refers to a $e_R^- e_L^+ \to \mu_L^- \mu_R^+$. For the case where the electrons are 100 % left-handed, $P(e^-) = -1$, the spin-averaged matrix element (averaging over the positron helicity states) is

$$\begin{aligned} \langle|\mathcal{M}_L|^2\rangle &= \frac{1}{2}\left[|\mathcal{M}_{LR\to RL}|^2 + |\mathcal{M}_{LR\to LR}|^2\right] \\ &= \frac{1}{2}|P_Z(s)|^2\, g_Z^4 s^2 (c_L^e)^2 \times \left[(c_R^\mu)^2(1-\cos\theta)^2 + (c_L^\mu)^2(1+\cos\theta)^2\right]. \end{aligned}$$

Similarly

$$\begin{aligned} \langle|\mathcal{M}_R|^2\rangle &= \frac{1}{2}\left[|\mathcal{M}_{RL\to RL}|^2 + |\mathcal{M}_{RL\to LR}|^2\right] \\ &= \frac{1}{2}|P_Z(s)|^2\, g_Z^4 s^2 (c_R^e)^2 \times \left[(c_R^\mu)^2(1+\cos\theta)^2 + (c_L^\mu)^2(1-\cos\theta)^2\right]. \end{aligned}$$

The left-right asymmetry is then given by

$$A_{LR} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} = \frac{\int_{-1}^{+1} \langle |\mathcal{M}_L|^2 \rangle \, \mathrm{d}(\cos\theta) - \int_{-1}^{+1} \langle |\mathcal{M}_R|^2 \rangle \, \mathrm{d}(\cos\theta)}{\int_{-1}^{+1} \langle |\mathcal{M}_L|^2 \rangle \, \mathrm{d}(\cos\theta) + \int_{-1}^{+1} \langle |\mathcal{M}_R|^2 \rangle \, \mathrm{d}(\cos\theta)},$$

Since the integral over $\cos\theta$ is the same for four terms,

$$\int_{-1}^{+1} (1 \pm \cos\theta)^2 \, \mathrm{d}(\cos\theta) = \int_{-1}^{+1} (1 \pm x)^2 \, \mathrm{d}x = \left[x \pm x^2 + \frac{x^3}{3} \right]_{-1}^{+1} = \frac{8}{3},$$

the left-right asymmetry becomes

$$\begin{aligned} A_{LR} &= \frac{(c_L^{\mathrm{e}})^2 \left[(c_R^{\mu})^2 + (c_L^{\mu})^2 \right] - (c_R^{\mathrm{e}})^2 \left[(c_R^{\mu})^2 + (c_L^{\mu})^2 \right]}{(c_L^{\mathrm{e}})^2 \left[(c_R^{\mu})^2 + (c_L^{\mu})^2 \right] + (c_R^{\mathrm{e}})^2 \left[(c_R^{\mu})^2 + (c_L^{\mu})^2 \right]} \\ &= \frac{(c_L^{\mathrm{e}})^2 - (c_R^{\mathrm{e}})^2}{(c_L^{\mathrm{e}})^2 + (c_R^{\mathrm{e}})^2} \\ &\equiv \mathcal{A}_{\mathrm{e}}. \end{aligned}$$

Hence the polarised left-right asymmetry provides a direct measurement of $\mathcal{A}_{\mathrm{e}}$.

16.5 From the expressions for the matrix elements given in (16.8), show that:

a) the average polarisation of the tau leptons produced in the process $\mathrm{e^+e^-} \to \mathrm{Z} \to \tau^+\tau^-$ is

$$\langle P_{\tau^-} \rangle = \frac{N_\uparrow - N_\downarrow}{N_\uparrow + N_\downarrow} = -\mathcal{A}_\tau,$$

where $N_\uparrow$ and $N_\downarrow$ are the respective numbers of τ^- produced in RH and LH helicity states;

b) the tau polarisation where the τ^- is produced at an angle θ with respect to the initial-state e^- is

$$P_{\tau^-}(\cos\theta) = \frac{N_\uparrow(\cos\theta) - N_\downarrow(\cos\theta)}{N_\uparrow(\cos\theta) + N_\downarrow(\cos\theta)} = -\frac{\mathcal{A}_\tau(1+\cos^2\theta) + 2\mathcal{A}_{\mathrm{e}}\cos\theta}{(1+\cos^2\theta) + \frac{8}{3}A_{FB}\cos\theta}.$$

a) In the limit $\sqrt{s} \gg m_\tau$, the matrix-elements for the different helicity combinations in the process $\mathrm{e^+e^-} \to \mathrm{Z} \to \tau^+\tau^-$ are given by equations (16.9)- (16.12)

$$\begin{aligned} |\mathcal{M}_{RL\to RL}|^2 &= |P_{\mathrm{Z}}(s)|^2 \, g_{\mathrm{Z}}^4 s^2 (c_R^{\mathrm{e}})^2 (c_R^{\tau})^2 (1+\cos\theta)^2, \\ |\mathcal{M}_{RL\to LR}|^2 &= |P_{\mathrm{Z}}(s)|^2 \, g_{\mathrm{Z}}^4 s^2 (c_R^{\mathrm{e}})^2 (c_L^{\tau})^2 (1-\cos\theta)^2, \\ |\mathcal{M}_{LR\to RL}|^2 &= |P_{\mathrm{Z}}(s)|^2 \, g_{\mathrm{Z}}^4 s^2 (c_L^{\mathrm{e}})^2 (c_R^{\tau})^2 (1-\cos\theta)^2, \\ |\mathcal{M}_{LR\to LR}|^2 &= |P_{\mathrm{Z}}(s)|^2 \, g_{\mathrm{Z}}^4 s^2 (c_L^{\mathrm{e}})^2 (c_L^{\tau})^2 (1+\cos\theta)^2, \end{aligned}$$

where $|P_{\mathrm{Z}}(s)|^2 = 1/[(s-m_{\mathrm{Z}}^2)^2 + m_{\mathrm{Z}}^2\Gamma_{\mathrm{Z}}^2]$ and $RL \to LR$ refers to a $\mathrm{e}_R^-\mathrm{e}_L^+ \to \tau_L^-\tau_R^+$. The number of events where the τ^- is produced in a left-handed/right-handed helicity state at an angle $\cos\theta$ will be proportional to

$$\begin{aligned}
\mathrm{d}N_\uparrow(\cos\theta) &\propto |\mathcal{M}_{RL\to RL}|^2 + |\mathcal{M}_{LR\to RL}|^2 \\
&\propto (c_R^{\mathrm{e}})^2(c_R^{\tau})^2(1+\cos\theta)^2 + (c_L^{\mathrm{e}})^2(c_R^{\tau})^2(1-\cos\theta)^2\,, \\
&\propto (c_R^{\tau})^2\left[(c_R^{\mathrm{e}})^2(1+\cos\theta)^2 + (c_L^{\mathrm{e}})^2(1-\cos\theta)^2\right] \\
\mathrm{d}N_\downarrow(\cos\theta) &\propto |\mathcal{M}_{RL\to LR}|^2 + |\mathcal{M}_{LR\to LR}|^2 \\
&\propto (c_R^{\mathrm{e}})^2(c_L^{\tau})^2(1-\cos\theta)^2 + (c_L^{\mathrm{e}})^2(c_L^{\tau})^2(1+\cos\theta)^2 \\
&\propto (c_L^{\tau})^2\left[(c_R^{\mathrm{e}})^2(1-\cos\theta)^2 + (c_L^{\mathrm{e}})^2(1+\cos\theta)^2\right]\,.
\end{aligned}$$

As in the previous, integrating over $\cos\theta$ leads to the same factor in all terms and thus

$$\begin{aligned}
N_\uparrow &\propto (c_R^{\tau})^2\left[(c_R^{\mathrm{e}})^2 + (c_L^{\mathrm{e}})^2\right]\,, \\
N_\downarrow &\propto (c_L^{\tau})^2\left[(c_R^{\mathrm{e}})^2 + (c_L^{\mathrm{e}})^2\right]\,.
\end{aligned}$$

Hence the mean tau polarisation is given by:

$$\begin{aligned}
\langle P_{\tau^-}\rangle &= \frac{N_\uparrow - N_\downarrow}{N_\uparrow + N_\downarrow} \\
&= \frac{(c_R^{\tau})^2 - (c_L^{\tau})^2}{(c_R^{\tau})^2 + (c_L^{\tau})^2} \\
&\equiv -\mathcal{A}_\tau\,.
\end{aligned}$$

b) In part a) the mean tau polarisation averaged over all solid angle was determined. However, the degree of tau polarisation is a function of $\cos\theta$. This can be measured by determining the numbers of LH and RH tau leptons for the case where the τ^- is detected in a (small) range of $\cos\theta$. In this case

$$P_{\tau^-}(\cos\theta) = \frac{\mathrm{d}N_\uparrow(\cos\theta) - \mathrm{d}N_\downarrow(\cos\theta)}{\mathrm{d}N_\uparrow(\cos\theta) + \mathrm{d}N_\downarrow(\cos\theta)}\,.$$

Using the expressions derived in part a) and writing $\cos\theta = x$,

$$\begin{aligned}
P_{\tau^-}(\cos\theta) &= \frac{\mathrm{d}N_\uparrow(\cos\theta) - \mathrm{d}N_\downarrow(\cos\theta)}{\mathrm{d}N_\uparrow(\cos\theta) + \mathrm{d}N_\downarrow(\cos\theta)} \\
&= \frac{(c_R^{\tau})^2\left[(c_R^{\mathrm{e}})^2(1+x)^2 + (c_L^{\mathrm{e}})^2(1-x)^2\right] - (c_L^{\tau})^2\left[(c_R^{\mathrm{e}})^2(1-x)^2 + (c_L^{\mathrm{e}})^2(1+x)^2\right]}{(c_R^{\tau})^2\left[(c_R^{\mathrm{e}})^2(1+x)^2 + (c_L^{\mathrm{e}})^2(1-x)^2\right] + (c_L^{\tau})^2\left[(c_R^{\mathrm{e}})^2(1-x)^2 + (c_L^{\mathrm{e}})^2(1+x)^2\right]} \\
&= \frac{\left[(c_R^{\tau})^2 - (c_L^{\tau})^2\right]\left[(c_R^{\mathrm{e}})^2 + (c_R^{\mathrm{e}})^2\right](1+x^2) + 2x\left[(c_R^{\mathrm{e}})^2 - (c_L^{\mathrm{e}})^2\right]\left[(c_R^{\tau})^2 + (c_L^{\tau})^2\right]}{\left[(c_R^{\tau})^2 + (c_L^{\tau})^2\right]\left[(c_R^{\mathrm{e}})^2 + (c_R^{\mathrm{e}})^2\right](1+x^2) + 2x\left[(c_R^{\mathrm{e}})^2 - (c_L^{\mathrm{e}})^2\right]\left[(c_R^{\tau})^2 - (c_L^{\tau})^2\right]}\,.
\end{aligned}$$

Dividing each term by $\left[(c_R^\tau)^2 + (c_L^\tau)^2\right]\left[(c_R^e)^2 + (c_R^e)^2\right]$ and remembering that

$$\mathcal{A} = \frac{c_L^2 - c_R^2}{c_L^2 + c_R^2},$$

gives

$$P_{\tau^-}(\cos\theta) = \frac{-\mathcal{A}_\tau(1+x^2) - 2x\mathcal{A}_e}{1 + x^2 + 2x\mathcal{A}_\tau\mathcal{A}_e}.$$

This expression can be simplified using

$$A_{FB}^\tau = \frac{3}{4}\mathcal{A}_e\mathcal{A}_\tau,$$

thus

$$P_{\tau^-}(\cos\theta) = -\frac{\mathcal{A}_\tau(1+\cos^2\theta) + 2\mathcal{A}_e\cos\theta}{(1+\cos^2\theta) + \frac{8}{3}A_{FB}^\tau\cos\theta},$$

as required.

16.6 The average tau polarisation in the process $e^+e^- \to Z \to \tau^+\tau^-$ can be determined from the energy distribution of π^- in the decay $\tau^- \to \pi^-\nu_\tau$. In the τ^- rest frame, the π^- four-momentum can be written $p = (E^*, p^*\sin\theta^*, 0, p^*\cos\theta^*)$ where θ^* is the angle with respect to the τ^- spin, and the differential partial decay width is

$$\frac{d\Gamma}{d\cos\theta^*} \propto \frac{(p^*)^2}{m_\tau}(1+\cos\theta^*).$$

a) Without explicit calculation, explain this angular dependence.

b) For the case where the τ^- is right-handed, show that the observed energy distribution of the π^- *in the laboratory frame* is

$$\frac{d\Gamma_{\tau^-_\uparrow}}{dE_{\pi^-}} \propto x,$$

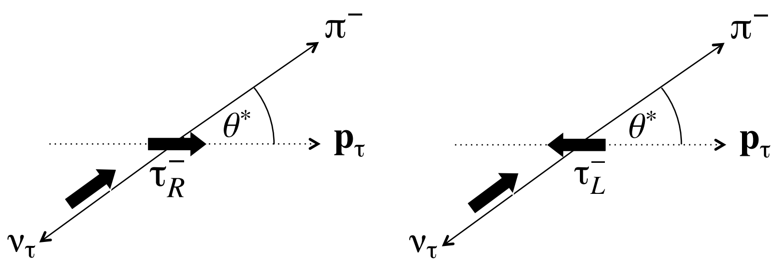
where $x = E_\pi/E_\tau$.

c) What is the corresponding π^- energy distribution for the decay of a LH helicity τ^-.

d) If the observed pion energy distribution is consistent with

$$\frac{d\Gamma}{dx} = 1.14 - 0.28x \equiv 0.86x + 1.14(1-x),$$

determine $\mathcal{A}_\tau$ and the corresponding value of $\sin^2\theta_W$.



a) Consider the decay in the tau rest frame with the angle θ^* defined with respect to the spin of the τ^-, as shown in the left-hand plot above. Since the neutrino will be left-handed, conservation of angular moment (just spin-half here) implies

$$\frac{\mathrm{d}N}{\mathrm{d}(\cos\theta^*)} \propto \cos^2\left(\frac{\theta^*}{2}\right) \propto 1 + \cos\theta^* \,.$$

b) Consider decay of a RH-helicity τ^- as shown in the left-hand plot above. With

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\cos\theta^*} \propto \frac{(\mathrm{p}^*)^2}{m_\tau}(1 + \cos\theta^*)\,.$$

In the laboratory frame the system will be boosted along the direction of the τ^- momentum and the laboratory-frame energy of the π^- will be

$$\begin{aligned} E_\pi &= \gamma_\tau E^* + \gamma_\tau\beta_\tau p_z^* \\ &= \gamma_\tau E^* + \gamma_\tau\beta_\tau \mathrm{p}^* \cos\theta^* \,. \end{aligned}$$

The energy distribution of the pion in the laboratory frame is related to the angular distribution in the tau rest frame by

$$\begin{aligned} \frac{\mathrm{d}\Gamma}{\mathrm{d}E_\pi} &= \frac{\mathrm{d}\Gamma}{\mathrm{d}\cos\theta^*}\frac{\mathrm{d}\cos\theta^*}{\mathrm{d}E_\pi} \\ &\propto \frac{(\mathrm{p}^*)^2}{m_\tau}(1+\cos\theta^*)\cdot\frac{1}{\gamma_\tau\beta_\tau\mathrm{p}^*} \\ &\propto \frac{\mathrm{p}^*}{\beta_\tau E_\tau}(1+\cos\theta^*)\,. \end{aligned}$$

This can be expressed in terms of the pion energy using

$$\begin{aligned} E_\pi &= \gamma_\tau E^* + \gamma_\tau\beta_\tau\mathrm{p}^*\cos\theta^*\,, \\ \Rightarrow \quad \cos\theta^* &= \frac{E_\pi - \gamma_\tau E^*}{\gamma_\tau\beta_\tau\mathrm{p}^*}\,, \end{aligned}$$

thus

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}E_\pi} \propto \frac{1}{\beta_\tau^2\gamma_\tau E_\tau}(E_\pi + \gamma_\tau\beta_\tau p^* - \gamma_\tau E^*)\,. \tag{16.1}$$

In the laboratory frame the energy of the τ^- is just $m_Z/2$ and therefore $\gamma_\tau = m_Z/2m_\tau = 25.7$ and $\beta_\tau = 0.9992$. The momentum of the pion in the tau rest frame is easily shown to be

$$p^* = \frac{m_\tau^2 - m_\pi^2}{2m_\tau} = 0.88\,\mathrm{GeV} \approx m_\tau/2$$
$$\Rightarrow \quad E^* = 0.89\,\mathrm{GeV} \approx m_\tau/2\,.$$

Consequently $\beta_\tau p^* - E^* < 0.02\,\mathrm{GeV}$ and to a good approximation (16.1) can be approximated as

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}E_\pi} \propto \frac{1}{\beta_\tau^2\gamma_\tau E_\tau}(E_\pi + \gamma_\tau\beta_\tau p^* - \gamma_\tau E^*)$$
$$\approx \frac{1}{\beta_\tau^2\gamma_\tau}\frac{E_\pi}{E_\tau}\,.$$

c) The case of the decay of a LH-helicity τ^- (as shown on the right in the above plot), the decay distribution in the tau rest frame can be obtained by replacing θ^* in the original decay distribution by $\pi - \theta^*$

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\cos\theta^*} \propto \frac{(\mathrm{p}^*)^2}{m_\tau}(1 + \cos[\pi - \theta^*])$$
$$= \frac{(\mathrm{p}^*)^2}{m_\tau}(1 - \cos\theta^*)\,.$$

Following the previous calculation the energy distribution of the decay pion will be

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}E_\pi} = \frac{\mathrm{d}\Gamma}{\mathrm{d}\cos\theta^*}\frac{\mathrm{d}\cos\theta^*}{\mathrm{d}E_\pi}$$
$$\propto \frac{\mathrm{p}^*}{\beta_\tau E_\tau}(1 - \cos\theta^*)\,.$$

where, as before, $\cos\theta^*$ is given by

$$E_\pi = \gamma_\tau E^* + \gamma_\tau\beta_\tau \mathrm{p}^* \cos\theta^*\,,$$
$$\Rightarrow \quad \cos\theta^* = \frac{E_\pi - \gamma_\tau E^*}{\gamma_\tau\beta_\tau \mathrm{p}^*}\,,$$

and this

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}E_\pi} \propto \frac{1}{\beta_\tau^2\gamma_\tau E_\tau}(\gamma_\tau\beta_\tau p^* + \gamma_\tau E^* - E_\pi)$$
$$\approx \frac{1}{\beta_\tau^2\gamma_\tau E_\tau}\gamma_\tau(m_\tau/2 + m_\tau/)2 - E_\pi$$
$$\propto \frac{1}{\beta_\tau^2\gamma_\tau}\left(1 - \frac{E_\pi}{E_\tau}\right),$$

where the following relations were used $\beta_\tau p^* \approx E^* \approx m_\tau/2$ and $\gamma_\tau m_\tau = E_\tau$.

d) From parts b) and c) the $\tau^- \to \pi^-\nu_\tau$ decays of RH and LH tau leptons give very different pion energy distributions, reflecting the different angular distributions of the decay relative to the tau line of flight:

$$\frac{d\Gamma_R}{dE_\pi} \propto x \quad \text{and} \quad \frac{d\Gamma_R}{dE_\pi} \propto (1-x)\,.$$

where $x = E_\pi/E_\tau = 2E_\pi/m_Z$. If the average τ^- polarisation is

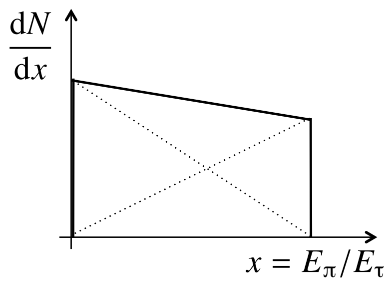
$$P_\tau = \frac{N_\uparrow - N_\downarrow}{N_\uparrow + N_\downarrow}\,,$$

and there are a total of $N = N_\uparrow + N_\downarrow$ decays, then

$$N_\uparrow = (1+P_\tau)N/2 \quad \text{and} \quad N_\uparrow = (1-P_\tau)N/2\,.$$

The corresponding pion energy distribution, which is correctly normalised, will be

$$\begin{aligned}\frac{dN}{dx} &= 2N_\uparrow x + 2N_\downarrow(1-x)\\ &= N(1+P_\tau)x + N(1-P_\tau)(1-x)\\ &= N\left[(1-P_\tau) + 2P_\tau x\right]\end{aligned}$$



The observed distribution (shown schematically above) of

$$\frac{d\Gamma}{dx} \propto 1.14 - 0.28x \equiv 0.86x + 1.14(1-x)\,,$$

has contributions from LH and RH $\tau^- \to \pi^-\nu_\tau$ decays and implies that $P_\tau = -0.14$ and therefore (from the previous question)

$$\mathcal{A}_\tau = -P_\tau = 0.14\,.$$

The tau asymmetry parameter is related to the couplings of the Z to tau leptons by

$$\begin{aligned}\mathcal{A}_\tau = \frac{(c_L^\tau)^2 - (c_R^\tau)^2}{(c_L^\tau)^2 + (c_R^\tau)^2} &\equiv \frac{2c_V^\tau c_A^\tau}{(c_V^\tau)^2 + (c_A^\tau)^2} \\ &= \frac{2c_V^\tau / c_A^\tau}{(c_V^\tau / c_A^\tau)^2 + 1} \\ &= \frac{2x}{x^2+1}\,,\end{aligned}$$

where $x = c_V^\tau / c_A^\tau$. Hence the measured value gives the quadratic equation

$$\begin{aligned} 0.14 &= \frac{2x}{x^2+1} \\ \Rightarrow \quad x^2 - 14.28x + 1 &= 0\,, \\ \Rightarrow \quad x &= 0.067\,. \end{aligned}$$

In the Standard Model

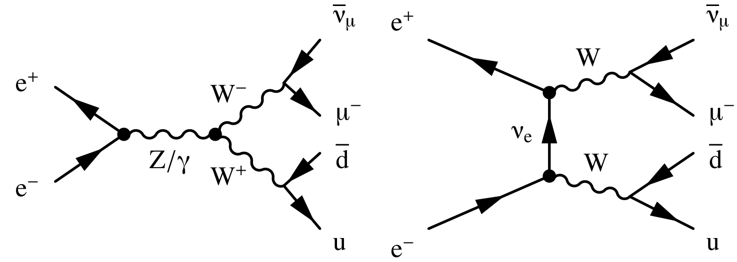
$$x = \frac{c_V^\tau}{c_A^\tau} = 1 - 4\sin^2\theta_{\mathrm{W}}\,,$$

and therefore the measurement can be interpreted as

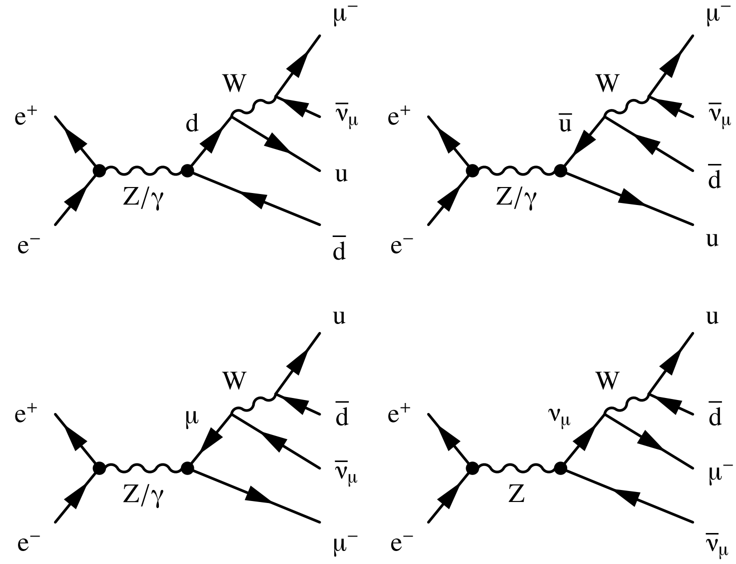
$$\begin{aligned} 1 - 4\sin^2\theta_{\mathrm{W}} &= 0.067 \\ 4\sin^2\theta_{\mathrm{W}} &= 0..933 \\ \sin^2\theta_{\mathrm{W}} &= 0.233\,. \end{aligned}$$

16.7 There are ten possible lowest-order Feynman diagrams for the process $e^+e^- \to \mu^-\bar{\nu}_\mu u\bar{d}$, of which only three involve a W^+W^- intermediate state. Draw the other seven diagrams (they are all s-channel processes involving a single virtual W).

The first three diagrams (CC03) involve the production of two W bosons, either through the s-channel production of a Z or γ, or through the t-channel exchange of a neutrino.

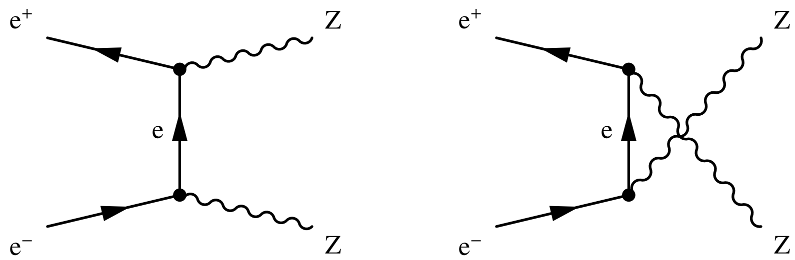


The remaining seven diagrams, all arise from pair production of quarks or leptons through Z or γ exchange with a W radiated from one of the final state particles.



16.8 Draw the two lowest-order Feynman diagrams for $e^+e^- \to ZZ$.

Because there are two identical particles in the final state, there are two (NC02) diagrams.



16.9 In the OPAL experiment at LEP, the efficiencies for selecting $W^+W^- \to \ell\nu q_1\overline{q}_2$ and $W^+W^- \to q_1\overline{q}_2q_3\overline{q}_4$ events were 83.8 % and 85.9 % respectively. After correcting for background, the observed numbers of $\ell\nu q_1\overline{q}_2$ and $q_1\overline{q}_2q_3\overline{q}_4$ events were respectively 4192 and 4592. Determine the measured value of the W-boson hadronic branching ratio $BR(W \to q\overline{q}')$ and its statistical uncertainty.

If N events are observed, the best estimate of the statistical uncertainty is $\sqrt{N}$ (from Poisson statistics). Assuming the backgrounds are relatively small, the observed numbers of events are

$$N(\ell\nu q_1\overline{q}_2) = 4192 \pm 64.7 \quad \text{and} \quad N(q_1\overline{q}_2q_3\overline{q}_4) = 4592 \pm 67.8\,.$$

Accounting for the efficiencies, ϵ_i, the numbers of produced events in the two categories are estimated by $n_i = N_i/\epsilon_i$ and the uncertainties are scaled in the same way:

$$n(\ell\nu q_1\overline{q}_2) = 5002.4 \pm 77.2 \quad \text{and} \quad n(q_1\overline{q}_2q_3\overline{q}_4) = 5345.8 \pm 78.9\,.$$

Since the expected numbers of events are

$$x(\ell\nu q_1\overline{q}_2) \propto 2BR(W \to \ell\nu)\times BR(W \to q\overline{q}') \quad \text{and} \quad x(q_1\overline{q}_2q_3\overline{q}_4) \propto BR(W \to q\overline{q}')^2\,.$$

The ratio of the two event categories gives

$$r \equiv \frac{n(\ell\nu q_1\overline{q}_2)}{n(q_1\overline{q}_2q_3\overline{q}_4)} = \frac{2BR(W \to \ell\nu)}{BR(W \to q\overline{q}')} = \frac{2[1 - BR(W \to q\overline{q}')])}{BR(W \to q\overline{q}')}\,.$$

Writing the hadronic branching ratio as b and rearranging the above expression gives

$$b = \frac{2}{r+2}\,.$$

Here r = 5002.4/5345.8 = 0.9357 and thus

$$b = 0.6812\,.$$

The uncertainty on b is related to the uncertainty on r by

$$\sigma_b^2 = \left(\frac{\partial b}{\partial r}\right)^2 \sigma_r^2 = \frac{\sigma_r^2}{r+2},$$

and the uncertainty on $r = n_1/n_2$ is given by

$$\begin{aligned}
\sigma_r^2 &= \left(\frac{\partial r}{\partial n_1}\right)^2 \sigma_1^2 + \left(\frac{\partial r}{\partial n_2}\right)^2 \sigma_2^2 \\
\Rightarrow \quad \frac{\sigma_r^2}{r^2} &= \frac{\sigma_1^2}{n_1^2} + \frac{\sigma_2^2}{n_2^2} \\
&= 4.56 \times 10^{-4} \\
\Rightarrow \quad \sigma_r &= 0.01998 \\
\Rightarrow \quad \sigma_b &= \frac{\sigma_r}{\sqrt{r+2}} = 0.0117\,.
\end{aligned}$$

Hence the observed numbers of events give

$$BR(\mathrm{W} \to \mathrm{q\overline{q}'}) = 68.1 \pm 1.2\,\%$$

consistent with the expected value.

16.10 Suppose the four jets in an identified $\mathrm{e^+e^-} \to \mathrm{W^+W^-}$ event at LEP are measured to have momenta,

$$\mathrm{p}_1 = 82.4 \pm 5\,\text{GeV}, \quad \mathrm{p}_2 = 59.8 \pm 5\,\text{GeV}, \quad \mathrm{p}_3 = 23.7 \pm 5\,\text{GeV} \quad \text{and} \quad \mathrm{p}_4 = 42.6 \pm 5\,\text{GeV},$$

and directions given by the Cartesian unit vectors,

$$\begin{aligned}
\hat{\mathbf{n}}_1 &= (0.72, 0.33, 0.61), \quad \hat{\mathbf{n}}_2 = (-0.61, 0.58, -0.53), \\
\hat{\mathbf{n}}_3 &= (-0.63, -0.72, -0.25), \quad \hat{\mathbf{n}}_4 = (-0.14, -0.96, -0.25).
\end{aligned}$$

Assuming that the jets can be treated as massless particles, find the most likely association of the four jets to the two W bosons and obtain values for the invariant masses of the (off-shell) W bosons in this event. Optionally, calculate the uncertainties on the reconstructed masses assuming that the jet directions are perfectly measured.

The invariant mass of a system of two particles is

$$\begin{aligned}
m_{ij}^2 &= (p_i + p_j) = (E_i + E_j)^2 - (\mathbf{p}_i + \mathbf{p}_j)^2 \\
&= E_i^2 + 2E_iE_2 + E_j^2 - \mathrm{p}_i^2 - \mathrm{p}_j^2 - 2\mathbf{p}_i \cdot \mathbf{p}_j \qquad = 2E_iE_j - 2\mathrm{p}_1\mathrm{p}_2\cos\theta_{ij}\,.
\end{aligned}$$

Here, where the jets are treated as massless ($\mathrm{p} = E$) and therefore

$$\begin{aligned}
m_{ij}^2 &= 2(E_iE_j - \mathrm{p}_i\mathrm{p}_j\cos\theta_{ij}) \\
&= 2E_iE_j\left(1 - \hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j\right).
\end{aligned}$$

For four jets, there are three possible associations to the two W bosons: (12)(34), (13)(24) or (14)(23). Using the values given the masses under these three different hypotheses are:

$$
\begin{aligned}
(12)(34)\ :&\quad m_{12} = 65.1\,\mathrm{GeV} \quad \text{and} \quad m_{34} = 17.8\,\mathrm{GeV}\,,\\
(13)(24)\ :&\quad m_{13} = 84.8\,\mathrm{GeV} \quad \text{and} \quad m_{24} = 82.6\,\mathrm{GeV}\,,\\
(14)(23)\ :&\quad m_{14} = 105.1\,\mathrm{GeV} \quad \text{and} \quad m_{23} = 50.5\,\mathrm{GeV}\,,
\end{aligned}
$$

and the most likely jet pairing is (13)(24).

The uncertainties on the measured masses are given by

$$
\begin{aligned}
\sigma_m^2 &= \left(\frac{\partial m}{\partial E_1}\right)^2 \sigma_{E_1}^2 + \left(\frac{\partial m}{\partial E_1}\right)^2 \sigma_{E_1}^2\\
&= \left(\frac{1}{2}\frac{E_2^{1/2}}{E_1^{1/2}}\left[1-\hat{\mathbf{n}}_i\cdot\hat{\mathbf{n}}_j\right]\right)^2 \sigma_{E_1}^2 + \left(\frac{1}{2}\frac{E_1^{1/2}}{E_2^{1/2}}\left[1-\hat{\mathbf{n}}_i\cdot\hat{\mathbf{n}}_j\right]\right)^2 \sigma_{E_2}^2\\
&= \frac{1}{4}\left(\frac{\sigma_{E_1}^2}{E_1^2} + \frac{\sigma_{E_1}^2}{E_1^2}\right) m^2\,.
\end{aligned}
$$

For the jet pairing most consistent with being from the process $e^+e^- \to W^+W^-$ this gives:

$$
(13)(24)\ :\quad m_{13} = (84.8 \pm 9.3)\,\mathrm{GeV} \quad \text{and} \quad m_{24} = (82.6 \pm 5.9)\,\mathrm{GeV}\,,
$$

consistent with the expected value of 80.3 GeV.

16.11 Show that the momenta of the final-state particles in the decay $t \to W^+b$ are

$$
p^* = \frac{m_t^2 - m_W^2}{2m_t}\,,
$$

and show that the decay rate of (16.31) leads to the expression for Γ_t given in (16.32).

In the rest frame of the top, the momenta of the two daughter particles are equal

and conservation of energy implies $m_t = E_b + E_W$. Writing this as $m_t - E_b = E_W$ and squaring gives

$$
\begin{aligned}
m_t^2 - 2m_tE_b + E_b^2 &= E_W^2\\
m_t^2 - 2m_tE_b + m_b^2 + p^{*2} &= m_W^2 + p^{*2}\\
\Rightarrow \qquad m_t^2 + (m_b^2 - m_W^2) &= 2m_tE_b\,.
\end{aligned}
$$

Squaring again to eliminate E_b leads to

$$\begin{aligned}
m_t^4 + 2m_t^2(m_b^2 - m_W^2) + (m_b^2 - m_W^2)^2 &= 4m_t(m_b^2 + p^{*2}) \\
m_t^4 - 2m_t^2(m_b^2 + m_W^2) + (m_W - m_b)^2(m_W + m_b)^2 &= 4m_t p^{*2} \\
m_t^4 - 2m_t^2\left[(m_W + m_b)^2 + (m_W - m_b)^2\right] + (m_W - m_b)^2(m_W + m_b)^2 &= 4m_t p^{*2} \\
\left[m_t^2 - (m_W + m_b)^2\right]\left[m_t^2 - (m_W - m_b)^2\right] &= 4m_t p^{*2}\,,
\end{aligned}$$

thus showing that

$$p^* = \frac{1}{2m_t}\sqrt{\left[(m_t^2 - (m_W + m_b)^2\right]\left[m_t^2 - (m_W - m_b)^2\right]}\,.$$

Since $m_b \ll m_W$ the term inside the square root can be approximated to

$$\begin{aligned}
\left[m_t^2 - (m_W + m_b)^2\right]\left[m_t^2 - (m_W - m_b)^2\right] &= \left[m_t^2 - m_W^2\left(1 + \frac{m_b}{m_W}\right)^2\right]\left[m_t^2 - m_W^2\left(1 - \frac{m_b}{m_W}\right)^2\right] \\
&\approx \left[m_t^2 - m_W^2 - 2m_W m_b\right]\left[m_t^2 - m_W^2 + 2m_W m_b\right] \\
&= \left[m_t^2 - m_W^2\right]^2 - 4m_W^2 m_b \\
&\approx \left[m_t^2 - m_W^2\right]^2\,.
\end{aligned}$$

Hence to a good approximation

$$p^* \approx \frac{m_t^2 - m_W^2}{2m_t}\,.$$

The same result could have been obtained much more quickly by simply neglecting the b-quark mass.

Substituting this expression into (16.31)

$$\begin{aligned}
\Gamma(t \to bW^+) &= \frac{g_W^2 p^{*2}}{16\pi m_t}\left(2 + \frac{m_t^2}{m_W^2}\right) \\
&= \frac{g_W^2}{64\pi m_t^3}(m_t^2 - m_W^2)^2\left(2 + \frac{m_t^2}{m_W^2}\right) \\
&= \frac{g_W^2 m_t^3}{64\pi m_W^2}\left(1 - \frac{m_W^2}{m_t^2}\right)^2\left(2\frac{m_W^2}{m_t^2} + 1\right) \\
&= \frac{G_F m_t^3}{8\sqrt{2}\pi}\left(1 - \frac{m_W^2}{m_t^2}\right)^2\left(1 + \frac{2m_W^2}{m_t^2}\right).
\end{aligned}$$

17 The Higgs Boson

17.1 By considering the form of the polarisation four-vector for a longitudinally polarised massive gauge bosons, explain why the t-channel neutrino-exchange diagram for $e^+e^- \to W^+W^-$, when taken in isolation, is badly behaved at high centre-of-mass energies.

The longitudinally polarised W travelling in the z-direction has polarisation four-vector

$$\epsilon_L^\mu = \frac{1}{m_W}(p_z, 0, 0, E)\,.$$

Consequently, at high energy ($E \gg m_W$) the matrix element for the t-channel process $e^+e^- \to W_L^+ W_L^-$ alone, will scale as

$$\mathcal{M}^2 \propto \left(\frac{E_W}{m_W}\right)^4\,,$$

and increases without limit. This is not the case for the transverse polarisation states

$$\epsilon_-^\mu = \frac{1}{\sqrt{2}}(0, 1, -i, 0) \quad \text{and} \quad \epsilon_+^\mu = -\frac{1}{\sqrt{2}}(0, 1, i, 0)\,.$$

17.2 The Lagrangian for the Dirac equation is

$$\mathcal{L} = i\overline{\psi}\gamma_\mu\partial^\mu\psi - m\overline{\psi}\psi\,,$$

Treating the eight fields ψ_i and $\overline{\psi}_i$ as independent, show that the Euler-Lagrange equation for the component ψ_i leads to

$$i\partial_\mu\overline{\psi}\gamma^\mu + m\overline{\psi} = 0\,.$$

The Lagrangian (density) for the Dirac equation is

$$\mathcal{L}_D = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi\,.$$

The partial derivatives with respect to each of the four components of the spinor ψ_i are

$$\frac{\partial\mathcal{L}_D}{\partial\partial(\partial_\mu\psi_i)} = i\overline{\psi}\gamma^\mu \quad \text{and} \quad \frac{\partial\mathcal{L}_D}{\partial\psi_i} = -m\overline{\psi}\,,$$

and the Euler-Lagrange equation gives:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\right)-\frac{\partial\mathcal{L}}{\partial\phi_i}=0$$
$$\partial_\mu i\overline{\psi}\gamma^\mu+m\overline{\psi}=0$$
$$i(\partial_\mu\overline{\psi})\gamma^\mu+m\overline{\psi}=0\,.$$

17.3 Verify that the Lagrangian for the free electromagnetic field,

$$\mathcal{L}=-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}\,,$$

is invariant under the gauge transformation $A_\mu\to A'_\mu=A_\mu-\partial_\mu\chi$.

The field strength tensor

$$F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu\,,$$

transforms to

$$\begin{aligned}F^{\mu\nu\prime}&=\partial^\mu(A^\nu-\partial^\nu\chi)-\partial^\nu(A^\mu-\partial^\mu\chi)\\&=\partial^\mu A^\nu-\partial^\mu\partial^\nu\chi-\partial^\nu A^\mu+\partial^\nu\partial^\mu\chi\\&=\partial^\mu A^\nu-\partial^\nu A^\mu\\&=F^{\mu\nu}\,,\end{aligned}$$

and consequently $F^{\mu\nu}F_{\mu\nu}$ is invariant under the gauge transformation.

17.4 The Lagrangian for the electromagnetic field in the presence of a current j^μ is

$$\mathcal{L}=-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}-j^\mu A_\mu\,.$$

By writing this as

$$\begin{aligned}\mathcal{L}&=-\tfrac{1}{4}(\partial^\mu A^\nu-\partial^\nu A^\mu)(\partial_\mu A_\nu-\partial_\nu A_\mu)-j^\mu A_\mu\\&=-\tfrac{1}{2}(\partial^\mu A^\nu)(\partial_\mu A_\nu)+\tfrac{1}{2}(\partial^\nu A^\mu)(\partial_\mu A_\nu)-j^\mu A_\mu\,,\end{aligned}$$

show that the Euler-Lagrange equation gives the covariant form of Maxwell's equations,

$$\partial_\mu F^{\mu\nu}=j^\nu\,.$$

The Lagrangian density

$$\begin{aligned}\mathcal{L}&=-\tfrac{1}{4}(\partial^\mu A^\nu-\partial^\nu A^\mu)(\partial_\mu A_\nu-\partial_\nu A_\mu)-j^\mu A_\mu\\&=-\tfrac{1}{4}(\partial^\mu A^\nu)(\partial_\mu A_\nu)-\tfrac{1}{4}(\partial^\nu A^\mu)(\partial_\nu A_\mu)+\tfrac{1}{4}(\partial^\mu A^\nu\partial_\nu A_\mu)+\tfrac{1}{4}(\partial^\nu A^\mu)(\partial_\mu A_\nu)-j^\mu A_\mu\\&=-\tfrac{1}{2}(\partial^\mu A^\nu)(\partial_\mu A_\nu)+\tfrac{1}{2}(\partial^\nu A^\mu)(\partial_\mu A_\nu)-j^\mu A_\mu\\&=-\tfrac{1}{2}(\partial^\nu A^\mu)(\partial_\nu A_\mu)+\tfrac{1}{2}(\partial^\nu A^\mu)(\partial_\mu A_\nu)-j^\mu A_\mu\,.\end{aligned}$$

Taking the partial derivatives with respect to the field A_μ and its derivatives $\partial_\nu A_\mu$ gives

$$\frac{\partial\mathcal{L}}{\partial[\partial(\partial_\nu A_\mu)]} = -\partial^\nu A^\mu + \partial^\mu A^\nu \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial A_\mu} = -j^\mu\,,$$

and the Euler-Lagrange equation gives:

$$\begin{aligned} \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)}\right) - \frac{\partial\mathcal{L}}{\partial A_\mu} &= 0 \\ \partial_\nu\left[\partial^\mu A^\nu - \partial^\nu A^\mu\right] + j^\mu &= 0 \\ \partial_\nu\left[\partial^\nu A^\mu - \partial^\mu A^\nu\right] &= j^\mu \\ \partial_\nu F^{\nu\mu} &= j^\mu\,, \end{aligned}$$

or equivalently

$$\partial_\mu F^{\mu\nu} = j^\nu\,.$$

17.5 Explain why the Higgs potential can only contain terms with even powers of the field ϕ.

Introducing odd powers of the field ϕ into the Higgs potential would break the underlying gauge invariance of the Lagrangian, which is the whole point of introducing the Higgs mechanism in the first place.

17.6 Verify that substituting (17.30) into (17.29) leads to

$$\mathcal{L} = \tfrac{1}{2}(\partial^\mu\eta)(\partial_\mu\eta) - \lambda v^2\eta^2 + \tfrac{1}{2}(\partial^\mu\xi)(\partial_\mu\xi)\,, -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}g^2v^2B^\mu B_\mu - V_{int} + gvB_\mu(\partial^\mu\xi)\,.$$

Substituting

$$\phi(x) = \frac{1}{\sqrt{2}}(v + \eta(x) + i\xi(x))$$

into

$$\begin{aligned} \mathcal{L} = &-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + (\partial_\mu\phi)^*(\partial^\mu\phi) - \mu^2\phi^2 - \lambda\phi^4 \\ &- igB_\mu\phi^*(\partial^\mu\phi) + ig(\partial_\mu\phi^*)B^\mu\phi + g^2B_\mu B^\mu\phi^*\phi\,. \end{aligned}$$

and using $\mu^2 = -\lambda v^2$gives

$$\begin{aligned}\mathcal{L} =& -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}(\partial_\mu\eta - i\partial_\mu\xi)(\partial^\mu\eta + i\partial^\mu\xi)\\ &+ \tfrac{1}{2}\lambda v^2(v+\eta)^2 + \tfrac{1}{2}\lambda v^2\xi^2 - \tfrac{1}{4}\lambda(v+\eta)^4 - \tfrac{1}{2}\lambda(v+\eta)^2\xi^2 - \tfrac{1}{4}\lambda\xi^4\\ &- i\tfrac{1}{2}gB_\mu(v+\eta-i\xi)(\partial^\mu\eta + i\partial^\mu\xi) + i\tfrac{1}{2}g(\partial_\mu\eta - i\partial_\mu\xi)B^\mu(v+\eta+i\xi)\\ &+ \tfrac{1}{2}g^2B_\mu B^\mu(v+\eta)^2 + \tfrac{1}{2}g^2B_\mu B^\mu\xi^2\\ =& -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) + \tfrac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi)\\ &+ \tfrac{1}{4}\lambda v^4 - \lambda v^2\eta^2 - \lambda v\eta^3 - \tfrac{1}{4}\lambda\eta^4 - \tfrac{1}{4}\lambda\xi^4 - \lambda v\eta\xi^2 - \tfrac{1}{2}\eta^2\xi^2\\ &+ gB_\mu(\partial^\mu\xi)(v+\eta) - gB_\mu\xi(\partial^\mu\eta)B^\mu\\ &+ \tfrac{1}{2}g^2v^2B_\mu B^\mu + g^2vB_\mu B^\mu\eta + \tfrac{1}{2}B_\mu B^\mu\eta^2 + \tfrac{1}{2}g^2B_\mu B^\mu\xi^2\,.\end{aligned}$$

Interpreting the terms with three or four fields (B, η or ξ) as three- and four-point interaction terms, leaves

$$\begin{aligned}\mathcal{L} =& -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) + \tfrac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi)\\ &- \lambda v^2\eta^2 + gvB_\mu(\partial^\mu\xi) + \tfrac{1}{2}g^2v^2B_\mu B^\mu - V_{int}(B,\eta,\xi)\,.\end{aligned}$$

which, when the terms are grouped together gives the required expression

$$\mathcal{L} = \underbrace{\tfrac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) - \lambda v^2\eta^2}_{\text{massive }\eta} + \underbrace{\tfrac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi)}_{\text{massless }\xi} - \underbrace{\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}g^2v^2B_\mu B^\mu}_{\text{massive gauge field}} - V_{int} + gvB_\mu(\partial^\mu\xi)\,.$$

17.7 Show that the Lagrangian for a complex scalar field ϕ,

$$\mathcal{L} = (D_\mu\phi)^*(D^\mu\phi)\,,$$

with the covariant derivative $D_\mu = \partial_\mu + igB_\mu$, is invariant under local U(1) gauge transformations,

$$\phi(x) \to \phi'(x) = e^{ig\chi(x)}\phi(x)\,,$$

provided the gauge field transforms as

$$B_\mu \to B'_\mu = B_\mu - \partial_\mu\chi(x)\,.$$

This proof just requires care. The original Lagrangian is

$$\begin{aligned}\mathcal{L} &= (D_\mu\phi)^*(D^\mu\phi) = (\partial_\mu\phi_i gB_\mu\phi)^*(\partial^\mu\phi_i gB^\mu\phi)\\ &= (\partial_\mu\phi^*)(\partial^\mu\phi) + ig(\partial_\mu\phi^*)B^\mu\phi - ig(\partial^\mu\phi)B_\mu\phi^* + g^2B_\mu B^\mu\phi\phi^*\,.\end{aligned}$$

Under the transformation

$$\phi(x) \to \phi'(x) = e^{ig\chi(x)}\phi(x) \quad \text{and} \quad B_\mu \to B'_\mu = B_\mu - \partial_\mu\chi(x)\,,$$

this Lagrangian becomes

$$\begin{aligned}
\mathcal{L}' =& \left[\partial_\mu\phi^* - ig(\partial_\mu\chi)\phi^*\right][\partial^\mu\phi + ig(\partial^\mu\chi)\phi] \\
&+ ig\left[\partial_\mu\phi^* - ig(\partial_\mu\chi)\phi^*\right][B^\mu - (\partial^\mu\chi)]\,\phi \\
&- ig\,[\partial^\mu\phi + ig(\partial^\mu\chi)\phi]\left[B_\mu - (\partial_m u\chi)\right]\phi^* \\
&+ g^2B_\mu B^\mu\phi\phi^* - g^2[B_\mu(\partial^\mu\chi) + B^\mu(\partial_\mu\chi)]\phi\phi^* + g^2(\partial_\mu\chi)(\partial^\mu\chi)\phi\phi^* \\
=&(\partial_\mu\phi^*)(\partial^\mu\phi) - ig(\partial_\mu\chi)(\partial^\mu\phi)\phi^* + ig(\partial_\mu\phi^*)(\partial^\mu\chi)\phi + g^2(\partial_\mu\chi)(\partial^\mu\chi)\phi\phi^* \\
&+ ig(\partial_\mu\phi^*)B^\mu\phi + g^2(\partial_\mu\chi)B^\mu\phi^*\phi - ig(\partial_\mu\phi^*)(\partial^\mu\chi)\phi - g^2(\partial_\mu\chi)(\partial^\mu\chi)\phi^*\phi \\
&- ig(\partial^\mu\phi)B_\mu\phi^* + g^2(\partial^\mu\chi)B_\mu\phi\phi^* + ig(\partial^\mu\phi)(\partial_\mu\chi)\phi^* - g^2(\partial^\mu\chi)(\partial_\mu\chi)\phi\phi^* \\
&+ g^2B_\mu B^\mu\phi\phi^* - g^2B_\mu(\partial^\mu\chi)\phi\phi^* - g^2B^\mu(\partial_\mu\chi)\phi\phi^* + g^2(\partial^\mu\chi)(\partial_\mu\chi)\phi\phi^* \\
&= (\partial_\mu\phi^*)(\partial^\mu\phi) + ig(\partial_\mu\phi^*)B^\mu\phi - ig(\partial^\mu\phi)B_\mu\phi^* + g^2B_\mu B^\mu\phi\phi^* \\
&= \mathcal{L}\,.
\end{aligned}$$

17.8 From the mass matrix of (17.40) and its eigenvalues (17.41), show that the eigenstates in the diagonal basis are

$$A_\mu = \frac{g'W_\mu^{(3)} + g_\mathrm{W}B_\mu}{\sqrt{g_\mathrm{W}^2 + g'^2}} \quad \text{and} \quad Z_\mu = \frac{g_\mathrm{W}W_\mu^{(3)} - g'B_\mu}{\sqrt{g_\mathrm{W}^2 + g'^2}}\,,$$

where A_μ and Z_μ correspond to the physical fields for the photon and Z.

The eigenvalues of the mass matrix of (17.40) are $\lambda_1 = 0$ and $\lambda_2 = g_\mathrm{W}^2 + g'^2$. The corresponding eigenvectors can be found by solving the equations

$$\mathbf{MX} = \begin{pmatrix} g_\mathrm{W}^2 & -g_\mathrm{W}g' \\ -g_\mathrm{W}g' & g'^2 \end{pmatrix}\mathbf{X} = \lambda\mathbf{X}\,,$$

where $\mathbf{M}$ is the mass matrix giving mass terms of the form

$$\begin{pmatrix} W_\mu^{(3)} & B_\mu \end{pmatrix}\mathbf{M}\begin{pmatrix} W^{(3)\mu} \\ B^\mu \end{pmatrix}.$$

For $\lambda = \lambda_1 = 0$ the eigenvalue equation gives

$$\begin{aligned}
\begin{pmatrix} g_\mathrm{W}^2 & -g_\mathrm{W}g' \\ -g_\mathrm{W}g' & g'^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \lambda_1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \\
\Rightarrow \quad g_\mathrm{W}^2x_1 &= g'g_\mathrm{W}x_2\,,
\end{aligned}$$

and thus the corresponding eigenvalue is:

$$\mathbf{X}_1 = \frac{1}{\sqrt{g_\mathrm{W}^2 + g'^2}}\begin{pmatrix} g' \\ g_\mathrm{W} \end{pmatrix}$$

For the second eigenvalue,

$$\begin{pmatrix} g_{\mathrm{W}}^2 & -g_{\mathrm{W}}g' \\ -g_{\mathrm{W}}g' & g'^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (g_{\mathrm{W}}^2 + g'^2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$\Rightarrow \quad g_{\mathrm{W}}^2 x_1 - g_{\mathrm{W}}g' x_2 = g_{\mathrm{W}}^2 x1 + g'^2 x_1$$
$$\Rightarrow \quad -g_{\mathrm{W}} x_2 = g' x_1 \,,$$

and thus the corresponding eigenvalue is:

$$\mathbf{X}_2 = \frac{1}{\sqrt{g_{\mathrm{W}}^2 + g'^2}}\begin{pmatrix} g_{\mathrm{W}} \\ g' \end{pmatrix} .$$

The mass eigenstates in the diagonal basis are therefore

$$\begin{pmatrix} A \\ Z \end{pmatrix} = \begin{pmatrix} g' & g_{\mathrm{W}} \\ g_{\mathrm{W}} & -g' \end{pmatrix}\begin{pmatrix} W \\ B \end{pmatrix}$$
$$\Rightarrow \quad A_\mu = \frac{g' W_\mu^{(3)} + g_{\mathrm{W}} B_\mu}{\sqrt{g_{\mathrm{W}}^2 + g'^2}} \quad \text{and} \quad Z_\mu = \frac{g_{\mathrm{W}} W_\mu^{(3)} - g' B_\mu}{\sqrt{g_{\mathrm{W}}^2 + g'^2}} .$$

17.9 By considering the interaction terms in (17.38), show that the HZZ coupling is given by

$$g_{\mathrm{HZZ}} = \frac{g_{\mathrm{W}}}{\cos\theta_{\mathrm{W}}} m_{\mathrm{Z}} .$$

The interaction terms in the Lagrangian arise from

$$\begin{aligned}(D_\mu\phi)^\dagger(D^\mu\phi) =& \tfrac{1}{2}(\partial_\mu h)(\partial^\mu h) + \tfrac{1}{8}g_{\mathrm{W}}^2(W_\mu^{(1)} + iW_\mu^{(2)})(W^{(1)\mu} - iW^{(2)\mu})(v+h)^2 \\ &+ \tfrac{1}{8}(g_{\mathrm{W}}W_\mu^{(3)} - g'B_\mu)(g_{\mathrm{W}}W^{(3)\mu} - g'B^\mu)(v+h)^2 \\ =& \tfrac{1}{2}(\partial_\mu h)(\partial^\mu h) + \tfrac{1}{8}g_{\mathrm{W}}^2(W_\mu^{(1)} + iW_\mu^{(2)})(W^{(1)\mu} - iW^{(2)\mu})(v+h)^2 \\ &+ \tfrac{1}{8}(g_{\mathrm{W}}^2 + g'^2)Z^\mu Z_\mu(v+h)^2 .\end{aligned}$$

Considering the part of the expression involving the Z fields and using the relation $g' = g_{\mathrm{W}}\tan\theta_{\mathrm{W}}$,

$$\begin{aligned}\mathcal{L}_{\mathrm{Z}} &= \tfrac{1}{8}(g_{\mathrm{W}}^2 + g'^2)Z^\mu Z_\mu(v+h)^2 \\ &= \frac{g_{\mathrm{W}}^2}{8}\left(1 + \tan^\theta\right)Z^\mu Z_\mu(v+h)^2 \\ &= \frac{g_{\mathrm{W}}^2}{8\cos^2\theta_{\mathrm{W}}}Z^\mu Z_\mu(v+h)^2 \\ &= \frac{g_{\mathrm{W}}^2}{8\cos^2\theta_{\mathrm{W}}}\left(v^2 Z^\mu Z_\mu + 2vhZ^\mu Z_\mu + h^2 Z^\mu Z_\mu\right) .\end{aligned} \tag{17.1}$$

The first term in the brackets gives the Z mass:

$$\tfrac{1}{2}m_Z^2 Z^\mu Z_\mu = \frac{g_W^2}{8\cos^2\theta_W} v^2 Z^\mu Z_\mu$$
$$\Rightarrow \quad m_Z = \tfrac{1}{2}\tfrac{g_W}{\cos\theta_W} v = \frac{m_W}{\cos\theta_W}\,.$$

The second term in the brackets of (17.1) gives the trilinear coupling between a physical Higgs field and two Z fields:

$$\begin{aligned}\mathcal{L}_{ZZh} &= \frac{g_W^2}{4\cos^2\theta_W} v h Z^\mu Z_\mu \\ &= \tfrac{1}{2}\frac{g_W}{\cos\theta_W} m_Z h Z^\mu Z_\mu \\ &= \tfrac{1}{2} g_Z m_Z h Z^\mu Z_\mu\,.\end{aligned}$$

Hence the HZZ coupling is

$$g_{HZZ} = \tfrac{1}{2}\frac{g_W}{\cos\theta_W} m_Z\,.$$

17.10 For a Higgs boson with $m_H > 2m_W$, the dominant decay mode is into two on-shell W bosons, $H \to W^+W^-$. The matrix element for this decay can be written

$$\mathcal{M} = -g_W m_W g_{\mu\nu} \xi^\mu(p_2)^* \xi^\nu(p_3)^*\,,$$

where p_2 and p_3 are respectively the four-momenta of the W^+ and W^-.

a) Taking $\mathbf{p}_2$ to lie in the positive z-direction, consider the nine possible polarisation states of the W^+W^- and show that the matrix element is only non-zero when both W bosons are left-handed ($\mathcal{M}_{\downarrow\downarrow}$), both W bosons are right-handed ($\mathcal{M}_{\uparrow\uparrow}$), or both are longitudinally polarised ($\mathcal{M}_{LL}$).

b) Show that

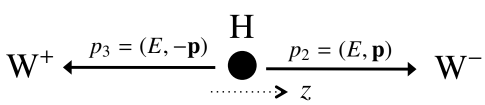
$$\mathcal{M}_{\uparrow\uparrow} = \mathcal{M}_{\downarrow\downarrow} = -g_W m_W \quad \text{and} \quad \mathcal{M}_{LL} = \frac{g_W}{m_W}\left(\tfrac{1}{2}m_H^2 - m_W^2\right).$$

c) Hence show that

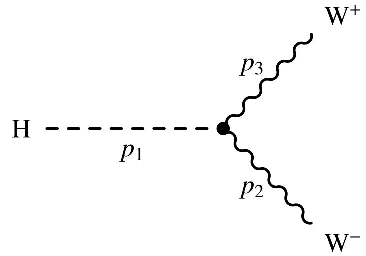
$$\Gamma(H \to W^+W^-) = \frac{G_F m_H^3}{8\pi\sqrt{2}}\sqrt{1-4\lambda^2}\,(1 - 4\lambda^2 + 12\lambda^4)\,,$$

where $\lambda = m_W/m_H$.

a) and b) Consider the decay $H \to W^+W^-$



Denoting the respective four-momenta of the W^- and W^+ as $p_2 = (E, 0, 0, \mathrm{p})$ and $p_3 = (E, 0, 0, -\mathrm{p})$, the corresponding is



The relevant Feynman rules are a factor $ig_{\mathrm{W}}m_{\mathrm{W}}g_{\mu\nu}$ at the HWW vertex and factors of $\epsilon^{\mu}(p_2)^*$ and $\epsilon^{\nu}(p_3)^*$ for the final-state W bosons: The matrix element for the decay is

$$\begin{aligned} -i\mathcal{M}_{fi} &= ig_{\mathrm{W}}m_{\mathrm{W}}g_{\mu\nu}\,\epsilon^{\mu}(p_2)^*\,\epsilon^{\nu}(p_3)^* \\ \Rightarrow \quad \mathcal{M}_{fi} &= -g_{\mathrm{W}}m_{\mathrm{W}}g_{\mu\nu}\,\epsilon^{\mu}(p_2)^*\,\epsilon^{\nu}(p_3)^*\,. \end{aligned}$$

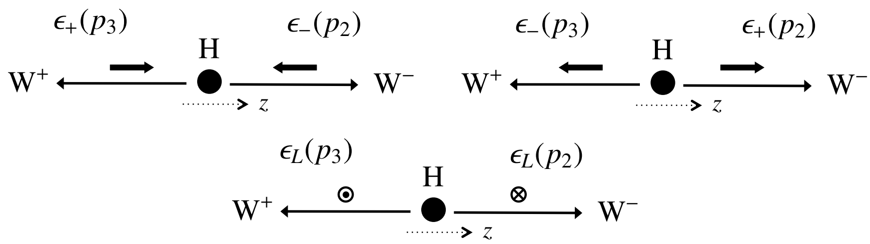
Thus the matrix element depends on the four-vector scalar product of the W boson polarisation four-vectors. The possible polarisation states are:

$$\begin{aligned} \epsilon^{\mu}_{+}(p_2)^* &= \tfrac{1}{\sqrt{2}}(0,-1,i,0)\,, \;\; \epsilon^{\mu}_{-}(p_2)^* = \tfrac{1}{\sqrt{2}}(0,1,i,0) \;\text{ and }\; \epsilon^{\mu}_{L}(p_2)^* = \tfrac{1}{m_{\mathrm{W}}}(\mathrm{p},0,0,E)\,, \\ \epsilon^{\nu}_{+}(p_3)^* &= \tfrac{1}{\sqrt{2}}(0,-1,i,0)\,, \;\; \epsilon^{\nu}_{-}(p_3)^* = \tfrac{1}{\sqrt{2}}(0,1,i,0) \;\text{ and }\; \epsilon^{\nu}_{L}(p_3)^* = \tfrac{1}{m_{\mathrm{W}}}(-\mathrm{p},0,0,E)\,, \end{aligned}$$

where the $\pm$ refer to the gauge boson spin pointing in either the $+\pm$ z-direction. Of the nine possible combinations, only three give non-zero four-vector scalar products:

$$\begin{aligned} \epsilon_{+}(p_2)^*\cdot\epsilon_{-}(p_3)^* &= +1\,, \\ \epsilon_{-}(p_2)^*\cdot\epsilon_{+}(p_3)^* &= +1\,, \\ \epsilon_{L}(p_2)^*\cdot\epsilon_{L}(p_3)^* &= -\tfrac{1}{m_{\mathrm{W}}^2}(\mathrm{p}^2+E^2)\,. \end{aligned}$$

The spin orientations for these three cases all correspond to spin-0 states as shown below.



The energy of each W is $E = m_\mathrm{H}/2$:

$$\begin{aligned} E^2 = \frac{m_\mathrm{H}^2}{4} &= \mathrm{p}^2 + m_\mathrm{W}^2\,, \\ \Rightarrow \quad \mathrm{p}^2 &= \tfrac{1}{4}m_\mathrm{H}^2 - m_\mathrm{W}^2\,, \\ \Rightarrow \quad E^2 + \mathrm{p}^2 &= \tfrac{1}{2}m_\mathrm{H}^2 - m_\mathrm{W}^2\,, \end{aligned}$$

Therefore the three non-zero matrix elements are:

$$\begin{aligned} \mathcal{M}_{\downarrow\downarrow} &= -g_\mathrm{W} m_\mathrm{W}\,, \\ \mathcal{M}_{\uparrow\uparrow} &= -g_\mathrm{W} m_\mathrm{W}\,, \\ \mathcal{M}_{LL} &= +g_\mathrm{W}\frac{1}{m_\mathrm{W}}(\mathrm{p}^2 + E^2) = +g_\mathrm{W}\frac{1}{m_\mathrm{W}}\left(\tfrac{1}{2}m_\mathrm{H}^2 - m_\mathrm{W}^2\right)\,, \end{aligned}$$

where the arrows indicate the helicity states of the W bosons.

c) Since the Higgs boson is a spin-0 scaler, the spin-averaged matrix element squared is just:

$$\begin{aligned} \langle|\mathcal{M}_{fi}|^2\rangle &= \mathcal{M}_{\downarrow\downarrow} + \mathcal{M}_{\uparrow\uparrow} + \mathcal{M}_{LL} \\ &= 2g_\mathrm{W}^2 m_\mathrm{W}^2 + \frac{g_\mathrm{W}^2}{m_\mathrm{W}^2}\left(\tfrac{1}{2}m_\mathrm{H}^2 - m_\mathrm{W}^2\right)^2 \\ &= g_\mathrm{W}^2 m_\mathrm{W}^2\left[2 + \left(\frac{m_\mathrm{H}^2}{2m_\mathrm{W}^2} - 1\right)^2\right]. \end{aligned}$$

The total decay rate is therefore

$$\begin{aligned} \Gamma(\mathrm{H} \to \mathrm{W}^+\mathrm{W}^-) &= \frac{\mathrm{p}}{8\pi m_\mathrm{H}^2}\langle|\mathcal{M}_{fi}|^2\rangle \\ &= \frac{g_\mathrm{W}^2 m_\mathrm{W}^2}{8\pi m_\mathrm{H}^2}\left(\tfrac{1}{4}m_\mathrm{H}^2 - m_\mathrm{W}^2\right)^{1/2}\left[2 + \left(\frac{m_\mathrm{H}^2}{2m_\mathrm{W}^2} - 1\right)^2\right] \\ &= \frac{g_\mathrm{W}^2 m_\mathrm{W}^2}{16\pi m_\mathrm{H}}\left(1 - \frac{4m_\mathrm{W}^2}{m_\mathrm{H}^2}\right)^{1/2}\left[\frac{m_\mathrm{H}^4}{4m_\mathrm{W}^4} - \frac{m_\mathrm{H}^2}{m_\mathrm{W}^2} + 3\right] \\ &= \frac{m_\mathrm{H}^3 g_\mathrm{W}^2}{64\pi m_\mathrm{W}^2}\left(1 - \frac{4m_\mathrm{W}^2}{m_\mathrm{H}^2}\right)^{1/2}\left[1 - 4\frac{m_\mathrm{W}^2}{m_\mathrm{H}^2} + 12\frac{m_\mathrm{W}^2}{m_\mathrm{H}^2}\right] \\ &= \frac{G_\mathrm{F}\, m_\mathrm{H}^3}{8\pi\sqrt{2}}(1 - \lambda^2)^{1/2}(1 - 4\lambda^2 + 12\lambda^4)\,, \end{aligned}$$

where $\lambda = m_\mathrm{W}/m_\mathrm{H}$.

17.11 Assuming a total Higgs production cross section of 20 pb and an integrated luminosity of 10 fb^{-1}, how many $\mathrm{H} \to \gamma\gamma$ and $\mathrm{H} \to \mu^+\mu^-\mu^+\mu^-$ events are expected in each of the ATLAS and CMS experiments.

The number of Higgs bosons produced in both the ATLAS and CMS experiments is:

$$N_{\mathrm{H}} = L \times \sigma \mathrm{pp} \to \mathrm{H} + X\,,$$

where L is the integrated luminosity:

$$L = \int \mathcal{L}\, \mathrm{d}t\,,$$

giving

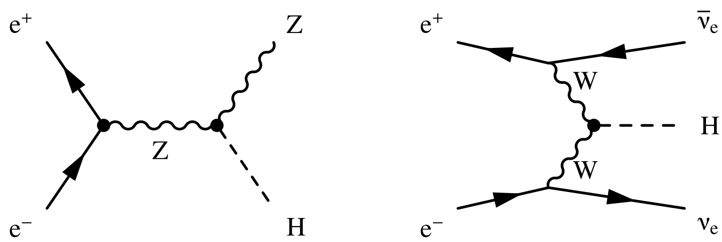
$$N_{\mathrm{H}} = 20 \times 10^3\,\mathrm{fb} \times 10\,\mathrm{fb}^{-1} = 200\,000\,.$$

The relevant Standard Model branching fractions are $BR(\mathrm{H} \to \gamma\gamma) \approx 0.002$, $BR(\mathrm{H} \to \mathrm{ZZ}^*) = 0.027$ and $BR(\mathrm{Z} \to \mu^+\mu^-) = 0.035$, and thus

$$N(\mathrm{H} \to \gamma\gamma) = 2 \times 10^5 \times 0.002 = 400\,,$$
$$N(\mathrm{H} \to \mathrm{ZZ}^* \to \mu^+\mu^-) = 2 \times 10^5 \times 0.027 \times 0.035^2 = 7\,.$$

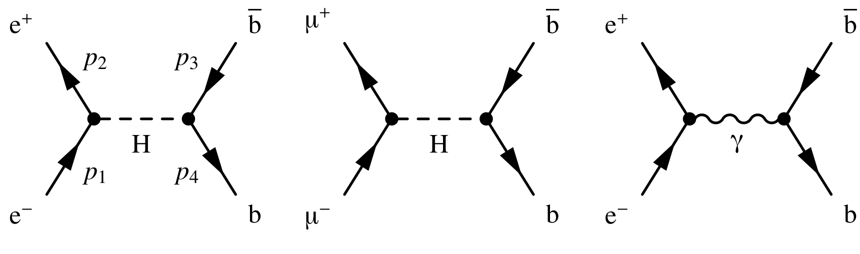
Despite the fact that very few $\mathrm{H} \to \mathrm{ZZ}^* \to \mu^+\mu^-$ are produced, such events leave a very clear signature in the detectors are relatively easily identified.

17.12 Draw the lowest-order Feynman diagrams for the processes $\mathrm{e}^+\mathrm{e}^- \to \mathrm{HZ}$ and $\mathrm{e}^+\mathrm{e}^- \to \mathrm{H}\nu_e\overline{\nu}_e$, which are the main Higgs production mechanism at a future high-energy linear collider.



17.13 In the future, it might be possible to construct a muon collider where the Higgs boson can be produced directly through $\mu^+\mu^- \to \mathrm{H}$. Compare the cross sections for $\mathrm{e}^+\mathrm{e}^- \to \mathrm{H} \to \mathrm{b}\overline{\mathrm{b}}$, $\mu^+\mu^- \to \mathrm{H} \to \mathrm{b}\overline{\mathrm{b}}$ and $\mu^+\mu^- \to \gamma \to \mathrm{b}\overline{\mathrm{b}}$ at $\sqrt{s} = m_{\mathrm{H}}$.

The Feynman diagrams for the three process are all s-channel processes.



The matrix element for the process $e^+e^- \to H \to b\overline{b}$ can be obtained from the Feynman rules where the couplings at the two $Hf\overline{f}$ vertices are given by

$$-i\frac{m_f}{v} = -i\frac{f}{2m_W}g_W\,,$$

and the propagator for the Higgs boson with total decay width Γ_H is

$$\frac{1}{q^2 - m_H^2 + im_H\Gamma_H},.$$

Hence the matrix element is given by

$$\begin{aligned}
-i\mathcal{M} &= \overline{v}(p_2)\left[-i\frac{m_e}{v}\right]u(p_1)\cdot\frac{1}{q^2 - m_H^2 + im_H\Gamma_H}\cdot\overline{u}(p_3)\left[-i\frac{m_b}{v}\right]v(p_4)\\
\mathcal{M} &= \frac{m_e m_b}{v^2 m_H\Gamma_H}[\overline{v}(p_2)u(p_1)]\cdot[\overline{u}(p_3)v(p_4)] \qquad \text{for } q^2 = m_H^2\,.
\end{aligned}$$

Since we are in the limit where the masses of the particles can be neglected and can be written

$$\begin{aligned}
p_1 &= (E, 0, 0, E)\,,\\
p_2 &= (E, 0, 0, -E)\,,\\
p_3 &= (E, E\sin\theta, 0, E\cos\theta)\,,\\
p_4 &= (E, -E\sin\theta, 0, -E\cos\theta)\,.
\end{aligned}$$

The corresponding spinors for the two possible helicity states of each particle are:

$$u_\uparrow(p_1) = \sqrt{E}\begin{pmatrix}1\\0\\1\\0\end{pmatrix},\ u_\downarrow(p_1) = \sqrt{E}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix},\ v_\uparrow(p_2) = \sqrt{E}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix},\ v_\downarrow(p_2) = \sqrt{E}\begin{pmatrix}0\\-1\\0\\-1\end{pmatrix}.$$

and

$$u_\uparrow(p_3) = \sqrt{E}\begin{pmatrix}c\\s\\c\\s\end{pmatrix},\ u_\downarrow(p_3) = \sqrt{E}\begin{pmatrix}-s\\c\\s\\-c\end{pmatrix},\ v_\uparrow(p_4) = \sqrt{E}\begin{pmatrix}c\\s\\-c\\-s\end{pmatrix},\ v_\downarrow(p_4) = \sqrt{E}\begin{pmatrix}s\\-c\\s\\-c\end{pmatrix}.$$

There are two possible combinations of intiial-state spinors that will give non-zero values for $\overline{v}(p_2)u(p_1) = v^\dagger(p_2)\gamma^0 u(p_1)$:

$$\overline{v}_\uparrow(p_2)u_\uparrow(p_1) = \overline{v}_\downarrow(p_2)u_\downarrow(p_1) = 2E\,.$$

Similarly, there are two possible combinations of intiial-state spinors that will give non-zero values for $\overline{u}(p_3)v(p_4)$:

$$\overline{u}_\uparrow(p_3)v_\uparrow(p_4) = -\overline{u}_\downarrow(p_3)v_\downarrow(p_4) = 2E\,.$$

Thus, for each of the four helicity combinations giving a non-zero matrix element at $q^2 = s = 4E = m_{\mathrm{H}}^2$

$$\begin{aligned}\mathcal{M} &= \pm\frac{m_{\mathrm{e}}m_{\mathrm{b}}}{v^2 m_{\mathrm{H}}\Gamma_{\mathrm{H}}}4E \\ &= \pm\frac{m_{\mathrm{e}}m_{\mathrm{b}}m_{\mathrm{H}}}{v^2\Gamma_{\mathrm{H}}}\,.\end{aligned}$$

The spin-averaged matrix element squared is just

$$\langle|\mathcal{M}|^2\rangle = \tfrac{1}{4}\times 4\times\mathcal{M} = \left(\frac{m_{\mathrm{e}}m_{\mathrm{b}}m_{\mathrm{H}}}{v^2\Gamma_{\mathrm{H}}}\right)^2\,.$$

Note the isotropic distribution of the final state particles, which is characteristic of a scalar interaction. The differential cross section (at the peak of the Higgs resonance) is given by:

$$\begin{aligned}\frac{\mathrm{d}\sigma^0}{\mathrm{d}\Omega} &= \frac{1}{64\pi^2 s}\langle|\mathcal{M}|^2\rangle \\ &= \frac{1}{64\pi^2 m_{\mathrm{H}}^2}\left(\frac{m_{\mathrm{e}}m_{\mathrm{b}}m_{\mathrm{H}}}{v^2\Gamma_{\mathrm{H}}}\right)^2\,, \\ \Rightarrow\quad \sigma^0 &= \frac{m_{\mathrm{e}}^2 m_{\mathrm{b}}^2}{16\pi v^4\Gamma_{\mathrm{H}}^2}\,.\end{aligned}$$

The corresponding expression for $\mu^+\mu^- \to \mathrm{H} \to \mathrm{b}\bar{\mathrm{b}}$ is obtained by replacing m_{e} by m_μ. The cross section for the QED process $\mathrm{e}^+\mathrm{e}^- \to \gamma \to \mathrm{b}\bar{\mathrm{b}}$ at $\sqrt{s} = m_{\mathrm{H}}$ is the now familiar

$$\sigma_{\mathrm{QED}} = \frac{4\pi\alpha^2}{3s}Q_{\mathrm{b}}^2 = \frac{16\pi\alpha^2}{27m_{\mathrm{H}}^2}\,.$$

Taking $m_{\mathrm{H}} = 126\,\mathrm{GeV}$, $v = 246\,\mathrm{GeV}$, $\Gamma_{\mathrm{H}} = 0.004\,\mathrm{GeV}$, $\alpha = 1/128$, $m_{\mathrm{b}} = 5\,\mathrm{GeV}$:

$$\begin{aligned}\sigma^0_{\mathrm{e}^+\mathrm{e}^-} : \sigma^0_{\mu^+\mu^-} : \sigma_{\mathrm{QED}} &= \frac{m_{\mathrm{e}}^2 m_{\mathrm{b}}^2}{16\pi v^4\Gamma_{\mathrm{H}}^2} : \frac{m_\mu^2 m_{\mathrm{b}}^2}{16\pi v^4\Gamma_{\mathrm{H}}^2} : \frac{16\pi\alpha^2}{27m_{\mathrm{H}}^2} \\ &= 2.2\times 10^{-12}\,\mathrm{GeV}^{-2} : 9.5\times 10^{-8}\,\mathrm{GeV}^{-2} : 7.1\times 10^{-9}\,\mathrm{GeV}^{-2}\,.\end{aligned}$$

Therefore, the cross section for $\mathrm{e}^+\mathrm{e}^- \to \mathrm{H} \to \mathrm{b}\bar{\mathrm{b}}$ is there orders of magnitude smaller than the QED process and it would be almost impossible to detect Higgs production in s-channel $\mathrm{e}^+\mathrm{e}^-$ annihilation. On the contrary, due to the larger muon mass, the cross section $\mathrm{e}^+\mathrm{e}^- \to \mathrm{H} \to \mathrm{b}\bar{\mathrm{b}}$ is an order of magnitude greater than the pure QED process, motivating the idea of muon collider (whether it is actually possible to construct such a machine is a different question).

Errata

p 56: *Question 2.8*: The reaction should (of course) read:

$$\mathrm{p} + \overline{\mathrm{p}} \rightarrow \mathrm{p} + \mathrm{p} + \overline{\mathrm{p}} + \overline{\mathrm{p}}\,.$$

p 56: *Question 2.9*: The question should ask for the *minimum* opening angle; the maximum opening angle is (rather trivially) π.

p 57: the factor of $\frac{1}{4}$ in the last line of *Question 2.16* should be removed, *i.e.* Find the eigenvalue(s) of the operator $\hat{\mathbf{S}}^2 = (\hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2)$, and deduce that the eigenstates of $\hat{S}_z$ are a suitable representation of a spin-half particle.

p 78: the mass of the pion in *Question 3.1* should be 140 MeV, not 140 GeV.

p146: In the matrix in the *footnote* $B_{22} \rightarrow B_{21}$.

p177: *Question 7.2* should be ignored. There was an error in my original solution, whereby finding a closed form was relatively straightforward - it isn't!

p231: there is a typo in the equation at the bottom of the page:

$$\tfrac{1}{2}\left[\mathbf{S}^2 - \mathbf{S_1}^2 - \mathbf{S_1}^2\right] \rightarrow \tfrac{1}{2}\left[\mathbf{S}^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2\right].$$

p312: In *Figure 12.5*, the arrows on the d and ν_μ are the wrong-way around, only left-handed chiral states participate in the weak charged-current.

p315: Line four contains a typo, the third reaction should read $\overline{\nu}_\mu \mathrm{u} \rightarrow \mu^+ \mathrm{d}$

p337: In *Figure 13.16*, the bottom two diagrams should (of course) show a π^+.

p341: There is a typo ($\mathrm{p}_1 \rightarrow \mathrm{p}_2$) in Equation (13.13), which should read

$$\Delta\phi_{12} = (E_1 - E_2)\left[T - \left(\frac{E_1 + E_2}{\mathrm{p}_1 + \mathrm{p}_2}\right)L\right] + \left(\frac{m_1^2 - m_2^2}{\mathrm{p}_1 + \mathrm{p}_2}\right)L\,.$$

This typo is repeated in *question 13.1*.

p362: In *question 13.2*, there is a spurious 4 in the denominator of the argument of the $\sin^2(...)$ in the second equation, it should read

$$\sin^2(2\theta)\sin^2\left(\frac{\Delta m^2[\mathrm{GeV}^2]L[\mathrm{GeV}^{-1}]}{4E_\nu[\mathrm{GeV}]}\right) \rightarrow \sin^2(2\theta)\sin^2\left(1.27\frac{\Delta m^2[\mathrm{eV}^2]L[\mathrm{km}]}{E_\nu[\mathrm{GeV}]}\right).$$

The expression in the main text is correct.

p363: Part d) of *question 13.9* should be ignored - it is poorly worded. The intention was to get the student to consider the case where the decay products of the pion were close to being perpendicular to the direction of the boost. Close to $\theta^* \sim \pi/2$ the transverse momentum is approximately p* and the longitudinal momentum is primarily due to the Lorentz boost.

p427: The last matrix element should read $\mathcal{M}_{LR}^2$ not $\mathcal{M}_{RR}^2$.

p458: *Question 16.7* should read $\mu^-\overline{\nu}_\mu u\overline{d}$.

p498: *Question 17.8* the expression for the fields should read:

$$A_\mu = \frac{g' W_\mu^{(3)} + g_W B_\mu}{\sqrt{g_W^2 + g'^2}} \quad \text{and} \quad Z_\mu = \frac{g_W W_\mu^{(3)} - g' B_\mu}{\sqrt{g_W^2 + g'^2}},$$