from the previous lectures ...

Graphical description

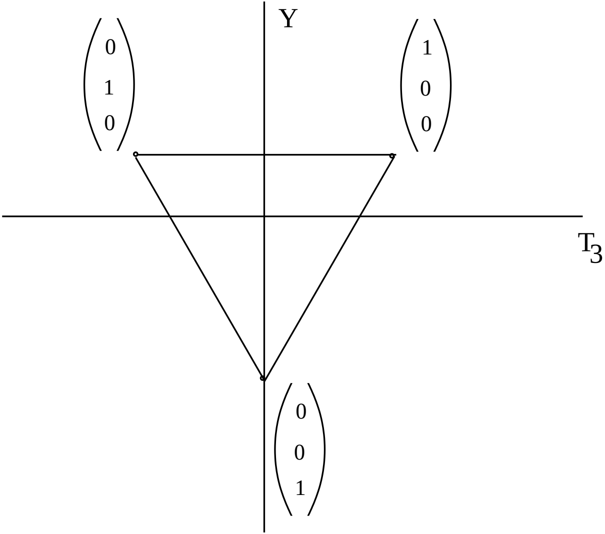
For $SU(3)$ we characterise the states by T_3 & Y.

$$\rightarrow \qquad (T_3, Y) \text{ plane}:$$

$$(T_3, Y) : \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (\frac{1}{2}, \frac{1}{3})$$

$$(T_3, Y) : \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (-\frac{1}{2}, \frac{1}{3})$$

$$(T_3, Y) : \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = (0, -\frac{2}{3})$$



$\rightarrow$ graphical representation of the fundamental triplet representation of $SU(3)$

The physical quantities

T_3 – third component of Isospin (same as for $SU(2)$).

Y – hypercharge

We can exchange the τ_1, τ_2, τ_3 generators of $SU(2)$ with

$$\tau_\pm = \frac{1}{2}(\tau_1 \pm i\tau_2) \quad \& \quad \tau_3$$

Similarly in $SU(3)$ define $\tau_\pm = \frac{1}{2}(\lambda_1 \pm i\lambda_2)$

$$\tau_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \tau_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tau_+ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0 \qquad \tau_+ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \tau_+ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$\tau_- \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \tau_- \begin{pmatrix}1\\0\\0\end{pmatrix} = \quad 0 \qquad \tau_- \begin{pmatrix}0\\0\\1\end{pmatrix} = 0$$

The three proints on the graphic triangular representation of the triplet of $SU(3)$ form a doublet $\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\}$ and a singlet $\begin{pmatrix}0\\0\\1\end{pmatrix}$ of $SU(2)$.

use $\lambda_1,\ \lambda_2,\ \lambda_3$ to exchange $\begin{pmatrix}1\\0\\0\end{pmatrix}$ & $\begin{pmatrix}0\\1\\0\end{pmatrix}$ but cannot act on $\begin{pmatrix}0\\0\\1\end{pmatrix}$

$$\begin{array}{cccccc} SU(3) & \rightarrow & SU(2)_I \times U(1)_Y & & \\ 3 & = & 2_{\frac{1}{3}} & + & 1_{-\frac{2}{3}} \\ \text{triplet} & & \text{doublet} & & \text{singlet} \end{array}$$

The representations of $SU(3)$ decompose under $SU(2) \times U(1)$.

In $SU(3)$ we can form generators that exchange $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$(\lambda_4 \pm i\lambda_5) \text{ exchanges } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \ \& \ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$(\lambda_6 \pm i\lambda_7) \text{ exchanges } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \ \& \ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

we use here the other $SU(3)$ generators that are not in $SU(2)_I$.

For every particle we know both T_3 & Y

Hence $SU(3) \supset SU(2)_I \times U(1)_Y$

in $SU(2)$ we can have higher order representations

For example

$$3: \quad \begin{array}{ccc} -1 & 0 & +1 \\ \begin{pmatrix}0\\0\\1\end{pmatrix} & \begin{pmatrix}0\\1\\0\end{pmatrix} & \begin{pmatrix}1\\0\\0\end{pmatrix} \end{array} \qquad \begin{array}{ccc} T_1 & T_2 & T_3 \\ \begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} & \begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix} & \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix} \end{array}$$

Check that the generators of the triplet representation $T_1,\ T_2,\ T_3$ satisfy the $SU(2)$ algebra $[T_i, T_j] = i\epsilon_{ijk}T_k$.

Any 3 $n \times n$ matrices that satisfy $[T_i, T_j] = i\epsilon_{ijk}T_k$ form a representation of the $SU(2)$ algebra.

The proton & neutron formed a doublet of $SU(2)_I$.

In the fundamental representation of $SU(3)$

There isn't a third particle with $m(?) \sim m(P) \sim m(N)$ that fits

$\Rightarrow \quad P,\ N$ form an Isospin doublet but are not part of an $SU(3)$ triplet

Can $P,\ N$ form an Isospin doublet in a higher order representation of $SU(3)$?

The $SU(3)$ generators obey the algebra
$$[T_i, T_j] = if_{ijk} T_k \tag{1}$$

with f_{ijk} totally antisymmetric under exchange of any two indices

$$\text{and} \quad f_{123} = 1 \quad f_{147} = \frac{1}{2} \quad f_{156} = -\frac{1}{2} \quad f_{246} = \frac{1}{2} \quad f_{257} = \frac{1}{2}$$
$$f_{345} = \frac{1}{2} \quad f_{367} = -\frac{1}{2} \quad f_{458} = \frac{\sqrt{3}}{2} \quad f_{678} = \frac{\sqrt{3}}{2}$$

All others vanish

$\rightarrow$ matrices of higher order representations satisfy the algebra in eq. (1)

For $SU(2)$ we have a solution at any order with matrices $(2\ell + 1) \times (2\ell + 1)$

For $SU(3)$ there isn't a solution at every order (*e.g.* order 2)

To find the higher order representations we use a different method.

Similar to the addition of angular momentum for $SU(2)$.

For $SU(2)$ $\psi^\alpha = \{|\uparrow\rangle, |\downarrow\rangle\}$, $S^2|\uparrow\rangle = \frac{1}{2}(\frac{1}{2}+1)|\uparrow\rangle$ $S^2|\downarrow\rangle = \frac{1}{2}(\frac{1}{2}+1)|\downarrow\rangle$

$S_z|\uparrow\rangle = +\frac{1}{2}|\uparrow\rangle$ $S_z|\downarrow\rangle = -\frac{1}{2}|\downarrow\rangle$

$\psi,\ \phi$ two spin $\frac{1}{2}$ wave functions

$$\psi^\alpha \otimes \phi^\beta : \qquad \text{Triplet} \begin{cases} |\uparrow_\psi\uparrow_\phi\rangle & T=1 \quad T_3=+1 \\ \frac{1}{\sqrt{2}}(|\uparrow_\psi\downarrow_\phi\rangle + |\downarrow_\psi\uparrow_\phi\rangle) & T=1 \quad T_3=\ 0 \\ |\downarrow_\psi\downarrow_\phi\rangle & T=1 \quad T_3=-1 \end{cases}$$

these are the symmetric combinations

$$\text{Singlet}\,\frac{1}{\sqrt{2}}(|\uparrow_\psi\downarrow_\phi\rangle - |\downarrow_\psi\uparrow_\phi\rangle) \qquad T=0 \quad T_3=\ 0$$

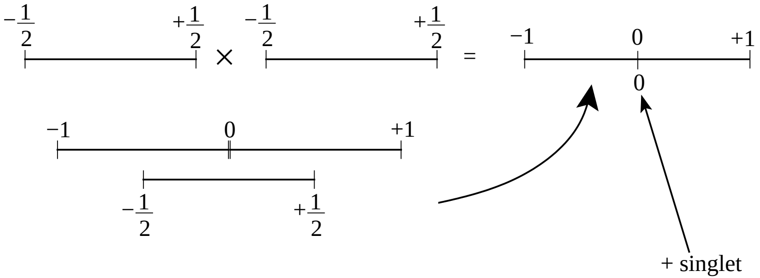
this is the antisymmetric combination

by taking the product of two Isospin doublets we get states with Isospin 1 or 0.

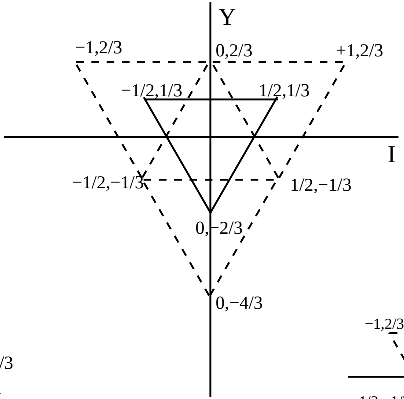
$$\underline{2} \times \underline{2} = \underline{3}_{\,\mathrm{S}} + \underline{1}_{\,\mathrm{A}}$$

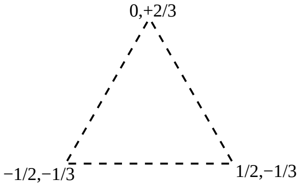
We built a $\underline{3}$ representation with spin 1 from the $\underline{2}$ representation with spin $\frac{1}{2}$.

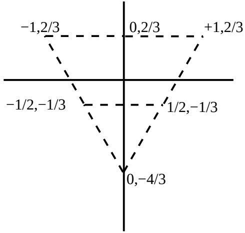
Graphically:



We can repeat the graphic analysis for $SU(3)$ with the product $\psi^\alpha \phi^\beta$ where ψ^α and ϕ^β are two triplets of $SU(3)$, and $\alpha,\ \beta = 1, 2, 3$.







The product $3 \times 3 = 6 + \bar{3} \quad \rightarrow \quad$ the sextet and $\bar{3}$ representations of $SU(3)$.

Inside the sextet we have

$$6 \;=\; \begin{cases} (-1,\frac{2}{3}) \qquad (0,\frac{2}{3}) \qquad (+1,\frac{2}{3}) & \rightarrow \quad SU(2)_I \text{ triplet} \\ \quad (-\frac{1}{2},-\frac{1}{3}) \qquad (\frac{1}{2},-\frac{1}{3}) & \rightarrow \quad SU(2)_I \text{ doublet} \\ \qquad (0,-\frac{4}{3}) & \rightarrow \quad SU(2)_I \text{ singlet} \end{cases}$$

which decomposes under $SU(3) \supset SU(2) \times U(1)$ as

$$6 = 3_{\frac{2}{3}} + 2_{-\frac{1}{3}} + 1_{-\frac{4}{3}}$$

and the $\bar{3}$ decomposes as

$$\bar{3} = 2_{-\frac{1}{3}} + 1_{\frac{2}{3}}$$

we get the $\bar{3}$ representation. Note that $\bar{3} \neq 3$, whereas in $SU(2)$ $\bar{2} = 2$.