from the previous lecture ...

<u>Lagrangian & Hamiltonian Mechanics</u>

In Newtonian mechanics we specify initial conditions for the position and velocity, and solve Newton's second order differential equations for the position and velocity as a function of time.

In Modern particle physics, we measure and compare initial and final energy and momentum. Energy and momentum are constants of the motion. Extract quantities that remain constant in the initial and final time and measure them experimentally.

In modern particle physics calculations are done in the framework of quantum field theories. A bridge between the "old" Newtonian mechanics and the "modern" particle physics is provided by the classical Lagrangian & Hamiltonian formulations of classical mechanics.

The Lagrangias $L = L(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n, t)$

The 2^{nd} order Euler–Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

We define the conjugate momentum

$$p_i = \frac{\partial L}{\partial \dot{q}_i}(\vec{q}, \dot{\vec{q}}, t)$$

An alternative formulation is provided by the Hamiltonian formulation

Hamilton change variables from configuration space to phases space

Configuration space : $(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n)$,

Phase space : $(q_1, \cdots, q_n, p_1, \cdots, p_n)$,

where $p_i = \dfrac{\partial L}{\partial \dot{q}_i}$

The tranformation is made by a Legendre transformation

$$H = \sum_{k=1}^{n} p_k \dot{q}_k - L(q_1, \cdots, q_{n-1}, \dot{q}_1, \cdots, \dot{q}_n, t) = H(q_k, p_k),$$

$$\text{where} \quad p_k = \frac{\partial L}{\partial \dot{q}_k}$$

Example: a particle in one dimension with constant energy E.

$$L = \frac{1}{2} m \dot{q}^2 - V(q)$$

$$p = \frac{\partial L}{\partial \dot{q}} = m \dot{q} \quad \Rightarrow \quad \dot{q} = \frac{p}{m}$$

$$H = p\dot{q} - L(q, \dot{q}) = p\dot{q} - \left(\frac{1}{2} m \dot{q}^2 - V(q) \right)$$

$$= \frac{p^2}{m} - \frac{m}{2} \frac{p^2}{m^2} + V(q) = \frac{p^2}{2m} + V(q)$$

In the Hamiltonian formalism we perform a Legendre transformation from configuration space to phase space that consist of the $2n$ Independent variables (q_i, p_i). The Hamiltionian is a function of these $2n$ variables and possibly of time, $H = H(q_i, p_i, t)$.

Legendre transformation from configuration to phase space

$$(q_i, p_i) \rightarrow 2n \text{ independent variables}$$
$$H = H(q_i, p_i, t).$$

The Hamilton equation of motion are obtained by taking

$$dH = \frac{\partial H}{\partial q_k} dq_k + \frac{\partial H}{\partial p_k} dp_k + \frac{\partial H}{\partial t} dt \tag{1}$$

Whereas from[1] $H = p_k \dot{q}_k - L$, we have

$$dH = p_k d\dot{q}_k + \dot{q}_k dp_k - \frac{\partial L}{\partial q_k} dq_k - \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k - \frac{\partial L}{\partial t} dt \tag{2}$$

[1] summation of k is implied.

with $\quad p_k = \frac{\partial L}{\partial \dot{q}_k} \quad$ and with Euler–Lagrange equations $\quad \frac{dp_k}{dt} - \frac{\partial L}{\partial q_k} = 0$

we get,

$$dH = \dot{q}_k dp_k - \frac{\partial L}{\partial q_k} dq_k - \frac{\partial L}{\partial t} dt \tag{3}$$

Comparing the coefficients of eqs. (1) and (3) yields the Hamilton equations of motion

$$\begin{aligned}
\frac{\partial H}{\partial p_k} &= \dot{q}_k \\
\frac{\partial H}{\partial q_k} &= -\frac{\partial L}{\partial q_k} = -\frac{dp_k}{dt} = -\dot{p}_k \\
\frac{\partial H}{\partial t} &= -\frac{\partial L}{\partial t}
\end{aligned}$$

We get $2n + 1$ 1^{st} order differential equations which are the Hamilton equations of motion.

When L does not depend explicitly on time $\Rightarrow$ $\frac{\partial L}{\partial t} = 0$

$$\Rightarrow \quad \frac{dH}{dt} = \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial t}$$
$$= \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} = 0$$

when $L \neq L(t) \quad \Rightarrow \quad H \neq H(t) \quad \Rightarrow \quad$ H is a constant of the motion.

When H does not depend explicitly on t,

H is identified with the conserved energy $H = E = \mathrm{constant}.$

Poisson brackets

Using Hamilton equations we can write for any canonical function $G(q_i, p_i, t)$

$$\begin{aligned}\frac{dG}{dt} &= \frac{\partial G}{\partial q_i}\dot{q}_i + \frac{\partial G}{\partial p_i}\dot{p}_i + \frac{\partial G}{\partial t}\\ &= \frac{\partial G}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial G}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial G}{\partial t}\end{aligned}$$

the Possion brackets are defined as

$$\{A, B\} = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}$$

<u>we can write</u> $\frac{dG}{dt} = \{G, H\} + \frac{\partial G}{\partial t}$

$$\Rightarrow \quad \text{if} \quad \frac{\partial G}{\partial t} = 0 \;\; \& \;\; \{G, H\} = 0 \Rightarrow \frac{dG}{dt} = 0$$

i.e. if G does not depend explicitly on time and the Poisson brackets of G with H vanish then G is conserved in time.

In Quantum mechanics the Poisson brackets are replaced by the commutator of the hermitian operators A and B, *i.e.*

$$\{A, B\} \;\rightarrow\; [A, B] \quad \text{where} \quad A\ , B \text{ are hemitian operators}$$

$$\Rightarrow \quad \frac{dA}{dt} = [A, H] + \frac{\partial \langle A \rangle}{\partial t} \quad \Rightarrow \quad \text{if} \;\; A \neq A(t) \;\; \& \;\; [A, H] = 0 \quad \Rightarrow \quad \frac{dA}{dt} = 0$$

$$\Rightarrow \quad A \quad A \text{ is a conserved operator}$$