MATH431 Modern Particle Physics Solutions 2

1.

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2$$

a.

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \quad , \quad g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix}$$

b.

$$\theta^\mu \to \theta^\mu + \epsilon\zeta^\mu(\theta,\phi)$$

We want to find functions

$$A(\theta,\phi) = \zeta^1(\theta,\phi)$$
$$\text{and } B(\theta,\phi) = \zeta^2(\theta,\phi)$$

such that ds^2 remains invariant under the transformations.

$$\theta \to \theta + \epsilon A(\theta,\phi)$$
$$\phi \to \phi + \epsilon B(\theta,\phi)$$
$$\sin\theta \to \sin(\theta+\epsilon A) \approx \sin\theta + \epsilon A\cos\theta$$
$$\sin^2\theta \to \sin^2\theta + 2\epsilon A\sin\theta\cos\theta + O(\epsilon^2)$$

Keeping terms to first order in ϵ

$$d\theta \to (1+\epsilon\frac{\partial A}{\partial\theta})d\theta + \epsilon\frac{\partial A}{\partial\phi}d\phi$$
$$d\phi \to \epsilon\frac{\partial B}{\partial\theta}d\theta + (1+\epsilon\frac{\partial B}{\partial\phi})d\phi$$
$$d\theta^2 \to (1+2\epsilon\frac{\partial A}{\partial\theta})d\theta^2 + 2\epsilon\frac{\partial A}{\partial\phi}d\theta d\phi$$
$$d\phi^2 \to (1+2\epsilon\frac{\partial B}{\partial\phi})d\phi^2 + 2\epsilon\frac{\partial B}{\partial\theta}d\theta d\phi$$
$$\sin^2\theta d\phi^2 \to \sin^2\theta(1+2\epsilon\frac{\partial B}{\partial\phi})d\phi^2 + \sin^2\theta 2\epsilon\frac{\partial B}{\partial\theta}d\theta d\phi + 2\epsilon A\sin\theta\cos\theta d\phi^2$$

we demand that ds^2 remains invariant.

$$d\theta^2 + \sin^2\theta d\phi^2 \to d\theta^2 + \sin^2\theta d\phi^2 + \underbrace{\cdots\cdots\cdots\cdots\cdots}_{\text{terms that vanish}}$$

demanding that the additional terms vanish we obtain the following constraints

$$d\theta^2 : \frac{\partial A}{\partial\theta} = 0 \Rightarrow A = A(\phi) = f'(\phi) = \frac{df}{d\phi}$$
$$d\phi^2 : \sin^2\theta\frac{\partial B}{\partial\phi} + A\sin\theta\cos\theta = 0 \Rightarrow \frac{\partial B}{\partial\phi} = -\frac{df}{d\phi}\frac{\cos\theta}{\sin\theta} \Rightarrow B = -f(\phi)\frac{\cos\theta}{\sin\theta} + g(\theta)$$
$$d\theta d\phi : \frac{\partial A}{\partial\phi} + \sin^2\theta\frac{\partial B}{\partial\theta} = 0 \Rightarrow f'' + \sin^2\theta\left(\frac{f(\phi)}{\sin^2\theta} + g'(\theta)\right) = 0$$

$$\Rightarrow f''(\phi) + f(\phi) = -\sin^2\theta g'(\theta) = \text{constant} = c$$

we solve the two differential equations for f and g

$$f'' + f = c \Rightarrow f = a\sin\phi + b\cos\phi + c$$
$$\sin^2\theta\frac{dg}{d\theta} = -c \Rightarrow g = c\frac{\cos\theta}{\sin\theta} + d$$

and obtain for A and B

$$A = f'(\phi) = a\cos\phi - b\sin\phi$$
$$B = -\frac{\cos\theta}{\sin\theta}(a\sin\phi + b\cos\phi + c) + c\frac{\cos\theta}{\sin\theta} + d = -\frac{\cos\theta}{\sin\theta}(a\sin\phi + b\cos\phi) + d$$

We are left with three parameters a, b and d, which we denote as α, β and γ respectively,

$$A = \alpha\cos\phi + \beta\sin\phi$$
$$B = -\frac{\cos\theta}{\sin\theta}(\alpha\sin\phi - \beta\cos\phi) + \gamma$$

we are left with three degrees of freedom. Note that we set $\beta = -b$.

c. we want to find the generator associated with each degree of freedom. The generators are given by summing

$$J \equiv \zeta^\mu \frac{\partial}{\partial\theta^\mu}$$
$$\alpha : J_1 = \cos\phi\frac{\partial}{\partial\theta} - \frac{\cos\theta}{\sin\theta}\sin\phi\frac{\partial}{\partial\phi}$$
$$\beta : J_2 = \sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\theta}{\sin\theta}\cos\phi\frac{\partial}{\partial\phi}$$
$$\gamma : J_3 = \frac{\partial}{\partial\phi}$$

calculating the commutation relations with the generators defined by $\tilde{J}_i = -iJ_i$ we then obtain

$$[\tilde{J}_i, \tilde{J}_j] = i\epsilon ijk\tilde{J}_k$$

which is the $SU(2)$ algebra.

d. the metric ds^2 is the metric on a surface of a sphere.

2.

a.

$$\vec{J}_\pm = \frac{1}{2}(\vec{J} + i\vec{K})$$

J_+ and J_- generate the algebra $SU(2)\otimes SU(2)^\dagger$. J_+^2 and J_-^2 are the Casimir operators of $SU(2)$ and $SU(2)^\dagger$, respectively, and are therefore invariants of the Lorentz group.

$$J_+^2 = \frac{1}{4}(J^2 - K^2 + 2i\vec{J}\cdot\vec{K})$$
$$J_-^2 = \frac{1}{4}(J^2 - K^2 - 2i\vec{J}\cdot\vec{K})$$

$$
\begin{aligned}
J^2 - K^2 &= 2(J^+2 + J_-^2) \\
\vec{J} \cdot \vec{K} &= -i(J_+^2 - J_-^2)
\end{aligned}
$$

Therefore $J^2 - K^2$ and $\vec{J} \cdot \vec{K}$ are Lorentz invariants as well, being the sum and difference of Lorentz invariants.

b. For the representation (j_1, j_2) of the $SU(2) \otimes SU(2)^\dagger$ algebra the number of states is $(2j_1 + 1)(2j_2 + 2)$.

The total spin is given by $j_1 + j_2$. Therefore the composition $j_1 \otimes j_2$ breaks under $SU(2)_J$ with the following spin states

$$
j_1 + j_2 \oplus j_1 + j_2 - 1 \oplus \cdots \oplus |j_1 - j_2|
$$

3.

In the rest frame of the particle, $E = m$ and $\vec{p} = 0$. Performing a boost along the x–axis, E and p^x transform as t and x, with $v = \tanh \eta$, $\gamma = \cosh \eta$ and the Lorentz transformations take the form

$$
\begin{aligned}
t &\longrightarrow (\cosh \eta) t + (\sinh \eta) x \\
x &\longrightarrow (\sinh \eta) t + (\cosh \eta) x
\end{aligned}
$$

The variable η is called the rapidity. Then, after a boost $E = m \cosh \eta$ and $p = m \sinh \eta$. Therefore,

$$
\frac{E + p}{E - p} = \mathrm{e}^{2\eta}.
$$

Performing another boost with rapidity η' in the same direction,

$$
\begin{aligned}
E &\longrightarrow E \cosh \eta' + p \sinh \eta' \\
p &\longrightarrow E \sinh \eta' + p \cosh \eta'
\end{aligned},
$$

so

$$
\begin{aligned}
\mathrm{e}^{2\eta} \longrightarrow & \frac{(E \cosh \eta' + p \sinh \eta') + (E \sinh \eta' + p \cosh \eta')}{(E \cosh \eta' + p \sinh \eta') + (E \sinh \eta' + p \cosh \eta')} \\
&= \mathrm{e}^{2\eta'} \left(\frac{E + p}{E - p} \right) = \mathrm{e}^{2\eta + 2\eta'}
\end{aligned}
$$