① $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = dx^2 + dy^2$

a) $g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

b) $x \rightarrow x + \epsilon A(x,y)$

$y \rightarrow y + \epsilon B(x,y)$

$$dx \rightarrow dx + \epsilon\left(\frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy\right) = \left(1 + \epsilon \frac{\partial A}{\partial x}\right) dx + \epsilon \frac{\partial A}{\partial y} dy$$

$$dy \rightarrow dy + \epsilon\left(\frac{\partial B}{\partial x} dx + \frac{\partial B}{\partial y} dy\right) = \left(1 + \epsilon \frac{\partial B}{\partial y}\right) dy + \epsilon \frac{\partial B}{\partial x} dx$$

$$dx^2 \Rightarrow \cancel{dx^2 + 2\epsilon \frac{\partial A}{\partial x}\frac{\partial A}{\partial y} dx dy} \quad \left(1 + 2\epsilon \frac{\partial A}{\partial x}\right) dx^2 + 2\epsilon \frac{\partial A}{\partial y} dx dy$$

$$dy^2 \rightarrow \left(1 + 2\epsilon \frac{\partial B}{\partial y}\right) dy^2 + 2\epsilon \frac{\partial B}{\partial x} dx dy$$

$$ds^2 = dx^2 + dy^2 = \left(1 + 2\epsilon \frac{\partial A}{\partial x}\right) dx^2 + 2\epsilon \frac{\partial A}{\partial y} dx dy$$

$$+ \left(1 + 2\epsilon \frac{\partial B}{\partial y}\right) dy^2 + 2\epsilon \frac{\partial B}{\partial x} dx dy$$

$\therefore$ to be invariant,

$$\left(1 + 2\epsilon \frac{\partial A}{\partial x}\right) = 1 \qquad \left(1 + 2\epsilon \frac{\partial B}{\partial y}\right) = 1$$

$$2\epsilon \frac{\partial A}{\partial y} + 2\epsilon \frac{\partial B}{\partial x} = 0$$

$$\Rightarrow \frac{\partial A}{\partial x} = 0 \qquad \therefore \; A \neq A(x) \qquad A = A(y)$$

$$\frac{\partial B}{\partial y} = 0 \qquad \therefore \; B \neq B(y) \qquad B = B(x)$$

$$\frac{\partial A}{\partial y} = -\frac{\partial B}{\partial x} = \text{Constant} = c$$

$$\therefore \; A(y) = cy + a$$

$$B(x) = -cx + b$$

~~'a' is a shift in the y-direction.~~

~~'b' is a shift in the x-direction~~

'a' is a shift in space

'b' is also a shift in space

'c' is a rotation.

(2) a) $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu$$

$$= -\frac{1}{4}(A_{\mu,\nu} - A_{\nu,\mu})(A^{\mu,\nu} - A^{\nu,\mu}) - j^\mu A_\mu$$

$$= -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(A_{\beta,\alpha} - A_{\alpha,\beta})(A_{\mu,\nu} - A_{\nu,\mu}) - j^\mu A_\mu$$

The Euler-Lagrange equations of motion,

$$\frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial A_{\mu,\nu}}\right) - \frac{\partial L}{\partial A_\mu} = 0$$

$$\frac{\partial L}{\partial A_\mu} = -j^\mu$$

$$\frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial A_{\mu,\nu}}\right) \Rightarrow \frac{\partial L}{\partial A_{p,q}} = -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(\delta^p_\alpha\delta^q_\beta - \delta^p_\beta\delta^q_\alpha)(A_{\mu,\nu} - A_{\nu,\mu})$$

$$+ (A_{\alpha,\beta} - A_{\beta,\alpha})(\delta^p_\mu\delta^q_\nu - \delta^p_\nu\delta^q_\mu)$$

$$= -\frac{1}{4}(\eta^{p\mu}\eta^{q\nu} - \eta^{q\mu}\eta^{p\nu})(A_{\mu,\nu} + A_{\nu,\mu}) + (\eta^{\alpha p}\eta^{\beta q} - \eta^{\alpha q}\eta^{\beta p})(A_{\beta,\alpha} - A_{\alpha,\beta})$$

$$= -\frac{4}{4}(A^{q,p} - A^{p,q}) = -F^{pq}$$

$$\therefore \cancel{\frac{\partial}{\partial x^\mu}(-F^{q}} - \partial_\mu F^{\mu\nu} = -j^\nu \Rightarrow \partial_\mu F^{\mu\nu} = j^\nu$$

$\partial_\mu j^\mu = \partial_\mu \partial_\nu F^{\mu\nu} = 0 \quad \therefore$ Current is Conserved

② b) New Legrangian,

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu + \frac{1}{2} M^2 A_\mu A^\mu$$

new $$\frac{\partial L}{\partial A_\mu} = -j^\nu \cancel{+\frac{1}{2} M^2 \eta^{ac} A_a A_c} + \frac{1}{2} M^2 \eta^{ac} \partial_\mu (A_a A_c)$$

$$= -j^\nu + \frac{1}{2} M^2 \eta^{ac} \left[\delta^a_{\ \mu} A_c + \delta^c_\mu A_a \right]$$

$$= -j^\nu + M^2 A^\mu$$

as $$\frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial A_{\mu,\nu}}\right) = -\partial_\mu F^{\mu\nu}$$

$$\therefore\ -\partial_\mu F^{\mu\nu} + j^\nu - M^2 A^\mu = 0$$

$$\partial_\mu F^{\mu\nu} + M^2 A^\mu - j^\nu = 0$$

$$\partial_\mu F^{\mu\nu} + M^2 A^\mu = j^\nu$$

$$\Rightarrow (\partial_\mu \partial^\mu + M^2) A^\mu = j^\nu$$