MATH 431 Class Test

1) $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = dx^2 + dy^2$

a) $g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

~~$x\varepsilon$~~

b) $x \to x + \varepsilon A(x, y)$

$y \to y + \varepsilon B(x, y)$

$$dx \to dx + \varepsilon\left(\frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy\right)$$

$$dy \to dy + \varepsilon\left(\frac{\partial B}{\partial x} dx + \frac{\partial B}{\partial y} dy\right)$$

$\Rightarrow \cancel{ds^2\,\varepsilon}$

$$dx^2 \to dx^2 + 2\varepsilon\left(\frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy\right) dx + O(\varepsilon^2)$$

$$dy^2 \to dy^2 + 2\varepsilon\left(\frac{\partial B}{\partial x} dx + \frac{\partial B}{\partial y} dy\right) dy + O(\varepsilon^2)$$

$= \cancel{ds\,\varepsilon^2}$

$\Rightarrow \cancel{ds^2 \to dx^2 + dy^2 + 2\varepsilon\left(\frac{\partial A}{\partial x} dx^2 + \frac{\partial B}{\partial y} dy^2\right.}$

$$\Rightarrow ds^2 \to dx^2 + dy^2 + 2\varepsilon\left(\frac{\partial A}{\partial x} dx^2 + \frac{\partial B}{\partial y} dy^2 + \left(\frac{\partial A}{\partial y} + \frac{\partial B}{\partial x}\right) dx\,dy\right) \cancel{+ O(\varepsilon^2)}$$

$+ O(\varepsilon^2)$

for dx^2: $\frac{\partial A}{\partial x} = 0 \Rightarrow A = A(y)$

for dy^2: $\frac{\partial B}{\partial y} = 0 \Rightarrow B = B(x)$

for $dx\,dt$: $\frac{\partial A}{\partial y} + \frac{\partial B}{\partial x} = 0$

$\Rightarrow \frac{\partial A}{\partial y} = -\frac{\partial B}{\partial x} = c$

$\Rightarrow A(y) = cy + a$

$B(x) = -cx + b$

for $c = b = 0$ and $a \neq 0$

$x \to x + \varepsilon a$

$y \to y$

which corresponds to a translation along the x-axis (in space)

for $c = a = 0$ $b \neq 0$

$x \to x$

$y \to y + \varepsilon b$

which corresponds to a translation along the y axis (In space)

for $a = b = 0$ $c \neq 0$

$x \to x + \varepsilon c y$

$y \to y - \varepsilon c x$

which corresponds to a translation in the x-y plane (In space)

2) $L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^{\mu} A_{\mu}$

$$L = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - j^\mu A_\mu$$

$$= -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial_\mu A_\nu - \partial_\nu A_\mu) - j^\mu A_\mu$$

$$\cancel{\frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial(\partial_\mu A_\nu)}\right) - \frac{\partial L}{\partial A_\mu} = 0}$$

$$\cancel{\frac{\partial L}{\partial A_\mu}} \quad \frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial(\partial_\mu A_\nu)}\right) - \frac{\partial L}{\partial A_\nu} = 0$$

$$\frac{\partial L}{\partial A_\nu} = -j^\nu$$

$$\cancel{\frac{\partial L}{\partial(\partial_\mu A_\nu)}}$$

$$\frac{\partial L}{\partial(\partial_\sigma A_\rho)} = -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}\Big((\delta^\rho_\alpha\delta^\sigma_\beta - \delta^\rho_\beta\delta^\sigma_\alpha)(\partial_\nu A_\mu - \partial_\mu A_\nu) + (\partial_\beta A_\alpha - \partial_\alpha A_\beta)(\delta^\rho_\mu\delta^\sigma_\nu - \delta^\rho_\nu\delta^\sigma_\mu)\Big)$$

$$= -\frac{1}{4}\Big((\eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu})(\partial_\nu A_\mu + \partial_\mu A_\nu) + (\eta^{\alpha\rho}\eta^{\beta\sigma} - \eta^{\alpha\sigma}\eta^{\beta\rho})(\partial_\alpha A_\beta - \partial_\beta A_\alpha)$$

$$= -(\partial^\rho A^\sigma - \partial^\sigma A^\rho) = -F^{\rho\sigma}$$

$$\Rightarrow \frac{\partial L}{\partial(\partial_\mu A_\nu)} = F^{\nu\mu} \quad \Rightarrow \frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial(\partial_\mu A_\nu)}\right) = \partial_\mu F^{\nu\mu}$$

$$\Rightarrow \partial_\mu F^{\nu\mu} = -\partial_\mu F^{\mu\nu}$$

$$-\partial_\mu F^{\mu\nu} + j^\nu = 0 \quad \Rightarrow \quad \partial_\mu F^{\mu\nu} = j^\nu$$

to show that j^ν is conserved it can be shown that $F^{\mu\nu}$ is invariant under a transformation

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

consider transformation $A^\mu \to \tilde{A}^\mu = A^\mu + \partial^\mu \Lambda$

$$\Rightarrow \tilde{F}^{\mu\nu} = \partial^\mu \tilde{A}^\nu - \partial^\nu \tilde{A}^\mu = \partial^\mu(A^\nu - \partial^\nu \Lambda) - \partial^\nu(A^\mu - \partial^\mu \Lambda)$$

$$= \partial^\mu A^\nu - \partial^\nu A^\mu - \partial^\mu \partial^\nu \Lambda + \partial^\nu \partial^\mu \Lambda = \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu}$$

$\Rightarrow \partial_\mu F^{\mu\nu}$ is invariant under transformation, and j^ν is conserved

b) $L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu + \frac{1}{2} m^2 A_\mu A^\mu$

$$\frac{\partial L}{\partial A_\nu} = -j^\nu + \frac{1}{2} m^2 A^\nu$$

$$\frac{\partial L}{\partial x^\mu}\left(\frac{\partial L}{\partial(\partial_\mu A_\nu)}\right) = \partial_\mu F^{\nu\mu} = -\partial_\mu F^{\mu\nu}$$

$$\Rightarrow \partial_\mu F^{\mu\nu} + \frac{1}{2} m^2 A^\nu = j^\nu$$

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu$$

$$\partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu + \frac{1}{2} m^2 A^\nu = j^\nu$$