MATH431 - Modern Particle Physics
Set Work: Sheet 4;

1. Prove

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\mathbf{j} = 0,$$

$$\text{where} \quad \mathbf{j} = -\frac{i\hbar}{2m}\left[\psi^* \nabla \psi - \psi \nabla \psi^*\right]$$

using the standard non-relativistic Schrödinger equation.

2. (i) The Lagrangian density for a real scalar field is given by

$$\mathcal{L}_0 = \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}m^2\phi^2$$

Show that the Hamiltonian H_0 is given by

$$H_0 = \tfrac{1}{2}\int [\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2] d^3x.$$

(ii) Show that if we take the usual canonical commutation relations,

$$[\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = i\hbar\delta(\mathbf{x} - \mathbf{x}'),$$
$$[\phi(\mathbf{x}, t), \phi(\mathbf{x}', t)] = 0,$$
$$[\pi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = 0,$$

the equations of motion are obtained from

$$i\hbar\dot{\phi} = [\phi, H_0] \quad \text{and} \quad i\hbar\dot{\pi} = [\pi, H_0] \ .$$

(iii) Consider the Lagrangian density

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I,$$

with

$$\mathcal{L}_I = -\frac{\lambda_3}{3!}\phi^3 - \frac{\lambda_4}{4!}\phi^4 \ .$$

Derive the equation of motion from

$$\partial^\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)}\right] - \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$

3. Assume a five-dimensional space-time $(t, \mathbf{x}, y)$, where $x = (t, \mathbf{x})$ are the usual four-dimensional space-time coordinates and y is the coordinate of an additional compact extra dimension, $-R/2 \le y \le R/2$.

Consider the free Klein-Gordon equation (KG) in this space-time,

$$(\partial_\sigma \partial^\sigma + m^2)\phi \ = \ 0\,,$$

where $\sigma = 0, 1, 2, 3, 4$ and $\mathbf{x} = \{x^1, x^2, x^3\}$ and $y = \{x^4\}$, *i.e.* $\partial_\sigma\partial^\sigma \equiv \partial_\mu\partial^\mu - \partial^2/\partial y^2$, with $\partial_\mu\partial^\mu = \partial_0^2 - \nabla^2$ the usual d'Alembert operator. The general solution of KG equation is obtained by a Fourier expansion and by imposing appropriate boundary conditions.

Show that

$$\phi(x,y) \;=\; \sum_{n=1}^{\infty} \phi_n(x)\,\mathrm{cs}\left(\frac{n\pi y}{R}\right) ,$$

where $\mathrm{cs}(n\pi y/R) = \cos(n\pi y/R)$ if n is odd and $\mathrm{cs}(n\pi y/R) = \sin(n\pi y/R)$ for even n, is a solution of the KG equation, provided that the Fourier coefficients $\phi_n(x)$ are solutions of a four-dimensional Klein-Gordon equation. Determine the mass spectrum as seen by a four-dimensional observer, and show that for $m = 0$ the masses are equally spaced. What is this infinite set of massive particles called?