May 2018 EXAMINATIONS

Introduction to Modern Particle Physics

TIME ALLOWED: Two and half hours

INSTRUCTIONS TO CANDIDATES: In this paper bold-face quantities like **x** represent three-dimensional vectors.

Full marks can be obtained for complete answers to FIVE questions. Only the best FIVE answers will be counted.

1. Consider the infinitesimal line element on a two dimensional surface

$$ds^2 = g_{\mu\nu} d\theta^\mu d\theta^\nu = d\theta^2 + \sin^2\theta d\phi^2$$

(a convenient notation is $\theta^\mu \equiv (\theta, \phi)$ $(\mu = 1, 2)$)

(a) Write the metric $g_{\mu\nu}$ in an explicit matrix form.
Write $g^{\mu\nu}$ in matrix form. [4 marks]

(b) Consider the set of infinitesimal transformations of the form

$$\theta^\mu \rightarrow \theta^\mu + \epsilon \zeta^\mu(\theta, \phi) \ ,$$

for which the line element ds^2 is invariant and where ϵ is an infinitesimal constant. Derive the conditions that the functions ζ^1 and ζ^2 must satisfy for ds^2 to remain invariant.

[16 marks]

2. (a) Show that if the Hamiltonian is independent of a generalised coordinate q_0, then the conjugate momentum p_0 is a constant of the motion. Such coordinates are called *cyclic coordinates*. Briefly describe two examples of physical systems which have a cyclic coordinate.

[6 marks]

(b) Show that in three-dimensional spherical polar coordinates the Hamiltonian of a particle of mass m moving in a potential $V(\vec{x})$ is

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + V(\vec{x})\,.$$

Now show that $p_\phi = constant$ if $\partial V/\partial\phi \equiv 0$ and give a physical interpretation of this result.

[14 marks]

3. (a) Two-particle states are defined by $|\mathbf{p}_1, \mathbf{p}_2\rangle \; = \; a^\dagger(\mathbf{p}_1) a^\dagger(\mathbf{p}_2)|0\rangle$. Show that

$$\langle \mathbf{p}'_1, \mathbf{p}'_2 | \mathbf{p}_1, \mathbf{p}_2\rangle \; = \; (2\pi)^6 (2p_1^0)(2p_2^0) \left[\delta(\mathbf{p}_1 - \mathbf{p}'_1)\delta(\mathbf{p}_2 - \mathbf{p}'_2) + \delta(\mathbf{p}_1 - \mathbf{p}'_2)\delta(\mathbf{p}_2 - \mathbf{p}'_1) \right] .$$

[6 marks]

(b) Show that the number operator, defined by

$$N \; := \; \frac{1}{(2\pi)^3} \int \frac{\mathrm{d}^3\mathbf{p}}{2p^0} \, a^\dagger(\mathbf{p}) a(\mathbf{p}) \, ,$$

satisfies

$$[N, a^\dagger(\mathbf{p})] \; = \; a^\dagger(\mathbf{p}) \, ,$$

and hence

$$N \, |\mathbf{p}_1 \ldots \mathbf{p}_n\rangle \; = \; n \, |\mathbf{p}_1 \ldots \mathbf{p}_n\rangle \, .$$

[14 marks]

4. (a) Given the four-vector current density $J_V^\mu = \bar{\psi}\gamma^\mu\psi$, derive the current conservation equation, $\partial_\mu J_V^\mu = 0$, by using the covariant form of the Dirac equation and the relation $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$.

[6 marks]

(b) Show that the axial four-vector current density $J_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$ is not conserved but instead satisfies the covariant equation

$$\partial_\mu J_A^\mu \; = \; 2im \, \bar{\psi}\gamma^5\psi \, .$$

[6 marks]

(c) Derive the *Gordon decomposition* of the Dirac transition current,

$$\bar{\psi}_f \gamma^\mu \psi_i \; = \; \frac{1}{2m} \, \bar{\psi}_f \left[(p_f + p_i)^\mu + i\sigma^{\mu\nu}(p_f - p_i)_\nu \right] \psi_i \, ,$$

where $\sigma^{\mu\nu} = \frac{1}{2} i (\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$.

[8 marks]

5. Let A_μ be the electromagnetic vector potential. The electromagnetic field strength tensor is then defined as

$$F_{\mu\nu} \;=\; \partial_\mu A_\nu - \partial_\nu A_\mu \,.$$

(a) Show that the covariant Maxwell equations (in the absence of sources) in four-vector notation can be derived from the electromagnetic Lagrangian given by

$$L_{\text{e.m.}} \;=\; -\frac{1}{4}\, F_{\alpha\beta}F^{\alpha\beta} \,.$$

[10 marks]

(b) Show that $L_{\text{e.m.}}$ is invariant under the transformation $A_\mu \to A_\mu - \partial_\mu \Lambda$, where Λ is a scalar function.

[4 marks]

(c) Show that by imposing a local $U(1)$ symmetry a mass term for the photon is forbidden.

[6 marks]

6.

(a.) Show that orbital angular momentum is a constant of the motion for a free nonrelativistic particle. [7 marks]

(b.) Show that orbital angular momentum is not a constant of the motion for a free Dirac particle. [7 marks]

(c.) Show that total angular momentum of a free Dirac particle is a constant of the motion. [6 marks]

7. Consider the simple unitary group $SU(3)$.

(a.) How many diagonal generators of the Lie algebra are there? Write down a representation of the diagonal generators in terms of 3×3 hermitian matrices.

[3 marks]

(b.) What is the dimension of the group? Write down a representation of all generators in terms of 3×3 hermitian matrices.

[3 marks]

(c.) What is the fundamental representation of $SU(3)$? Write down its decomposition in terms of a maximal subgroup.

[3 marks]

(d.) Draw the graphic illustration of the fundamental representation, indicating clearly the eigenvalues of each state under the diagonal generators.

[3 marks]

(e.) Find the product and the decomposition under the maximal subgroup of the fundamental times the anti-fundamental representations of $SU(3)$.

[4 marks]

(f.) Find the product and the decomposition under the maximal subgroup of the fundamental times the fundamental representations of $SU(3)$.

[4 marks]