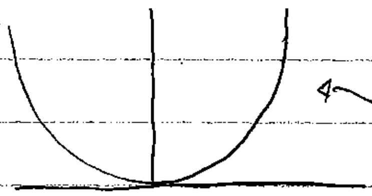
In the case of the KG. E.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$$

This looks like $V(\phi) \sim \frac{1}{2} k \phi^2$

$V(\phi)=\frac{1}{2}m^2\phi^2$



the potential is a function of Φ

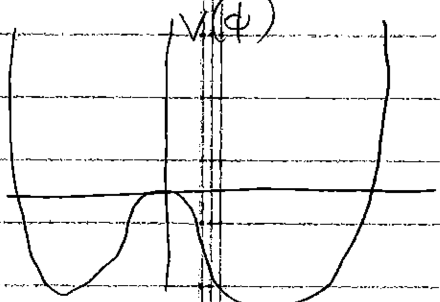
classically the particle is in a potential well. if it has initial energy E it will oscillate about the vacuum $V(\phi)=0$

The lowest point that the field can be in is the vacuum.

So far we are describing a free field

Suppose that we want to describe a field which is not free in a potential.

$$V(\phi) = A\phi^4 + B\phi^2 \qquad A>0,\ B<0.$$



In Newtonian mechanics this would correspond to a double well.

Here the point $V(\phi)=0$ is no longer the vacuum.

$$\frac{\partial V(\phi)}{\partial \phi} = 4A\phi^3 + 2B\phi = 0$$

$$(2A\phi^2 + B)\phi = 0$$

$$\phi = 0, \quad \phi = \pm\sqrt{\frac{-B}{2A}}$$

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The connection between the quantized field and its particle interpretation is seen by looking at the Fourier transformed field.

$$\phi(x) = \frac{1}{(2\pi)^4} \int d^4p \; \tilde{\phi}(p) \, e^{-ip\cdot x}.$$

$$\tilde{\phi}(p) = \int d^4x \; e^{ip\cdot x} \; \phi(x).$$

for $\phi(x)$ to satisfy the K.G. equation we must have,

$$(\partial^2 + m^2)\phi = \frac{1}{(2\pi)^4} \int (m^2 - p^2) \, \tilde{\phi}(p) e^{-ip\cdot x} d^4p =$$

i.e. $(p^2 - m^2)\,\tilde{\phi}(p) = 0$

i.e $\tilde{\phi}(p) \neq 0$ only when $p^2 = m^2$

$$\tilde{\phi}(p) = (2\pi)\,\delta(p^2 - m^2)\, f(p) =$$

$$p^0 = \pm\sqrt{\underline{p}^2 + m^2}$$

we may set: $f(\vec{p}) = \Theta(p^0) f_+(\vec{p}) + \Theta(-p^0) f_-(\vec{p})$

$$\Theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0. \end{cases}$$

From: the properties of the Delta function we have, 17/2/06. 2 / MPS. 63 / Friday / Abingdon /

$$\int f(x)\,\delta(x-a)\,dx = f(a)$$

$$\int f(x)\,\delta(\lambda x-\lambda a)\,dx = \int f\left(\frac{y}{\lambda}\right)\delta(y-\lambda a)\frac{dy}{\lambda} = \frac{f(a)}{\lambda}.$$

$y=\lambda x$

$\frac{1}{\lambda}dy = dx$

$$\int_\alpha^\beta f(x)\,\delta(F(x))\,dx = \qquad y=F(x) \quad x=F^{-1}(y)$$

$$dy = F'(x)dx \quad dx=\frac{1}{F'(F^{-1}(y))}dy.$$

$$= \int_{F(\alpha)}^{F(\beta)} f(F^{-1}(y))\,\delta(y)\frac{1}{F'(F^{-1}(y))}dy = \sum_i \frac{F(a_i)}{|F'(a_i)|} \quad \text{where } y=F(a_i)=0$$

$y=F(a)=0$

hence $\delta(P^2-m^2) = \delta(P_0^2-(\vec{P}^2+m^2)) =$

$$= \frac{1}{2P_0}\delta\left(P_0-(\vec{P}^2+m^2)^{1/2}\right) + \frac{1}{2P_0}\delta\left(P_0+(\vec{P}^2+m^2)^{1/2}\right).$$

$$\phi(x,t) = \frac{1}{(2\pi)^4}\int dt \int \frac{d^3\vec{p}}{2p^0}\left(e^{-ip\cdot x}f_+(\vec{p}) + e^{ip\cdot x}f_-(\vec{p})\right),$$

Here $p^0 = \sqrt{\vec{p}^2+m^2}$

$$f_+(\vec{p}) = f\left(+\sqrt{\vec{p}^2+m^2},\ +\vec{p}\right)$$

$$f_-(\vec{p}) = f\left(-\sqrt{\vec{p}^2+m^2},\ -\vec{p}\right)$$

The f_+ term correspond to positive energy states

The f_- term correspond to negative energy states.

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so far the fourier decomposition that we discussed is classical.
i.e. we didn't yet impose commutation relations.

if we impose the commutation relations $[\Pi, \phi] = \delta(x - x')$

$$[\phi, \phi] = 0 = [\Pi, \Pi].$$

Then the amplitudes of the fourier modes become annihilation and creation operators.

$$f_-(\vec{p}) = (f_+(\vec{p}))^+ = a^+(\vec{p})$$
$$f_+(\vec{p}) = a(\vec{p}).$$

it can then be shown that $a(\vec{p})$ and $a^+(\vec{p})$ obey the the commutation relation.

$$[a(\vec{p}), a^+(\vec{p}')] = 2p^0 \delta(\vec{p} - \vec{p}')(2\pi)^3$$
$$[a(\vec{p}), \vec{a}(\vec{p}')] = 0 = [a^+(\vec{p}), a^+(\vec{p}')]$$

This are similar to the commutation relation of harmonic o.

$[a, a^+] = 1 \quad [a, a] = 0 \quad [a^+, a^+] = 0.$

i.e. the quantum field $\phi(x)$ creats and annihilates particle states with momentum $\vec{p}$,

The vacuum is defined by

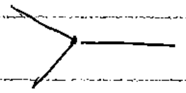
$$a(\vec{p})|0\rangle = 0 \qquad \forall \vec{p} \qquad \langle 0|0\rangle = 1$$

1-particle state $\quad |\vec{p}\rangle = a^+(\vec{p})|0\rangle$

2-particle states $\quad |\vec{p}_1, \vec{p}_2\rangle = a^+(p_1)\, a^+(p_2)|0\rangle$

17/2/06, 4 / and so forth.

We can now see how we can use this formalism to particles and their interactions.

$\phi(x)$: $a(p)$ $a^\dagger(p)\rangle$ $\quad \phi^3$  $\quad \phi^4$ 

We developed a diagramatic representation of interactions

→ Feynman diagrams.

The interactions that we described so far are by using a single scalar field.

In nature we are familiar so far with

Gravity, E & M, weak, strong

$s=$	$+2$	$+1$	$+1$	$+1$
	Graviton	Photon	$W^{\pm}, Z$	gluons
$m=$	0	0	$80\,\text{GeV}$	0
		$U(1)$ ×	$SU(2)$ ×	$SU(3)$

how can we describe these interactions?

S.M. → $SU(3)_c \times SU(2)_L \times U(1)_Y$ → local gauge interaction

In the modern language of elementary particles interactions correspond to invariances of the Lagrangian under some symmetry.

Interaction ⟷ invariance under a local gauge symmetry

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symmetries of the Lagrangian correspond to conserved currents.

consider. $\mathcal{L}_{KG} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m\phi^2$

it is invariant under $\phi \to -\phi$. $\Rightarrow$ discrete symmetry.

consider 2 scalar fields with mass m.

(**) $$L = \frac{1}{2}\left[\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2\right] - \frac{1}{2}m\left(\phi_1^2 + \phi_2^2\right),$$

with the transformation: $\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$, $\alpha \to$ global constant.

The Lagrangian (**) is invariant under this rotation $\to$

$\alpha \to$ continuous parameter. $\to$ global continuous symmetry $\alpha \neq \alpha(x)$

consider the complex field $\Phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$

$\Phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$

The Lagrangian in terms of Φ. $\Phi^\dagger\Phi = \frac{1}{2}(\phi_1^2 + \phi_2^2)$.

(***) $$L = (\partial_\mu\Phi)^\dagger\partial^\mu\Phi - \Phi^\dagger\Phi \qquad (\phi_1, \phi_2 \text{ Real})$$

in terms of Φ the rotation becomes.

$$\Phi = \frac{1}{\sqrt{2}}\left[(\cos\alpha\,\phi_1 + \sin\alpha\,\phi_2) + i(-\sin\alpha\,\phi_1 + \cos\alpha\,\phi_2)\right] =$$

$$= \frac{1}{\sqrt{2}}\left[(\cos\alpha - i\sin\alpha)\phi_1 + i(\cos\alpha - i\sin\alpha)\phi_2\right]$$

$$= e^{-i\alpha}\,\Phi$$

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hence! $\Phi \to e^{-i\alpha}\,\Phi$

$\Phi^* \to e^{i\alpha}\,\Phi^*$

The continuous symmetry in terms of Φ is a

under phase transformation.

The symmetry is a global $U(1)$ symmetry

$\to$ arbitrary choice of the phase. $\to$ continuous global symmetry

what happens if $\alpha = \alpha(x)$?

$\Phi(x) \to e^{-i\alpha(x)}\,\Phi(x)$.

$\partial_\mu \Phi = \left(-i\partial_\mu \alpha(x)\,\phi(x) + \partial_\mu \phi(x)\right) e^{-i\alpha(x)}$.

$\hookrightarrow g^{\mu\nu}\left[\partial_\mu\left(e^{-i\alpha(x)}\Phi\right)\right]^+\left[\partial_\nu\left(e^{-i\alpha(x)}\phi\right)\right] + m^2\left(e^{-i\alpha(x)}\Phi\right)^+\left(e^{i\alpha(x)}\Phi\right)$.

$= g^{\mu\nu}\left[\partial_\mu\phi(x) - i\left(\partial_\mu\alpha(x)\right)\phi(x)\right]^+\left[\partial_\nu\phi - i\left(\partial_\nu\alpha(x)\right)\Phi\right] + m^2\phi^+\phi$

if $\alpha \neq \alpha(x)$ the derivative terms $\partial\alpha(x)$ drops

we have to fix the original Lagrangian to get a Lagrangian which is invariant under local phase transformations $\alpha = \alpha(x)$.

we redefine the derivative as $\partial_\mu \to \partial_\mu + a_\mu(x)$

where $a_\mu(x)$ is a function of x.

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Under the local phase transformations

$$a_\mu \to a'_\mu(x) = a_\mu(x) + i\,\partial_\mu \alpha(x)$$ $\alpha(x) \to$ scalar function of x.

Then $(\partial_\mu + a_\mu(x))\Phi(x) \to (\partial_\mu + a'_\mu(x))(e^{-i\alpha(x)}\Phi) =$

$$= e^{-i\alpha(x)}(\partial_\mu + a_\mu(x) + i\partial_\mu\alpha(x) - i\partial\alpha(x))\Phi(x)$$

$$= e^{-i\alpha(x)}(\partial_\mu + a_\mu(x))\Phi(x)$$

Hence we now get that the Lagrangian in invariant under the local phase transformation

$$\Phi(x) \to \Phi'(x) = e^{-i\alpha(x)}\Phi(x)$$

Requiring local phase invariance $\to$ introduce $a_\mu(x)$
$\to$ local gauge field.

$\to$ The electromagnetic field.

All interactions in the S.M. are gauge interactions.

$\to$ invariance under local phase transformations + internal symmetries.

electromagnetic interaction $\leftrightarrow$ continuous local symmetry.

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The Electromagnetic field.

Maxwell's equations $(\epsilon_0 = \mu_0 = c = 1)$.

$$\vec{\nabla}\cdot\vec{E} = \rho_{em}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\nabla\times\vec{B} = \vec{J}_{em} + \frac{\partial \vec{E}}{\partial t}.$$

define $J^{\mu}_{e.m} = (\rho_{em}, \vec{J}_{e.m})$.

In terms of scalar and vector potential $V, \vec{A}$,

$$\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} \qquad \vec{B} = \vec{\nabla}\times A$$

So $(\vec{\nabla}\times\vec{\nabla}\times A)_i = \epsilon_{ijk}\partial_j\epsilon_{klm}\partial_l A_m =$

$= \epsilon_{kij}\epsilon_{klm}\partial_j\partial_l A_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j\partial_l A_m =$

$= \partial_i\partial_j A_j - \partial_j\partial_j A_i = (\vec{\nabla}(\vec{\nabla}\cdot\vec{A}))_i - \vec{\nabla}^2 A_i =$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A} = \vec{J}_{em} - \frac{\partial^2\vec{A}}{\partial t^2} - \frac{\partial}{\partial t}\vec{\nabla}V$$

$$\nabla\cdot\vec{E} = -\nabla^2 V - \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = \rho_{e.m.}$$

OR $$\left(\frac{\partial^2 A}{\partial t^2} - \nabla^2 A\right) + \nabla\frac{\partial V}{\partial t} + \vec{\nabla}(\vec{\nabla}\cdot A) = J_{e.m.}$$

$$\left(\frac{\partial^2}{\partial t^2}V - \nabla^2 V\right) - \frac{\partial}{\partial t}\frac{\partial V}{\partial t} - \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = \rho_{e.m.}$$

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We can write this in four vector notation.

defining 4-vector potential $A^{\mu} = (V, \vec{A})$; $A_{\mu} = (V, -\vec{A})$

$$\partial_{\nu}\partial^{\nu} A^{\mu} - \partial^{\mu}(\partial_{\nu} A^{\nu}) = \partial_{\nu}(\partial^{\nu}A^{\mu} - \partial^{\mu}A^{\nu})$$

$$= \partial_{\nu} F^{\nu\mu} = J^{\mu}_{em}.$$

where we defined the electromagnetic field strength tensor

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} = -F_{\nu\mu}.$$

$$F_{0i} = \partial_0 A_i - \partial_i A_0 = (-\vec{\nabla} V - \partial_t \vec{A})_i = E_i$$

$$F_{ij} = \partial_i A_j - \partial_j A_i = -(\vec{\nabla} \times A)_k = -\epsilon_{kij}\partial_i A_j = -B_k$$

hence

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

The electromagnetic field strength tensor and hence Maxwell's equations are invariant under the gauge transformations

$$A^{\mu} \to A'^{\mu} = A^{\mu} + \partial^{\mu}\chi. \qquad \chi \to \text{scalar function}$$

$$F^{\mu\nu} \to F'^{\mu\nu} = \partial^{\mu}(A^{\nu} + \partial^{\nu}\chi) - \partial^{\nu}(A^{\mu} + \partial^{\mu}\chi)$$

$$= \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu} + \partial^{\mu}\partial^{\nu}\chi - \partial^{\nu}\partial^{\mu}\chi$$

$$= \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu} = F^{\mu\nu}$$

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$= F^{\mu\nu}$

$\Rightarrow$ we can always choose $\partial_\mu A^\mu = 0$ (Lorentz gauge),

if $\partial_\mu A^\mu = f \neq 0 \rightarrow \partial_\mu (A^\mu + \partial^\mu \chi) = 0 \Rightarrow \partial_\mu \partial^\mu \chi = -f$.

$\Rightarrow$ in free space ($J^\mu = 0$) we have $\partial_\nu \partial^\nu A^\mu = 0$

$\rightarrow$ Massless K.G.E for each component of A^μ.

$\rightarrow$ A^μ is a 'wave function' of Photon.

$\rightarrow$ A^μ is a four vector $\rightarrow$ photon has spin 1.

Plane-wave solutions: $A^\mu = \epsilon^\mu e^{-ik\cdot x} = \epsilon^\mu e^{-i(\omega t - \vec{k}\cdot\vec{r})}$.

ϵ^μ = polarization four vector.

$k^\mu = (\omega, \vec{k}) \rightarrow$ wave-4 vector.

From the wave equation $k \cdot k = 0 \Rightarrow \omega^2 = k^2 \Leftrightarrow E^2 = p^2 c^2$ (m=0)

From the Lorentz gauge condition.

$\partial_\mu A^\mu = 0 \Rightarrow \epsilon \cdot k = 0 \Rightarrow \epsilon^0 = \dfrac{\vec{\epsilon}\cdot\vec{k}}{\omega}$

$\Rightarrow \epsilon'^\mu = \epsilon^\mu + a k^\mu$ is equivalent to ϵ^μ for any a = constant

$\Rightarrow$ choose $\epsilon_0 = 0$. $\Rightarrow \epsilon_\mu k^\mu = \vec{\epsilon}\cdot\vec{k} = 0$

$\Rightarrow$ for $\vec{k} = (0, 0, k_z) \Rightarrow \epsilon^\mu_x = (0, 1, 0, 0)$ $\epsilon^\mu_y = (0, 0, 1, 0)$

2-polarization states. $\epsilon^\mu_{R,L} = (0, 1, \pm i, 0)/\sqrt{2}$ ← circular polarization.

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Electromagnetic interactions.

We introduce e.m. interactions via
the minimal substitution in the eq. of motion.

$$E \to E - eV \qquad \vec{P} \to \vec{P} - e\vec{A}$$

Relativistically $\quad p^\mu \to p^\mu - eA^\mu \qquad \partial^\mu \to \partial^\mu + ieA^\mu$.

The K.G. equation becomes.

$$(\partial_\mu + ieA_\mu)(\partial^\mu + ieA^\mu)\phi + m^2\phi = 0.$$

$$(\partial_\mu\partial^\mu + m^2)\phi = -ie\left[A_\mu\partial^\mu\phi + \partial_\mu(A^\mu\phi)\right] + e^2A_\mu A^\mu\phi$$

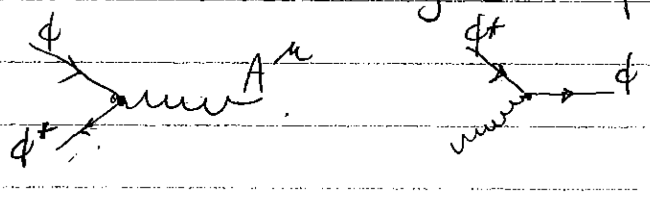
we saw that the minimal coupling prescription and the gauge condition $A^\mu \to A^\mu + \partial^\mu\chi$ are the same as the local phase invariance

$$\phi(x) \to e^{-i\alpha(x)}\phi(x) \quad \to \text{local } U(1) \text{ symmetry}$$

The conserved current is now.

$$J^\mu = i\left(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\right) - 2e\,A^\mu(x)\,\phi^*(x)\,\phi(x).$$

The second term provides the coupling of the scalar field to the electromagnetic field



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The Dirac eq.

The Schrödinger eq. $i\hbar \frac{\partial}{\partial t}\psi = H\psi$

is linear in $i\hbar\frac{\partial}{\partial t}$

since $P_\mu = i\hbar\partial_\mu$ & $P^2 = E^2 - \vec{P}^2 = m^2$

The K.G.E is quadratic in $\frac{\partial}{\partial t}$

Dirac wanted to find a relativistically covariant equation which is linear in $i\hbar\frac{\partial}{\partial t}$ i.e. linear in Energy → Hamiltonian → translation of time.

→ linear in time + Relativistically covariant ⇒ linear in $-i\hbar\vec{\nabla}$

$-i\hbar\vec{\nabla}$ → the quantum generator of spatial translations.

Relativistically $E^2 = \vec{p}^2c^2 + m^2c^4 \Rightarrow E = \pm\sqrt{p^2c^2 + m^2c^4}$

$$i\hbar\frac{1}{c}\frac{\partial}{\partial t}\psi(\vec{r},t) = \pm\sqrt{\vec{p}^2 + m^2c^2}$$

We have to get rid of the $\sqrt{}$

Write $\sqrt{p^2 + m^2c^2} = \alpha_i p_i + \beta m \quad (*)$

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DIRAC → find α_i, β such that (*) hold ?

⇒ covariant eq. linear in $i\hbar\frac{\partial}{\partial t}$ & $-i\hbar\vec{\nabla}$?

Take: Square of (*).

$$\vec{p}^2 + m^2c^2 = \sum p_i p_i + m^2 c^2 = (\alpha_i p_i + \beta mc)^2 =$$

$$= (\alpha_i p_i + \beta mc)(\alpha_j p_j + \beta mc) =$$

$$= \alpha_i \alpha_j p_i p_j + (\alpha_i \beta + \beta \alpha_i) p_i mc + \beta^2 m^2 c^2$$

For the equation to hold we must impose the following requirements

① $\beta^2 = 1$

② $\alpha_i \beta + \beta \alpha_i = 0$ ← no linear term in p_i in the square.

These conditions can hold only if α_i, β are matrices.

with α_i, β → anti-commuting matrices.

③ $\alpha_i \alpha_j + \alpha_j \alpha_i = 0$ $\quad i \neq j$.

④ $\alpha_i^2 = 1$ $\quad i = j$.

Can we find α_i, β that satisfy these conditions?

1) 2×2 matrices

Check Pauli matrices $\quad \alpha_i = \sigma_i$

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$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\sigma_i^2 = \underline{1}$

$\sigma_i \sigma_j + \sigma_j \sigma_i = 0$ for $i \neq j$

But: we lack a 2×2 β matrix that satisfies ① & ②

2) 3x3 matrices

no solution $\longleftrightarrow$ solution must be of even order.

Proof: Assume an odd order solution.

Assume A β matrix which is diagonal; as in the 2x2 case we can always diagonalize at least one matrix.

$\beta = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ $\beta^2 = 1 \Rightarrow \lambda_i = \pm 1$

we don't know how many λ_i are positive or negative?

Proove that $\overline{Tr}\,\beta = 0$

In that case: # of +1 eigenvalues = # of −1 eigenvalues

$\Rightarrow$ β must be even.

$\alpha_i \beta + \beta \alpha_i = 0$ / $\cdot\, \alpha_i$

$\alpha_i \beta \alpha_i + \beta \alpha_i^2 = 0 \Rightarrow \alpha_i \beta \alpha_i + \beta = 0$

$\Rightarrow \overline{Tr}\,\beta = -\overline{Tr}\,\alpha_i \beta \alpha_i = -\overline{Tr}\,\alpha_i \alpha_i \beta = -\overline{Tr}\,\beta$

$\Rightarrow \overline{Tr}\,\beta = -\overline{Tr}\,\beta = 0 \Rightarrow \beta_{n\times n}$ with n-even.

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At order 4 i.e. 4×4 matrices there is a solution.

$$\alpha_i = \left(\begin{array}{c|c} 0 & \sigma_i \\ \hline \sigma_i & 0 \end{array}\right) \qquad \beta = \left(\begin{array}{c|c} I_{2\times 2} & \\ \hline & I_{2\times 2} \end{array}\right)$$

where σ_i are 2×2 Pauli matrices

The Dirac equation

(**) $$i\hbar \frac{1}{c}\frac{\partial}{\partial t}\psi(\vec{x},t) = \left(-i\hbar\, \vec{\alpha}\cdot\vec{\nabla} + \beta mc\right)\psi(\vec{x},t)$$

The wave function $\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$ is a 4-component vector

2 components are spin ↑ spin ↓ particles
2 components are spin ↑ spin ↓ antiparticles

consequences: (1) Dirac equation predicts the existence of antiparticles
(2) time and space derivatives are linear.

multiply (**) by β. ; $\beta^2 = 1$

$$i\hbar\beta\frac{1}{c}\frac{\partial}{\partial t}\psi = \left(-i\hbar\,\beta\vec{\alpha}\cdot\vec{\nabla} + \beta^2 mc\right)\psi$$

OR. $$i\hbar\Big(\underbrace{\beta}_{\gamma^0}\frac{1}{c}\frac{\partial}{\partial t}\psi + \underbrace{\beta\vec{\alpha}}\cdot\vec{\nabla}\Big)\psi = mc\psi$$

γ^0 $\quad$ $+\gamma^i = \beta\alpha^i$ $\quad$ $\gamma^i = +\beta\alpha^i = \alpha^i\beta$

hence $$i\hbar\left(\gamma^0\frac{\partial}{\partial ct} + \gamma^i\partial_i\right)\psi = mc\psi.$$

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OR $i\hbar(\eta^{\mu\nu}\gamma_\mu \partial_\nu)\psi = i\hbar\gamma^\nu\partial_\nu\psi = i\hbar\not{\partial}\psi = mc\psi$

$$\gamma^\mu p_\mu \psi = mc\psi$$

$$\not{p}\psi = mc\psi \to (\not{p} - mc)\psi = 0 \leftarrow \text{free Dirac eq.}$$

setting $\hbar = c = 1 \Rightarrow (i\not{\partial} - m)\psi = 0$

Lowest order Dirac γ^μ matrices $\to$ 4x4 $\to$ massive particles

massless particles $\to$ no constraint on $\beta \Rightarrow$ 2x2 solution $\to$ Pauli matrices

The Dirac eq. is of the form $i\hbar \frac{\partial}{\partial t}\psi = H\psi$

$$H = -i\hbar c\,\vec{\alpha}\cdot\vec{\nabla} + \beta mc^2$$

hermiticity $\to H = H^+ \iff \alpha_i = \alpha_i^+ \quad \beta = \beta^+$

$$\Rightarrow \gamma^{0+} = \gamma^0 \quad \alpha^i = \gamma^0\gamma^i = (\gamma^0\gamma^i)^+ = \gamma^{i+}\gamma^{0+} = \gamma^{i+}\gamma^0$$

$$\Rightarrow \gamma^0\gamma^{i+}\gamma^0 = \gamma^i$$

Summarize: $\gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu+} \qquad \mu = 0,1,2,3$

Together with $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$

are the two properties that define the Dirac γ-matrices

Representation: $\gamma^0 = \begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \qquad i = 1,2,3,$

$\sigma_i \to$ Pauli matrices

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solutions of Dirac eq. $\to$ 4-component objects $\to$ spinors. $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$
$\to$ (not 4-vectors).

Each component obeys the K.G.E. by construction.

$$E^2 = H^2 = \vec{p}^{\,2} + m^2 \Rightarrow E^2 \, I \, \psi = (\vec{p}^{\,2} + m^2) \, I \, \psi$$

<u>spin of Dirac particles</u>.

How do we prove that the Dirac eq. correspond spin ½ ?

<u>Show</u>: Exist operator $\vec{S}$ such that $\vec{J} = \vec{L} + \vec{S}$ is a constant of the motion.

And $(\hbar = 1)$ $\quad \vec{S}^{\,2} = S(S+1) = \frac{3}{4} I \quad (\hbar^2)$

<u>Note</u>: $\vec{L} = \vec{r} \times \vec{p}$ is not a constant of the motion:

$$H = \beta m + \vec{\alpha} \cdot \vec{p} = \beta m + \alpha_i p_i$$

$$\vec{L} = L_x \hat{\imath} + L_y \hat{\jmath} + L_z \hat{k} \quad \text{e.g. } L_z = (x p_y - y p_x)$$

$$[L_z, H] = [x, H] p_y - [y, H] p_x = i \alpha_x p_y - i \alpha_y p_x = i (\vec{\alpha} \times \vec{p})_z$$

<u>in general</u> $\quad [\vec{L}, H] = i \vec{\alpha} \times \vec{p} \neq 0.$

$\Rightarrow$ we need $[\vec{S}, H] = -i \vec{\alpha} \times \vec{p}$

This is true if $\vec{S} = \frac{1}{2} \vec{\Sigma}$ with $\Sigma_j = \begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}$

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<u>we want</u>: $[S_j \, , \alpha_i p_i + \beta m] = -i(\vec{\alpha} \times P)_j$.

$$= [S_j, \alpha_i] p_i + [S_j, \beta] m$$

$$[S_j, \beta] = \begin{pmatrix} A & \\ & A \end{pmatrix}\begin{pmatrix} I & \\ & -I \end{pmatrix} - \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} A & \\ & A \end{pmatrix} = 0 \Rightarrow S_j = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

$$[S_j, \alpha_i] \sim \alpha_k \sim \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}\begin{pmatrix} A & \\ & A \end{pmatrix} = \begin{pmatrix} 0 & \sigma_j\sigma_i - \sigma_i\sigma_j \\ \sigma_j\sigma_i - \sigma_i\sigma_j & 0 \end{pmatrix}$$

$$= -i \begin{pmatrix} 0 & 2\sigma_k \\ 2\sigma_k & 0 \end{pmatrix} = -2i \, \alpha_k .$$

Defining: $S_j = \frac{1}{2}\begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}$ we get the desired result.

<u>Then</u>, $\vec{S}^2 = \frac{1}{4}\left(\Sigma_x^2 + \Sigma_y^2 + \Sigma_z^2\right) = \frac{3}{4} I \Rightarrow$ spin $= 1/2$.

<u>Furthermore</u>, $[L_i \, , S_j] = 0$

<u>Magnetic Moment of the Dirac eq</u>

In an electromagnetic field we make the usual minimal substitutions

$$H \to H - eV \qquad \vec{p} \to \vec{p} - e\vec{A}$$ electric charge.

In the Dirac eq. we obtain

$$H - eV = \vec{\alpha} \cdot (\vec{p} - e\vec{A}) + \beta m,$$

Note: we no longer get the K.G. eq. when squaring

$$(H - eV)^2 = \sum_{jk} \alpha_j \alpha_k (p_j - eA_j)(p_k - eA_k) + m^2 =$$

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$$(\vec{p}-e\vec{A})^2+m^2-e\sum_{j\neq k}(\alpha_j\alpha_k\, p_j A_k + \alpha_j\alpha_k\, A_j p_k)$$

For $j\neq k$: $\alpha_j\alpha_k = \begin{pmatrix}0 & \sigma_j\\ \sigma_j & 0\end{pmatrix}\begin{pmatrix}0 & \sigma_k\\ \sigma_k & 0\end{pmatrix} = \begin{pmatrix}\sigma_j\sigma_k & 0\\ 0 & \sigma_j\sigma_k\end{pmatrix} = i\epsilon_{jkl}\begin{pmatrix}\sigma_l & \\ & \sigma_l\end{pmatrix} = i\epsilon_{jkl}\Sigma_l$

$$p_j A_k f = (-i\nabla_j A_k)f = A_k(-i\nabla_j f) - i(\nabla_j A_k)\, f = A_k p_j - i\nabla_j A_k$$

$$\epsilon_{jkl}\Sigma_l \nabla_j A_k = \Sigma_l\, \epsilon_{ljk}\nabla_j A_k = \Sigma_l(\vec{\nabla}\times\vec{A})_l = \vec{\Sigma}\cdot\vec{B}$$

Thus we get: $-e\sum_{j\neq k}\alpha_j\alpha_k\, p_j A_k + \alpha_j\alpha_k\, A_j p_k =$

$$= -e\sum_{j\neq k}\alpha_j\alpha_k\, A_j p_k + \alpha_j\alpha_k\, A_k p_j - e\vec{\Sigma}\cdot\vec{B}$$

$$= -e\sum_{j\neq k}(\alpha_j\alpha_k + \alpha_k\alpha_j)A_j p_k - e\vec{\Sigma}\cdot\vec{B} = -e\vec{\Sigma}\cdot\vec{B}$$

Hence: $(H-eV)^2 = (\vec{p}-e\vec{A})^2+m^2-e\vec{\Sigma}\cdot\vec{B}$

$$(H-eV) = m\left(1+\frac{(\vec{p}-e\vec{A})^2-e\vec{\Sigma}\cdot\vec{B}}{m^2}\right)^{1/2}$$

NR-limit: $\simeq m + \frac{1}{2m}(\vec{p}-e\vec{A})^2 - \frac{e}{2m}\vec{\Sigma}\cdot\vec{B}$

This correspond to a magnetic moment.

$$\mu = \frac{e}{m}\vec{S} = g_e\left(\frac{e}{2m}\right)\vec{S}$$

where $g_e = 2$ (Experiment => 2.0023193...) quantum field theory corrections

Superb success of Dirac's eq. (g-2) of muon -> contemporary

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Dirac density and current

To give a probabilistic interpretation to the Dirac wave-function ψ we have to construct a conserved current j^μ.

we have, $(i\gamma^\mu \partial_\mu - m)\psi = 0$ $\quad (*)$

$$\psi^+(-i\gamma^{\mu+}\overleftarrow{\partial}_\mu - m) = 0$$

using $\gamma^{\mu+} = \gamma^0\gamma^\mu\gamma^0$

$\Rightarrow \psi^+(-i\gamma^0\gamma^\mu\gamma^0\overleftarrow{\partial}_\mu - m) = 0 \quad *\gamma^0$

$\Rightarrow \psi^+\gamma^0(-i\gamma^\mu\overleftarrow{\partial}_\mu - m\gamma^0) = 0$

we define the Dirac adjoint. $\overline{\psi} = \psi^+\gamma^0$

$\Rightarrow \overline{\psi}(i\gamma^\mu\overleftarrow{\partial}_\mu + m) = 0$ $\quad (**)$

multiplying $(*)$ by $\overline{\psi}$ from the left and $(**)$ by ψ from the right.

we get. $i\overline{\psi}\gamma^\mu\overrightarrow{\partial}_\mu\psi + i\overline{\psi}\gamma^\mu\overleftarrow{\partial}_\mu\psi = i\partial_\mu(\overline{\psi}\gamma^\mu\psi) = 0.$

So $\quad j^\mu = \overline{\psi}\gamma^\mu\psi \Rightarrow \partial_\mu j^\mu = 0.$

$\Rightarrow j^\mu$ is our conserved current.

Then : $\rho = j^0 = \overline{\psi}\gamma^0\psi = \psi^+(\gamma^0)^2\psi = \psi^+\psi = \sum^4 |\psi_\alpha|^2 \geq 0$

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$$J^j = \bar{\psi}\,\gamma^j \psi = \psi^\dagger \gamma^0 \gamma^j \psi = \psi^\dagger \alpha^j \psi.$$

→ ρ is positive definite; but we still get negative energy solutions → antiparticle → multi-state - quantum field.

end of lecture (19)

Solutions of the Dirac equation $\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$.

Each component obeys the K.G.E.

$$\psi = u(E,\vec{p})\, e^{-ip\cdot x} = u(E,\vec{p})\, e^{-i(Et - \vec{p}\cdot\vec{x})}.$$

→ Positive energy plane wave.

$$(i\gamma^\mu \partial_\mu - m)\psi = (\gamma^\mu p_\mu - m)\, u = 0.$$

Writing $u = \begin{pmatrix}\phi\\ \chi\end{pmatrix} = \begin{pmatrix}\phi_1\\ \phi_2\\ \chi_1\\ \chi_2\end{pmatrix}$

$\gamma^j = \gamma_0 \alpha_j = \begin{pmatrix}I & \\ & -I\end{pmatrix}\begin{pmatrix}0 & \sigma_j\\ \sigma_j & 0\end{pmatrix} = \begin{pmatrix}0 & \sigma_j\\ -\sigma_j & 0\end{pmatrix}$

$$(\gamma^0 E - \vec{\gamma}\cdot\vec{p} - m)\begin{pmatrix}\phi\\ \chi\end{pmatrix} = \left[\begin{pmatrix}I & \\ & -I\end{pmatrix}E - \begin{pmatrix}0 & \sigma_j\\ -\sigma_j & 0\end{pmatrix}p_j - m\begin{pmatrix}I & \\ & I\end{pmatrix}\right]\begin{pmatrix}\phi\\ \chi\end{pmatrix}$$

$$= \begin{pmatrix}E-m & -\vec{\sigma}\cdot\vec{p}\\ \vec{\sigma}\cdot\vec{p} & -E-m\end{pmatrix}\begin{pmatrix}\phi\\ \chi\end{pmatrix} = 0 \Rightarrow (E-m)\phi = \vec{\sigma}\cdot\vec{p}\,\chi$$

$$\vec{\sigma}\cdot\vec{p}\,\chi = +(E+m)\phi.$$

$$\Rightarrow \chi = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,\phi.$$

Recall that $\vec{S} = \frac{1}{2}\vec{\Sigma} = \frac{1}{2}\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}$ $\quad S_z = \begin{pmatrix}1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1\end{pmatrix}$

hence. $\phi = N\begin{pmatrix}1\\ 0\end{pmatrix}$ for spin up along z-axis

$\phi = N\begin{pmatrix}0\\ 1\end{pmatrix}$ for spin down along z axis.

Also have: $\vec{\sigma}\cdot\vec{p}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}p_x + \begin{pmatrix}0 & -i\\ +i & \end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}p_y + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}p_z = \begin{pmatrix}p_z\\ p_x + ip_y\end{pmatrix}$

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similarly $\vec{\sigma}\cdot\vec{p}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}p_x - ip_y\\ p_z\end{pmatrix}$

Thus $u^{\uparrow} = N\begin{pmatrix}1\\0\\ \frac{p_z}{E+m}\\ \frac{p_x+ip_y}{E+m}\end{pmatrix}$ $\quad u^{\downarrow} = N\begin{pmatrix}0\\1\\ \frac{p_x-ip_y}{E+m}\\ \frac{p_z}{E+m}\end{pmatrix}$.

Normalization is calculated from $\rho = \psi^{+}\psi = u^{+}u = 2E$

$2E$ - particles per unit volume.

This gives $N^2\cdot\left[1 + \frac{p_x^2+p_y^2+p_z^2}{(E+m)^2}\right] = 2E$

Using $\vec{p}^{\,2} = E^2 - m^2$ gives $N^2\left[1 + \frac{(E-m)\cancel{(E+m)}}{(E+m)^{\cancel{2}}}\right] = 2E$

$= N^2\left(\frac{2E}{E+m}\right) = 2E \implies N = \sqrt{E+m}$

For a particle in the rest frame $p^{\mu} = (m, 0, 0, 0)$. $\vec{p} = 0$.

we get $U^{\uparrow} = N\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ $\quad U^{\downarrow} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$

in NR-limit $U^{\uparrow} \to$ spin-up $\quad u^{\downarrow}$ - spin down. fields

For antiparticle of 4-momentum $(E, \vec{p})$ we need A solution with $p^{\mu} \to (-E, -\vec{p})$.

$$\psi = v(E,\vec{p})\, e^{+ip_{\mu}x^{\mu}} = v(E,\vec{p})\, e^{i(Et - \vec{p}\cdot\vec{x})}$$

From the Dirac equation: $(i\gamma^{\mu}\partial_{\mu} - m)\psi = (-\gamma^{\mu}p_{\mu} - m)\, v(E,\vec{p}) = \left[\begin{pmatrix}-I & 0\\ 0 & I\end{pmatrix}E + \begin{pmatrix}0 & \sigma_j\\ -\sigma_j & 0\end{pmatrix}p_j - m\right]$

$$= \begin{pmatrix}-E-m & \vec{\sigma}\cdot\vec{p}\\ -\vec{\sigma}\cdot\vec{p} & E-m\end{pmatrix}\begin{pmatrix}\phi\\ \chi\end{pmatrix} = 0$$

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$$\Rightarrow \vec{\sigma}\cdot\vec{p}\,\chi = (E+m)\,\phi \quad \Rightarrow \quad \phi = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi.$$

$$\vec{\sigma}\cdot\vec{p}\,\phi = (E-m)\,\chi$$

ie the 4-momentum spin is reversed thus

$$v^{\uparrow} = N\begin{pmatrix} \frac{p_x - i p_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \qquad v^{\downarrow} = N\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + i p_y}{E+m} \\ 1 \\ 0 \end{pmatrix}.$$

Charge conjugation

consider the K.G.E for a charged particle in an electromagnetic field

$$\vec{p} \to \vec{p} - e\vec{A}$$

$$E \to E - eV$$

$$(E - eV)^2 = (\vec{p} - e\vec{A})^2 + m^2$$

In quantum mechanics. $E \to i\hbar\partial_t \quad \vec{p} \to -i\hbar\vec{\nabla}$.

$$\Rightarrow \left(i\hbar\partial_t - eV\right)^2 \phi(x,t) = \left(-i\hbar\vec{\nabla} - e\vec{A}(x,t)\right)^2 \phi(x,t) + m^2\phi(x,t)$$

The complex conjugate eq.

$$\left(-i\hbar\partial_t - eV\right)\phi^* = \left(i\hbar\vec{\nabla} - e\vec{A}\right)^2\phi^* + m^2\phi^*$$

$$\Rightarrow \left(i\hbar\partial_t + eV\right)\phi^* = \left(-i\hbar\vec{\nabla} + e\vec{A}\right)\phi^* + m^2\phi^*$$

Hence if $\phi = e^{-i(Et - \vec{p}\cdot x)} = e^{-iEt + i p\cdot x}$.

is a solution of the K.G.E with $E > 0$ $\vec{p}$

26/2/06.2 | MPS.85 | Sunday/Abingdon $\phi^* = e^{-i(E(-t) - \vec{p}\cdot\vec{x})}$

ϕ^* is a solution of the K.G.E with $E > 0$

$p \to -\vec{p}$ And $e \to -e$ $m = m_0$

This operation is: $\phi \to \phi^*$

$e \to -e$

called charge conjugation, C

The K.G.E is invariant under charge conjugation

The Dirac equation is also invariant under charge conjugation.

$\psi \to$ negative energy solution.

Transformation $\psi \to \psi^c = C\psi^* \to$ charge conjugation C.

such that ψ^c is a positive energy solution with $e \to -e$.

To find C!

Write!

$$\gamma^\mu (i\partial_\mu - eA_\mu)\psi - m\psi = 0 \leftarrow \text{Dirac eq.}$$

$$\to \gamma^\mu(\partial_\mu + ieA_\mu)\psi + im\psi = 0$$

$$\Rightarrow \gamma^{\mu *}(\partial_\mu - ieA_\mu)\psi^* - im\psi^* = 0$$

$$-C\gamma^{*\mu}C^{-1}(\partial_\mu - ieA_\mu)\psi^c + im\psi^c = 0$$

Hence we need: $C\gamma^{*\mu}C^{-1} = -\gamma^\mu$

i.e. $\gamma^\mu C = -C\gamma^{*\mu}$

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Since all γ^μ are real except γ^2 (which is purely imaginary) in our standard representation, we can take,

$$C = i\gamma^2 = i\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & \begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix} \\ \begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

For free particles we have $\quad v^{\uparrow c} = u^{\uparrow} \qquad v^{\downarrow c} = -u^{\downarrow}$

Parity Invariance

Similarly to charge conjugation we can also extract the transformation properties of the Dirac wave function under Parity transformations.

define: $$\psi(\vec{r},t) \to \psi^P(\vec{r},t) = \underline{P}\,\psi(-\vec{r},t)$$

$\overset{?}{\text{invariance}}$ → we want to find $\underline{P}$ such that ψ^P is also a solution

Dirac eq. → $$(i\gamma^\mu\partial_\mu - m)\,\psi(\vec{r},t) = 0$$

→ $$(\gamma^\mu\partial_\mu + im)\,\psi(\vec{r},t) = 0.$$

→ $$(\gamma^0\partial_0 \mp \vec{\gamma}\cdot\vec{\nabla} + im)\,\psi(r,t) = 0.$$

$\vec{r}\to-\vec{r}$ → $$(\gamma^0\partial_0 - \vec{\gamma}\cdot\vec{\nabla} + im)\,\psi(-\vec{r},t) = 0$$

$\psi\to\psi^P$ → $$\left(P\gamma^0P^{-1}\frac{\partial}{\partial t} - P\vec{\gamma}\cdot P^{-1}\cdot\vec{\nabla} + im\right)\psi^P(r,t) = 0.$$

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hence for invariance to hold we need

$$P\gamma^0 P^{-1} = \gamma^0$$
$$P\gamma^j P^{-1} = -\gamma^j \qquad (j = 1,2,3)$$

$\Rightarrow$ $\underline{P}\gamma^0 = \gamma^0 \underline{P} \qquad P\gamma^j = -\gamma^j \underline{P}$

These relations are satisfied by $\underline{P} = \gamma_0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

For a particle at rest $\psi = u(m, \vec{0})e^{-imt}$

$$\psi^P = \underline{P}\psi = +\psi$$

For an antiparticle at rest

$$\psi = v(m, \vec{0})\, e^{+imt}$$
$$\psi^P = \underline{P}\psi = -\psi$$

$\Rightarrow$ Particles and antiparticles have opposite intrinsic parity.

For K.G.E the parity transformation is,

$$\phi(\vec{r}, t) \rightarrow \underline{\phi}^P(\vec{r}, t) = \phi(-\vec{r}, t)$$

Since $\phi(\vec{r}, t)$ is a scalar function under L.T. $\phi'(\vec{r}', t') = \phi(\vec{r}, t)$

follows from K.G.E $\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right) \phi(\vec{r}, t) = 0$

$\underline{P} \Rightarrow \left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right) \phi'(-\vec{r}, t) = 0$

hence ϕ, ϕ' are solutions of the same eq

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In the case of the Dirac eq. the scalar is:

$$\Phi = \bar{\psi}\psi = \psi^{+}\gamma^{0}\psi$$

Check: $\Phi(\vec{r},t) = \psi^{+}(\vec{r},t)\gamma^{0}\psi(\vec{r},t)$

$$\Phi^{P}(\vec{r},t) = \psi^{+}(-\vec{r},t)\gamma^{0+}\gamma^{0}\underbrace{\gamma^{0}\psi}_{P\psi(-\vec{r},t)}(-\vec{r},t) =$$

$$= \psi^{+}(-\vec{r},t)\ \gamma^{0}\ \psi(-\vec{r},t) = \Phi(-\vec{r},t).$$

$\rightarrow$ $\bar{\psi}\psi$ transforms as a scalar under parity trans.

Similarly. J^{μ} is a true vector.

$$J^{\mu}(\vec{r},t) = \psi^{+}(\vec{r},t)\gamma^{0}\gamma^{\mu}\psi(\vec{r},t).$$

$$J^{P\mu}(\vec{r},t) = \psi^{+}(-\vec{r},t)\gamma^{0+}\gamma^{0}\gamma^{\mu}\gamma^{0}\psi(-\vec{r},t)$$

but $\gamma^{0+}\gamma^{0}\gamma^{\mu}\gamma^{0} = \gamma^{\mu}\gamma^{0} = \gamma^{0}\gamma^{\mu}$ for $\mu = 0$.

$= -\gamma^{0}\gamma^{\mu}$ for $\mu = 1,2,3$.

Hence! $J^{P0}(\vec{r},t) = J^{0}(-\vec{r},t)$ $\quad \vec{J}^{P}(\vec{r},t) = -\vec{J}(-\vec{r},t).$

As we would expect from a true 4-vector. $(t,\vec{x}) \xrightarrow{P} (t,-\vec{x})$

we define the matrix

$$\gamma^{5} = i\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3}\gamma^{4} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \leftarrow \text{in our Representati}$$

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We will see that weak interactions involve the axial current.

$$J_A^{\mu} = \bar{\psi}\,\gamma^{\mu}\gamma^5\psi = \psi^{\dagger}\gamma^0\gamma^{\mu}\gamma^5\psi$$

under Parity transformations

$$J_A^{P\mu} = \psi^{\dagger}(-\vec{r},t)\,\gamma^{0\dagger}\gamma^0\gamma^{\mu}\gamma^5\gamma^0\,\psi(-\vec{r},t) = \psi^{\dagger}(-\vec{r},t)\,\gamma^{\mu}\gamma^5\gamma^0\psi$$

now $\{\gamma^{\mu},\gamma^5\} = 0$ for $\mu = 0, 1, 2, 3.$

$$(\gamma^5)^2 = I.$$

Hence

$$J_A^{P0}(\vec{r},t) = -J_A^{0}(-\vec{r},t) \qquad \vec{J}_A^{P}(\vec{r},t) = \vec{J}_A(-\vec{r},t)$$

As expected for an axial vector

similarly: $\Phi_p = \bar{\psi}\gamma^5\psi$ is a pseudo-scalar.

$$\Phi_p^P(\vec{r},t) = \psi^{\dagger}(-\vec{r},t)\,\overset{\gamma^0\gamma^0}{\downarrow}\,\gamma^5\gamma^0\,\psi(-\vec{r},t) =$$

$$= -\bar{\psi}(-\vec{r},t)\,\gamma^5\,\psi(-\vec{r},t) =$$

$$= -\Phi_p(-\vec{r},t)$$

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$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$

Massless Dirac Particles

The Dirac eq. $H\psi = i\hbar \frac{\partial}{\partial t}\psi = (-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi$

For $m=0$ the positive energy free particle solutions are

$$\psi = u(E,\vec{p})\, e^{-i(Et - \vec{p}\cdot\vec{x})}$$

For $m = 0 \Rightarrow E = |\vec{p}|$ and $u = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$ gives

$$E\cdot I\, u = \vec{\alpha}\cdot\vec{p}\; u = \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix} u$$

hence $\begin{pmatrix} |\vec{p}| & -\vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & |\vec{p}| \end{pmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0 \Rightarrow \begin{array}{l} \vec{\sigma}\cdot\vec{p}\,\chi = |\vec{p}|\,\phi \\ \vec{\sigma}\cdot\vec{p}\,\phi = |\vec{p}|\,\chi \end{array}$

$\Rightarrow \chi = \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\phi$ & $\phi = \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\chi \Rightarrow \chi = \left(\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\right)^2 \chi$

$\Rightarrow \left(\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\right)^2 = I$

$\Rightarrow \Lambda = \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}$ is the helicity operator with $\Lambda^2 = 1$

and eigenvalues $\Lambda = \pm 1 \to$ spin $\frac{\text{along}}{\text{Against}}$ $\vec{p}$ $\to$ $\frac{\text{Right}}{\text{Left}}$ handed

helicity operators projection of spin on $\frac{\vec{p}}{|\vec{p}|}$

$\Rightarrow$ For massless particles $m=0$ the 2 component spinor χ, ϕ are eigenstates of the helicity operators

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For massless particles. helicity is Lorentz invariant,

note that if ψ represents a massless particle the

$$\gamma^5 \psi = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} \Lambda\phi \\ \phi \end{pmatrix} = \Lambda \psi \qquad (\Lambda^2 = 1)$$

hence γ^5 is the helicity operator for massless particles. (minus helicity for massless antiparticles).

in the case of massless particles we can decompose the Dirac eq. into two equations for the two helicity eigenstates.

we can introduce the basis:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$$

in this basis (chiral).

$$\left(\gamma^0 E - \gamma^j p_j\right)\begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} 0 & |\vec{p}| - \vec{\sigma}\cdot\vec{p} \\ |\vec{p}| + \vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0$$

hence (in this basis) $\quad \dfrac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\chi = +1\chi \qquad \dfrac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\phi = -\phi.$

therefore in this basis χ and ϕ are the eigenstates of the helicity operator with eigenvalue +1 and −1 respectively.

$$\chi = \psi_R \qquad \phi = \psi_L.$$

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ψ_L and ψ_R are two component spinors.

they transform as $(\frac{1}{2}, 0)$ and $(0\ \frac{1}{2})$

Representations of the Lorentz group.

A Dirac spinor can be written as

$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ ψ_L, ψ_R are called Weyl spinors

we define the operators $P_{L,R} = \left(\frac{1 \pm \gamma_5}{2}\right)$

In the chiral basis we have.

$P_L = \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}$ $P_L^2 = P_L$

$P_R = \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $P_R^2 = P_R$.

and $P_R P_L = P_L P_R = 0$, $P_L + P_R = \frac{1+\gamma_5}{2} + \frac{1-\gamma_5}{2} = 1$

$P_L \gamma^\mu = \gamma^\mu P_R$ $P_R \gamma^\mu = \gamma^\mu P_L$

hence we have $P_L \psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix} = \psi_L$

$P_R \psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix} = \psi_R$.

Furthermore from $P_R P_L = 0$ we have $P_R \psi_L = 0$

$P_L \psi_R = 0$

The Weak interations were observed experimentally to have the Vector – Axial vector form, (V–A) i.e.

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$$(J^\mu - J_A^\mu)_{fi} = \bar{\psi}_f \gamma^\mu (1-\gamma^5)\psi_i$$

If i is a massless particle then $(1-\gamma_5)\psi_i$ vanishes for helicity $+1$, i.e. only left-handed fields interact. The same applies to particle f since

$$\bar{\psi}_f \gamma^\mu (1-\gamma^5)\psi_i = \psi_f^+ \gamma^0 (1+\gamma_5)\gamma^\mu \psi_i = \psi_f^+ (1-\gamma_5)\gamma^0\gamma^\mu \psi_i$$

$$= \left[(1-\gamma_5)\psi_f\right]^+ \gamma^0\gamma^\mu \psi_i.$$

Hence this is non vanishing only if the f-particle is a left-handed field.

In the Standard Model only left-handed fields interact via the weak interactions.

The Lagrangian density that gives the Dirac eq. of motion.

$$\mathcal{L}_D = \bar{\psi} i \gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi.$$

$$= \bar{\psi}(P_L^2 + P_R^2) i\gamma^\mu \partial_\mu \psi - m\bar{\psi}(P_L^2 + P_R^2)\psi =$$

$$= \psi^+ P_R \gamma^0 i \gamma^\mu \partial_\mu P_R \psi + \psi^+ P_L \gamma^0 i \gamma^\mu \partial_\mu P_L \psi - m\psi^+ P_R \gamma^0 P_L \psi$$

$$- m\psi^+ P_L \gamma^0 P_R \psi =$$

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$$= \psi_R^\dagger \gamma^0 i\gamma^\mu \partial_\mu \psi_R + \psi_L^\dagger \gamma^0 i\gamma^\mu \partial_\mu \psi_L - m\left(\psi_R^\dagger \gamma^0 \psi_L + \psi_L^\dagger \gamma^0 \psi_R\right)$$

$$= \bar{\psi}_R i\gamma^\mu \partial_\mu \psi_R + \bar{\psi}_L i\gamma^\mu \partial_\mu \psi_L - m\left(\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R\right),$$ Dirac mass term

we see that the kinetic terms containing the Derivatives involve $L \leftrightarrow L$ $R \leftrightarrow R$ terms

<u>whereas:</u> the mass terms involve

$L \leftrightarrow R$ $R \leftrightarrow L$ terms, ← Dirac mass term.

This is a crucial result for modern particle physics

Majorana fields

So far we encountered Weyl and Dirac spinors.

A Majorana spinor is a Dirac spinor in which ψ_L and ψ_R are not independent,

<u>Rather</u> | $\psi_M = \begin{pmatrix} \psi_L \\ i\sigma^2 \psi_L^* \end{pmatrix}$ → same # of D.o.f. as Weyl spinor.

A Majorana spinor is invariant under charge conjugation.

$$\psi_M^c = \psi_M.$$

with a Majorana spinor $\psi_R = i\sigma_2 \psi_L^*$

Majorana spinor → neutral field → its own anti particle

⇒ $m \bar{\psi}_M \psi_M$.

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classification of elementary particles

we assembled some of the ingredients that are needed to classify elementary particles

We saw: { spin: 0 - scalars ½ - fermions +1 gauge bosons +2 - gravit

Poincare { mass

charge - electric - weak - strong.

The charges depend on the particle interactions...

Additional properties : light | heavy

don't interact strongly → leptons | Hadrons → interact strongly

1940's | electron, neutrino | Proton neutron → Chadwi

fermions | e ν_e | P ← stable $\tau \geq 10^{32}$ year | N ~ $\tau_{1/2}$ ~15 min

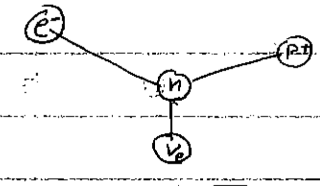
bosons | γ

beta decay

global local

τ - lifetime depends on conservation rules.

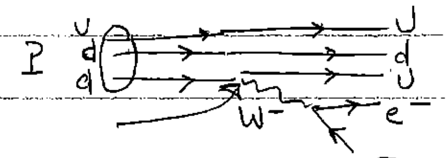
interactions.



in fermi theory $G_F \bar{P} \gamma^\mu n \bar{e} \gamma^\nu \nu_e$ ← four fermii interact

$G_F \sim 10^{-5}$

in modern particle physics.



$G_F \sim \frac{1}{M_W^2}$

interaction mediated by a heavy vector boson.

M_W. 80 GeV

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Isospin symmetry → classify elementary particles → broken symmetry

$S = 1/2$ N•——•P → proton & neutron form a doublet.

M = 939 MeV, 938 MeV →

e-charge 0 +1 if we turn off E-interactions can't distinguish P, N

SU(2) - Isospin - global continuous SU(2) symmetry
Exact if we ignore E-M- interactions
Approximate in nature.
versus E&M which is exact.

In the 1950's a slew of particles (Resonances) have been discovered

All the particles that interacted strongly formed families of isospin interactions.

Examples

1) Pions π^- •——• π^0 ——• π^+ $m(\pi^0) \sim 139$ MeV ; $m(\pi^{\pm}) = 139.5$ MeV,
spin = 0
Isospin = +1 → triplet.

2) Kions K^0 •——• K^+ $m(K^+) \sim 493.7$ MeV ; $m(K^0) = 497.8$ MeV,
$S = 0$
Isospin = 1/2, → doublet.

3) $\bar{P}^-$ •——• $\bar{N}^0$ $m(\bar{P}^-) \sim 938$ MeV $m(N) \sim 939$ MeV.
$S = 1/2$
Isospin = 1/2

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(4) Σ^- — Σ^0 — Σ^+

$m(\Sigma^+) = 1189.36$ MeV
$m(\Sigma^0) = 1192.46$ MeV
$m(\Sigma^-) = 1197.34$ MeV,
$s = \frac{1}{2}$
Isospin $= +1$

(5) Λ^0 $m(\Lambda^0) = 1115.6$ MeV $s = \frac{1}{2}$
Isospin $= 0$.

(6) η $m(\eta) = 548.8$ MeV $s = 0$
Isospin $= 0$

Classification: states with same spin and comparable mass form Isospin families

Relations between decay products.

The observed Resonances decay via their strong electromagnetic, weak interactions. The decays are typified by the: decay rates $\sim \frac{1}{\text{lifetime}}$.

Strong $\sim 10^{-24}$ sec
E+M $\sim 10^{-18}$ sec
Weak $\sim 10^{-8}$ sec.

Hadrons – strongly interacting → baryons spin $= n + \frac{1}{2}$ $n=0$
→ mesons spin $= n$ $n=0,$

lepton – not strongly interacting → charged → e, μ, τ.
→ neutral → ν_e, ν_μ, ν_τ

gauge bosons spin 1 γ, $W^\pm$ Z, G.

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All interactions respect familiar conservation laws.

e.g. → charge, energy, momentum, angular momentum.

In Addition. some additional conservation laws must be imposed.

e.g. $P \rightarrow e^+ \pi^0$

→ Respects conservation of charge, angular momentum, energy

but Not observed in nature $\tau_p \geq 10^{32}$ years → on going search

"we know it in our bones"

→ introduce conserved baryon charge $B(P) = +1$ $B(\bar{P})$=
$B(L) = 0 = B(M)$.

⇒ $P \not\rightarrow e^+ \pi^0$

similarly for leptons. $\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu$ allowed.

$\mu^- \rightarrow e^- \gamma$ forbidden

introduce. L_e, L_μ, L_τ.

$L_e = +1$ for e^- and ν^e
$L_e = -1$ for e^+ and $\bar{\nu}^e$
$L_e = 0$ for everyone else.

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	μ^-	$\rightarrow$	e^-	$\bar{\nu}_e$	ν_μ	
Thus	μ^-	$\rightarrow$	e^-	$\bar{\nu}_e$	ν_μ.	
spin	$\frac{1}{2}$	$\rightarrow$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$J_f = j_1 + j_2 - j_3 = \frac{1}{2}$ ✓
charge	-1	$\rightarrow$	-1	0	0	✓
L_μ	$+1$	$\rightarrow$	0	0	$+1$	✓
L_e	0	$\rightarrow$	1	-1	0	✓

Baryon & Lepton number are exactly conserved in nature (As observed).

Some Conservation laws may be approximate, e.g.

$K^+ \rightarrow \pi^+ \pi^0$ $\rightarrow \sim 20\%$ branching ratio.

$\tau(K^+) \sim 10^{-8}$ sec $\rightarrow$ weak decay. ?

Why not strong K^+ decay?

Gellmann & Nishijima $\rightarrow$ A new additive conserved quantum number

$S \rightarrow$ strangeness

$S(P) = S(\pi) = 0$ / $S(K^+) = S(K^0) = +1$ / $S(\Lambda, \Sigma) = -1$ / $S(\Xi) = -2$

Strong & electromagnetic interactions conserve strangeness
weak interaction violates strangeness.

$\rightarrow$ classification by Isospin is not sufficient to classify hadronic states

$\rightarrow$ need a larger symmetry group. G such that

$SU(2)_I \subset G$ $\leftarrow$ $SU(2)_I$ subgroup of G.

$G = \ ? \rightarrow G = SU(3)_{flavor} \rightarrow$ Gellmann & Neeman
$\rightarrow$ the eightfold way.

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Unitary groups $SU(n)$.

S ← simple; U ↓ unitary; n → Rank → # of diagonal generators mutually commuting.

Unitarity: $U^+U = I \Rightarrow U^+ = U^{-1}$

Simple: $\det U = 1$

Examples

$N=1$ $\quad \psi \to U\psi$. $\quad U^+ = U^{-1} \to U = e^{i\alpha}$ $\quad U^+ = e^{-i\alpha} = U$

$N=2$ $\quad \psi \to U\psi$

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \to \begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

$$U = \begin{pmatrix}A & B\\ C & D\end{pmatrix} \qquad U^+ = \begin{pmatrix}A^* & C^*\\ B^* & D^*\end{pmatrix}. \qquad U^{-1} = \frac{1}{(AD-BC)}\begin{pmatrix}D & -B\\ -C & A\end{pmatrix}$$

$U^+ = U^{-1}$ $\Rightarrow$ $A^* = \frac{D}{\det U}$ $\quad D^* = \frac{A}{\det U}$ $\quad C^* = -\frac{B}{\det U}$ $\quad B^* = -\frac{C}{\det U}$ (**)

Unitary matrices: $U^+ = U^{-1} \Rightarrow U^+U = I$

$$\det U^{-1} = (\det U)^{-1} \; ; \; \det U^+ = (\det U)^*$$

$$U^+ = U^{-1} \Rightarrow (\det U)^* = \frac{1}{\det U} \Rightarrow |\det U| = \pm 1$$

since $|\det U| = 1$ we get from the above eqs. that

$|A| = |D|$ & $|B| = |C|$.

The most general 2×2 unitary matrix can therefore be written as

$$U = \begin{pmatrix}A & B\\ C & D\end{pmatrix} = \begin{pmatrix}e^{i\alpha} & 0\\ 0 & e^{i\beta}\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}e^{i\gamma} & 0\\ 0 & e^{i\delta}\end{pmatrix}$$

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This product is the solution of the eqs. in (~~***~~)

\# of D.O.F. in $\bar{U}$ is 4 : θ + three of $\underbrace{\alpha, \beta, \gamma, \delta}$.

The phases appearing in U are $\alpha+\gamma$, $\alpha+\delta$, $\beta+\gamma$, $\beta+\delta$.

in a 2×2 unitary matrix there are 4 D.o.f.

generalization. an $N\times N$ unitary matrix has N^2 D.o.f.

Theorem : A unitary matrix $\bar{U}$ can be written as $\bar{U} = e^{iH}$

where H is hermitian. $(H^+ = H)$

Assume an hermitian $N\times N$ matrix : $H = S + iA$

real symmetric matrix real anti symmetric matrix

The # of D.o.f. in H : $S \rightarrow N + \left(\frac{N^2 - N}{2}\right) = \frac{N(N+1)}{2}$

diagonal off Diagonal $\rightarrow S$

$$A \rightarrow \frac{N^2 - N}{2} = \frac{N(N-1)}{2}$$

Hence: # of D.o.f. in H : $\frac{N(N+1)}{2} + \frac{N(N-1)}{2} = N^2$

A unitary matrix also has N^2 D.o. freedom.

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○ In U(2): write the most general 2×2 unitary matrix.

we need 4 independent hermitian matrices;

the space of 2×2 hermitian matrices is spanned by the

BASIS: $H = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \vec{\tau} \right]$ Pauli matrices $\quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\tau_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

general H-matrix $H = \beta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \vec{\alpha} \cdot \vec{\tau}$.

general U(2) matrix: $U = e^{i(\beta I + \vec{\alpha} \cdot \vec{\tau})}$.

○ SU(2) → $\det U = 1$

simple

if $U = e^{iH}$ then $\det U = e^{i \operatorname{tr} H}$.

in general, $\det A \neq 0 \Rightarrow \det A = \det PAP^{-1} = \det A_D = \lambda_1 \cdots \lambda_n$.

if $\det A \neq 0 \Rightarrow \operatorname{tr} A = \operatorname{tr} PAP^{-1} = \operatorname{tr} A_D = \lambda_1 + \cdots + \lambda_n$.

hence: if $U = e^A$ $\quad \operatorname{Det} U = \det e^A = \det\left(I + A + \frac{A^2}{2} + \cdots\right) =$

$= \det\left(I + PAP^{-1} + \frac{PAP^{-1}PAP^{-1}}{2} + \cdots\right) =$

$= \det\left(I + A_D + \frac{A_D^2}{2} + \cdots\right) = \det\left(e^{\lambda_1} \cdots e^{\lambda_n}\right) =$

$= e^{\lambda_1} \cdots e^{\lambda_n} = e^{\lambda_1 + \cdots + \lambda_n} = e^{\operatorname{tr} A}$.

○ $\Rightarrow \det e^A = e^{\operatorname{tr} A}$.

hence $\det U = e^{i \operatorname{tr} H} \iff \det U = \underline{1} \Rightarrow \operatorname{tr} H = 0$.

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$\Rightarrow \beta = 0.$ $\quad tr\, I = 2 \quad tr\, \tau_j = 0$

Therefore the most general SU(2) matrix $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$

$U = e^{i \vec{\alpha} \cdot \vec{\tau}} \simeq I + i \vec{\alpha} \cdot \vec{\tau}$

end of lecture 24 →

Find the most general hermitian 3x3 matrix. $H_{3\times3}$.

$\#(D.o.f.) = 9$

if $tr\, H = 0 \Rightarrow \#(D.o.f.) = 8.$ for $\det U = 1$

In SU(2) we had 3-matrices & 3 coefficients.

In SU(3) we have 8-matrices & 8 coefficients

such that $\det \bar{U} = e^{i\, tr\, H} = 1.$

$\bar{U} = e^{i \vec{\alpha} \cdot \vec{\lambda}}$ $\vec{\lambda}$ are hermitian matrices with $tr \lambda_j = 0$

$\lambda_1, \cdots \lambda_8$

The number of SU(3) group generators is eight.
" " " SU(2) " " " three.

For SO(2) we have only one diagonal hermitian matrix with trace $H_{2\times2} = 0$. $\rightarrow \tau_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

The other two matrices are $\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

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For $SU(2) \rightarrow$ only one diagonal matrix.

in $SU(3)$ find $\vec{\lambda}$.

# of Diagonal matrices, $\lambda = \begin{pmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{pmatrix}$ $\quad \mathrm{tr}\,\lambda = 0 \Rightarrow \lambda = \begin{pmatrix} \alpha & & \\ & \beta & \\ & & -\alpha-\beta \end{pmatrix}$

$\Rightarrow$ two diagonal matrices:

$$\lambda_3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \qquad \lambda_8 = \begin{pmatrix} 1 & & \\ & +1 & \\ & & -2 \end{pmatrix}$$

The importance of the diagonal matrices is that they provide the maximal set of mutually commuting operators whose eigenvalues characterize the elementary particles.

The other $\vec{\lambda}$ matrices are not diagonal

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}.$$

$(\lambda_1, \lambda_2, \lambda_3)$ form an $SU(2)$ subgroup of $SU(3)$.

subgroup: a subgroup of generators that satisfy commutation relations among themselves, $[\lambda_1, \lambda_2] = i\lambda_3$. etc.

In $SU(2)$

$\psi \rightarrow U\psi$. $\qquad U \rightarrow 2\times2$ matrix ψ - 2-vector.

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Take $\frac{1}{2}\sigma_3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$: $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -1/2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1/2\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The spinors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are eigenvectors with eigenvalues $+1/2$ & $-1/2$

These are the eigenstates of σ_3.

In SU(2) we cannot characterize these states with an additional eigenvalue

we can characterize the spin exactly only in one direction, say along the z axis.

In SU(3) the analog of σ_3 is λ_3.

$$\frac{1}{2}\lambda_3 = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The three eigenvectors of $1/2\,\lambda_3$ are:

$$\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1/2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -1/2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

similarly to SU(2) the physical states can be characterized by the eigenvalues of the operator $(1/2\,\lambda_3)$.

The eigenvectors of $1/2\,\lambda_3$ are also eigenvectors of λ_8.

$$1/2\,\lambda_8 = \begin{pmatrix} \frac{1}{2\sqrt{3}} & & \\ & \frac{1}{2\sqrt{3}} & \\ & & -\frac{1}{\sqrt{3}} \end{pmatrix}.$$

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$$\frac{1}{2}\begin{pmatrix}1/\sqrt{3} & & \\ & 1/\sqrt{3} & \\ & & -2/\sqrt{3}\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{1}{\sqrt{12}}\begin{pmatrix}1\\0\\0\end{pmatrix} \quad \frac{1}{2}\begin{pmatrix}1/\sqrt{3} & & \\ & 1/\sqrt{3} & \\ & & -2/\sqrt{3}\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \frac{1}{\sqrt{12}}\begin{pmatrix}0\\1\\0\end{pmatrix} \quad \frac{1}{2}\begin{pmatrix}1/\sqrt{3} & & \\ & 1/\sqrt{3} & \\ & & -2/\sqrt{3}\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = -\frac{1}{\sqrt{3}}$$

There are no additional diagonal matrices among $\lambda_1 \cdots \lambda_8$.
Therefore these are eigenvectors of λ_3 and λ_8 only.

$\Rightarrow$ Eigenstates of SU(3) are characterized by eigenvalues of λ_3 & λ

In SU(2) we classified particles according to the value of τ_3.

Graphical description,

we define $\quad T_3 = \frac{1}{2}\lambda_3$

$\qquad\qquad Y = \frac{1}{\sqrt{3}}\lambda_8$

in SU(2) we had a doublet with 2 eigenvalues $\frac{1}{2}, -\frac{1}{2}$

graph 1 $\quad \begin{pmatrix}0\\1\end{pmatrix}$ ——— $\begin{pmatrix}1\\0\end{pmatrix}$

$T_3 = \frac{1}{2}\lambda_3 \quad T_3 = -\frac{1}{2} \qquad T_3 = +\frac{1}{2}$ $\quad$ using $\frac{1}{2}(\tau_1 \pm i\tau_2)$ we can move from $\begin{pmatrix}1\\0\end{pmatrix} \longleftrightarrow \begin{pmatrix}0\\1\end{pmatrix}$

$$\tau_+ = \frac{1}{2}(\tau_1 + i\tau_2) = \frac{1}{2}\left[\begin{pmatrix}0&1\\1&0\end{pmatrix} + i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\right] = \begin{pmatrix}0&1\\0&0\end{pmatrix} \qquad \tau_+\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\tau_- = \frac{1}{2}(\tau_1 - i\tau_2) = \frac{1}{2}\left[\begin{pmatrix}0&1\\1&0\end{pmatrix} - i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\right] = \begin{pmatrix}0&0\\1&0\end{pmatrix} \qquad \tau_-\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\tau_+\downarrow = \uparrow \qquad \tau_+\uparrow = 0 \qquad \tau_-\uparrow = \downarrow \qquad \tau_-\downarrow = 0$$

The proton and the neutron form an Isospin doublet.

$\Rightarrow$ Therefore $\quad T_3(P) = \frac{1}{2} \quad T_3(n) = -\frac{1}{2}$

For SU(3) we characterize the states by T_3 & Y.

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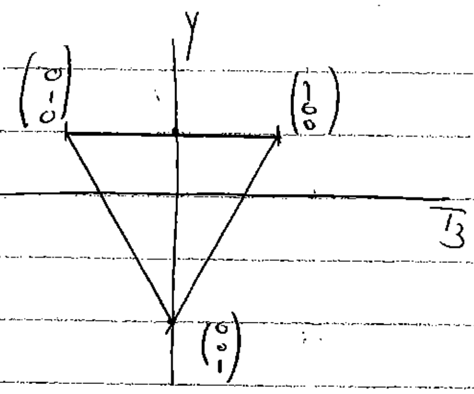
→ (T_3, Y) plane:



$(T_3, Y): \begin{pmatrix}1\\0\\0\end{pmatrix} = \left(\tfrac{1}{2}, \tfrac{1}{3}\right)$

$(T_3, Y): \begin{pmatrix}0\\1\\0\end{pmatrix} = \left(-\tfrac{1}{2}, \tfrac{1}{3}\right)$,

$(T_3, Y): \begin{pmatrix}0\\0\\1\end{pmatrix} = \left(0, -\tfrac{2}{3}\right)$

→ Graphical representation of the fundamental triplet representation of SU(3)

The physical quantities

T_3 - third component of Isospin (same as for SU(2))

Y - hypercharge.

We can exchange the τ_1, τ_2, τ_3 generators of SU(2)

with $\tau_\pm = \frac{1}{2}(\tau_1 \pm i\tau_2)$ & τ_3

Similarly in SU(3) define $\tau^\pm = \frac{1}{2}(\lambda_1 \pm i\lambda_2)$

$\tau^+ = \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix} \quad \tau^- = \begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}$

$\tau^+\begin{pmatrix}1\\0\\0\end{pmatrix} = 0 \quad \tau^+\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad \tau^+\begin{pmatrix}0\\0\\1\end{pmatrix} = 0$

$\tau^-\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad \tau^-\begin{pmatrix}0\\1\\0\end{pmatrix} = 0 \quad \tau^-\begin{pmatrix}0\\0\\1\end{pmatrix} = 0.$

The tree points on the graphic triangular representation of the triplet of SU(3) form a doublet $\begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix}$ and a singlet $\begin{pmatrix}0\\0\\1\end{pmatrix}$ of SU(2).

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If we use only $\lambda_1, \lambda_2, \lambda_3$ we can only exchange $\begin{pmatrix}1\\0\\0\end{pmatrix}$ & $\begin{pmatrix}0\\1\\0\end{pmatrix}$ but cannot act on $\begin{pmatrix}0\\0\\1\end{pmatrix}$.

→ Therefore we have a doublet & a singlet of SU(2)

$$SU(3) \rightarrow SU(2)_I \times U(1)_Y$$

$$\underset{\text{triplet}}{3} = \underset{\text{doublet}}{2_{1/3}} + \underset{\text{singlet}}{1_{-3/2}}$$

The representations of SU(3) decompose under $SU(2) \times U(1)$.

In SU(3) we can form generators that exchange $\begin{pmatrix}1\\0\\0\end{pmatrix}$ & $\begin{pmatrix}0\\0\\1\end{pmatrix}$.

$(\lambda_4 \pm i\lambda_5)$ exchanges $\begin{pmatrix}1\\0\\0\end{pmatrix}$ & $\begin{pmatrix}0\\0\\1\end{pmatrix}$

$(\lambda_6 \pm i\lambda_7)$ exchanges $\begin{pmatrix}0\\1\\0\end{pmatrix}$ & $\begin{pmatrix}0\\0\\1\end{pmatrix}$.

We used here the other SU(3) generators that are not in $SU(2)_I$.

For every particle we know both T_3 & Y.

hence $SU(3) \supset SU(2)_I \times U(1)_Y$.

in SU(2) we can have higher order representations.

For Example 3: $\underset{\begin{pmatrix}0\\0\\1\end{pmatrix}}{-1}$ —— $\underset{\begin{pmatrix}0\\1\\0\end{pmatrix}}{0}$ —— $\underset{\begin{pmatrix}1\\0\\0\end{pmatrix}}{+1}$

$$T_1 = \begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} \quad T_2 = \begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix} \quad T_3 = \begin{pmatrix}1&&\\&0&\\&&-1\end{pmatrix}$$

Check $[T_i, T_j] = i\epsilon_{ijk} T_k$

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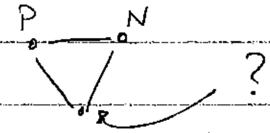
Any 3 $n \times n$ matrices that satisfy $[T_i, T_j] = i\epsilon_{ijk} T_k$ form a Representation of the SU(2) algebra.

The Proton & the neutron formed a doublet of $SU(2)_I$

In the fundamental Rep. of SU(3)



There isn't a third particle with with $m(?) \sim m(P) \sim m(N)$ that fi

$\Rightarrow$ P, N form an Isospin doublet but are not part of an SU(3) triplet

Can it be that P, N form an Isospin doublet in a higher order Representation of SU(3).

we want to find higher reps of SU(3),

The SU(3) generators obey the algebra

$$(***) \qquad [T_i, T_j] = i f_{ijk} T_k$$

With f_{ijk} totally antisymmetric under exchange of any 2 indices

And: $f_{123} = 1$ $f_{147} = 1/2$ $f_{156} = -1/2$ $f_{246} = 1/2$ $f_{257} = 1/2$
$f_{345} = 1/2$ $f_{357} = -1/2$ $f_{458} = \sqrt{3}/2$ $f_{678} = \sqrt{3}/2$

All the others vanish.

$\rightarrow$ matrices of higher order Reps satisfy the (***) algebra.

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For $SU(2)$ we have a solution for any with matrices $(2\ell+1)\times(2\ell+1)$

For $SU(3)$ there isn't a solution at every order (e.g. order 2).

To find the higher Reps we use a different method, similar to the addition of angular momentum for $SU(2)$.

For $SU(2)$ $\psi^\alpha = \{ |\uparrow\rangle, |\downarrow\rangle \}$. $S^2|\uparrow\rangle = \frac{1}{2}(\frac{1}{2}+1)|\uparrow\rangle$ $S^2|\downarrow\rangle = \frac{1}{2}(\frac{1}{2}+1)|\downarrow\rangle$

$S_z|\uparrow\rangle = +\frac{1}{2}|\uparrow\rangle$ $S_z|\downarrow\rangle = -\frac{1}{2}|\downarrow\rangle$.

$\psi, \phi \rightarrow$ two spin $\frac{1}{2}$ wave functions.

$\psi^\alpha \otimes \phi^\beta$:

Triplet:
$$\begin{cases} |\uparrow_\psi \uparrow_\phi\rangle & T=1 \quad T_3 = 1 \\ \frac{1}{\sqrt{2}}(|\uparrow_\psi \downarrow_\phi\rangle + |\downarrow_\psi \uparrow_\phi\rangle) & T=1 \quad T_3 = 0 \\ |\downarrow\downarrow\rangle & T=1 \quad T_3 = -1 \end{cases}$$

symmetric combinations

Singlet $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ $T=0$ $T_3 = 0$

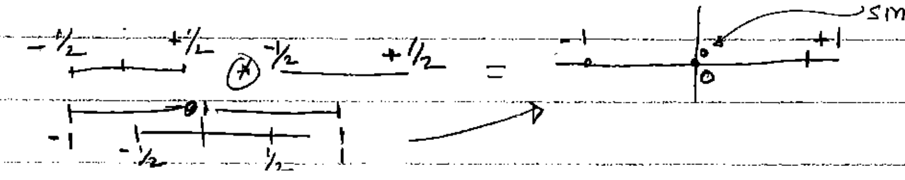
anti symmetric.

by taking the product of 2 isospin doublets we get states with Isospin $= 1$ or 0.

$\underline{2} \times \underline{2} = \underline{3}_S + \underline{1}_A$

we built a higher $\underline{3}$ representation with spin 1 from the $\underline{2}$ representation with spin $\frac{1}{2}$.

Graphically



$= 3 + 1$.

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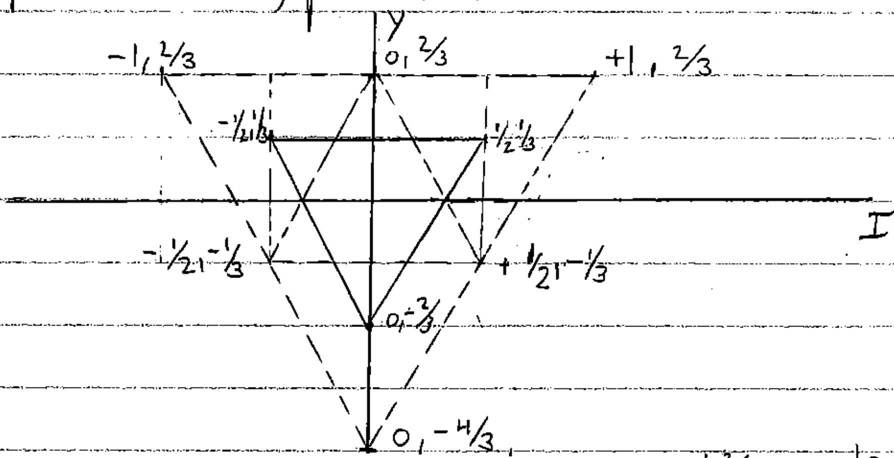
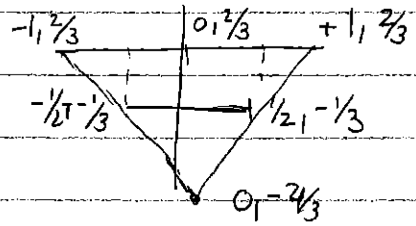
we can repeat the graphic analysis for SU(3)

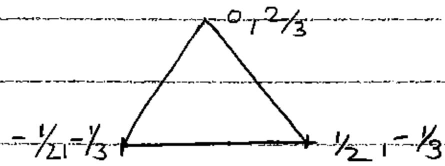
$\psi^\alpha \quad \phi^\beta \qquad \alpha, \beta = 1, 2, 3$



we get the following SU(3) reps.



And



$3 \times 3 = 6 + \bar{3}$

inside the

$$6 = \begin{cases} (-1, 2/3) \quad (0, 2/3) \quad (+1, 2/3) & \to SU(2)_I \text{ triplet} \\ (-1/2, -1/3) \quad (1/2, -1/3) & \to SU(2)_I \text{ double} \\ (0, -2/3) & \to SU(2)_I \text{ singlet} \end{cases}$$

$$SU(3) \supset SU(2) \times U(1)$$

$$6 = 3_{2/3} + 2_{-1/3} + 1_{-4/3}$$

$$\bar{3} = 2_{-1/3} + 1_{2/3}$$

we got the $\bar{3}$ representation $\bar{3} \neq 3$ in $SU(2)$ $\bar{2} = 2$.

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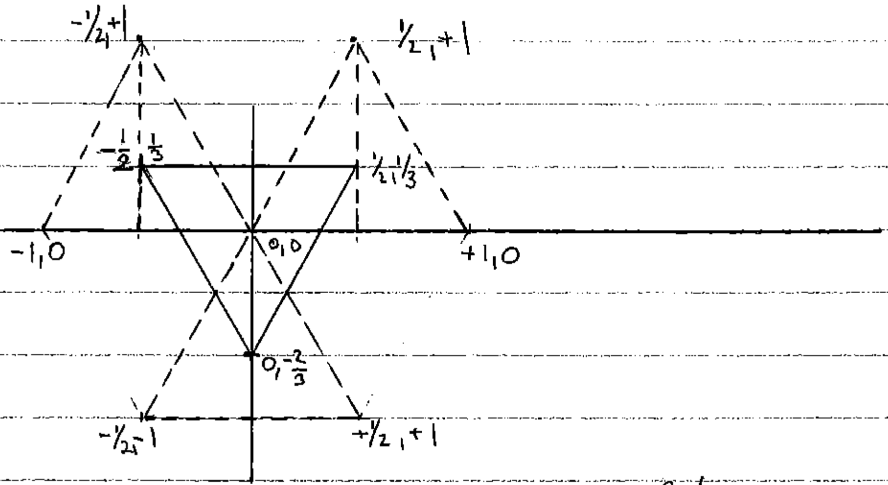
Are there physical particles that fit the 6 & $\bar{3}$ together with P, ...
not yet. Look at another possibility. $3 \times \bar{3}$

multiply $\bar{3}$ at every point of 3 →

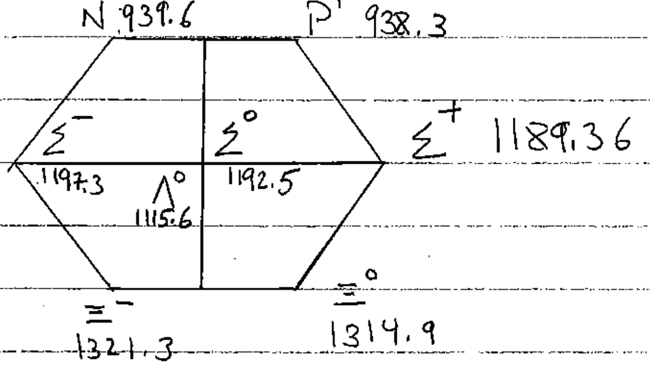


$3 \times \bar{3} = 8 + 1 \rightarrow$ octet + singlet.

under $SU(2) \times U(1)$ the octet decomposes as

$$8 = 2_{+1} + 3_0 + 2_{-1} + 1_0$$

The octet has a physical assignment.



masses in MeV

The spin of the particles is $S = ½$
Baryon number $B = +1$

We can find a relation between the electric charge and the Isospin T_3 and hypercharge Y.

$Q = \alpha T_3 + \beta Y$ $\quad Q(P) = +1 = \alpha \cdot \frac{1}{2} + \beta \cdot 1$ $\Big\} \Rightarrow \alpha = 1 \quad \beta = ½$
$Q(N) = 0 = \alpha \cdot -½ + \beta \cdot 1$

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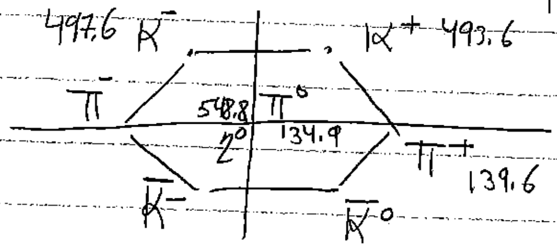
Second Example

$Q = T_3 + \frac{1}{2} Y$ → Holds for the other particles in the octet.



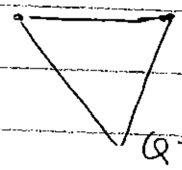
Mesons: $S = 0$ $B = 0$

question: which Representations of SU(3) can represent particles and w

8 - Baryons, mesons.

3, 3*, 6, 6* - don't represent integrally charged particles

$Q = -\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3} = -\frac{1}{3}$ ← $Q = -\frac{1}{3}$ $Q = \frac{+2}{3}$ → $Q = T_3 + \frac{1}{2}Y = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{3}\right) = \frac{2}{3}$



$Q = -\frac{1}{3}$ → $Q = 0 + \frac{1}{2}\left(-\frac{2}{3}\right) = -\frac{1}{3}$.

Perhaps the physical Representations are those that yield integrally charged states.

Another physical Rep. | 10

Δ^- x	Δ^0 x	Δ^+ x	Δ^{++} x	$I = 3/2$	$Y = 1$
x Y^{*-}	x Y^{*0}	x Y^{*+}		$I = 1$	$Y = 0$
x Ξ^{*-}	x Ξ^{*0}			$I = \frac{1}{2}$	$Y = -1$
x Ω^-				$I = 0$	$Y = -2$

$S = 3/2$
$B = 3/2$

$m(\Delta) \sim 1230 - 1234$ MeV
$m(Y^*) \sim 1382$ MeV } 150 MeV
$m(\Xi^*) \sim 1531$ MeV } 150 MeV,

"The eight fold way"

The Ω^- was predicted by Gellmann with $m(\Omega^-) \sim 1680$ MeV
And was discovered in 1964.

The SU(3) representation are obtained from products of $3, \bar{3}$

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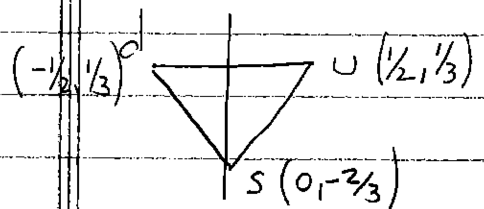
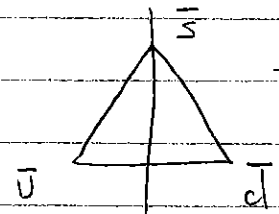
Physical	unphysical
$3 \times \bar{3} = 8 + 1$	$3 \times 3 = 6 + \bar{3}$
$3 \times 3 \times 3 = 10 + 8 + 8 + 1$	
$\bar{3} \times \bar{3} \times \bar{3} = \overline{10} + 8 + 8 + 1$	

questions: only some products of $3; \bar{3}$ are physical, why?
is there a physical meaning to the $3, \bar{3}$?

Gellmann & Zweig: The baryons & mesons are made of more elementary building blocks → quarks.

quarks





antiquarks

$Q(\text{up}) = \frac{2}{3}$ $\qquad$ $Q(\text{antiup}) = -\frac{2}{3}$
$Q(\text{down}) = -\frac{1}{3}$ $\qquad$ $Q(\text{antidown}) = -\frac{1}{3}$
$Q(\text{strange}) = -\frac{1}{3}$ $\qquad$ $Q(\text{antistrange}) = +\frac{1}{3}$.

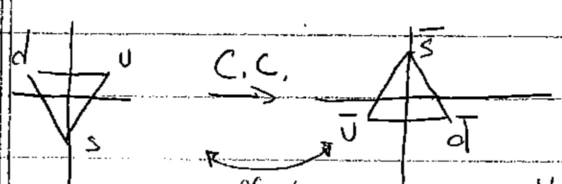
The spin of the quarks must be $S = \frac{1}{2} \Rightarrow S(P, N) = \frac{1}{2}$

The physical states therefore correspond to bound states of

quark-antiquark → $3 \times \bar{3}$
quark-quark-quark → $3 \times 3 \times 3$
antiquark-antiquark-antiquark → $\bar{3} \times \bar{3} \times \bar{3}$

	T_3	Y	$T_3 + \frac{1}{2}Y = Q$
u	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$
d	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{3}$
s	0	$-\frac{2}{3}$	$-\frac{1}{3}$

	T_3	Y	$T_3 + \frac{Y}{2} = Q$
$\bar{u}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{2}{3}$
$\bar{d}$	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{3}$
$\bar{s}$	0	$\frac{2}{3}$	$\frac{1}{3}$



Reflection about the origin

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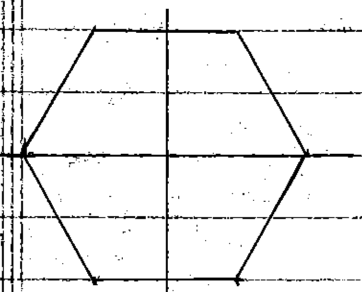
In $SU(2)$ $\quad 2 = \bar{2}$

In $SU(3)$ $\quad 3 \neq \bar{3}$

Meson & baryons in the quark model.

We can now see how the proton & neutron and all the other plew of hadron resonanses fit in the quarkmodel.

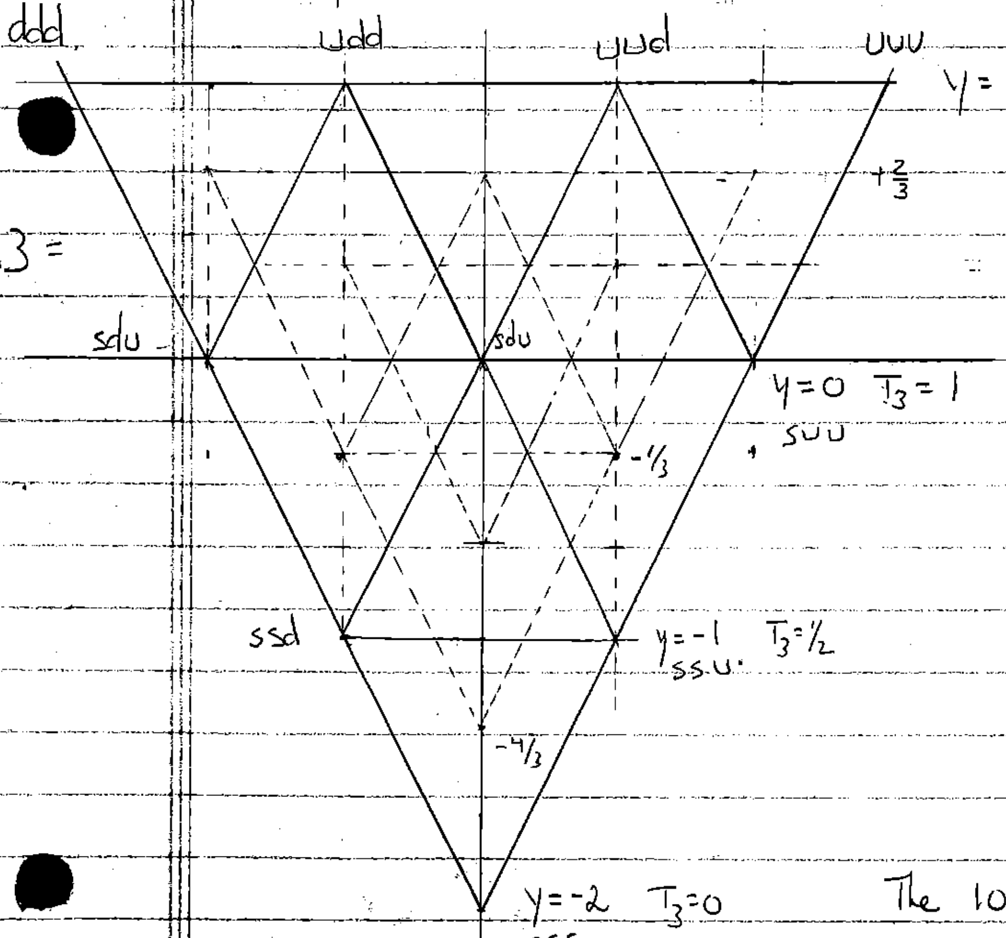
baryons.



$$3 \times 3 \times 3 = (6 + \bar{3}) \times 3 = 6 \times 3 + \bar{3} \times 3 =$$

$$= 10 + 8 + 8 + 1$$

$6 \times 3 =$



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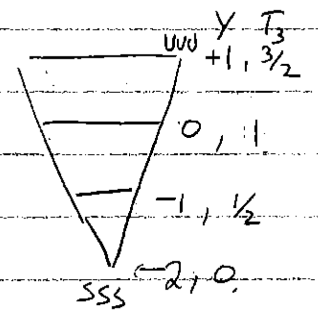

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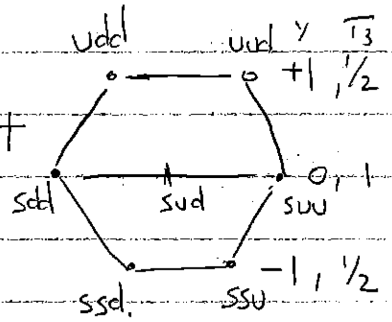
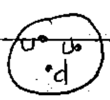


The 10 is a symmetric rep.

uuu → spin 3/2.



so P = uud $\qquad$ N = udd.

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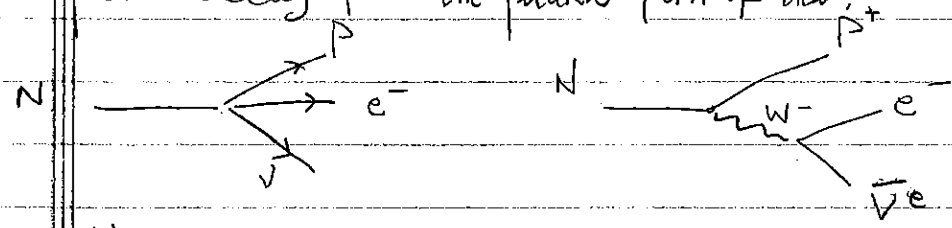
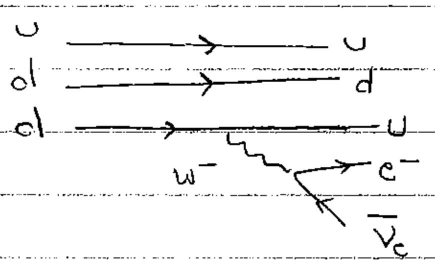
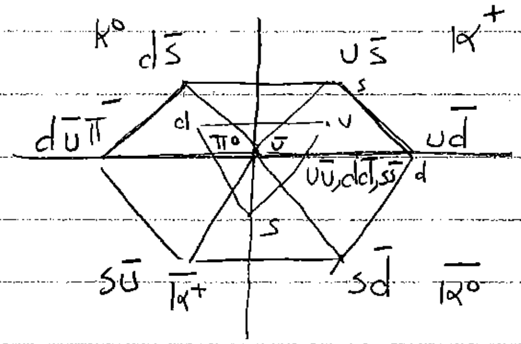
Beta decay from the quark point of view.



quark $\longrightarrow$ picture



mesons, $\underset{q}{3} \times \underset{\bar q}{\bar 3} = 8 + 1$



$$\pi^0 = \frac{1}{\sqrt2}(u\bar u - d\bar d)$$
$$\eta = \frac{1}{\sqrt6}(u\bar u + d\bar d - 2s\bar s)$$ $\Big\} \in 8$
$$\eta' = \frac{1}{\sqrt3}(u\bar u + d\bar d + s\bar s) \quad \} \to 1.$$

The decuplet is in a fully symmetric rep of SU(3) and spin

$$\Delta^{++} = u^\uparrow u^\uparrow u^\uparrow \qquad \frac{1}{\sqrt2}(\lambda_1 - i\lambda_2)$$
$$\Delta^{+} = \frac{1}{\sqrt3}(u^\uparrow u^\uparrow d^\uparrow + u^\uparrow d^\uparrow u^\uparrow + d^\uparrow u^\uparrow u^\uparrow).$$
$$\Delta^{0} = \frac{1}{\sqrt3}(udd + dud + ddu).$$
$$\Delta^{-} = d^\uparrow d^\uparrow d^\uparrow$$

To go down the axis we operate with a lowering operator $(\lambda_4 - i\lambda_5)$ etc.

$$Y^{+} = \frac{1}{\sqrt3}(uus + usu + suu).$$

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$$\Upsilon^0 = \frac{1}{\sqrt{6}} (uds + dus + dsu + usd + sdu + sud)$$

$$\Omega^- = s^\uparrow s^\uparrow s^\uparrow$$

What is the wave function of $\underline{P}$?

P is orthogonal to Δ^+. both combinations of uud.

$$P = (\alpha uud + \beta udu + \gamma duu)$$

$P \cdot \Delta^+ = 0 \Rightarrow \alpha + \beta + \gamma = 0 \Rightarrow 2$ solutions.

$P_A = \sqrt{\frac{1}{2}}(ud - du)u$ Asymmetric under $1 \longleftrightarrow 2$

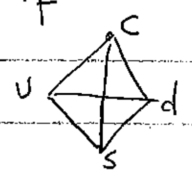
$P_S = \sqrt{\frac{1}{6}}\left[(ud + du)u - 2uud\right]$ symmetric under $1 \longleftrightarrow 2$.

Nice.... but...

Problems: 1) $\Delta^{++} = u^\uparrow u^\uparrow u^\uparrow$ fully symmetric $\rightarrow$ conflict with Pauli's spin-statistics Relation.

2) charm was discovered $\rightarrow SU(3)_f \rightarrow$ not enough.

$SU(4)_f \longrightarrow 4$



In 1974 J/ψ particle with $m \sim 3.1$ GeV. Spin = 0 / $c\bar{c}$.

bottom was discovered in 1978 $m(\Upsilon) \approx 10$. GeV. / $b\bar{b}$

top was discovered in 1994 $m_t \sim 175$ GeV.

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To resolve the conflict with Pauli's Exclusion principle. A new quantum attribute is introduced - color.

All color bound states for color singlets, hence the Δ^{++} wave function is Asymmetric under color and symmetric under flavor x spin x space quantum numbers.

$$(qqq)_{\text{col. singlet}} = \sqrt{\tfrac{1}{6}}\,(RGB - RBG + BRG - BGR + GBR - GRB).$$

This wave function is asymmetric under exchange of any two colors.

→ quarks are in the fundamental of $SU(3)_{\text{color}}$ $q = \begin{pmatrix} R \\ G \\ B \end{pmatrix}$, $\bar{q} = \begin{pmatrix} \bar{R} \\ \bar{G} \\ \bar{B} \end{pmatrix}$.

$SU(3)_{\text{color}}$ is a new degree of freedom + different from $SU(3)_F$.

$SU(3)_{\text{color}}$ is exact $SU(3)_F$ is approximate and accidental.

similarly to $SU(3)_F$ we can introduce $SU(4)_F$, u, d, s, c.

$SU(3)_F$ violated by mass differences of $O(100\,\text{MeV})$.

$SU(4)_F$ violated by mass differences of $O(1\,\text{GeV})$.

Reps. $4\,4$ → mesons. $(3\,\bar{3})_{\text{color}} = 8 + 1$

$4 \cdot 4 \cdot 4$ → baryons. $(3 \cdot 3 \cdot 3)_{\text{color}} = 10 + 8 + 8 + 1$

quarks have spin ½ → fermions.

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$SU(3)_{color} \to$ Exact symmetry $\to$ strong interactions

$U(1)_{em} \to$ Exact symmetry $\to$ E&M interactions.

under color: $q \to \bar{U} q = e^{i\vec{\alpha}(x)\cdot\vec{\lambda}} q$ $\quad \lambda = (\lambda_1 \cdots \lambda_8)$

$\alpha = (\alpha_1 \cdots \alpha_8)$

Gauge symmetry $\to$ local phase invariance $\to$ gauge bosons.

$\psi (\partial_\mu + i \vec{A}_\mu \cdot \vec{\lambda}) \psi$.

Here $\vec{A} = (A_1, \ldots, A_8)$.

The strong interactions correspond to local Phase invariance under $SU(3)_{color}$.

quarks are observed only confined states inside hadrons.

$\to$ not observed as free quarks.

observed electric charge is integral $\quad Q(u) = 2/3 \quad Q(d) = -1/3$

$Q(c) = 2/3 \quad Q(s) = -1/3$

The bound states are (qqq) and $(q\bar{q}) \Rightarrow$ only integral combinations are observed.

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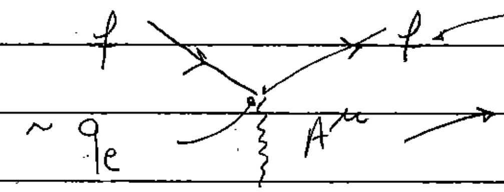
What about the weak interactions?

E&M → U(1) local symmetry → Abelian.

Weak → SU(2) local symmetry } → Non Abelian
Strong → SU(3) " }

Consider E&M interactions

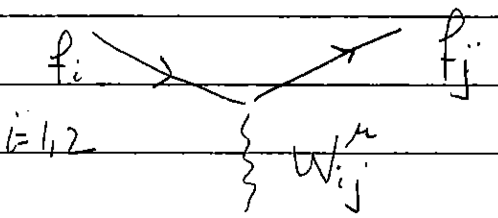
U(1)



← fermions, scalars.

Photon

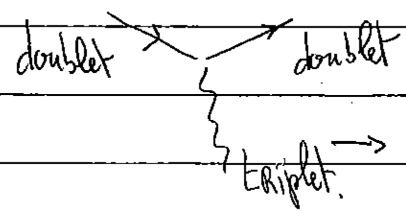
SU(2)



$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ → doublet.

The E&M interactions don't change the identity of the particle

Weak interactions f_1 may be different from f_2.



→ W^μ are in the adjoint of SU(2).

The gauge bosons are always in the adjoint Rep. $n \times \bar{n} = \underbrace{(n^2-1)}_{\text{adj.}} + 1$

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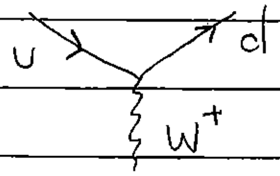
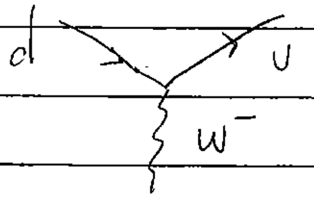
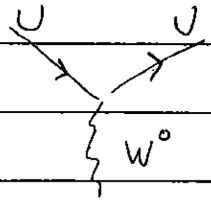
$$W^\mu_\alpha \qquad \mu = 0,1,2,3 \ \text{Lorentz index} \qquad \alpha = 1,2,3$$

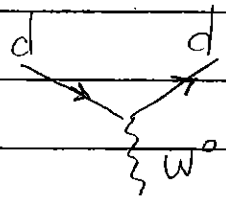
Couplings



$$Q(u) = 2/3 \longrightarrow Q(d) = -1/3$$







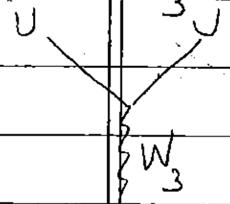
The current in the Lagrangian has the form,

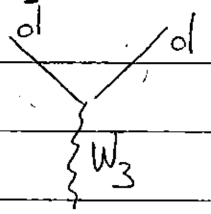
$$(\bar u, \bar d)\begin{pmatrix} & \end{pmatrix}\begin{pmatrix} u \\ d \end{pmatrix} \qquad \sum_{i=1}^{3} \tau_i W^\mu_i = \vec\tau \cdot \vec W^\mu$$

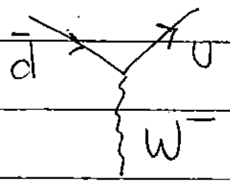
$$= (\bar u, \bar d)\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} W_1^\mu + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} W_2^\mu + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} W_3^\mu\right]\begin{pmatrix} u \\ d \end{pmatrix}$$

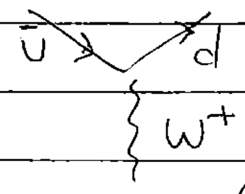
$$= (\bar u, \bar d)\begin{bmatrix} W_3^\mu & W_1^\mu - iW_2^\mu \\ W_1^\mu + iW_2^\mu & -W_3^\mu \end{bmatrix}\begin{pmatrix} u \\ d \end{pmatrix} =$$

$$= W_3^\mu(\bar u u - \bar d d) + (W_1^\mu - iW_2^\mu)\bar u d + (W_1^\mu + iW_2^\mu)\bar d u$$









In SU(3) $(\bar u_1, \bar u_2, \bar u_3)\begin{pmatrix} & \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $\vec\lambda \cdot \vec A^\mu = \lambda_j A_j^\mu \quad j = 1,..,8$

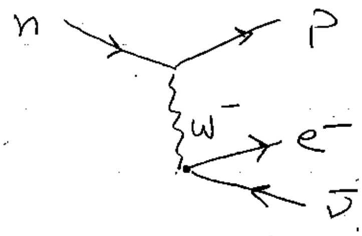
color

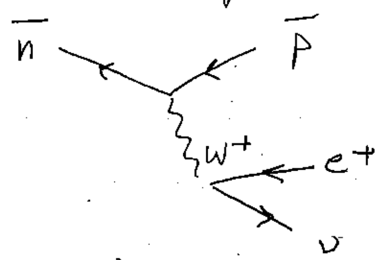
Gellmann matrices

17/3/06.8 | MPS.122 | Friday | Abingdon

Unification of E&M & weak interactions — Glashow 1961, Weinberg 1968, Salam

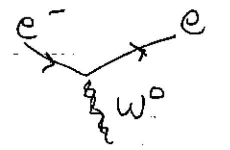
Problems Weak interactions also involve leptons.

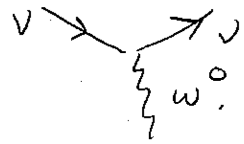




However, if $W^-, W^+ \in SU(2)$

we must also have $W^0 \in SU(2)$

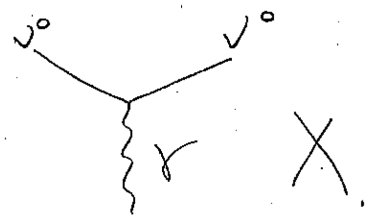




Do these currents exist in nature?

? identify $W^0 = \gamma \rightarrow$ photon?

but



ν is neutral → does not couple to γ.

must have a new W^0 that couples to ν.

The new W^0 must be a mixture of W^3 and γ

such that $Q(W^{\pm}) = \pm 1$.

18/3/06.1 | MPS.123 | Saturday | Abingdon |

we saw: Weak interactions only couple to left-handed fields

E&M couples to both left & right handed fields,

→ only (e_L, ν_L) (u_L, d_L) interact weakly.

(e_L, e_R) (u_L, u_R, d_L, d_R) interact E&M.

$Q(\nu_L) = 0$

so far no need for (ν_R)

i.e. no strong, weak or E&M interactions for ν_R.

Left-handed fields form doublets of $SU(2)_W$.
Right handed fields are singlets of $SU(2)_W$.

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \quad \begin{pmatrix} u \\ d \end{pmatrix}_L \quad e_R \quad u_R \quad d_R$$

$$\begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L \quad \begin{pmatrix} c \\ s \end{pmatrix}_L \quad \mu_R \quad c_R \quad s_R$$

$$\begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L \quad \begin{pmatrix} t \\ b \end{pmatrix}_L \quad \tau_R \quad t_R \quad b_R.$$

The quarks are triplets of $SU(3)_{color}$.

The leptons are singlets of $SU(3)_{color}$.

18.3.06.2 | MPS, 124 | Saturday | Abingdon

The SU(2) doublets have T_3^W quantum number.

for The SU(2) singlets $T_3^W = 0$.

The $SU(2)_W$ does not contain E&M interaction

We have to introduce a U(1) symmetry to incorporate E&M charges

$SU(2)_W \times U(1)_Y$

But the new $U(1)_Y \neq U(1)_{em}$.

All SU(2) representations must have the same.

$U(1)_Y$ charge But $Q(e_L) \neq Q(\nu_L)$

hence $U(1)_Y \neq U(1)_{e.m.}$

combination $Q_{em} = T_{3W} + \frac{1}{2} Y$

Find values for Y such that $Q_{e.m.}$ is reproduced for the different particle state.

	T_3	$\frac{1}{2}Y$	Q_{em}		T_3	$\frac{1}{2}Y$	Q_{em}
ν_L	$\frac{1}{2}$	$-\frac{1}{2}$	0	u	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{2}{3}$
e_L	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	d	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{3}$
e_R	0	-1	-1	u_R	0	$\frac{2}{3}$	$\frac{2}{3}$
ν_R	0	0	0.	d_R	0	$-\frac{1}{3}$	$-\frac{1}{3}$

24/3/06.1 | MPS.125 | Friday | Abingdon

how can we write a four vector current j^μ that will incorporate both the weak & Electromagnetic interactions

$$J_\mu^+(x) = \bar{\chi}_L \gamma_\mu \tau_+ \chi_L$$

$$J_\mu^-(x) = \bar{\chi}_L \gamma_\mu \tau_- \chi_L$$

$$\tau_\pm = \frac{1}{2}(\tau_1 \pm i\tau_2)$$

$$\chi_L = \begin{pmatrix} \nu \\ e^- \end{pmatrix}_L$$

$$J_\mu^3(x) = \bar{\chi}_L \gamma_\mu \frac{1}{2}\tau_3 \chi_L$$

$$= (\bar{\nu}, \bar{e}^-)_L \gamma_\mu \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \nu \\ e^- \end{pmatrix}_L =$$

$$= \frac{1}{2}\bar{\nu}_L \gamma_\mu \nu_L - \frac{1}{2}\bar{e}_L \gamma_\mu e_L$$

we can write these in the form.

$$J_\mu^i(x) = \bar{\chi}_L \gamma_\mu \frac{1}{2}\tau_i \chi_L \qquad \text{with } i = 1, 2, 3.$$

These currents couple to the vector bosons.

$$W^{\mu\mp} = \frac{1}{\sqrt{2}}(W^1 \mp i W^2).$$

using the identity $\frac{1}{2}(\tau_1 W^1 + \tau_2 W^2) = \sqrt{\frac{1}{2}}(\tau^+ W^+ + \tau^- W^-)$

we can write

$$\underbrace{\frac{1}{\sqrt{2}}\left(J_\mu^+ W^{+\mu} + J_\mu^- W^{-\mu}\right)}_{\text{Charged currents}} + \underbrace{J_\mu^3}_{\text{neutral current.}} W^{3\mu} = \sum_{i=1}^{3} J_\mu^i W^{i\mu}$$

The electromagnetic current.

$$j_{em}^\mu A_\mu = \underbrace{eQ}_{\text{electric charge coupling}} \bar{\psi}\gamma^\mu \psi \overset{\text{photon.}}{A_\mu} = \underset{\text{coupling.}}{e}\, \bar{\psi}\gamma^\mu \underset{\text{charge.}}{Q} \psi A_\mu.$$

24/3/06.2 | MPS. 126 / Friday / Abingdon.

The new U(1) current $\frac{g'}{2}\bar{\Psi}\gamma^{\mu} Y \Psi B_{\mu}$

gauge coupling — hyper charge — gauge field.

The neutral SU(2) current is.

$$g\bar{\Psi}\gamma^{\mu} T_3 \Psi W^3_{\mu}.$$

with $T^3 = \tau_3/2$.

To get consistency with charge assignment we should have.

$$Q = T_3 + \frac{1}{2}Y \Rightarrow J^{em}_{\mu} = J^3_{\mu} + \frac{1}{2}J^Y_{\mu}$$

we obtain this by ~~making~~ a rotation on B_{μ}, W^3_{μ}

$$\begin{pmatrix} A_{\mu} \\ Z_{\mu} \end{pmatrix} = \begin{pmatrix} \cos\theta_w & \sin\theta_w \\ -\sin\theta_w & \cos\theta_w \end{pmatrix}\begin{pmatrix} B_{\mu} \\ W^3_{\mu} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta_w B_{\mu} + \sin\theta W^3_{\mu} \\ -\sin\theta_w B_{\mu} + \cos\theta_w W^3_{\mu} \end{pmatrix}$$

OR inversely.

$$\begin{pmatrix} B^{\mu} \\ W^3_{\mu} \end{pmatrix} = \begin{pmatrix} \cos\theta_w & -\sin\theta_w \\ \sin\theta_w & \cos\theta_w \end{pmatrix}\begin{pmatrix} A_{\mu} \\ Z_{\mu} \end{pmatrix}$$

Weinberg angle.

We get. $g J^3_{\mu} W^{3\mu} + \frac{g'}{2} J^Y_{\mu} B^{\mu} =$

$$= g J^3_{\mu}\left(\sin\theta_w A^{\mu} + \cos\theta_w Z^{\mu}\right) + \frac{g'}{2} J^Y_{\mu}\left(\cos\theta_w A^{\mu} - \sin\theta_w Z^{\mu}\right)$$

24/3/06.3 | MPS, 127 | FRIday | Abingdon |

$$= \left(g \sin\theta_w \, \bar{J}_\mu^{\,3} + g' \cos\theta_w \frac{\bar{J}_\mu^{\,Y}}{2} \right) A^\mu + \left(g \cos\theta_w \, \bar{J}_\mu^{\,3} - g' \sin\theta_w \frac{\bar{J}_\mu^{\,Y}}{2} \right) Z^\mu .$$

The first term is the electromagnetic interaction

$$e \, j_\mu^{em} A^\mu = e \left(j_\mu^{3} + \frac{1}{2} j_\mu^{Y} \right) A^\mu$$

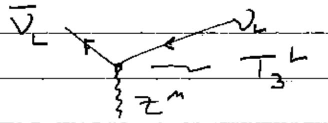
Therefore we have to impose

$$e = g \sin\theta_w = g' \cos\theta_w \Rightarrow \tan\theta_w = g'/g$$

We can express the neutral current interaction in the form

$$\frac{g}{\cos\theta_w} \left(\bar{j}_\mu^{\,3} - \sin^2\theta_w \, j_\mu^{em} \right) Z^\mu = \frac{g}{\cos\theta_w} \bar{J}_\mu^{\,NC} Z^\mu .$$

We now have a new neutral current coupling neutrinos to Z



This neutral current was proposed by Glashow in 1961 and observed at CERN in 1970.

→ we still have a problem.

→ E/M interactions → long range → $m_\gamma = 0$

→ weak interactions → short range → $M_{W^\pm Z} \neq 0$

now → symmetry breaking.

Lagrange - invariant, but, Vacuum → symmetry broken

22/4/06.1 | MPS.128 | Saturday | Abingdon |

The vacuum → the state of lowest energy.

The Higgs mechanism

consider a scalar field with the Lagrangian.

$$\mathcal{L} \equiv T - V = \frac{1}{2}(\partial_\mu \phi)^2 - \left(\frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4\right).$$

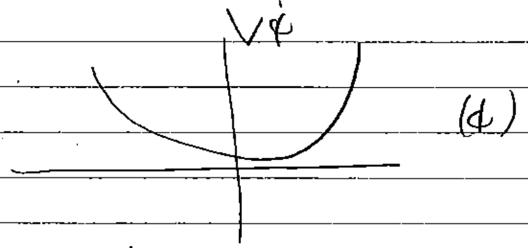
$$V(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4.$$

with $\lambda > 0$.

This Lagrangian is invariant under the transformation,

$$\phi \longrightarrow -\phi$$

For $\mu^2 > 0$ the potential looks like

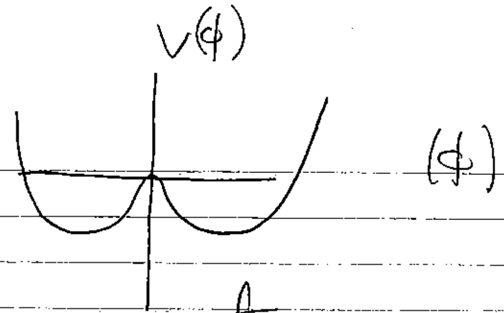


The Lagrangian describes a self-interacting Scalar field with coupling λ and mass μ

The ground states correspond to $\langle\phi\rangle = 0$.

and it obeys the Reflection symmetry of the Lagrangian.

12/4/06. 4 / MPS. 129 / Saturday / Abingdon /



For $\mu^2 < 0$

The potential has two minima at

$$\frac{\partial V}{\partial \phi} = \phi\left(\mu^2 + \lambda \phi^2\right) = 0$$

$$\Rightarrow \langle \phi \rangle = \pm v \quad \text{with} \quad v = \sqrt{\frac{-\mu^2}{\lambda}}$$

The extremum $\phi = 0$ does not correspond to the minimum of the energy.

We perform perturbative calculation around the classical minimum.

$$\phi = v \qquad \text{OR} \qquad \phi = -v.$$

<u>we write</u>

$$\phi(x) = v + \eta(x)$$

$\eta(x)$ represents quantum fluctuations about this minimum.

Substituting into the Lagrangian we obtain.

$$\mathcal{L}' = \frac{1}{2}\partial_\mu(v+\eta)\,\partial^\mu(v+\eta) - \frac{1}{2}\left(\mu^2(v+\eta)^2 + \frac{\lambda}{4}(v+\eta)^4\right)$$

$$= \frac{1}{2}(\partial_\mu \eta)^2 - \left(\frac{\mu^2}{2}(v^2 + 2\eta v + \eta^2) + \frac{\lambda}{4}(v^4 + 4v^3\eta + 6v^2\eta^2\right.$$

$\boxed{\mu^2 = -\lambda v^2}$ $$\left. + 4v\eta^3 + \eta^4)\right) = \frac{1}{2}(\partial_\mu \eta)^2 + \frac{\lambda v^2}{2}(v^2 + \cancel{2\eta v} + \eta^2)$$

2/4/06.3 | MPS, 130

$$-\frac{\lambda}{4}\left(v^4+\cancel{4v^3\eta}+6v^2\eta^2+4v\eta^3+\eta^4\right)=$$

$$=\frac{1}{2}(\partial_\mu\eta)^2-\lambda v^2\eta^2-\lambda v\,\eta^3-\frac{\lambda}{4}\eta^4+\text{const},$$

The field η has a mass term with the correct sign.

$$m_\eta=\sqrt{2\lambda v^2}=\sqrt{-2\mu^2}$$

The higher order terms in η are self-interaction terms.

we do perturbation theory around a stable minimum,

$\phi=+v+\eta$. ϕ

η is a massive field

The reflection symmetry is broken by the choice of the vacuum.

Consider now a complex scalar field.

$$\phi=(\phi_1+i\phi_2)/\sqrt{2}.$$

$$\mathcal{L}=(\partial_\mu\phi)^*(\partial^\mu\phi)-\mu^2\phi^*\phi-\lambda(\phi^*\phi)^2$$

$\mathcal{L}$ is invariant under the global $U(1)$ symmetry.

$$\phi\to e^{i\alpha}\phi.$$

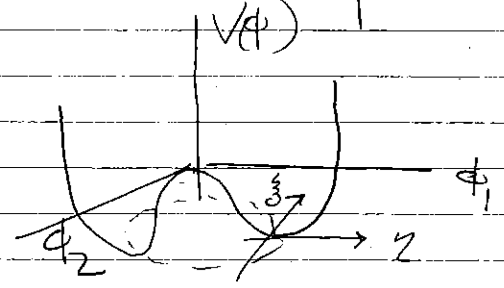
For λ and $\mu^2<0$

we rewrite the Lagrangian

$$\mathcal{L}=\frac{1}{2}(\partial_\mu\phi_1)^2+\frac{1}{2}(\partial_\mu\phi_2)^2-\frac{1}{2}\mu^2(\phi_1^2+\phi_2^2)-\frac{1}{4}\lambda(\phi_1^2+\phi_2^2)^2$$

22/4/06, 4 | MPS. 131 |

The is now a circle of minima



at $\phi_1^2 + \phi_2^2 = v^2$ with $v^2 = -\frac{\mu^2}{\lambda}$.

We translate the field ϕ to, $\phi_1 = v$ $\phi_2 = 0$.
Expand the Lagrangian around the vacuum,

with $\phi(x) = \sqrt{\tfrac{1}{2}}\,[v + \eta(x) + i\,\xi(x)]$

$$\mathcal{L}' = \frac{1}{2}(\partial_\mu \eta)^2 + \frac{1}{2}(\partial_\mu \xi)^2 + \mu^2 \eta^2 + \text{const.} + \text{cubic \& quartic terms in } \eta, \xi.$$

The term $\mu^2\eta^2$ is a mass term for the η field $m_\eta = \sqrt{-2\mu^2}$ as before.

There is no corresponding mass term for ξ.
→ massless scalar field.
Goldstone theorem → spontaneously broken continuous global symmetry → Goldstone boson.

22.4.06. 5 / MPS. B 2 Saturday (Abingdon)

Consider now a complex scalar field coupled to a continuous $U(1)$ symmetry.

$$\phi(x) \longrightarrow e^{i\alpha(x)}\phi(x)$$

As we saw local invariance requires that we replace ∂_μ by

$$D_\mu = \partial_\mu - ieA_\mu.$$

and the gauge field A_μ transforms as

$$A_\mu \longrightarrow A_\mu + \frac{1}{e}\partial_\mu \alpha.$$

The gauge invariant Lagrangian is

$$\mathcal{L} = (\partial^\mu + ieA^\mu)\phi^*(\partial_\mu - ieA_\mu)\phi - \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2 - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

For $\mu^2 > 0$
Lagrangian for charged selfinteracting scalar field with mass μ.
For $\mu^2 < 0$ Expand

$$\phi(x) = \frac{1}{\sqrt{2}}\left(v + \eta(x) + i\xi(x)\right).$$

$$\Rightarrow \mathcal{L}' = \frac{1}{2}(\partial_\mu\xi)^2 + \frac{1}{2}(\partial_\mu\eta)^2 - v^2\lambda\eta^2 + \frac{1}{2}e^2v^2A^\mu A_\mu - evA_\mu\partial^\mu\xi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \text{interaction terms}.$$

2.7.06, 6 / MPS, 133 / Saturday / Abingdon /

The particle spectrum appears to be

massless goldstone boson ξ. $\quad m_\xi = 0$

massive scalar $\quad \eta$, $\quad m_\eta = \sqrt{2\lambda v^2}$

a massive vector boson $\quad A_\mu$. $\quad m_A = ev$

However the interpretation should be revised.

massless $\quad A_\mu \to 2_T$ physical degrees of freedom.

massive $\quad A_\mu \to 2_T + 1_L$ physical degrees of freedom.

where did the third D.O.F come from. ?

note that to lowest order.

$$\phi = \sqrt{\tfrac{1}{2}}\left(v + \eta(x) + i\xi(x)\right) \cong \sqrt{\tfrac{1}{2}}\left(v + \eta(x)\right) e^{i\xi(x)/v},$$

$\to$ use a different set of fields. $\quad h, \theta, A_\mu$.

$$\phi \to \frac{1}{\sqrt{2}}\left(v + h(x)\right) e^{i\theta(x)/v}$$

$$A_\mu \to A_\mu + \frac{1}{ev}\partial_\mu \theta$$

substitute into $\mathcal{L}$.

we get
$$\mathcal{L}'' = \frac{1}{2}(\partial_\mu h)^2 - \lambda v^2 h^2 + \frac{1}{2}e^2 v^2 A_\mu A^\mu - \lambda v h^3 - \frac{1}{4}\lambda h^4 + \frac{1}{2}e^2 A_\mu^2 h^2 + v e^2 A_\mu^2 h - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

The Goldstone boson disappeared altogether.

$\to$ the Goldstone boson is absorbed as the longitudinal mode of A^μ $\to$ only 2 physical fields $\quad h, A^\mu$.

22.4.06,7 / MPS, 134 / Saturday / Abingdon /

We are ready to see how the Higgs mechanism operates in the standard model.

Take $\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial_\mu \phi) - \mu^2 (\phi^\dagger \phi) - \lambda (\phi^\dagger \phi)^2$

with $\phi = \begin{pmatrix} \phi_\alpha \\ \phi_\beta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}$.

The Lagrangian is invariant under the gauge transformation

$$\phi \to \phi' = e^{i \alpha_a \tau_a / 2} \phi$$

The covariant Derivative.

$$D_\mu = \partial_\mu + ig \frac{\tau_a}{2} W_\mu^a \qquad a = 1, 2, 3.$$

under

$$\phi(x) \to \phi'(x) = \left(1 + \alpha(x) \cdot \frac{\vec{\tau}}{2}\right) \phi(x)$$

$$\vec{W}_\mu \to \vec{W}_\mu - \frac{1}{g} \partial_\mu \vec{\alpha} - \vec{\alpha} \times \vec{W}_\mu.$$

The gauge invariant Lagrangian is

$$\mathcal{L} = \left(\partial_\mu \phi + ig \frac{1}{2} \vec{\tau} \cdot \vec{W}_\mu \phi\right)^\dagger \left(\partial^\mu \phi + ig \frac{1}{2} \vec{\tau} \cdot \vec{W}^\mu \phi\right) - V(\phi)$$

$$- \frac{1}{4} \vec{W}_{\mu\nu} \cdot \vec{W}^{\mu\nu}$$

where $\vec{W}_{\mu\nu} = \partial_\mu \vec{W}_\nu - \partial_\nu \vec{W}_\mu - g \vec{W}_\mu \times \vec{W}_\nu$

22.4.06 | MPS, 135 | Saturday | Abingdon

Take $\mu^2 < 0 \qquad \lambda > 0$.

The potential has a minimum at

$$\phi^\dagger \phi = \frac{1}{2}\left(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2\right) = -\frac{\mu^2}{2\lambda}.$$

Expand about a minimum. choose

$$\phi_1 = \phi_2 = \phi_4 = 0 \qquad \phi_3^2 = -\frac{\mu^2}{\lambda} \equiv v^2$$

Expand $\phi(x)$ about the vacuum,

$$\phi_0 = \sqrt{\frac{1}{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}.$$

$$\phi(x) = \sqrt{\frac{1}{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$$

The three additional degrees of freedom are absorbed as the longitudinal components of $W_1^\mu, W_2^\mu, W_3^\mu$.

The mass term.

$$\left| i g \frac{1}{2} \vec{\tau} \cdot \vec{W}_\mu \phi \right|^2 =$$

$$\frac{g^2}{8} \left| \begin{pmatrix} W_\mu^3 & W_\mu^1 - i W_\mu^2 \\ W_\mu^1 + i W_\mu^2 & W_\mu^3 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \right|^2$$

$$= \frac{g^2 v^2}{8} \left[(W_\mu^1)^2 + (W_\mu^2)^2 + (W_\mu^3)^2 \right]$$

12.4.96.9 | MPS.136 | Saturday | Abingdon

Now consider the Lagrangians of the $SU(2)_L \times U(1)_Y$ of the Standard Model coupled to ϕ.

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \quad \text{with} \quad \phi^+ = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

$$\phi^0 = \frac{1}{\sqrt{2}}(\phi_3 + i\phi_4)$$

$$\vec{T} = \frac{\vec{\tau}}{2}$$

$$\mathcal{L}_2 = \left|\left(i\partial_\mu - g\vec{T}\cdot\vec{W}_\mu - g'\frac{Y}{2}B_\mu\right)\phi\right|^2 - V(\phi)$$

$$V(\phi) = \mu^2(\phi^+\phi) + \lambda(\phi^+\phi)^2$$

with $\mu^2 < 0 \quad \lambda > 0.$

Expand about $\quad \phi_0 = \sqrt{\frac{1}{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}$

The mass terms for the gauge bosons.

$$\left|\left(-ig\frac{\vec{\tau}}{2}\cdot\vec{W}_\mu - i\frac{g'}{2}B_\mu\right)\phi\right|^2 =$$

$$= \frac{1}{8}\left|\begin{pmatrix} gW^3_\mu + g'B_\mu & g(W^1_\mu - iW^2_\mu) \\ g(W^1_\mu + iW^2_\mu) & -gW^3_\mu + g'B_\mu \end{pmatrix}\begin{pmatrix} 0 \\ v \end{pmatrix}\right|^2$$

$$= \frac{1}{8}v^2g^2\left[(W^1_\mu)^2 + (W^2_\mu)^2\right] + \frac{v^2}{8}(g'B_\mu - gW^3_\mu)(g'B^\mu - gW^{3\mu})$$

$$= \frac{1}{2}(vg)^2 W^+_\mu W^{\mu -} + \frac{1}{8}v^2(W^3_\mu, B_\mu)\begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix}\begin{pmatrix} W^{3\mu} \\ B^\mu \end{pmatrix}.$$

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The first term is the mass term for W^+, W^-

$$M_{W^\pm} = \frac{1}{2} v g.$$

The second term gives

$$\frac{1}{8} v^2 [g W_\mu^3 - g' B_\mu]^2 + 0 [g' W_\mu^3 + g B_\mu]^2$$

$$= \frac{1}{2} M_Z^2 Z_\mu^2 + \frac{1}{2} M_A^2 A_\mu^2$$

massive　　　　massless.

normalizing the fields.

$$A_\mu = \frac{g' W_\mu^3 + g B_\mu}{\sqrt{g^2 + g'^2}} \quad \text{with } M_A = 0.$$

$$Z_\mu = \frac{g W_\mu^3 - g' B_\mu}{\sqrt{g^2 + g'^2}} \quad \text{with } M_Z = \frac{1}{2} v \sqrt{g^2 + g'^2}$$

using the relation $g'/g = \tan\theta_W$.

we have. $\frac{M_W}{M_Z} = \cos\theta_W$.

→ verified Experimentally to high precision