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Lorentz group.

A contravariant 4-vector $x^\mu = (ct, \vec{x})$ $\mu = 0,1,2,3$

In particle physics we often set $c = 1$ $\hbar = 1$

In this convention $m_{electron} = 0.5 \frac{MeV}{c^2} = 0.5 MeV$

etc. $c \neq 1$ may be restored if needed, otherwise all quatities are expressed in energy units.

hence

$$x^0 = t \qquad x^i = \vec{x}.$$

the length of the four vector is given by.

$$x \cdot x = c^2 t^2 - x^2 - y^2 - z^2 \underset{c=1}{=} t^2 - x^2 - y^2 - z^2$$

This is symbolized by.

$$x \cdot x = \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} x^\mu x^\nu = x_\mu x^\mu$$

where

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

is the Minkowski metric.

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In general the scalar product of two 4-vectors X^μ; y^μ is given by $X \cdot y = \sum_{\mu,\nu} \eta_{\mu\nu} X^\mu y^\nu$

We will use Einstein's summation convention

a down index is summed with an up index and the summation symbol is dropped.

Scalar - no free indecies. - (all are summed in the scalar product).

Vector - one free index (X^μ).

tensor - two and more ($g^{\mu\nu}$, $R_{\mu\nu\rho\sigma}$, ...)

The Lorentz transformations are the transformation that preserve the scalar product.

In particular they preserve the length of a 4-vector

$$t^2 - \vec{X}^2 = X \cdot X = \eta_{\mu\nu} X^\mu X^\nu = X' \cdot X' = \eta_{\mu\nu} X'^\mu X'^\nu = t'^2 - \vec{X}'^2$$

where X and X' are related by a Lorentz transformation.

In general we write the metric as $g^{\mu\nu}$ and its components can be functions of space-time $\Rightarrow$ general relativity. curved space

In flat-space $g^{\mu\nu} \rightarrow \eta^{\mu\nu}$

$g^{\mu\nu} g_{\nu\sigma} = \delta^{\mu}{}_{\sigma}$ $(g^{\mu\nu})^{-1} =$

We write: $X_\mu = \sum_{\nu=0}^{3} g_{\mu\nu} X^\nu$

X_μ is a covariant vector.

$X_\mu = (t, -\vec{x})$.

The Lorentz invariant can be written as $X_\mu X^\mu$.

Given a vector $X^\mu = (t, \vec{x})$.

There are two differential quantities of interest.

1. $dX^\mu \rightarrow$ differential $\rightarrow$ contravariant 4-vector.
2. $\frac{\partial}{\partial X^\mu} = \partial_\mu \rightarrow$ gradient $\rightarrow$ covariant 4-vector.

How do they behave under coordinate transformations $X^\mu \rightarrow X'^\mu$:

$$dX^\mu \rightarrow dX'^\mu = \frac{\partial X'^\mu}{\partial X^\nu} dX^\nu$$

$$\frac{\partial}{\partial X^\mu} \rightarrow \frac{\partial}{\partial X'^\mu} = \frac{\partial X^\nu}{\partial X'^\mu} \frac{\partial}{\partial X^\nu}$$

The differential and the gradient transform differently.

Vector that transforms like the differential is a contra-variant vector.

$$V'^{\mu} = \frac{\partial X'^{\mu}}{\partial X^{\nu}} V^{\nu}$$

Vector that transforms like the gradient is a covariant vector,

$$V'_{\mu} = \frac{\partial X^{\nu}}{\partial X'^{\mu}} V_{\nu} \qquad \boxed{\partial_{\mu} = \frac{\partial}{\partial X^{\mu}}}$$

Example the momentum vector $P^{\mu} = (E, \vec{P})$
$P_{\mu} = (E, -\vec{P})$.

in relativistic quantum mechanics $P_{\mu} \sim \frac{\partial}{\partial X^{\mu}}$

$P_{\mu} X^{\mu} = Et - \vec{P} \cdot \vec{X}$

Properties of Lorentz transformations

1. Transformation that preserve the scalar product and the length of 4-vectors

$g_{\mu\nu} X^{\mu} X^{\nu}$ is invariant under Lorentz transformation
no change in size and shape.

The invariance implies the existence of a symmetry
The symmetry is generated by a GROUP.

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2. Assume $X^\mu \to X'^\mu$ under some Lorentz trans.

$$X^\mu \to X'^\mu = \Lambda^\mu{}_\nu X^\nu$$

$$g_{\mu\nu} X^\mu X^\nu \to g_{\mu\nu} X'^\mu X'^\nu = g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta X^\alpha X^\beta$$

$$= g_{\alpha\beta} X^\alpha X^\beta$$

$$\Rightarrow (*) \quad g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = g_{\alpha\beta}$$

in matrix notation $\Lambda^T g \Lambda = g$

$\to$ Defines the Lorentz transformations.

$$\Rightarrow (\det \Lambda)^2 = 1 \Rightarrow \det \Lambda = \pm 1$$

The physical case $\det \Lambda = 1 \to$ continuousely connected to the identity.

$\det \Lambda = 1 \to$ Proper Lorentz transformations

$\det \Lambda = -1 \to$ improper " "

look at the 00 component of $\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g_{\mu\nu} = g_{\alpha\beta}$

$$g_{\mu\nu} \Lambda^\mu{}_0 \Lambda^\nu{}_0 = +1$$

$$+(\Lambda^0{}_0)^2 - \sum_i (\Lambda^i{}_0)^2 = +1$$

$$(\Lambda^0{}_0)^2 = 1 + \sum_i (\Lambda^i{}_0)^2 \geq 1$$

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$$(\Lambda^0{}_0)^2 \geq 1 \quad \begin{cases} \Lambda^0{}_0 \geq 1 & \text{orthochronos} \\ \Lambda^0{}_0 \leq -1 & \text{non orthochronos} \end{cases}$$

An example of a nonorthochronos Lorentz transformation is given by:

<u>Reflection:</u> $\Lambda^\mu{}_\nu = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

The Lorentz transformations that are continuously connected to the ident.

ⓐ $\det \Lambda = 1$ ← Proper.

ⓑ $\Lambda^0{}_0 \geq 1$ ← orthochronos.

<u>Examples:</u> $\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ $\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$.

ⓐ <u>Rotation</u> $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & \vec{R} \end{pmatrix}$

$\det \Lambda = \det R$

$\det R = \pm 1$

$\det R = +1 \rightarrow$ Proper.

ⓑ boosts in X-direction: $\Lambda = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\det \Lambda = \cosh^2\eta - \sinh^2\eta = 1$$

$$\Lambda^0{}_0 = \cosh\eta \geq 1.$$

ⓒ time inversion $\Lambda = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

$\det \Lambda = -1$ $\Lambda^0{}_0 = -1 \Rightarrow$ improper non-orthochronos.

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(d) Full inversion $\Lambda = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

$\det \Lambda = +1 \quad \Lambda^0{}_0 = -1 \Rightarrow$ Proper non-orthochronos

$\rightarrow$ All L.T. are generated by the above.

The proper orthochronos Lorentz transformations correspond to Rotations & boosts.

$\rightarrow$ 6 parameters = 3 angles + 3 boosts.

L.T. that are proper orthochronos are continuously connected to the identity

Performing an infinitesimal proper orthochronos L.T.

(**) $$\Lambda^\mu{}_\nu = \underbrace{\delta^\mu{}_\nu}_{\text{identity trans.}} + \underbrace{\omega^\mu{}_\nu}_{\text{an infinitesimal transformation}}$$

$\delta^\mu{}_\nu = +1$ for $\mu = \nu$; $= 0$ for $\mu \neq \nu$

calculate to order $O(\omega)$

for continuous transformations the properties of infinitesimal transformations fixes the transformation properties of the Lorentz group.

finite transformations are obtained by integration

- substitute $(**)$ in $g_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} = g_{\alpha\beta}$

$$g_{\mu\nu}\left(\delta^{\mu}{}_{\alpha}+W^{\mu}{}_{\alpha}\right)\left(\delta^{\nu}{}_{\beta}+W^{\nu}{}_{\beta}\right) = g_{\alpha\beta}$$

$$g_{\mu\nu}\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta} + \underbrace{g_{\mu\nu}W^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta}}_{g_{\mu\beta}W^{\mu}{}_{\alpha}} + \underbrace{g_{\mu\nu}\delta^{\mu}{}_{\alpha}W^{\nu}{}_{\beta}}_{g_{\alpha\nu}W^{\nu}{}_{\beta}} + g_{\mu\nu}W^{\mu}{}_{\alpha}W^{\nu}{}_{\beta} = g_{\alpha\beta}$$

$$g_{\alpha\beta} \quad + \quad W_{\beta\alpha} \quad + \quad W_{\alpha\beta}$$

end of lecture 2.

$$W_{\beta\alpha} \quad + \quad W_{\alpha\beta}$$

- The tensor of infinitesimal transformations is antisymmetric.

<u>Example:</u> Rotation group in two dimensions.

$$\begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix} \simeq \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix} + \begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}\theta \quad \text{for small } \theta.$$

The number of degrees of freedom in a 4×4 antisymmetric matrix.

<u>For general n:</u> $\dfrac{n^2-n}{2} = \dfrac{n(n-1)}{2}$

for $n=4 \Rightarrow 4\cdot 3/2 = 6 \rightarrow$ 3 Rotation angles + 3 boosts.

- $$W_{\mu\nu} = \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & b_3 & -b_2 \\ -a_2 & -b_3 & 0 & b_1 \\ -a_3 & b_2 & -b_1 & 0 \end{pmatrix}$$

Exercise: show how a finite transformation is obtained from infinitesimal transformations.

Algebraic Properties of the Lorentz Group

we associate an operator $U(\Lambda)$ with the LT Λ.

for the special case $\Lambda = \delta \rightarrow U(\delta) = I$

we want to find the operator associated with $\Lambda = \delta + \omega$.

to order $O(\omega)$ $\quad U(\delta+\omega) = I + \frac{1}{2} i \underbrace{J_{\mu\nu}}_{\text{operators}} \underbrace{\omega^{\mu\nu}}_{\text{infinitesimal trans.}} + \cdots$

Look at the example of rotations $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & \alpha & -\beta \\ -\alpha & 1 & \gamma \\ \beta & -\gamma & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$A = I + \alpha \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

operators $J^{\mu\nu}$ $\qquad \alpha, \beta, \gamma \in \omega^{\mu\nu}$

These operators are non-hermitian $\rightarrow$ multiply by i to get hermitian operators

$$A = I + i\alpha \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + i\beta \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} + i\gamma \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

now the operators are hermitian.

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- From the definition of $U(\delta+\omega)$ we defined 16 $J_{\mu\nu}$ operators only the antisymmetric components are independent → only 6 are independent.

Define the operators

$$K_i = J_{i0} = -J_{0i} \quad i=1,2,3. \qquad \text{(3 operators)}$$

$$J_i = \frac{1}{2}\sum_{jk} \epsilon_{ijk} J_{ik} \qquad \text{(3 operators)}$$

This is a projection of the antisymmetry of ω on J.
These are the six physical generators of the L.G.

$$U(\delta+\omega) = I + i\vec{a}\cdot\vec{K} - i\vec{b}\cdot\vec{J} \qquad \omega^{\mu\nu} = \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & b_3 & -b_2 \\ -a_2 & -b_3 & 0 & b_1 \\ -a_3 & b_2 & -b_1 & 0 \end{pmatrix}$$

- Example: Regular Rotations $\left(A = I + i\alpha\vec{J} \right)$.

The operators $\vec{K}, \vec{J}$ are the basic operators of the L.G.
These are the generators of the L.G.
$\vec{J}$ are the generators of Rotations
K are the generators of boosts.

We will determine the commutation relations by imposing the group property on U

$$(*) \quad U(\Lambda_1\Lambda_2) = U(\Lambda_1)\,U(\Lambda_2) \leftarrow \text{the group property.}$$

- for Regular Rotations $\Lambda_1 = \theta_1 \quad \Lambda_2 = \theta_2 \quad \Lambda_1\Lambda_2 = \theta_1 + \theta_2$

$$\begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix}\begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{pmatrix} = \begin{pmatrix} \cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) \\ -\sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix}.$$

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$$U(\Lambda_1) = I + \frac{i}{2} J_{\mu\nu} W_1^{\mu\nu} \qquad \Lambda_1 = \delta + \omega_1$$
$$U(\Lambda_2) = I + \frac{i}{2} J_{\mu\nu} W_2^{\mu\nu} \qquad \Lambda_2 = \delta + \omega_2$$

$$\Lambda_1\Lambda_2 = (\delta + \omega_1)(\delta+\omega_2) \underset{\text{first order}}{=} \delta + (\omega_1 + \omega_2) + O(\omega^2)$$

$$\rightarrow \quad J(\Lambda_1\Lambda_2) = I + \frac{i}{2} J_{\mu\nu} (\omega_1^{\mu\nu} + \omega_2^{\mu\nu})$$

substituting into the group property (*) and keeping terms to second order in $O(\omega^2)$ we get the commutation relations.

Rotations $\rightarrow$ $[J_i, J_j] = i\epsilon_{ijk} J_k \qquad i,j,k = 1,2,3$

$$[J_i, K_j] = i\epsilon_{ijk} K_k$$

$$[K_i, K_j] = -i\epsilon_{ijk} J_k$$

The minus sign in the third relation distinguishes the Lorentz transformations from ordinary Rotations in four dimensions

define the combinations

$$\vec{J}_+ = \frac{1}{2}(\vec{J} + i\vec{K}) \qquad \vec{J}_- = \frac{1}{2}(\vec{J} - i\vec{K})$$

$\vec{J}_+, \vec{J}_-$ are not hermitian $\quad \vec{J}_+^{\,+} = \vec{J}_- \quad \vec{J}_-^{\,+} = \vec{J}_+$

The commutation relations for J_+, J_-

$$[J_i^+, J_j^+] = \frac{1}{4}[J_i + iK_i, J_j + iK_j] = \frac{1}{4} i\epsilon_{ijk}(J_k + iK_k + iK_k + J_k) = i\epsilon_{ijk} J_k^+$$

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similarly. $[J_i^+, J_j^+] = i\epsilon_{ijk} J_k^+$

$[J_i^-, J_j^-] = i\epsilon_{ijk} J_k^-$

$[J_i^-, J_j^+] = 0.$

we found two disjoint groups of generators each obeying an SU(2) algebra

$SU(2) \times SU(2)$

Each representation of the Lorentz group is labeled by the indecies of the two disjoint SU(2) algebras (j_1, j_2).

each representation has $(2j_1+1) \otimes (2j_2+1)$ components.

as $\vec{J} = \vec{J}_+ + \vec{J}_-$ spin is given by $j_1 + j_2$

Examples:

	(j_1, j_2)	spin	components	
(a)	$(0,0)$	0	1	singlet.
(b)	$(\frac{1}{2},0)$	$\frac{1}{2}$	2	Weyl-spinor
(c)	$(0,\frac{1}{2})$	$\frac{1}{2}$	2	Weyl spinor
(d)	$(\frac{1}{2},0)+(0,\frac{1}{2})$	$\frac{1}{2}$	4	Dirac spinor
(e)	$(\frac{1}{2},\frac{1}{2})$	$1,0$	4	Vector.

if we write $x^\mu = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$ under rotation t is a singlet. $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is a triplet

the generator of rotations $\vec{J} = \vec{J}_+ + \vec{J}_-$

under rotations a four vector decomposes into a singlet and a triplet.

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From the point of view of $(\frac{1}{2}, \frac{1}{2})$ Rep. $\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow$

$3 \to \uparrow\uparrow \; ; \; \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \; ; \; \downarrow\downarrow$ spin = 1 $j_3 = (-1, 0, 1)$

$1 \to \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$ spin 0 $j_3 = 0$

The Poincare group.

we saw that spin is a label of representations of the Lorentz group.

However we cannot yet classify elementary particles, which also have mass!

The Lorentz transformation are not the most general.

we should write down all the symmetries of a Relativistic line element.

$(*)$ $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$

So far we discussed rotations & boosts that form the symmetries of the Lorentz group.

We want to write the most general set of transformation that keep the line element invariant.

look at the two dimensional case: $ds^2 = -dt^2 + dx^2$

we perform the most general infinitesimal transformations.

$t \to t + \epsilon T(t, x)$ ϵ an infinitesimal number

$x \to x + \epsilon R(t, x)$

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We want to find the function T, R such that the line element remains invariant.

$$dt \to dt + \epsilon\left(\frac{\partial T}{\partial t}dt + \frac{\partial T}{\partial x}dx\right)$$

$$dx \to dx + \epsilon\left(\frac{\partial R}{\partial t}dt + \frac{\partial R}{\partial x}dx\right)$$

$$ds^2 \to -\left[\left(1+\epsilon\frac{\partial T}{\partial t}\right)dt + \epsilon\frac{\partial T}{\partial x}dx\right]^2 + \left[\left(1+\epsilon\frac{\partial R}{\partial x}\right)dx + \epsilon\frac{\partial R}{\partial t}dt\right]^2$$

we keep first order terms in ϵ and impose invariance

$$ds^2 = -dt^2 + dx^2 + \left(-\frac{\partial T}{\partial t}dt^2 + \frac{\partial R}{\partial x}dx^2 + \left(-\frac{\partial T}{\partial x} + \frac{\partial R}{\partial t}\right)dx\,dt\right)\epsilon + O(\epsilon^2)$$

$$\Rightarrow \quad \frac{\partial T}{\partial t} = 0 \quad \Rightarrow \quad T(x)$$

$$\frac{\partial R}{\partial x} = 0 \quad \Rightarrow \quad R(t).$$

$$-\frac{\partial T}{\partial x} + \frac{\partial R}{\partial t} = 0 \quad \Rightarrow \quad \frac{\partial T}{\partial x} = \frac{\partial R}{\partial t} = \text{const.}$$

$$\Rightarrow \quad T(x) = c\,x + a$$
$$R(t) = c\,t + b$$

We obtained 3 degrees of freedom that are represented by three constants of the motion a, b, c.

We check what are the three degrees of freedom.

Choose $c = b = 0$ $\begin{cases} x \to x \\ t \to t + \epsilon a = t + a^{\epsilon} \end{cases}$ ← translation in time.

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ⓑ fix $a = c = 0 \quad b \neq 0$ $\quad t \to t$, $x \to x + b^{\epsilon}$ — translation in space.

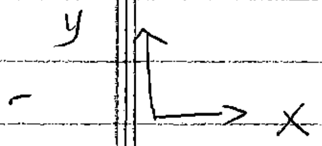
ⓒ fix $a = b = 0 \quad c \neq 0$. $\quad t \to t + cx$, $x \to x + ct$.

The third transformation is a boost.

If we had taken the line element to be $ds^2 = dt^2 + dx^2$ we would have obtained ordinary rotations.

$$t \to t + cx \qquad \begin{pmatrix} t \\ x \end{pmatrix} \to \begin{pmatrix} t \\ x \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} t \\ x \end{pmatrix}$$
$$x \to x - ct$$

As we saw before



$$ds^2 = dx^2 + dy^2 \quad \to \quad 2D \text{ Rotation}$$

boost is a generalization of Rotations to 4D Minkowski space-time.

Returning to 4D Minkowski space-time.

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2.$$

we have

4	translations	dt, dx, dy, dz
3	Rotations	$dxdy, dxdz, dydz$
3	boost	$dtdx, dtdy, dtdz$.

The group that describes these set of symmetries is

the Poincare group.

The total number of generators of the poincare group is 10.

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our aim is to find the algebra of the Poincare group. and its invariants,
The translation symmetries are important
The momentum operator generates translations.

we will therefore obtain momentum and mass ($P_\mu P^\mu = m^2$) which is second label of particle states.

let's look at an example: in one dimension

$$X \to X + a$$

$$\phi(x) \to \phi(x+a).$$

we are looking for an operator which induces this operation.

$$\phi(x+a) = U(a)\,\phi(x)$$

$$\phi(x) \to \phi(x+a) = \sum_n \frac{a^n}{n!}\left(\frac{\partial^n}{\partial x^n}\phi(x)\right)\Big|_{a=0} =$$

$$= \underbrace{\left(\sum_n \frac{a^n}{n!}\frac{\partial^n}{\partial x^n}\right)}_{\text{operator}}\phi(x) = e^{a\frac{\partial}{\partial x}}\,\phi(x).$$

$$U(a) = e^{a\frac{\partial}{\partial x}} \quad \leftarrow \text{ is the operator}$$

$$U(a) = e^{a\frac{\partial}{\partial x}} \simeq 1 + a\frac{\partial}{\partial x} + \cdots = 1 + i\left(-i a\frac{\partial}{\partial x}\right) + \cdots$$

$$= 1 + i a P$$

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$$\underline{P} = -i\frac{\partial}{\partial x}$$

The complex factor i arises from the hermiticity of $\overline{U}$

$$\overline{U} = \underline{I} + ia\underline{P}$$
$$\overline{U}^{+} = \underline{I} - ia\underline{P}$$
$$U^{-1} = \underline{I} - ia\underline{P}$$
$$U^{+}U = (\underline{I} + ia\underline{P})(\underline{I} - ia\underline{P}) = \underline{I} + a^2\underline{P}^2 \simeq \underline{I}$$

In four space-time dimensions

$$\overline{U}(a^\mu) \simeq \underline{I} + i a^\mu \underline{P}_\mu$$

The generators of the Poincare group are:

$$\underline{P}_\mu = -i\partial_\mu \quad \leftarrow \text{translations} \rightarrow 4$$
$$\underline{L}_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \quad \leftarrow \text{Rotations + boosts} \rightarrow 6$$

The most general transformation consistent with the Poincare group

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$$

we want to find:

(*) The commutation relations among the generators of the group.

(*) The maximal set of commuting operators

Example: Rotations $[L_i, L_j] = i\epsilon_{ijk} L_k$.

Casimir operator $\leftarrow$ $[L^2, L_i] = 0 \rightarrow$ 2 mutually commuting L^2, L_z.

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$L^2 |j,m\rangle = j(j+1)\, |j,m\rangle$

$L_z |j,m\rangle = m \quad |j,m\rangle$

The states are labelled by their eigenvalues under the commuting operators.

We saw that $P_\mu = -i \frac{\partial}{\partial x^\mu}$ ← 4 generators.

① $\Rightarrow [P_\mu, P_\nu] = \left[-i\frac{\partial}{\partial x^\mu}, -i\frac{\partial}{\partial x^\nu}\right] = 0$

If we perform two successive translations the order does not matter.

$x^\mu \to x'^\mu = x^\mu + a^\mu$ ① translate by a^μ

$x'^\mu \to x''^\mu = x'^\mu + b^\mu$ ② " " b^μ

$= x^\mu + a^\mu + b^\mu$

$x^\mu \to x'^\mu = x^\mu + b^\mu$ ① " by b^μ

$x'^\mu \to x''^\mu = x'^\mu + a^\mu$ ② " by a^μ

$= x^\mu + b^\mu + a^\mu$

$= x^\mu + a^\mu + b^\mu$

The order of the translation does not matter hence the commutator of the two operations (generator). commutes.

② $[P_\mu, X^\nu] = \left[-i\frac{\partial}{\partial x^\mu}, X^\nu\right] = -i\left[\frac{\partial}{\partial x^\mu}, X^\nu\right] = -i\delta^\nu_\mu.$

③ $[P_\mu, R_i] = ?$ $[P_\mu, J_i] = ?$

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we perform two successive Poincare trans.

① $X^\mu \to X'^\mu = \Lambda^{(1)\mu}{}_\nu X^\nu + a^{(1)\mu}$ first trans.

② $X'^\mu \to X''^\mu = \Lambda^{(2)\mu}{}_\nu X'^\nu + a^{(2)\mu} =$ second trans.

$= \Lambda^{(2)\mu}{}_\nu \left(\Lambda^{(1)\nu}{}_\lambda X^\lambda + a^{(1)\nu}\right) + a^{(2)\mu}$

$= \underbrace{\Lambda^{(2)\mu}{}_\nu \Lambda^{(1)\nu}{}_\lambda}_{\text{product of 2 successive L.T.}} X^\lambda + \underbrace{\Lambda^{(2)\mu}{}_\nu a^{(1)\nu} + a^{(2)\mu}}_{\text{translation that include 2 terms that also contains the L.T}}$

we symbolize: $U(\Lambda_2, a_2)\, U(\Lambda_1, a_1)$.

we perform the first trans ① and then the second ②

We Require | (*) $\bar{U}(\Lambda_2, a_2)\, \bar{U}(\Lambda_1, a_1) = U(\Lambda_2 \Lambda_1, \Lambda_2 a_1 + a_2)$

we make an infinitesimal transformation and expand to first order,

$\bar{U}(\Lambda, a) = 1 + i\vec{\alpha}\cdot\vec{J} - i\vec{\beta}\cdot\vec{K} + i a^\mu P_\mu$.

we substitute in (*) and keep terms to second order and from that we can derive the commutation relations.

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Rather than go through this in detail let me explain I general how this works this will then be valid in many other cases as well;

In quantum mechanics, which underlies quantum field theory and hence particle physics, operator are unitary.

$$U_{QM}^{-1} = U_{QM}^{+} \qquad U_{QM}^{+} U_{QM} = \underline{I}$$

The reason is that probability for a pure state should be preserved. (like money in a pure bank).

$$|\psi\rangle \rightarrow |\psi'\rangle = U\,|\psi\rangle$$

$$\underline{P} = \langle\psi'|\psi'\rangle = \langle\psi|U^{+}U|\psi\rangle = \langle\psi|\psi\rangle.$$

A convenient way to write a unitary operator

$$\bar{U} = e^{i\theta} \rightarrow \bar{U}^{+} = e^{-i\theta} \Rightarrow U^{+}U = e^{-i\theta} e^{i\theta} = \underline{I}$$

So we see the exponentiation is a good way to represent unitary operators.

a property of $e^{a} e^{b} = e^{a+b}$. $\qquad ab = ba$

a and b commute.

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What happens however if a and b do not commute;

write $e^A e^B \stackrel{?}{=} e^{A+B}$

$$\left(I + A + \frac{A^2}{2!}\right)\left(I + B + \frac{B^2}{2}\right) \stackrel{?}{=} I + A + B + \frac{(A+B)^2}{2}.$$

$$I + A + B + AB + \frac{A^2}{2!} + \frac{B^2}{2!} \stackrel{?}{=} I + A + B + \frac{A^2}{2} + \frac{AB}{2} + \frac{BA}{2} + \frac{B^2}{2}.$$

The two sides are not equal. !

To remedy the situation fix on the left hand side.

$$I + A + B + \frac{A^2}{2} + \frac{AB}{2} + \frac{BA}{2} + \frac{B^2}{2} + \frac{AB}{2} - \frac{BA}{2}$$

$$= I + (A+B) + \frac{(A+B)^2}{2} + \frac{1}{2}[A,B]$$

hence. $e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$ (to second order).

now suppose we have the exponential representation of a unitary group $e^{i\alpha_a X_a}$

where X_a are hermitian generators and form a vector space.

α_a are infinitesimal numbers

summation on repeated indices is implied.

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In general as we saw $e^{i\alpha_a X_a} e^{i\beta_b X_b} \neq e^{i(\alpha_a + \beta_a)X_a}$

but as the elements $e^{i\gamma_a X_a}$ form a group.

we must have,

$$e^{i\alpha_a X_a} e^{i\beta_b X_b} = e^{i\delta_a X_a}$$

for some δ (summation over repeated indicies).

digression: Properties of a group.

In general groups will crop up throughout the course so it is useful to some of their properties:

Assume group G under some group product $\otimes$.

① if $g_1, g_2 \in G$ then $g_1 \otimes g_2 = g_3 \in G$

② There exist an identity element $e \in G$ and $eg = ge = g$ for all $g \in G$.

③ for each $g \in G$ there exist an inverse $g^{-1} \in G$ and $g^{-1}g = e$

Example G = integers $\otimes = +$

① if $n, m \in G$ $n+m \in G$; ② $n+0 = 0+n = n$

③ $n + (-n) = 0$.

but G = integers is not a group under $\otimes = \times$.

There is no inverse.

$n \times m \in G$ $n \cdot 1 = 1 \cdot n = n$

but $n \times \frac{1}{n} = 1$ $\frac{1}{n} \notin G$.

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G = Rational numbers form a group under $\otimes = \times$

① $x, y \in G \quad x \cdot y = z \in G$

② $x \cdot 1 = 1 \cdot x = x \quad 1$ is the identity.

③ $x = \frac{m}{n} \quad x^{-1} = \frac{n}{m} \quad x \cdot x^{-1} = \frac{m}{n} \cdot \frac{n}{m} = 1 \Rightarrow x^{-1} \in G.$

etc.. groups & Lie algebras are the bedrock of modern particle physics.

Getting back to our problem at hand,

<u>we must have</u>:
$$e^{i\alpha_a X_a} e^{i\beta_b X_b} = e^{i\delta_a X_a} \qquad (**)$$

for some δ_a (summation over indeces is implied).

We expand the exponents up to quadratic order in α and β.

$$\left(I + i\alpha_a X_a + \frac{(i\alpha_a X_a)^2}{2}\right)\left(I + i\beta_b X_b + \frac{(i\beta_b X_b)^2}{2}\right) =$$

$$= I + i\alpha_a X_a + i\beta_b X_b - \frac{(\alpha_a X_a)^2}{2} - \underset{\substack{\uparrow \\ \text{complete the square.}}}{\alpha_a X_a \beta_b X_b} - \frac{(\beta_b X_b)^2}{2}$$

$$= I + i\alpha_a X_a + i\beta_b X_b +$$

$$- \frac{(\alpha_a X_a)^2}{2} - \frac{\alpha_a X_a \beta_b X_b}{2} - \frac{\beta_b X_b}{2}\frac{\alpha_a X_a}{2} - \frac{(\beta_b X_b)^2}{2} +$$

$$- \frac{\alpha_a X_a \beta_b X_b}{2} + \frac{\beta_b X_b \alpha_a X_a}{2} =$$

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$$= I + i\alpha_a X_a + i\beta_b X_b + \frac{\left(i\left(\alpha_a X_a + \beta_b X_b\right)\right)^2}{2} - \frac{[\alpha_a X_a, \beta_b X_b]}{2}$$

hence we get that

$$e^{i\alpha_a X_a} e^{i\beta_b X_b} = e^{i\alpha_a X_a + i\beta_b X_b - \frac{1}{2}[\alpha_a X_a, \beta_b X_b]}$$

OR, noting the group property (**) we have

$$i\delta_a X_a = i\alpha_a X_a + i\beta_b X_b - \frac{1}{2}[\alpha_a X_a, \beta_b X_b]$$

$$= i\alpha_a X_a + i\beta_b X_b - \frac{1}{2}\alpha_a\beta_b [X_a, X_b]$$

hence we must have that

$$[X_a, X_b] = i f_{abc} X_c \quad \text{for some } f_{abc}.$$

f_{abc} are called the structure constants and summarize the group multiplication law.

Getting back to the Poincare group!

The generators are:

$$P_\mu = -i\partial_\mu \leftarrow \text{translations } 4$$

$$L_{\mu\nu} = i(X_\mu\partial_\nu - X_\nu\partial_\mu) \leftarrow \text{Rotations plus boosts} \leftarrow 6.$$

They obey the commutation relations.

$[P^\mu, P^\nu] = 0$ that we saw.

(*) $[L_{\mu\nu}, L_{\rho\sigma}] = i g_{\nu\rho} L_{\mu\sigma} - i g_{\mu\rho} L_{\nu\sigma} - i g_{\nu\sigma} L_{\mu\rho} + i g_{\mu\sigma} L_{\nu\rho}$.

The Later one is the Lie Algebra of $SO(3,1)$ the most general rep. of the generators of $SO(3,1)$ that obeys (*) is given by

$$J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}$$

where $S_{\mu\nu}$ obeys (*) and commute with $L_{\mu\nu}$

Additionally: $[J_{\mu\nu}, P_\rho] = -i g_{\mu\rho} P_\nu + i g_{\nu\rho} P_\mu$

In terms of J_i, K_j we have.

$[J_i, P_j] = i \epsilon_{ijk} P_k$

end of lecture 5/6.

$[J_i, P_0] = 0$

$[K_i, P_j] = i H \delta_{ij}$ $\quad \boxed{H = P_0}$

$[J_i, H] = 0 \quad [P_i, H] = 0$

$[K_i, H] = i P_i \quad [P^0, J^{i0}] = i\eta^{0i}P^0 - i\eta^{00}P^i$

$[H, K_i] = -i P_i$

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Antisymmetric tensor in 3D : ϵ_{ijk} $i,j,k = 1,2,$

$\epsilon_{123} = +1$ $\epsilon_{132} = -1$

$\epsilon_{231} = +1$ $\epsilon_{213} = -1$ $\epsilon_{ijk} = 0$ if $i=j$ / $i=k$ / $j=k$

$\epsilon_{312} = +1$ $\epsilon_{321} = -1$

$\epsilon_{123} = +1 = \epsilon_{\text{even permutations}} = -\epsilon_{\text{odd permutations}}$

for A 3x3 matrix $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$\det A = \sum_{ijk} \epsilon_{ijk} a_i b_j c_k = \epsilon_{123} a_1 b_2 c_3 + \epsilon_{132} a_1 b_3 c_2$

$+ \epsilon_{213} a_2 b_1 c_3 + \epsilon_{231} a_2 b_3 c_1 + \epsilon_{312} a_3 b_1 c_2 + \epsilon_{321} a_3 b_2 c_1$

$= a_1(b_2 c_3 - b_3 c_2) - a_2(b_1 c_3 - b_3 c_1) + a_3(b_1 c_2 - b_2 c_1)$.

Useful identities: $\epsilon_{ijk}\epsilon_{ilm} = (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})$.

$\epsilon_{ijl}\epsilon_{ijk} = 2\delta_{lk}$.

facilitates vector calculus calculations in 3D.

Generalizes to nD $\epsilon_{\mu\nu\rho\ldots\sigma}$: $\epsilon_{12\ldots n} = +1 = \epsilon_{\text{e.p.}} = -\epsilon_{\text{o.p.}}$

A: nxn matrix $\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$.

$\text{Det} A = \sum_{ijk\ldots} \epsilon_{ijk\ldots p}\, a_{1i} a_{2j} \cdots a_{np}$

in 4D $\epsilon_{\mu\rho\sigma\tau}$ $\mu, \rho, \sigma, \tau = 0, 1, 2, 3$.

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A more elegant form of the commutation relations is obtained by defining the

<u>Pauli-Lubanski vector.</u> $W_\sigma = \frac{-1}{2} \epsilon_{\sigma\mu\nu\lambda} J^{\mu\nu} P^\lambda$

$$\Rightarrow W_\sigma P^\sigma = \frac{-1}{2} \epsilon_{\sigma\mu\nu\lambda} J^{\mu\nu} P^\lambda P^\sigma = \frac{-1}{2} \epsilon_{\mu\nu\lambda\sigma} P^\lambda P^\sigma = 0$$

$$W_0 = \frac{-1}{2} \epsilon_{0ijk} J^{ij} P^k = -J_k P^k = -\vec{J} \cdot \vec{P} \quad (\sim \text{helicity}).$$
(to be discussed)

$$W_i = \frac{-1}{2} \epsilon_{ijk0} J^{jk} P^0 + \frac{-1}{2} \epsilon_{ij0k} J^{j0} P^k + \frac{-1}{2} \epsilon_{i0jk} J^{0j} P^k =$$

$$= -P_0 \vec{J}_i + \epsilon_{ijk} P_j K_k$$

$$\vec{W} = -P_0 \vec{J} + \vec{P} \times \vec{K}$$

for $\vec{P} = 0$ $\quad P_0 = m \Rightarrow \vec{W} = -m \vec{J} = -m \vec{S}$ spin

<u>The commutation relations become:</u>

$$[J_{\mu\nu}, W_\rho] = i(g_{\nu\rho} W_\mu - g_{\mu\rho} W_\nu).$$

$$[W_\mu, P_\nu] = 0$$

$$[W_\mu, W_\nu] = i \epsilon_{\mu\nu\rho\sigma} W^\rho P^\sigma$$

<u>Casimir invariants of the Poincare group.</u>

a Casimir operator is one that commutes with all generators of the <u>group</u>

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eigenvalues of the casimir operators are labels of 1-particle states.

The casimir operator commutes with the Hamiltonian

$H = P_0 \Rightarrow [C, H] = 0$

$P_0 = +i\hbar \frac{\partial}{\partial t} \leftarrow$ translation in time.

Hence the eigenvalues of the Casimir operator are constants in time $\to$ constants of the motion.

<u>single particle state</u>: $\psi(x)$; $|\vec{P}, s\rangle$ $\vec{P}$ - momentum
s - other Q.#'s.

$P_\mu P^\mu$ is the first casimir operator of the Poincare group.

$P_\mu P^\mu |\vec{P}, S\rangle = m_0^2 |\vec{P}, S\rangle \to m_0$ - mass.

The mass of a particle is poincare invariant.

The second is casimir operator of the Poincare group is given by. $W_\mu W^\mu$

We saw that W_μ is a four vector

hence $W_\mu W^\mu$ is a Lorentz scalar

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And commutes with $J^{\mu\nu}$ the generators of the Lorentz group.

From the explicit form of W_μ it also follows that $[W^\mu, P^\nu] = 0$

(using the asymmetry $\epsilon^{\mu\nu\rho\sigma}$ and the symmetry of $P^\nu = +i\partial^\mu$

hence $[W_\mu W^\mu, P^\nu] = 0$.

Since $W_\mu W^\mu$ is Lorentz invariant we can compute it in a convenient frame.

if $m \neq 0$ it is convenient to choose the rest frame of the particle.

In this frame $p^\mu = (m, 0, 0, 0)$

$$W^\mu = -\frac{1}{2} \epsilon^{\mu\nu\rho 0} J_{\nu\rho} P_0 = -\frac{m}{2} \epsilon^{\mu\nu\rho 0} J_{\nu\rho}$$

$$= \frac{m}{2} \epsilon^{0\mu\nu\rho} J_{\nu\rho}$$

$\Rightarrow W^0 = 0$

$$W^i = \frac{m}{2} \epsilon^{0ijk} J_{jk} = \frac{m}{2} \epsilon^{ijk} J_{jk} = m J^i$$

Therefore on a one particle state with mass m and spin j we have

$$-W_\mu W^\mu = m^2 j(j+1) \qquad (m \neq 0)$$

- $m=0$ there is no rest frame $|\vec{v}|=c$

choose a frame with $P^\mu = (w, 0, 0, w)$

$$\Rightarrow W^0 = -\tfrac{1}{2}\epsilon^{0ijk} J_{ij} P_k = w J^3 = W^3$$

$$W^1 = -\tfrac{1}{2}\epsilon^{10jk} J_{0j} P_k - \tfrac{1}{2}\epsilon^{1j0k} J_{j0} P_k - \tfrac{1}{2}\epsilon^{1jk0} J_{jk} P_0 =$$

$$= w(J^1 - K^2)$$

Similarly $W^2 = w(J^2 + K^1)$

Therefore $-W_\mu W^\mu = w^2\left[(K^2 - J^1)^2 + (K^1 + J^2)^2\right] \quad (m=0)$

The limit $m \to 0$ is non-trivial

$\Rightarrow$ study separately massive and massless representations

end of lecture 7

Massive Reps

$$P^\mu P_\mu = m^2$$

$$W_\mu W^\mu = -m^2 j(j+1)$$

[redacted]

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- take $m^2 > 0$

$\Rightarrow$ massive Reps are labeled by mass m
spin j.

for massive states can choose $P^\mu = (m, 0, 0, 0)$.

$\Rightarrow$ invariant under spatial Rotations.

$\Rightarrow$ "little group" L.T. that leaves P^μ invariant. $P'^\mu = \Lambda^\mu{}_\nu(p) P^\nu$

$\Rightarrow$ little group for massive Reps is $SU(2)$.

- $\Rightarrow$ massive Reps of mass m are labeled

by spin $j = 0, \frac{1}{2}, 1, \ldots$

states within each rep. are labeled by,

$J_z = -j, -j+1, \ldots, j$.

$\Rightarrow$ massive particles of spin j have $\underline{2j+1}$ D.O.F.

<u>Massless Reps</u>: $P_\mu P^\mu = \underline{P}^2 = m^2 = 0$

- $\underline{P}^\mu = (w, 0, 0, w)$ or $P^\mu = (0, 0, 0, 0)$.

The second case is unphysical. it is unchanged by a Lorentz trans.

In this case there is no rest frame.

"Little group" - set of Lorentz transformations $\Lambda^\mu{}_\nu$ that leaves the momentum 4-vector invariant.

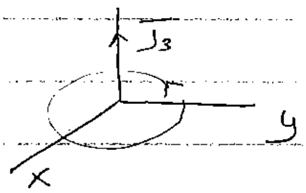
$$\Lambda^\mu{}_\nu(p)P^\nu = P^\mu$$

choose the frame. $P^\mu = (\omega, 0, 0, \omega)$.

<u>note that</u>. x, y Rotations leave this P^μ invariant.

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This is an $SO(2)$ subgroup generated by J^3.



To find the most general transformations that leave $P^\mu = (\omega, 0, 0, \omega)$ invariant it is sufficient to restrict to infinitesimal Lorentz transformations.

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

we look for the most general matrix $\omega^{\mu\nu}$

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which satisfies $\omega^{\mu\nu} = -\omega^{\nu\mu}$

and $\Lambda^{\mu\nu} P_\nu = (\delta^{\mu\nu} + \omega^{\mu\nu}) P_\nu = \delta^{\mu\nu} P_\nu + \omega^{\mu\nu} P_\nu = P^\mu$

Therefore $\omega^{\mu\nu} P_\nu = 0$

for $P_\nu = (\omega, 0, 0, -\omega)$

$$\Rightarrow \begin{pmatrix} 0 & \omega^{01} & \omega^{02} & \omega^{03} \\ -\omega^{01} & 0 & \omega^{12} & \omega^{13} \\ -\omega^{02} & -\omega^{12} & 0 & \omega^{23} \\ -\omega^{03} & -\omega^{13} & -\omega^{23} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = 0$$

This gives the set of constraints,

$$\omega^{03} = 0$$
$$\omega^{01} + \omega^{13} = 0$$
$$\omega^{02} + \omega^{23} = 0$$

Denoting $\omega^{01} = \alpha \quad \omega^{02} = \beta \quad \omega^{12} = \theta$

The most general transformation that leaves P^μ invariant

$$\rightarrow \quad \Lambda = e^{-i(\alpha A + \beta B + \theta C)}$$

where (with a lower second index),

$$A^\mu{}_\nu = i \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad B^\mu{}_\nu = i \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

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$$C^{\mu}{}_{\nu} = i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

we get: $C^{\mu}{}_{\nu} = (J^3)^{\mu}{}_{\nu}$.

similarly: $A^{\mu}{}_{\nu} = (K^1 + J^2)^{\mu}{}_{\nu}$

$B^{\mu}{}_{\nu} = (K^2 - J^1)^{\mu}{}_{\nu}$.

we showed that for massless states the Poincare invariant $-W_{\mu}W^{\mu}$ is given by

$$-W_{\mu}W^{\mu} = \omega^2\left[(K^2 - J^1)^2 + (K^1 + J^2)^2\right]$$

hence we obtained:

$$-W_{\mu}W^{\mu} = \omega^2\left[A^2 + B^2\right]$$

using the expressions for the matricies that we found or the commutation relations of the Lorentz algebra we find that the J^3, A, B generators close an algebra.

$$[J^3, A] = +iB \qquad [J^3, B] = -iA \qquad [A, B] = 0$$

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This is the same as the group generated by
P^x, P^y and $L^z = (xP^y - yP^x)$
i.e. translations and rotations in the Euclidean x, y plane
with A and B playing the role of the translation operators
The algebra is denoted by $ISO(2)$.
since A, B commute there eigenvalues are continuous and noncompact,
that is for massless particle we find that there exist a
continuous degree of freedom that is not realized physically
we demand therefore that A and B annihilate
the physical massless states. $A|\vec{p}, a, b\rangle = a|\vec{p}, a, b\rangle$
$B|\vec{p}, a, b\rangle = b|\vec{p}, a, b\rangle$
with $a = b = 0$ for physical massless states
Therefore $-W_\mu W^\mu = 0$ for physical massless states.
This agrees well with $\lim_{m\to 0} W_\mu W^\mu = \lim_{m\to 0} -m^2(j(j+1)) = 0$
that we found in the massive case.

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- on massless states with $a, b = 0$ the little group is $SO(2)$ or $U(1)$.

The generator of rotations in the x, y plane is J^3

The representations are labeled by the eigenvalue h of J^3, which is the angular momentum in the direction of propogation.

→ <u>helicity</u>: projection of the spin on the direction of the momentum.

- The helicity is quantized → proof based on topological properties of the Lorentz group.

→ massless states are labeled by their helicity.

$$h = \frac{\vec{P}}{|P|} \cdot \vec{J} \qquad \frac{\vec{P}}{|P|} \text{ a unit vector in the direction of momentum}$$

helicity is quantized, for massless states only two helicity states $h = 0, \pm 1/2, \pm 1, \pm 3/2, \pm 2 \ldots$

- Photon $m^2 = 0$ two polarization states $h = \pm 1$

Graviton $m^2 = 0$ two polarization states $h = \pm 2$

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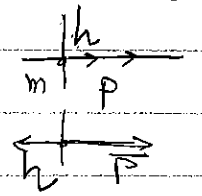
for massive particles $(2j+1)$ helicity states.

for massive particles we can go from $+h$ to $-h$ states by a Lorentz trans. by a boost.



for massless particles this is not possible → $c = 1$ in all frames.

helicity is a Lorentz invariant of massless states.

Lagrangian & Hamiltonian Mechanics

Newtonian mechanics: specify position & velocity at $t = t_i$

$$m\frac{d^2x}{dt^2} = F(x) \Longrightarrow X = X(t = t_f) \quad \dot{X} = \dot{X}(t = t_f)$$

Modern particle physics. speify Energy & momentum at $t = t_0$

Modern particle physics: calculations are done in the framework of quantum field theories.

A bridge between old & new is provided by the classical Lagrangian & Hamiltonian formulations of classical mechanics.

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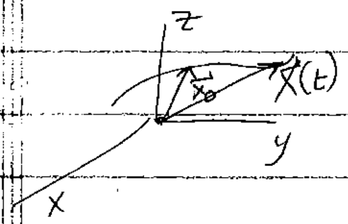
Newton: $\vec F = m\vec a \Rightarrow -\nabla V(\vec X) = m\frac{d^2\vec X}{dt^2}$ (for conserved forces)



$$\vec v(t) = \frac{d\vec X}{dt} \qquad \vec a(t) = \frac{d^2\vec X(t)}{dt^2}$$

$$\vec P = m\frac{d\vec v}{dt}$$

$$\vec F = \frac{d\vec P}{dt}$$

dynamical functions of $\vec X$ and $\vec v$

Examples: $\vec L = \vec X \times \vec P$

$$E = \frac{1}{2}m v^2 + V(\vec x, t) = T + V(\vec x, t)$$

constant of the motion

Energy is conserved if $V = V(\vec X) \neq V(t)$ $\boxed{F = -\vec\nabla V}$

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}m v^2 + V(\vec x)\right) = m\vec v \cdot \frac{d\vec v}{dt} + \frac{d}{dt}V(\vec x) =$$

$$= m\vec v \cdot \dot{\vec v} + \vec\nabla V(\vec x)\frac{d\vec x}{dt} = (\underbrace{m\dot{\vec v}}_{ma} + \underbrace{\vec\nabla V}_{\vec F})\cdot\vec v = 0$$

An alternative way to write Newton's equations.

Take $L = T - V = \frac{1}{2}m\dot v^2 - V(\vec x)$.

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$= \frac{1}{2} m \dot{\vec{x}}^2 - V(x, y, z)$

$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z)$

for x $\quad \frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x} \qquad \frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x}$.

we get $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = m\ddot{x} + \frac{\partial V}{\partial x} = 0$

This is a very important result and generalizes to many mechanical systems and modern field theories.

For a conserving mechanical system with n-degrees of freedom $q_1 \cdots q_n$ with potential. $V(q_1 \cdots q_n)$.

The Lagrangian.

$$L = L(q_1 \cdots q_n, \dot{q}_1 \cdots \dot{q}_n, t) = T(\dot{q}_1 \cdots \dot{q}_n, t) - V(q_1 \cdots q_n, t).$$

The 2nd order Euler Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

The motion of the physical system is solved by specifying an initial condition

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$q_1 \cdots q_n, \ \dot{q}_1 \cdots \dot{q}_n$ at $t = t_0$

OR $q_1 \cdots q_n$ at $t = t_0$ $t = t_f$

we define the conjugate momentum,

$$P_i = \frac{\partial L}{\partial \dot{q}_i}(\vec{q}, \dot{\vec{q}}, t).$$

Cyclic coordinate a coordinate that does not appear explicitly in L

$$L = L(q_1 \cdots q_{n-1}, \dot{q}_1 \cdots \dot{q}_n, t) \Rightarrow \frac{\partial L}{\partial q_n} = 0$$

and $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) = 0$ $\frac{d}{dt}(P_n) = 0 \Rightarrow P_n = \text{const.}$

$\Rightarrow$ Aim find cyclic coordinates $\Rightarrow$ constants of the motion,

In the Lagrange formulation we describe the system in terms of n 2^{nd} order differential eqs,

An alternative formulation is provided by the Hamiltonian formulation.

Hamilton $\Rightarrow$ describe the motion in terms of 1^{st} order diff. Eqs.

Still: need to specify $2n$ initial conditions,

$\Rightarrow$ Price $2n$ first order differential equations.

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Hamilton: change variables

configuration space $\to$ phase space

Configuration space: $(q_1 \cdots q_n, \dot{q}_1 \cdots \dot{q}_n)$,

Phase space $(q_1 \cdots q_n, p_1 \cdots p_n)$

Where $$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$

The transformation is made by a Legendre transform.

$$H = \sum_{k=1}^{n} p_k \dot{q}_k - L(q_1 \cdots q_n, \dot{q}_1 \cdots \dot{q}_n, t) = H(q_k, p_k, t) \qquad p_k = \frac{\partial L}{\partial \dot{q}_k}$$

H is the Hamiltonian and is a function of q_i, p_i, and t.

Example) a particle in one dimension, with E = const

$$L = \frac{1}{2} m \dot{q}^2 - V(q)$$

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} \Rightarrow \dot{q} = \frac{p}{m}$$

end of lecture 10.1

$$H = p\dot{q} - L(q,\dot{q}) = p\dot{q} - \left(\frac{1}{2} m \dot{q}^2 - V(q)\right) = \frac{p^2}{m} - \frac{m p^2}{2m^2} + V(q) =$$

$$= \frac{p^2}{2m} + V(q).$$

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○ Legendre transformation from configuration to phase space

$(q_i, p_i) \rightarrow 2n$ independent variables $H = H(q_i, p_i, t)$.

The Hamilton eqs. of motion are obtained by taking

(*) $dH = \frac{\partial H}{\partial q_k} dq_k + \frac{\partial H}{\partial p_k} dp_k + \frac{\partial H}{\partial t} dt$

From $H = p_k \dot{q}_k - L$

we have $dH = \cancel{p_k d\dot{q}_k} + \dot{q}_k dp_k - \frac{\partial L}{\partial q_k} dq_k - \cancel{\frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k} - \frac{\partial L}{\partial t} dt$

○ with $p_k = \frac{\partial L}{\partial \dot{q}_k}$ and $\frac{d}{dt}(p_k) - \frac{\partial L}{\partial q_k} = 0$

(**) $dH = +\dot{q}_k dp_k - \frac{\partial L}{\partial q_k} dq_k - \frac{\partial L}{\partial t} dt$

Comparing coefficients of (*) and (**)

$\frac{\partial H}{\partial p_k} = \dot{q}_k$

○ $\frac{\partial H}{\partial q_k} = -\frac{\partial L}{\partial q_k} = -\frac{d}{dt}(p_k) = -\dot{p}_k$

$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$

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We get $2n+1$ 1^{st} order differential eqs.

These are Hamilton Eqs. of motion.

When L is not explicitly dependent on time $\frac{\partial L}{\partial t} = 0$

Then: $\frac{dH}{dt} = \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial t} =$

$$= \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} = 0$$

When $L \neq L(t) \Rightarrow H \neq H(t) \Rightarrow$ H is a constant of the motion.

When H does not depend explicitly on t:

H is identified with the conserved energy $H = E =$ constant.

Poisson brackets

Using Hamilton eqs. we can write for any canonical function $G(p_i, q_i, t)$

$$\frac{dG}{dt} = \frac{\partial G}{\partial q_i}\dot{q}_i + \frac{\partial G}{\partial p_i}\dot{p}_i + \frac{\partial G}{\partial t} =$$

$$= \frac{\partial G}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial G}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial G}{\partial t}$$

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The Poisson brackets are defined by

$$\{A,B\} = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}$$

we can write $\frac{dG}{dt} = \{G,H\} + \frac{\partial G}{\partial t}$

$\Rightarrow$ if $\frac{\partial G}{\partial t} = 0$ & $\{G,H\} = 0 \Rightarrow \frac{dG}{dt} = 0$

G is conserved in time.

In Q.M. $\{A,B\} \rightarrow [A,B]$ A, B hermitian operators

$\Rightarrow \frac{d\langle A\rangle}{dt} = [A,H] + \frac{\partial\langle A\rangle}{\partial t} \Rightarrow$ if $A \neq A(t)$ & $[A,H] = 0 \Rightarrow \frac{dA}{dt} = 0$

$\Rightarrow A$ is conserved operators

The action principle

Euler Eq. of motion are derived from an action Principle

Given $L(q_i, \dot{q}_i) = \sum_i \frac{1}{2} m_i \dot{q}_i^2 - V(q_i)$

define $S = \int dt\, L(q, \dot{q})$

Action principle: for fixed values of $q(t_i) = q_{in}$, $q(t_f) = q_{out}$

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then the classical trajectory which satisfies these boundary conditions is an extremum of the action,

$$\delta \int_{t_{in}}^{t_f} dt\, L(q,\dot{q}) = 0.$$

$$\delta S = \int_{t_{in}}^{t_{out}} dt\, \delta L(q_i,\dot{q}_i) = \int dt \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i \right) =$$

$$= \int_{t_{in}}^{t_{out}} dt \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \frac{dq_i}{dt} \right) =$$

$$= \int_{t_{in}}^{t_{out}} dt \left(\frac{\partial L}{\partial q_i}\delta q_i - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i \right) + \underbrace{\frac{\partial L}{\partial \dot{q}_i}\delta q_i \Big|_{t_1}^{t_2}}_{0} = 0.$$

$$= \int_{t_{in}}^{t_{out}} dt \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right)\delta q_i = 0.$$

This must hold for any variation of δq_i hence

we must have $\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$ <— Euler-Lagrange eq

Classical field theory

So far we discussed systems with discrete and finite number of particles

classical field $\psi(\vec{r},t)$: function of $\vec{r}$, t specifying the value of the field at a space-time point

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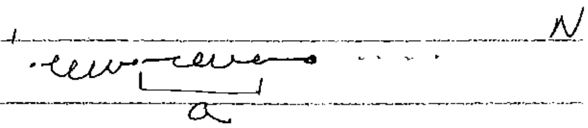
A field representation can simplify a many body mechanical problem

<u>Example:</u> $\frac{\partial^2 y(x,t)}{\partial t^2} \quad \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial x^2} = 0$ $\quad y(x,t) \sim 10^{23}$ molecules

Continuous dynamics in the Lagrangian - Hamiltonian formalism.

The Lagrangian for longitudinal motion of a N-particle linear elastic chain.

$$L = \sum_{i=1}^{N} \frac{1}{2} m \dot{y}_i^2 - \sum_{2} \frac{1}{2} K \left(y_{i+1} - y_i\right)^2$$



m_i - mass of i^{th} particle.
y_i - transverse displacement.
K - elastic constant.
a - equilibrium position
Na - total length of the chain.

$$L = \frac{1}{2} \sum_{i=1}^{N} a\left[\frac{m}{a} \dot{y}_i^2 - Ka\left(\frac{y_{i+1} - y_i}{a}\right)^2\right] = \sum_i a L_i$$

$$\lim_{a \to 0} \sum_{i=1}^{N} a L_i = \int dx\, \mathcal{L}(x) = L$$

x is a continuous index replacing i.

$$\lim_{a \to 0} \frac{m}{a} \to \frac{dm}{dx} = \mu - \text{linear mass density}$$

$$\lim_{\substack{a \to 0 \\ N \to \infty}} Ka \to \tau = \text{elastic tension}$$

Young's modulus.

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$$L = \lim_{a\to 0} \sum a\left\{\frac{1}{2}\frac{m}{a}\dot{y}_i^2 - \frac{1}{2}Ka\left(\frac{y_{i+1}-y_i}{a}\right)^2\right\} \to \int dx\left(\frac{1}{2}\mu\dot{y}^2 - \frac{1}{2}\tau y'^2\right).$$

Lagrangian density $\mathcal{L}(x) = \left\{\frac{1}{2}\mu\dot{y}^2 - \frac{1}{2}\tau y'^2\right\}$. per unit length

$\mathcal{L}$ is a function of the field velocity $\dot{y}$, the field coordinate $y(x,t)$, and the field gradient $y'(x,t)$.

x is an index. a point in the field.

The generalization to three dimensions.

$$L = \int \mathcal{L}(\dot{\phi}, \phi, \nabla\phi)\, dx\,dy\,dz \qquad \phi = \phi(x,y,z,t).$$

The action is given by:

$$S = \int_{t_1}^{t_2} dt\, L[\phi] = \int_{t_1}^{t_2} dt \int dx\,dy\,dz\, \mathcal{L}(\phi, \dot{\phi}, \nabla\phi) =$$

$$= \int d^4x\, \mathcal{L}(\phi, \partial^\mu\phi) \quad \leftarrow \text{Relativistic notation.}$$

The equation of motion for the field ϕ are obtained by requiring that the variation of the action vanishes and demanding that $\delta\phi = 0$ at t_1, t_2

$$\delta\int \overbrace{\mathcal{L}(\phi, \partial^\mu\phi)}^{\text{no explicit } x^\mu \text{ dependence in } \mathcal{L}.} d^4x =$$

$$= \int\left(\frac{\partial\mathcal{L}}{\partial\phi}(\phi,\partial^\mu\phi)\,\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta(\partial_\mu\phi)\right) d^4x.$$

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with the $\mathcal{L}(\partial^\mu \phi) = \partial^\mu(\delta\phi)$

we get $= \int_\Omega \left(\frac{\partial \mathcal{L}}{\partial \phi}(\phi, \partial^\mu \phi)\,\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\,\partial_\mu \delta\phi \right) d^4x$

$= \int_\Omega \delta\phi \left(\frac{\partial \mathcal{L}}{\partial \phi}(\phi, \partial^\mu \phi) \right) d^4x + \underbrace{\int_\Omega \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\,\delta\phi \right) d^4x}_{=0} - \int_\Omega \delta\phi\,\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) d^4x$

$\int v\,du = vu - \int u\,dv$

boundary term $\Rightarrow$ by the divergence theorem (Gauss law)

$= \int \delta\phi \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial_\mu \phi} \right) \right) d^4x = 0$

Since this holds for any $\delta\phi$ we have that

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$$

This gives the Euler-Lagrange Eqs. of motion for the field ϕ

$\Rightarrow$ In the case of the Harmonic chain

end of lecture 11/4/12

$\mathcal{L} = \frac{1}{2}\mu \dot{y}^2 - \frac{1}{2}\tau y'^2$

$\frac{\partial \mathcal{L}}{\partial \dot{y}} = \mu \dot{y} \qquad \frac{\partial \mathcal{L}}{\partial y'} = -\tau y' \qquad \frac{\partial \mathcal{L}}{\partial y} = 0$

$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu y)} - \frac{\partial \mathcal{L}}{\partial y} = \mu \frac{\partial^2 y}{\partial t^2} - \tau \frac{\partial^2 y}{\partial x^2} = 0$

the familiar wave eq.

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If $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$

$$= \frac{1}{2} g^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) - \frac{1}{2} m^2 \phi^2$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = \frac{1}{2}\left(g^{\mu\nu} \partial_\nu \phi + g^{\mu\nu} \partial_\nu \phi\right) + m^2 \phi$$

$$= (\partial_\mu \partial^\mu + m^2)\, \phi(x,t), \qquad = 0$$

This is the Klein-Gordon equation of a relativistic free scalar field that we will encounter in more detail later on.

The Hamiltonian for a field can be written as

$$H = \iiint \mathcal{h}(\phi, \pi)\, dx\, dy\, dz$$

where the generalized momentum density is given by

$$\pi(x,t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

The Hamiltonian density is defined by

$$\mathcal{h} = \dot{\phi}(x,t)\, \pi(x,t) - \mathcal{L}$$

and $H = \int d^3x\, \dot{\phi}(x,t)\, \pi(x,t) - L = \int \mathcal{h}\, d^3x$

For the Klein-Gordon field. $\mathcal{L} = \dot{\phi}^2 - (\nabla \phi)^2 - \frac{1}{2} m^2 \phi^2$

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I have used here: $\partial^\mu = (\partial_t, \vec{\nabla})$ $\partial_\mu = (\partial_t, -\vec{\nabla})$

hence $\Box = \partial_\mu \partial^\mu = (\partial_t^2 - \vec{\nabla}^2)$.

hence $\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$

$$h = \dot{\phi}^2 - \left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2\right) = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$

The total energy $\int h\, d^3x$ should be conserved if $H \neq H(t)$.

And be the zero component of some four vector P^μ

to construct this we introduce the energy-momentum tensor,

$$T^{\mu\nu} = \partial^\mu\phi \frac{\partial \mathcal{L}}{\partial(\partial_\nu\phi)} - \mathcal{L}\eta^{\mu\nu}$$

Then we have: $P^\mu = \int T^{\mu 0}\, d^3x$

And $H = P^0 = \int T^{00}\, d^3x$.

$T^{\mu\nu}$ is conserved in the sense of the continuity condition.

$$\partial_\nu T^{\mu\nu} = \partial_\nu\left(\partial^\mu\phi \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)} - \mathcal{L}\eta^{\mu\nu}\right) = \partial^\mu\phi\, \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\right) + \partial_\nu\partial^\mu\phi \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}$$

$$-\eta^{\mu\nu}\left(\frac{\partial\mathcal{L}}{\partial\phi}\partial_\nu\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi)}\partial_\nu\partial_\lambda\phi\right) = \left(\partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)} - \frac{\partial\mathcal{L}}{\partial\phi}\right)\partial^\mu\phi +$$

$$+\cancel{\frac{\partial\mathcal{L}}{\partial\partial_\nu\phi}\partial_\nu\partial^\mu\phi} - \cancel{\frac{\partial\mathcal{L}}{\partial\partial_\lambda\phi}\partial^\mu\partial_\lambda\phi} = 0.$$

The conservation of $T^{\mu\nu}$ implies that $P^\mu = \int T^{\mu 0} d^3x$ transforms as a 4-vector and is time independent. P^μ is the energy-momentum 4-vector.

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The Klein - Gordon equation

In non relativistic quantum mechanics

$\vec{P} \rightarrow -i\hbar\vec{\nabla} \qquad E \rightarrow i\hbar\frac{\partial}{\partial t}$ → quantum operators.

The Hamiltonian for an energy conserving system is

$$H = \frac{\vec{P}}{2m} + V(\vec{q}) = E$$

leads to the shrödinger equation by substitution.

$$\left(-\frac{\hbar^2}{2m}\vec{\nabla}^2\psi + V(\vec{q})\right)\psi(\vec{q},t) = i\hbar\frac{\partial}{\partial t}\psi(\vec{q},t).$$

In special relativity the four vector P^μ is given by

$$P^\mu = \left(\frac{E}{c}, \vec{P}\right)$$

We have $\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right)$; $\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right)$.

Therefore, we can identify $P^\mu = i\hbar\partial^\mu$

In special relativity $E = \frac{mc^2}{\sqrt{1-v^2/c^2}} \qquad \vec{p} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}}$

Therefore $P^2 = P_\mu P^\mu = \frac{E^2}{c^2} - \vec{p}^{\,2} = m^2c^2$

Following the example of non-relativistic quantum-mechanics

We use $P^\mu \rightarrow i\hbar\partial^\mu$

And obtain the wave eq. $\quad -\hbar^2\partial_\mu\partial^\mu\phi = m^2c^2\phi$

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OR $\left(\partial^2 + \frac{m^2c^2}{\hbar^2}\right)\phi(\vec{x},t) = 0 \leftarrow$ the Klein-Gordon Eq.

Intepretation as a single particle eq. is problematic

The eq. describes a scalar particle of mass m, but not in a single state but a multi-state, i.e. a field.

to find solutions of the K.G. eq. we put it in a box and impose that the wave function vanishes on the boundaries we then take the volume to infinity.

we assume that the particle is free i.e. $V(\vec{x}) = 0$

The solution in the box is a plane wave solutions of the form.

$$\phi(\vec{x},t) \sim e^{ik_\mu x^\mu}$$

$x^\mu = (t, \vec{x})$

$k^\mu = (\omega, \vec{k})$ $\qquad k_\mu = (\omega, -\vec{k}) \leftarrow$ constant.

$$\phi(\vec{x},t) \sim e^{i(\omega t - \vec{k}\cdot\vec{x})}$$

We want the solution to describe a free particle of mass m.

we substitute this solution in the K.G. eq. $\Rightarrow \omega^2 - |\vec{k}|^2 = m^2$

The particle is confined in the box $\xrightarrow{\text{Assume}}$ $\phi(x,t) = T(t)X(x)Y(y)Z(z)$

substituting in K.G.E. $\rightarrow -\frac{T''}{T} + \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = m^2 \rightarrow$ const.

all constants $\rightarrow$ $\omega^2 \quad -k_x^2 \quad -k_y^2 \quad -k_z^2$

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To satisfy the B.C. we impose $k_x = \frac{n_1\pi}{L}$ $k_y = \frac{n_2\pi}{L}$ $k_z = \frac{n_3\pi}{L}$

where n_1, n_2, n_3 are integers

we obtained $\omega^2 = m^2 + \frac{\pi^2}{L^2}(n_1^2 + n_2^2 + n_3^2)$ ← dispersion relation

The solutions of the K.G.E. are momentum eigenstates.

we have: $\phi = Ae^{-ik\cdot x} = Ae^{-ik^\mu x_\mu} \Rightarrow \partial^\mu\phi = \frac{\partial}{\partial x_\mu}\phi = -ik^\mu Ae^{-ik\cdot x} = -ik^\mu\phi$

$$\partial^2\phi = -k_\mu k^\mu\phi = -k^2\phi$$

$$\Rightarrow \left(\partial^2 + \frac{m^2c^2}{\hbar^2}\right)\phi = 0 \Rightarrow k^2 = \frac{m^2c^2}{\hbar^2} = \frac{1}{\lambda^2}$$

$\lambda = \frac{\hbar}{mc}$ has dimension of length → compton wavelength of particle of mass

we have $p^\mu\phi = i\hbar\partial^\mu\phi = \hbar k^\mu\phi$.

ϕ describes a momentum eigenstate with eigenvalue $\hbar k^\mu$.

The condition $k^2 = \frac{m^2c^2}{\hbar^2}$ is the same as $P^2 = m^2c^2$ → the mass shell condition.

we have: $P^2 = \frac{E^2}{c^2} - \vec{P}^2 = m^2c^2 \Rightarrow \frac{E}{c} = \pm\sqrt{m^2c^2 + \vec{P}^2}$

⇒ some solutions of K.G.E. correspond to negative energy states.

→ interpretation as single particle state is problematic.

Such interpretation is also in conflict with the probability interpretation of quantum mechanics. → non-relativistically $|\psi(x)|^2 = P(x)$.

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$\rho(x)$ - probability density $\rightarrow \int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$

The probability density in quantum mechanics obeys a continuity eq.

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

where $\vec{J} = -\frac{i\hbar}{2m}\left(\psi^* \vec{\nabla}\psi - \psi \vec{\nabla}\psi^*\right)$

Provided that the potential $V(\vec{r})$ is real, $\rho(x)$ is conserved.

we want to construct similar quantities for the K.G. eq.
Furthermore, we want the continuity condition to be covariant;
i.e. to hold in all inertial frames $\Rightarrow \partial_\mu j^\mu = 0 \leftarrow$ covariant

find 4-vector $j^\mu = (j^0, \vec{J})$ which obeys $\partial_\mu j^\mu = 0$

Start with K.G.E.

$$\phi^* \cdot / (\partial^2 + m^2)\phi = 0$$
$$\phi \cdot / (\partial^2 + m^2)\phi^* = 0$$

$$\rightarrow \phi^* \partial^2 \phi - \phi \partial^2 \phi = 0$$

$$g_{\alpha\beta}\left(\phi^* \partial^\alpha \partial^\beta \phi - \phi \partial^\alpha \partial^\beta \phi^*\right) = g_{\alpha\beta}\partial^\alpha\left(\phi^* \partial^\beta \phi - \phi \partial^\beta \phi^*\right) = \partial_\beta\left(\phi^* \partial^\beta \phi - \phi \partial^\beta \phi^*\right) = 0$$

$$\Rightarrow j^\mu \sim \left(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*\right) \Rightarrow \partial_\mu j^\mu = 0$$

in Schrödinger's case $\rho = \psi^* \psi$

here $\rho = \frac{i\hbar}{2mc^2}\left(\phi^* \frac{\partial}{\partial t}\phi - \phi \frac{\partial}{\partial t}\phi^*\right)$

$$\phi \sim e^{\pm i(Et - \vec{p}\cdot\vec{x})} = e^{\pm iEt} e^{\mp i \vec{p}\cdot\vec{x}}$$

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for $\phi \sim e^{+iEt}$ ( ) antiparticle case with $E \approx mc^2$ we have,

$$\rho = \frac{i\hbar}{2mc^2}\left(\frac{imc^2}{\hbar}\phi^*\phi + i\frac{mc^2}{\hbar}\phi^*\phi\right) = -\phi^*\phi < 0$$

Therefore we get negative probability associated with the antiparticle and possitive probability associated with the particle.

Does not make sense as probability density → multiply time charge e.

=> charge density $\rho = \frac{i\hbar e}{2mc^2}\left(\phi^*\frac{\partial}{\partial t}\phi - \phi\frac{\partial\phi^*}{\partial t}\right)$

charge current density $\vec{J} = \frac{e\hbar}{2im}\left[\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*\right]$

makes sense as charge density not probability density.

-> makes sense as a quantum field -> creating-anihilating particles

Quantization of the K.G. field (Real scalar field)

The Lagrangian: $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2$

leads to the K.G. equation: $(\partial^2 + m^2)\phi = 0$

we can write $\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2$

from which we derive $\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}$

In ordinary quantum mechanics we impose the relations: $[q_i, p_j] = i\hbar\delta_{ij}$

$$[q_i, q_j] = [p_i, p_j] = 0$$

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in classical field theory the coordinates are replaced by the fields

$$q_i \to \phi(x) \qquad P_i \to \pi(x).$$

analoguesly, in quantum field theory we impose the equal-time commutation relations

$$[\phi(\vec{x},t), \pi(\vec{x},t)] = i\hbar\delta(\vec{x}-\vec{x}')$$

$$[\phi(\vec{x},t), \phi(\vec{x}',t)] = [\pi(\vec{x},t), \pi(\vec{x}',t)] = 0$$

δ-function
$\int f(x)\delta(x-a) = f(a)$
$\int_\alpha^\beta f(x)\delta(F(x)) = \frac{f(a)}{|F'(a)|}$
where $F(a)=0$

These are equal time commutation relations $\to$ canonical commutation relation

in the Heisenberg picture The equations of motion are given by

$$i\hbar\dot{\alpha} = [\alpha, H] \quad \to \alpha \to \text{operator} \quad H - \text{Hamiltonian}, \quad \alpha \neq \alpha(t)$$

we have $H = \int \mathcal{H}\, d^3x = \int \left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right)$

The canonical commutation relations reproduce the E.O.M.

Consider: $[\phi(\vec{x},t), H] = \left[\phi(\vec{x},t), \int \left[\frac{1}{2}\pi(x',t)^2 + \frac{1}{2}(\nabla\phi(x',t))^2 + \frac{1}{2}m^2\phi(x',t)^2\right] d^3x'\right]$

$$= \frac{1}{2}\int d^3x' \left\{[\phi(\vec{x},t), \pi(x',t)^2] + [\phi(x), (\nabla'\phi(x'))^2] + m^2[\phi(x), \phi(x')^2]\right\}$$

now $[\phi(x), \phi(x')] = 0 \Rightarrow [\phi(x), \nabla'\phi(x')] = 0$

$$\Rightarrow [\phi(x), H] = \frac{1}{2}\int d^3x' [\phi(x), \pi(x')^2] = \frac{1}{2}\int d^3x' \left(\phi(x)\pi(x')^2 - \pi(x')^2\phi(x)\right) =$$

$$= \frac{1}{2}\int d^3x' \left(\phi(x)\pi(x')\pi(x') - \pi(x')\phi(x)\pi(x') + \pi(x')\phi(x)\pi(x') - \pi(x')\phi(x)\pi(x')\right] =$$

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$$= \frac{1}{2}\int d^3x' \left(\Pi(x')\,[\phi(x),\Pi(x')] + [\phi(x),\Pi(x')]\,\Pi(x')\right) =$$

$$= \int d^3x'\,\Pi(x')\,\delta(x-x') = i\hbar\,\Pi(x) = i\hbar\,\dot{\phi}(x) \quad \checkmark \text{ ok.}$$

We obtained the correct equation of motion!

So far we discussed the free Klein Gordon eq.

How do we incorporate interactions?

In Newtonian mechanics we describe interactions by adding a potential.

$$L = \frac{1}{2} m \sum_i \left(\frac{\partial x_i}{\partial t}\right)^2 - V(\vec{x})$$

$$m \sum \ddot{x}_i = -\vec{\nabla} V(\vec{x}) = \vec{F}(\vec{x})$$

If we have more than one particle

$$\sum_j m_j \sum_i \ddot{x}_{ji} = \sum_{ij} -\nabla_j V(\vec{x}_i - \vec{x}_j)$$

Typically we consider the interactions to be 2-body interactions and we sum over all the interacting particles.

For harmonic motion in one-dimension.

$$V(x) = \frac{1}{2} k x^2.$$

$$m\ddot{x} = -\frac{\partial V(x)}{\partial x} = -k x$$