November 2020

Liverpool Lectures on Modern Particle Physics
Introduction to Modern Particle Physics

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Abstract

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1 Overview

Physics : Mathematical modelling of experimental observations. Predict outcomes of experiments.

Initial conditions : Mathematical model → Predicted outcome ↔ Experimental data

Practical : An acceptable mathemcatical model is the one which is most successful in accounting for a wide range of experimental observations

Themes: Reductionism : Large to small

Celestial, Atomic, Nuclear, sub–nuclear, ...

Themes: Unification : Newton, Maxwell, Einstein, Glashow–Weinberg–Salam, Georgi–Glashow,

Earth & Skies, Electric & Magentic, Mechanics & ElectroMagnetism, Weak & EM, ElectroWeak & Strong

Inventory :

Forces : E&M, Weak, Strong: spin +1 particles

Gravity: spin 2 particle

Particles :

Quarks & leptons

The aim of these lectures is to provide an introduction to the theoretical foundations of modern particle physics. The course is aimed at third year undergraduate students in UK universities. Befitting with the UK environment I do not assume any prior knowledge. The material is introduced at an elementary level. The goal is to whet the student's appetatite for theoretical physics closer to the contemporary frontier. In this spirit the presentation aims to be elementary, though the concepts contains the basic elements the underlie our leading theory of elementary particle physics, the Standard Model. Physics is first and foremost an experimental science. The language that we use to encode the experimental data is mathematics. We may therefore regard physics as the mathematical modelling of experimental observations, where in this we may include terrestrial and extra–terrestrial observations. So, for example, Newton formulated his laws of mechanics by using observational data of planetary motion. Likewise today's observations include data from cosmological, astro-physical and terrestrial laboratories. Typically, in the mathematical modelling of the experimental data, we are given some initial conditions. The mathematical model is then used to calculate a predicted outcome of these initial conditions, which are tested against the experimental data. Physicists are practical people. An accepted mathematical model is the one which is most successfull in accounting for a wide range of experimental observations. The mathematical formulation of physical observations follows several themes. Partly this follows from the technological development of the experimental instruments. Thus, one theme is that of reductionism, *i.e.* from large to small. The initial observations since ancient times were made in the celestial domain, where astronomers mapped the motion of stars and planets. Progress is also made by posing general postulates, without firm observational evidence. In this regard, since ancient times both the geo–centric and helio–centric models of planetary motions were proposed. With advances in optical instrumentation Galileo was able to provide the decisive evidence in favour of the helio–centric model. Kepler and Newton formulated their laws of planetary motion and mechanics following these advances. With technological advances the electric and magnetical forces and the atomic world that they rule where explored and understood in the course of the 19th century, with key contributions by

Farady, Maxwell and Einstein. Further down the scale Rutherford and Fermi shed light on the nuclear world. Which brings us closer to the sub–atomic world, which has been probed since the fifties to the present day. To account for the experimental data emerging with the exposition of new layers of the physical, new mathematical models and tools were developed. Newton formulated the laws of mechanics and the inverse square law of the gravitational interactions; Maxwell encapsulated the dynamics of charged matter in electric and magnetic fields in the four Maxwell equations, whereas Einstein developed the theory of special relativity in an attempt to reconcile Maxwell's equations with Galilean mechanics. Quantum mechaics was developed to account for the observed quantum spectrum of atomic and nuclear energy spectra, and were morphed into quantum field theories, in order to account for detailed features of this spectra. The formalism of the Standard Model utilises the framework of quantum field theories, adapted to Abelian and non–Abelian gauge symmetries. The Standard Model of particle physics is a renormalisable quantum field theory with specified matter and gauge content. The aim of these lectures is to introduce the student to these structures at an elementary level.

Another prevailing theme in physics is that of unification. With the development of experimental instrumentation that can probe smaller and smaller scales, using higher and higher energies, or higher resolving power, different mathematical tools are introduced to model the emerging mathematical data. Synthesising disparate mathematical formalisms of experimental data into one coalescing framework is a guiding principle in theoretical physics. It enlarges the scope and the predictive power of the mathematical model. I note here a common misconception in regard to the term "predictive power". The point is not that a mathematical model needs to be able to predict something that has not been observed before, and will be observed by some future experiment. This is perhaps a prerequisite for those wishing to win a Nobel prize. For a physical theory, though, the terminology "predictive power" is used in the sense of "calculability". We want in our mathematical representation of the physical world to start with some boundary conditions, which can be boundary conditions in time or in space, and to be able to calculate outcomes that can be detected in observations. Now, we want the minimum number of free parameters in our mathematical model to be able to account for the maximal number of observations in our physical experiment. This is why unification is such a prevailing theme. Unifying different mathematical structures into one framework typically reduces the number of free parameters and enlarges the predictive power of the unified theory, in the sense that it is able to account for a wider range of observable phenomena. Indeed, unification runs as a constant theme through the history of the development of our understanding of the physical world over the past few centuries. Newton's laws of mechanics unified celestial and terrestrial observations of mechanical motion. Maxwell's equations unified the electric and magnetic forces in one electromagnetic theory. Einstein unified electromagnetism with mechanics. The Standard Glashow–Weinberg–Salam model unifies the electromagnetic and weak interactions, whereas Grand Unified Theories unify the Standard Model with Quantum Chromo Dynamics (QCD), the theory underlying the strong interactions. The final frontier is the synthesis of quantum mechanics, which accounts for the sub–atomic interactions, and general relativity, which describes the gravitational interaction in the celestial, galactical and cosmological spheres, into one overridding theory. These lectures will introduce the key ingredients of the Standard Model and will offer a glimpse into Grand Unified Theories. The subject of the final frontier, in the framework of string theories, is covered in the module "introduction to string theory", which is offered next year. To proceed in our quest we first make an inventory of what the Standard Model is composed of.

The Standard Model of particle physics is composed of three sectors: the forces; the matter and the Higgs sectors. The forces in the Standard Model are mediated by spin +1 particles that are transform in the adjoint representation of a unitary group. These forces correspond to invariances of the Standard Model Lagrangian under the local gauge group transformations. There are three forces in the Standard Model, corresponding to invariaces under three gauge sectors: the electromagnetic force, which correspond to invariance under the $U(1)$ transformations; the weak force, which corresponds to invriance under $SU(2)$ transformations; and the strong force, which corresponds to invriance under $SU(3)$ transformations. The gravitational interaction, which is not part of the Standard Model, is mediated by a Spin +2 field. The quantisation of the gravitational field constitute much of the contemporary interest in theoretical physics.

The matter sector of the Standard Model is composed of soin 1/2 particles that transform in representations of the Standard Model gauge symmetries. It is divided in the categories of quarks and leptons, where quarks are particles that transform under the $SU(3)$ gauge symmetry of the strong interactions, whereas leptons are those that do not. Hence, the quarks interact via the strong interactions, whereas leptons do not. The particles in the Standard Model appear in three replications that transform in indentical manner under the Standard Model gauge group. The first generation of quarks are the up– and down–quarks, the second are the charm and the strange quarks, and the third are the top and bottom quarks. The first generation of leptons are the electron–neutrino and the electron, the second are the muon–neutrino and the muon, and the third are the tau–neutrino and the tau–lepton. The three replications are typically referred to as three generations and carry identical charges under the Standard Model gauge groups. They differ in their respective mass scales, *i.e.* the first generation is the lightest, whereas the third one is the heaviest. This distinction is also reflected in the historical process of their experimental discovery. The electron was the first to be observed in 1897 by Thompson, whereas the last particle to be discovered, nearly a century later was the top–quark in 1995. The Higgs sector of the Standard Model is composed of a single spin 0 state that transform as a doublet under the weak interactions.

$$\begin{array}{cccc} \underline{\text{Quarks}} & \begin{pmatrix} \text{up} \\ \text{down} \end{pmatrix} & \begin{pmatrix} \text{charm} \\ \text{strange} \end{pmatrix} & \begin{pmatrix} \text{top} \\ \text{bottom} \end{pmatrix} \\ \underline{\text{Leptons}} & \begin{pmatrix} \nu_e \\ \text{electron} \end{pmatrix} & \begin{pmatrix} \nu_\mu \\ \text{muon} \end{pmatrix} & \begin{pmatrix} \nu_\tau \\ \text{tau} \end{pmatrix} \end{array}$$

The Standard Model is an accummulations of over two centuries of research that concluded in a model with 19 continuous free parameters that accounts for all observable atomic and sub–atomic phenomena. This is a remarkable feat that accounts for tens of thousands of experimental observations, in the most complex experimental apparatus devoleped to date. Yet, the Standard Model is not the end of the road. It leaves many yet unanswered questions. The Higgs particle, whose Vacuum Expectation Value (VEV) generates mass for the Standard Model particle, was detected at the Large Hadron Collider (LHC) in 2012. However, the origin or the mass scale of the Higgs particle and its stability with respect to corrections from higher scales remains an enigma.

The Standard Model particle spectrum suggests the embedding of its gauge and matter multiplets in larger groups in the framework of Grand Unified Theories (GUTs). This unification gives rise to interactions that mediate rapid proton decay. We know in our bones that is not the case, because if it was all the protons in our body would have decayed long ago. In fact there would not be any matter at all! We are indeed fortunate that the proton is stable. In fact, its lifetime exceeds that of the universe by many orders of magnitude. Dedicated experiments searching for proton decay indicate that $\tau_{proton} \geq 10^{34}$years. Furthermore, the Standard Model is an effective field theory that provides viable parameterisation of all observable sub–atomic data up to some scale in which new physics must be present. In the first place, we know that the quantisation of gravity cannot be formulated consistently as a quantum field theory. Gravity is not renormalisable. In quantum field theories, in general, we encouter some amplitudes that diverge. For instance, when calculating the self–energy of the electron. The way to counter that is to absorb the infinities in some quantities that are fixed in experiments. The physical parameters that are measured in experiments contain these infinities, which reflects our ignorance of their fundamental origin. If this can be done for a finite number of parameters, the physical theory is said to be renormalisable. Gravity on the other hand is not renormalisable because it contains an infinite number of infinities.

While gravity is extremely weak at the electroweak scale, at some scale, which is typically called the Planck scale, it becomes of comparable strength to the gauge interactions. At that scale its effect on the particle dynamics must be included and the Standard Model, which is particular quantum field theory model, ceases to provide viable parameterisation. So the Standard Model cannot be the end of the story. It must be augmented by new physics at some cutoff scale that we can denote generically as Λ. At that scale new interactions are induced among the Standard Model particles that are suppressed by the cutoff scale. Some of these oprators can induce rapid proton decay, which indicates that the new scale is above 10^{16}GeV. Fourteen orders of magnitude above the electroweak scale and merely two orders of magnitude below the reduced Planck scale. Why is there such a vast separation between the electroweak scale and the GUT and Planck scales, and what keeps it stable, *i.e.* what mechanism keeps the electroweak scale stable against corrections from the higher scales. Imagine living by the seaside and building a dyke to protect the shores from 5 meters waves. But that will provide protection from 50 meters waves in the case of a tsunami. This is called the hierarchy problem. To protect the EW scale the existence of physics beyond the SM have been contemplated. Among them a symmetry between fermions and bosons, called supersymmetry. We can resort to the seaside analogy again. When a tsunami occurs it is usually precedeed by a retraction of the sea waters, followed by the tsunami wave. Imagine that as the sea waters retract, doors in the seabed opens up to vast empty caverns. As the water come back they fill the empty spaces which negates the rising level of waters. Hence the shores are protected. This is more or less how supersymmetry works. In the case of supersymmetry bosonic corrections are balanced by fermionic corrections, with a net effect that the EW scale remains intact. Another mechanism that has been proposed is that of large extra dimensions, that in a sense also diverts the rising levels into the vast empty spaces of the extra dimensions.

The synthesis of the Standard Model with gravity cannot be achieved within the framework of point quantum field theories. String theory provides the most advanced contemporary framework to explore this unification. In string theory elementary particles are no longer point particles, but have an internal dimension. The result is that divergent amplitudes in quantum field theories are rendered finite in string theory.

There are additional questions that tell us that in order to understand basic features of the Standard Model, we have to study its fusion with gravity. Why are there three generations in nature? How do the mass parameters in the SM fixed? How did the universe came to be, and how were its basic constituents produced? These puzzles require inputs that go beyond the realm of quantum field theories.

In this course we will focus on the mathematical structures that make up the Standard Model of particle physics. The study of modern particle physics is the study of the symmetries that underlie the sub–atomic interactions among the basic constituents of matter. These symemtries are associated with conservation laws, for example, the conservation of angular momentum in classical mechanics is associated with invariance of the mechanical potential under rotations. We will also see how symmetries are violated. We can make an analogy with a spinning wheel on a car. As long as the wheel is symmetric it turns around soomthly and the car moves freely. If you make a puncture in it the car movement will be impedded. What we would like to do is to set up the mathematical formalism to link between symmetries and conservation laws, as well as their violation. This setup is used to interpret the experimental observations. The mathematical formalism involves groups, and their representations. We start with the Lorentz group.

2 Special relativity and the Lorentz group

Before turning to the Lorentz group and special relativity, consider rotations in a two dimensional plane. The transformation is induced by a 2x2 matrix

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \simeq \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{2.1}$$

for small θ. Similarly in three spatial dimensions

$$\vec{X}' = R\vec{X}$$

where R is a 3x3 matrix. The magnitude of vectors does not change when we rotate them *i.e.*,

$$\vec{X}' \cdot \vec{X}' = \vec{X} \cdot \vec{X} \tag{2.2}$$

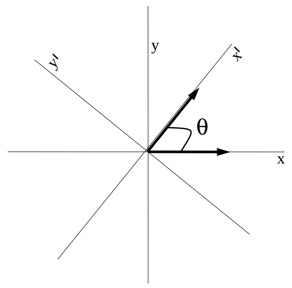
or

$$\vec{X}^T R^T R \vec{X} = \vec{X} \cdot \vec{X} \tag{2.3}$$

i.e. we have to impose the condition $R^T R = I$. The metric in this case is

$$\eta_{ij} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.4}$$

Rotations in 2D



2.1 Special relativity and the Lorentz group

Einstein's special relativity follows from two basic tenets:

- c, the speed of light in vacuum is constant in all inertial frames with $c = 3 \cdot 10^8 m/s$ (in these lectures we often set $c = 1$).
- The laws of physics are the same in all inertial frames.

Events are labelled by their time and position in intertial frames.

$$\longrightarrow \qquad \text{4vector} \;\; X^\mu \;=\; (ct, \vec{X}) \qquad \mu = 0,1,2,3$$
$$x^0 \;=\; ct \quad \vec{\mathbf{X}} \;=\; (x^1, x^2, x^3) \;=\; (x, y, z)$$

The length of the four vector is given by

$$X \cdot X \;=\; c^2 t^2 - x^2 - y^2 - z^2 \;=\; t^2 - x^2 - y^2 - z^2 \qquad \text{with } c = 1$$

This is written as

$$X \cdot X \;=\; \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} X^\mu X^\nu \;=\; \eta_{\mu\nu} X^\mu X^\nu \;=\; X_\mu X^\mu$$

where by Einstein's summation convention, indices that are repeated as lower and upper indices are summed over, *i.e.* over $\mu = 0,1,2,3$. Here $\eta_{\mu\nu}$ is the Minkowski metric given by

$$\eta_{\mu\nu} \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.5}$$

In general the scalar product of two 4–vectors X^μ, Y^μ is given by

$$X \cdot Y \;=\; \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} X^\mu Y^\nu$$

We will use Einstein's summation convention. A down index is summed with an up index, and the summation symbol is dropped. We will encounter different types of objects

- scalar – no free indices. All indices are summed over.
- vector – one free index (*e.g.* X^μ)
- tensor – two and more free indices ($g^{\mu\nu}$, $R_{\mu\nu\rho\sigma}$)

The Lorentz transformations are the transformations that preserve the scalar product. In particular, they preserve the length of a 4–vector.

$$t^2 - \vec{\mathbf{X}}^2 \;=\; X \cdot X \;=\; \eta_{\mu\nu} X^\mu X^\nu = X' \cdot X' \;=\; \eta_{\mu\nu} X'^\mu X'^\nu \;=\; t'^2 - \vec{\mathbf{X}}'^2$$

where X and X' are related by a Lorentz transformation

$$X^\mu \;\rightarrow\; X'^\mu \;=\; \Lambda^\mu_\nu X^\nu$$

In general we write the metric as $g^{\mu\nu}$ and its components can be functions of spacetime. This is the subject of general relativity in which spacetime can be curved. The metric satisfies

$$g^{\mu\nu}g_{\nu\sigma} = \delta^{\mu}_{\sigma} \quad \Rightarrow \quad (g^{\mu\nu})^{-1} \ = \ g_{\mu\nu}$$

In flat spacetime that we are dealing with in this lectures $g^{\mu\nu} \ \to \eta^{\mu\nu}$. We distinguish between 4–vectors with upper and lower indices. We write

$$X_\mu = \sum_{\nu=0}^{3} g_{\mu\nu} X^\nu$$

X_μ is a covariant vector and X^μ is a contravariant vector. For $X^\mu = (t, \vec{\mathbf{X}}) \to X_\mu = (t, -\vec{\mathbf{X}})$, for our choice of the Minkowski metric. The Lorentz invariant can be written as $X_\mu X^\mu$. Given a vector $X^\mu = (t, \vec{\mathbf{X}})$ there are two differential quantities of interest

- 1. $dX^\mu \qquad \to$ differential $\to$ contra variant four vector
- 2. $\frac{\partial}{\partial X^\mu} = \partial_\mu \ \to$ gradient $\to$ covariant four vector

How do they behave under coordinate transformations $X^\mu \to X'^\mu$?

- 1. $dX^\mu \to dX'^\mu \ = \ \frac{\partial X'^\mu}{\partial X^\nu} dX^\nu$
- 2. $\frac{\partial}{\partial X^\mu} \ \to \frac{\partial}{\partial X'^\mu} \ = \ \frac{\partial X^\nu}{\partial X'^\mu} \frac{\partial}{\partial X^\nu}$

We see that the differential and the grandient transform differently. A four vector that transforms like the differential is a contra–variant vector

$$V'^\mu \ = \ \frac{\partial X'^\mu}{\partial X^\nu} V^\nu$$

whereas four vector that transforms like the gradient is a covariant vector

$$V'_\mu \ = \ \frac{\partial X^\nu}{\partial X'^\mu} V_\nu$$

in the following we will use the notation

$$\partial_\mu = \frac{\partial}{\partial X^\mu}$$

Example of a four vector is the momentum vector $P^\mu = (E, \vec{\mathbf{P}})$ and $P_\mu = (E, -\vec{\mathbf{P}})$. In relativistic quantum mechanics the momentum four vector is proportional to the gradient

$$P_\mu \sim \frac{\partial}{\partial X^\mu}$$

and

$$P_\mu X^\mu = Et - \vec{\mathbf{P}} \cdot \vec{\mathbf{X}}$$

2.2 Properties of Lorentz transformations

Lorentz transformations are transformations that preserve the scalar product and the length of Lorentz four vectors, just like the rotations that preserve the length of three vectors in three dimensional spatial space. The only difference is that in three dimensions we are in Euclidean space with its Euclidean metric, whereas Lorentz transformations operate in Minkowski spacetime with its Minkowski metric. The length of four vectors in Minkowski space is given by

$$\eta_{\mu\nu} X^\mu X^\nu$$

which is invariant under Lorentz transformations, *i.e.* there no change in its size and shape. The invariance implies the existence of a symmetry, which is generated by a group, the Lorentz group.

Assume $X^\mu \to X'^\mu$ under some Lorentz transformation, *i.e.*

$$X^\mu \to X'^\mu = \Lambda^\mu{}_\nu X^\nu$$

Hence

$$\eta_{\mu\nu} X^\mu X^\nu \to \eta_{\mu\nu} X'^\mu X'^\nu = \eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta X^\alpha X^\beta = \eta_{\alpha\beta} X^\alpha X^\beta$$

$$\Rightarrow \eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta} \tag{2.6}$$

or in matrix notation

$$\Lambda^T \eta \Lambda = \eta$$

where η is the 4x4 Minkowski metric and Λ is the 4x4 matrix of Lorentz transformations. The identity in eq. (2.6) defines the Lorentz transformations.

$$\Rightarrow (\mathrm{Det}\Lambda)^2 = 1 \Rightarrow \mathrm{Det}\Lambda = \pm 1$$

The physical transformations correspond to those that can be continuously connected to the identity, for which $\mathrm{Det}\Lambda = +1$.

- $\mathrm{Det}\Lambda = +1 \to$ Proper Lorentz tranformations
- $\mathrm{Det}\Lambda = -1 \to$ Imroper Lorentz tranformations

We can look at the 00 component of the identity $\eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta}$

$$\begin{aligned} & \eta_{\mu\nu} \Lambda^\mu{}_0 \Lambda^\nu{}_0 = +1 \\ \Rightarrow \quad & (\Lambda^0{}_0)^2 - \sum_i (\Lambda^i{}_0)^2 = +1 \\ \Rightarrow \quad & (\Lambda^0{}_0)^2 = 1 + \sum_i (\Lambda^i{}_0)^2 \geq 1 \end{aligned}$$

We have that as,

$$\begin{aligned} (\Lambda^0{}_0)^2 \geq 1 \Rightarrow \qquad & \Lambda^0{}_0 \geq +1 \quad \text{orthochronos LT} \\ \text{or} \quad & \Lambda^0{}_0 \leq -1 \quad \text{non orthochronos LT} \end{aligned}$$

An example of a nonorthochronos Lorentz transformation is given by **reflections**:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The Lorentz transformations that are continuously connected to the identity are proper and orthochronos *i.e.*

- 1. $\mathrm{Det}\Lambda = 1 \leftrightarrow$ proper
- 2. $\Lambda^0{}_0 \geq 1 \leftrightarrow$ orthochronos

Examples:

- $$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- **rotations**:

  $$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 \\ 0 & \vec{R} \end{pmatrix}$$

  where $\vec{R}$ are 3x3 rotation matrices in the three spatial dimensions. We have that $\mathrm{Det}\Lambda = \mathrm{Det}R = \pm 1$ and $\mathrm{Det}R = +1$ for proper rotations.

- **boosts**: For a boost along the x–axis

  $$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

  we have $\mathrm{Det}\Lambda = \cosh^2\eta - \sinh^2\eta = 1$ and $\Lambda^0{}_0 = \cosh\eta \geq 1$.

- **time inversion**:

  $$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.7}$$

  $\mathrm{Det}\Lambda = -1$ and $\Lambda^0{}_0 = -1$. An improper non–orthochronos Lorentz transformation.

- **full inversion**:

  $$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.8}$$

  $\mathrm{Det}\Lambda = +1$ and $\Lambda^0{}_0 = -1$. A proper non–orthochronos Lorentz transformation.

All Lorentz transformations are generated by the above transformations. The proper and orthochronos Lorentz transformations correspond to rotations and boosts. These are the physical Lorentz transformations that can be continuously connected to the identity Lorentz transformation, *i.e.* we can write them in infinitesimal form, similar to the way that we expanded the rotation in two dimensions in eq. (2.1). The proper orthochronos Lorentz transformations form a group. We can parametrise these transformations in terms of six parameters, similar to the θ parameter in eq. (2.1)

$$6 \text{ parameters} = 3 \text{ angles} + 3 \text{ boosts}$$

We can write an infinitesimal proper orthochronos Lorentz tranformation in the form

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu} \tag{2.9}$$

where $\delta^{\mu}{}_{\nu}$ is the identity transformation and $\omega^{\mu}{}_{\nu}$ is a 4x4 matrix of the infinitesimal parameters. Hence,

$$\delta^{\mu}{}_{\nu} = +1 \text{ for } \mu = \nu \text{ ; } = 0 \text{ for } \mu \neq \nu$$

Next, we expand eq. (2.6) to first order in the infinitesimal parameters in $\omega^{\mu}{}_{\nu}$. For the continuous transformations the properties of the inifitesimal transformations fix the transformation properties of the Lorentz group, and finite transformations are obtained by integration. To obtain the constraints on the infinitesimal parameters we insert (2.9) into (2.6) and expand to first order in $O(\omega)$,

$$
\begin{aligned}
& \eta_{\mu\nu}(\delta^{\mu}{}_{\alpha} + \omega^{\mu}{}_{\alpha})(\delta^{\nu}{}_{\beta} + \omega^{\nu}{}_{\beta}) = \eta_{\alpha\beta} \\
\text{open brackets} \quad & \\
& \eta_{\mu\nu}\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta} + \eta_{\mu\nu}\omega^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta} + \eta_{\mu\nu}\delta^{\mu}{}_{\alpha}\omega^{\nu}{}_{\beta} + \eta_{\mu\nu}\omega^{\mu}{}_{\alpha}\omega^{\nu}{}_{\beta} = \eta_{\alpha\beta} \\
\Rightarrow \quad & \eta_{\alpha\beta} + \omega_{\beta\alpha} + \omega_{\alpha\beta} + O(\omega^2) = \eta_{\alpha\beta} \\
\Rightarrow \quad & \omega_{\beta\alpha} + \omega_{\alpha\beta} = 0 \\
\rightarrow \quad & \text{The tensor of infinitesimal transformations is antisymmetric.}
\end{aligned}
$$

<u>**rotations**</u>: rotation group in two dimensions

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{for small } \theta \tag{2.10}$$

The number of degrees of freedom in a 4x4 antisymmetric matrix?
<u>**for general** n:</u> $\frac{n^2-n}{2} = \frac{n(n-1)}{2}$

$$\text{for } n = 4 \rightarrow \frac{4\cdot 3}{2} = 6 \rightarrow 3 \text{ rotation angles} + 3 \text{ boosts}$$

$$\Rightarrow \omega_{\mu\nu} = \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & b_3 & -b_2 \\ -a_2 & -b_3 & 0 & b_1 \\ -a_3 & b_2 & -b_1 & 0 \end{pmatrix} \tag{2.11}$$

2.3 Algebraic properties of the Lorentz group

We associate an operator $U(\Lambda)$ with the Lorentz transformation Λ. For the special case $\Lambda = \delta \;\rightarrow\; U(\delta) = I$, the identity operator. We want to find an operator associated with $\Lambda = \delta + \omega$. To order $O(\omega)$

$$U(\delta + \omega) = I + \frac{1}{2} i J_{\mu\nu} \omega^{\mu\nu} + \cdots$$

where $J_{\mu\nu}$ are a set of operators and the $\omega^{\mu\nu}$ are the infinitesimal parameters. to see the relevant structure we can look at the example of rotations

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & \alpha & -\beta \\ -\alpha & 1 & \gamma \\ \beta & -\gamma & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{2.12}$$

we can write the matrix in the form

$$A = I + \alpha \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \tag{2.13}$$

where the 3x3 matrices are the operators *i.e.* the analogs of the operators $J^{\mu\nu}$ and $\alpha, \beta, \gamma \in \omega^{\mu\nu}$. The expansion in (2.13) is almost in the desired form. The caveat is that the operators are not hermitian, whereas for physics reasons we would like them to be hermitian. To obtain that we multiply by i, *i.e.*

$$A = I + i\alpha \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + i\beta \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} + i\gamma \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{2.14}$$

Now the operators are hemitian.

To extract the algebraic properties of the Lorentz group, we will first discuss the problem in general and then specialise to the generators of the Lorentz group. Our observations here will then be valid in different cases as well.

In quantum mechanics, which underlies quantum field theory, and hence modern particle physics, operators are unitary,

$$U_{QM}^{-1} = U_{QM}^{\dagger} \quad \Rightarrow \quad U_{QM}^{\dagger} U_{QM} = I$$

The reason that operators have to be unitary is that for a pure state probability should be preserved. Under a unitary transformation we have

$$|\psi\rangle \rightarrow |\psi'\rangle = U|\psi\rangle$$

$$P = \langle\psi'||\psi'\rangle = \langle\psi|U^{\dagger}U|\psi\rangle = \langle\psi|\psi\rangle.$$

Hence, the probability is preserved if U is unitary. A convenient way to write a unitary opetor is using exponentiation

$$\begin{aligned} & U = \mathrm{e}^{iO} \rightarrow U^{\dagger} = \mathrm{e}^{-iO^{\dagger}} \\ \Rightarrow \quad & U^{\dagger}U = \mathrm{e}^{-iO^{\dagger}} \mathrm{e}^{iO} = I \end{aligned}$$

We see that provided that the operator O is hermitian, exponentiation is a good way to represent unitary operators. A property of exponents is

$$\mathrm{e}^a \mathrm{e}^b = \mathrm{e}^{a+b}$$

provided that a and b commute *i.e.* $ab = ba$. What happens, however, if a and b do not commute? Write

$$\mathrm{e}^A \mathrm{e}^B \stackrel{?}{=} \mathrm{e}^{A+B} \tag{2.15}$$

and A and B are two operators that do not commute *i.e.* $[A, B] \neq 0$. To test the identity in (2.15) we can expand the two sides and compare at equal orders in the expansion

$$\begin{aligned}
(I + A + \frac{A^2}{2!})(I + B + \frac{B^2}{2!}) &\stackrel{?}{=} (I + A + B + \frac{(A+B)^2}{2!}) \\
I + A + B + AB + \frac{A^2}{2!} + \frac{B^2}{2!} &\stackrel{?}{=} I + A + B + \frac{A^2}{2!} + \frac{AB}{2!} + \frac{BA}{2!} + \frac{B^2}{2!}
\end{aligned}$$

We see that the two sides are not equal! To remedy the inequality we can fix the left hand side

$$\begin{aligned}
& I + A + B + \frac{A^2}{2!} + \frac{AB}{2!} + \frac{BA}{2!} + \frac{B^2}{2!} + \frac{AB}{2!} - \frac{BA}{2!} \\
= \quad & I + (A+B) + \frac{(A+B)^2}{2!} + \frac{[A, B]}{2!}
\end{aligned}$$

Hence, to second order we derived the identity

$$\mathrm{e}^A \mathrm{e}^B = \mathrm{e}^{A+B+\frac{1}{2}[A,B]} \tag{2.16}$$

Now, suppose we have the exponential representation of a unitary group

$$\mathrm{e}^{i\alpha_a X_a},$$

where X_a are hermitian generators and form a vector space; α_a are infinisetimal numbers; and summation over repeated indecis is implied. In general, as we saw,

$$\mathrm{e}^{i\alpha_a X_A} \mathrm{e}^{i\beta_b X_b} \neq \mathrm{e}^{i(\alpha_a X_a + \beta_b X_b)}$$

but as the elements

$$\mathrm{e}^{i\gamma_a X_a}$$

form a group, we must have

$$\mathrm{e}^{i\alpha_a X_A} \mathrm{e}^{i\beta_b X_b} = \mathrm{e}^{i\delta_a X_a} \tag{2.17}$$

for some δ_a, where summation over repeated indices is implied.

2.3.1 Digression: properties of a group

Groups will crop up throughout the lectures. It is useful to recall some of their properties. Assume group G under some product $\otimes$:

1. if $g_1,\ g_2 \in G$ then $g_1 \otimes g_2 = g_3 \in G$
2. there exist an identity element $e \in G$ and $eg = ge = g$ for all $g \in G$.

3. for each $g \in G$ there exist an inverse $g^{-1} \in G$ and $gg^{-1} = e$

Examples:

1

$$\begin{array}{ll}
 & G = \text{integers}; \quad \otimes = + \\
 & \text{if } n, m \in G \;\Rightarrow\; n + m \in G;\; n + 0 = 0 + n = n;\; n + (-n) = 0 \in G \\
\Rightarrow & \text{the group criteria are satisfied} \\
 & \text{but } G \;=\; \text{ integers is not a group under } \otimes = \times \text{ as there is no inverse} \\
 & i.e. \text{ if } n,\; m \;\in G \;\Rightarrow\; n \times 1 \;=\; 1 \times n \;=\; n \\
 & \text{but } n \times \dfrac{1}{n} \;=\; 1 \text{ and } \dfrac{1}{n} \notin G \\
 & \text{Hence, the group criteria are not satisfied.}
\end{array}$$

2

$$\begin{array}{ll}
 & G = \text{rational numbers form a group under}; \quad \otimes = \times \\
 & \text{if } X, Y \in G \;\Rightarrow\; X \times Y = Z \in G; \\
 & X \times 1 = 1 \times X = X; \;\rightarrow\; 1 \text{ is the identity} \\
 & X = \dfrac{n}{m} \;\Rightarrow\; X^{-} = \dfrac{m}{n} \;\Rightarrow\; X \times X^{-1} = \dfrac{m}{n}\dfrac{n}{m} = 1 \rightarrow X^{-1} \in G \\
\Rightarrow & \text{the group criteria are satisfied}
\end{array}$$

Groups and Lie algebras are the bedrock of modern particle physics. Getting back to our problem, we expand the exponents in eq. (2.17) up to quadratic order in α and β

$$\begin{array}{ll}
 & \left(I + i\alpha_a X_a + \dfrac{(i\alpha_a X_a)^2}{2}\right)\left(I + i\beta_b X_b + \dfrac{(i\beta_b X_b)^2}{2}\right) \\
= & I + i\alpha_a X_a + i\beta_b X_b - \dfrac{(\alpha_a X_a)^2}{2} - \alpha_a X_a \beta_b X_b - \dfrac{(\beta_b X_b)^2}{2} \\
 & (\quad \text{complete the square } of \qquad \alpha_a X_a \beta_b X_b) \\
= & I + i\alpha_a X_a + i\beta_b X_b + \\
 & -\dfrac{(\alpha_a X_a)^2}{2} - \dfrac{\alpha_a X_a \beta_b X_b}{2} - \dfrac{\beta_b X_b \alpha_a X_a}{2} - \dfrac{(\beta_b X_b)^2}{2} \\
 & \qquad\qquad - \dfrac{\alpha_a X_a \beta_b X_b}{2} + \dfrac{\beta_b X_b \alpha_a X_a}{2} \\
= & I + i(\alpha_a X_a + \beta_b X_b) + \dfrac{(i(\alpha_a X_a + \beta_b X_b))^2}{2} - \dfrac{[\alpha_a X_a, \beta_b X_b]}{2}
\end{array}$$

We get that

$$e^{i\alpha_a X_A} e^{i\beta_b X_b} = e^{i\alpha_a X_a + i\beta_b X_b - \frac{1}{2}[\alpha_a X_a, \beta_b X_B]} \tag{2.18}$$

or noting the group property eq. (2.17) we have

$$\begin{aligned}
i\delta_a X_a &= i\alpha_a X_a + i\beta_b X_b - \frac{1}{2}[\alpha_a X_a, \beta_b X_B] \\
&= i\alpha_a X_a + i\beta_b X_b - \frac{1}{2}\alpha_a \beta_b [X_a, X_B]
\end{aligned}$$

Hence, we must have

$$[X_a, X_b] = i f_{abc} X_c \ \text{ for some } f_{abc}$$

f_{abc} are called the structure constants and summarise the group multiplication law.

We can now turn our attention back to the special case of the Lorentz group. We saw that the matrix of infinitesimal transformation that are connected continuously to the identity is an antisymmetric second rank tensor, *i.e.* $\omega_{\mu\nu} = -\omega_{\nu\mu}$. In four spacetime dimensions that corresponds to three boosts and three rotations.

$$\to \ \Lambda = \mathrm{e}^{\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}$$

where $J^{\mu\nu}$ are generators of the Lorentz algebra. To first order in the infinisetimal parameters ω we have

$$\Lambda \sim I - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} \tag{2.19}$$

A representation of the Lorentz group transforms as

$$\phi^i \ \to \ \left[\mathrm{e}^{\frac{i}{2}\omega_{\mu\nu}J_R^{\mu\nu}}\right]^i_j \phi^j \ = \ U(\Lambda)\phi \ , \qquad\qquad j =, 1 \ldots, n$$

$J_R^{\mu\nu}$ are the generators in the R–representation of the Lorentz group as $n \times n$ matrices. As $\mu, \nu \ = \ 0, \cdots, 3$ we have 16 $J^{\mu\nu}$ matrices, but on six of those are independent as $J_{\mu\nu} = -J_{\nu\mu}$.

All elementary particles in nature transform in representations of the Lorentz group.

scalars $\to$ Higgs boson

fermions $\to$ matter

vectors $\to$ force mediators

The generators of the Lorentz group satisfy the Lie algebra.

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho})$$

which is the Lie algebra of $SO(1,3)$. We define the operators

$$\begin{aligned} K_i &= J_{i0} = -J_{0i} \qquad\qquad i = 1, 2, 3 \\ J_i &= \frac{1}{2}\sum_{j,k}\epsilon_{ijk}J_{jk} \end{aligned}$$

where $\epsilon_{123} = +1, \ \epsilon_{213} = -1, \ \epsilon_{112} = 0$, etc ... To first order in the infinitesimal parameters the Lorentz rotations and boosts are given by

$$\Lambda = I + i\vec{a} \cdot \vec{K} - i\vec{b} \cdot \vec{J}$$

The second rank tensor $\omega_{\mu\nu}$ is given in terms of the a_i and b_i as in eq. (2.11). For example, as we have seen in eq. (2.14) in the case of **rotations:** $A = I + i\vec{\alpha} \cdot \vec{J}$. Hence, the generators of the Lorentz group are:

$$\begin{aligned} \vec{J} \ &\to \ \text{generators of rotations} \\ \vec{K} \ &\to \ \text{generators of boost} \end{aligned}$$

satisfy the commutation relations

$$
\begin{aligned}
[J_i, J_j] &= i\epsilon ijkJ_k \\
[J_i, K_j] &= i\epsilon ijkK_k \\
[K_i, K_j] &= -i\epsilon ijkJ_k
\end{aligned}
$$

We **define the combinations:**

$$
\begin{aligned}
\vec{J}_+ &= \frac{1}{2}\left(\vec{J} + i\vec{K}\right) \\
\vec{J}_- &= \frac{1}{2}\left(\vec{J} - i\vec{K}\right)
\end{aligned}
$$

Note that the generators $\vec{J}_+$ and $\vec{J}_-$ are not hermitians as,

$$
\begin{aligned}
\vec{J}_+^\dagger &= \vec{J}_- \\
\vec{J}_-^\dagger &= \vec{J}_+
\end{aligned}
$$

The commutation relations of $\vec{J}_+$ and $\vec{J}_-$ are

$$
\begin{aligned}
&\left[J_i^+, J_j^+\right] = \frac{1}{4}\left[J_i + iK_i, J_j + iK_j\right] = \\
&\frac{1}{4}\left(J_k + iK_k + iK_k + J_k\right) = i\epsilon_{ijk}J_k^+
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\left[J_i^+, J_j^+\right] &= i\epsilon_{ijk}J_k^+ \\
\left[J_i^-, J_j^-\right] &= i\epsilon_{ijk}J_k^+ \\
\left[J_i^+, J_j^-\right] &= 0
\end{aligned}
$$

The J_i^+ and J_i^- generate two disjoint generators of an $SU(2)$ algebra, $SU(2) \times SU(2)^\dagger$. Each representation of the Lorentz group is labelled by the indecis of the two disjoint $SU(2)$ algebras (j_1, j_2).
Each representation has $(2j_1 + 1) \otimes (2j_2 + 1)$ components.
As $\vec{J} = \vec{J}_+ + \vec{J}_-$ spin is given by $j_1 + j_2$.

examples:	(j_1, j_2)	spin	components	
a	(0,0)	0	1	singlet
b	$(\frac{1}{2}, 0)$	$\frac{1}{2}$	2	Weyl spinor
c	$(0, \frac{1}{2})$	$\frac{1}{2}$	2	Weyl spinor
d	$(\frac{1}{2}, 0) + (0, \frac{1}{2})$	$\frac{1}{2}$	4	Dirac spinor
e	$(\frac{1}{2}, \frac{1}{2})$	1,0	4	vector

If we write $X^\mu = (t, x, y, z)$, under rotation t is a singlet and (x, y, z) is a triplet. Recalling that the generatorof rotations is given by $\vec{J} = \vec{J}_+ + \vec{J}_-$, we note that under rotations a Lorentz four vector decomposes into a singlet and a triplet.

From the point of view of the $(\frac{1}{2}, \frac{1}{2})$ representation, each spin $\frac{1}{2}$ state has two components that we can denote as spin up and spin down. In total we have four possibilities $\uparrow\uparrow$, $\uparrow\downarrow$, $\downarrow\uparrow$, $\downarrow\downarrow$, from which we can form three symmetric combinations and one asymmetric

$$\begin{array}{lll} 3 \leftrightarrow & \uparrow\uparrow;\ \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow);\ \downarrow\downarrow & \text{spin} = 1\ ;\ j_3 = (+1, 0, -1) \\ 1 \leftrightarrow & \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow); & \text{spin} = 0\ ;\ j_3 = 0 \end{array}$$

we see that the triplet is symmetric with respect to exchange of the two spin states, whereas the singlet is asymmetric.

3 The Poincare group

We saw that spin is a label of representations of the Lorentz group, which correspond to elementary particles in nature. However, we cannot yet classify elementary particles which also have mass. The reason is that the Lorentz transformations are not the most general infinitesinal transformations of the infinitesimal relativistic line element. We should write down the most general transformations of a relativistic line element

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \tag{3.1}$$

So far we discussed rotations & boosts that are symmetries of the Lorentz group. We want to write the most general set of transformations that keep the line element in eq. (3.1) invariant. To determine the most general set of transformations we can look at the two dimensional case

$$dS^2 = c^2dt^2 - dx^2 \tag{3.2}$$

The most general inifintesimal transformations are given by

$$\begin{aligned} t &\rightarrow t + \epsilon T(t,x) \\ x &\rightarrow x + \epsilon R(t,x) \end{aligned}$$

where ϵ is an infinitesimal parameters. We want to find the functions T and R such that the line element remains invariant

$$\begin{aligned} dt &\rightarrow dt + \epsilon(\frac{\partial T}{\partial t}dt + \frac{\partial T}{\partial x}dx) \\ dx &\rightarrow dx + \epsilon(\frac{\partial R}{\partial t}dt + \frac{\partial R}{\partial x}dx) \end{aligned}$$

$$ds^2 \rightarrow [(1+\epsilon\frac{\partial T}{\partial t})dt + \epsilon\frac{\partial T}{\partial x}dx]^2 - [(1+\epsilon\frac{\partial R}{\partial x})dx + \epsilon\frac{\partial R}{\partial t}dt]^2$$

we require invariance of ds^2. Expanding to first order in ϵ

$$ds^2 \ \rightarrow \ dt^2 - dx^2 + 2\epsilon\left(\frac{\partial T}{\partial t}dt^2 - \frac{\partial R}{\partial x}dx^2 + (\frac{\partial T}{\partial x} - \frac{\partial R}{\partial t})dxdt\right) + O(\epsilon^2)$$

we impose that the coefficients of the addtional terms vanish. These yield the constraints on the functions T and R.

$$
\begin{aligned}
dt^2 \quad &: \quad \frac{\partial T}{\partial t} = 0 \Rightarrow T = T(x) \\
dx^2 \quad &: \quad \frac{\partial R}{\partial x} = 0 \Rightarrow R = R(t) \\
dxdt \quad &: \quad \frac{\partial T}{\partial x} - \frac{\partial R}{\partial t} = 0 \Rightarrow \frac{dT}{dx} = \frac{dR}{dt} = \text{constant} = c
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow T(x) &= cx + a \\
R(t) &= ct + b
\end{aligned}
$$

We obtained three degrees of freedom that are respresented by the three constants of the motion $a,\ b\ , c$. We can check what are the three constants of motion.

a. choose $c = b = 0\ ;\ a \neq 0 \quad \Rightarrow \quad x \to x\ ;\ t \to t + \epsilon a$
hence this case corresponds to a translation in time

b. choose $a = c = 0\ ;\ b \neq 0 \quad \Rightarrow \quad t \to t\ ;\ x \to x + \epsilon b$
hence this case corresponds to a translation in space

c. choose $a = b = 0\ ;\ c \neq 0 \quad \Rightarrow \quad t \to t + \epsilon cx\ ;\ x \to \epsilon ct + x$
the third transformation corresponds to a boost

If we had taken the line element to be $ds^2 = dt^2 + dx^2$ we would have obtained orinary rotations.

$$
\begin{aligned}
t &\to t + \epsilon cx \\
x &\to x - \epsilon ct
\end{aligned}
$$

or in matrix form

$$
\begin{pmatrix} t \\ x \end{pmatrix} \to \begin{pmatrix} t \\ x \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}
$$

that we have seen before. Hence, the case of the infinitesimal line element $ds^2 = dt^2 + dx^2$ corresponds to rotaions in two dimensions. The boost transformation is a generalisation of rotations to four dimensional Minkowski spacetime. Returning to four dimensional Minkowski spacetime in eq. (3.1) $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$,

$$
\begin{aligned}
\underline{\text{we have}} \qquad & 4-\text{translations}\, dt, dx, dy, dz \\
& 3-\text{rotations} \quad dxdy, dxdz, dydz \\
& 3-\text{boosts} \quad\ \ dtdx, dtdy, dtdz
\end{aligned}
$$

The group that describes this set of symmetries is the Poincare group. The total number of generator of the Poincare group is 10. Our aim is to find the algebra of the Poincare group and its invariants. This will give us the labels of elementary particles. The translation symmetries are important. They relate to the momentum operator that generates translations. We will therefore obtain momentum and mass from the relativistic invariant $P_\mu P^\mu = m^2$, where P^μ is the particle momentum four vector. Mass is the second label of particle states.

To start let us look at the transformation of the function $\phi(x)$ in one dimension

$$\begin{aligned} x &\rightarrow x+a \\ \phi(x) &\rightarrow \phi(x+a) \end{aligned}$$

we are looking for an operator that induces this transformation.

$$\phi(x+a) = U(a)\phi(x)$$

$$\begin{aligned} \phi(x) \rightarrow \phi(x+a) &= \sum_n \frac{a^n}{n!}\left(\frac{\partial^n}{\partial x^n}\phi(x)\right)\Bigg|_{a=0} \\ &= \underbrace{\left(\sum_n \frac{a^n}{n!}\frac{\partial^n}{\partial x^n}\right)}_{\text{operator}}\phi(x) = \mathrm{e}^{a\frac{\partial}{\partial x}}\phi(x) \\ U(a) = \mathrm{e}^{a\frac{\partial}{\partial x}} &\leftarrow \quad \text{is the operator} \\ U(a) = \mathrm{e}^{a\frac{\partial}{\partial x}} &\simeq 1 + a\frac{\partial}{\partial x} + \cdots = 1 + i(-ia\frac{\partial}{\partial x}) + \cdots \\ &= 1 + iaP + \cdots \end{aligned}$$

i.e $P = -i\frac{\partial}{\partial x}$ is the operator that induces the translation in x. The complex factor i arises because we require that $U(a)$ is a unitary operator.

$$\begin{aligned} U &= I + iaP \\ U^\dagger &= I - iaP \\ U^{-1} &= I - iaP \\ U^\dagger U &= (I - iaP)(I + iaP) = I + a^2P^2 \simeq I \end{aligned}$$

In four spacetime dimensions

$$U(a^\mu) \simeq I + ia^\mu P_\mu$$

The generators of the Poincare group are:

$$\begin{aligned} P_\mu &= i\partial_\mu && \leftarrow 4 \text{ translations} && (3.3) \\ L_{\mu\nu} &= i(X_\mu\partial_\nu - X_\nu\partial_\mu) && \leftarrow 3 \text{ rotations} + 3 \text{ boosts} && (3.4) \end{aligned}$$

The most general transformation consistent with the Poincare group.

$$X^\mu \rightarrow X'^\mu = \Lambda^\mu_\nu X^\nu + a^\mu$$

We want to find:

- the commutation relations among the generators of the group
- the maximal set of commuting operators $\rightarrow$ physical labels

Example : Rotations

$$[L_i, L_j] = i\epsilon_{ijk}$$

Casimir operator L^2

$$\left[L^2, L_i\right] = 0 \quad \rightarrow \quad \text{2 mutually commuting operators } L^2,\ L_z.$$

$$L^2|j,m\rangle = j(j+1)|j,m\rangle$$

$$L_z|j,m\rangle = m|j,m\rangle$$

the states are labelled by their eigenvalues under the commuting operators.

We saw that $P_\mu = i\frac{\partial}{\partial X^\mu}$ $\leftarrow$ 4 generators

1 $$\Rightarrow \quad [P_\mu, P_\nu] \ = \ 0$$

the order does not matter if we perform two successive translations.

$$\begin{aligned} X^\mu \ \rightarrow \ X'^\mu &= X^\mu + a^\mu \quad \text{1. translate by } a^\mu \\ X'^\mu \ \rightarrow \ X''^\mu &= X'^\mu + b^\mu \quad \text{2. translate by } b^\mu \\ &= X^\mu + a^\mu + b^\mu \end{aligned}$$

$$\begin{aligned} X^\mu \ \rightarrow \ X'^\mu &= X^\mu + b^\mu \quad \text{1. translate by } b^\mu \\ X'^\mu \ \rightarrow \ X''^\mu &= X'^\mu + a^\mu \quad \text{2. translate by } a^\mu \\ &= X^\mu + a^\mu + b^\mu \end{aligned}$$

The order of the translations does not matter, hence the commutator of the two operations (generators) commutes.

2 $$[P_\mu, X^\nu] = \left[i\tfrac{\partial}{\partial X^\mu}, X^\nu\right] = i\left[\tfrac{\partial}{\partial X^\mu}, X^\nu\right] = i\delta^\nu{}_\mu$$

3 $$[P_\mu, K_i] = \ ? \qquad , \qquad [P_\mu, J_i] = \ ?$$

We perform successive Poincare transformations

$$\begin{aligned} \text{1.} \quad X^\mu \ \rightarrow \ X'^\mu &= {\Lambda_1}^\mu{}_\nu X^\nu + {a_1}^\mu \quad \text{first transformation} \\ \text{2.} \quad X'^\mu \ \rightarrow \ X''^\mu &= {\Lambda_2}^\mu{}_\nu X'^\nu + {a_2}^\mu \quad \text{second transformation} \\ &= {\Lambda_2}^\mu{}_\nu \left({\Lambda_1}^\nu{}_\lambda X^\lambda + {a_1}^\nu\right) + {a_2}^\mu \\ &= \underbrace{{\Lambda_2}^\mu{}_\nu {\Lambda_1}^\nu{}_\lambda X^\lambda}_{\text{2 successive L.T.}} \quad + \quad \underbrace{{\Lambda_2}^\mu{}_\nu a'^\nu + {a_2}^\mu}_{\text{translation that includes a L.T.}} \end{aligned}$$

we symbolise:

$$U(\Lambda_2, a_2)U(\Lambda_1, a_1)$$

We perform the first transformation 1. and then the second 2.

we require:

$$U(\Lambda_2, a_2)U(\Lambda_1, a_1) = U(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2) \tag{3.5}$$

where $U(\lambda, a)$ is given by

$$U(\Lambda, a) = 1 + i\vec{\alpha} \cdot \vec{J} - i\vec{\beta} \cdot \vec{K} + ia^{\mu} P_{\mu} \tag{3.6}$$

Here, α_i, β_i and a^{μ} are infinitesimal parameters. Hence, (3.6) is an expansion of $U(\Lambda, a)$ to first order in the infinitesimal parameters. Inserting (3.6) into (3.5) and keeping terms to second order in the inifinitesimal parameters, we derive the commutation relations. The generator of translations is given in eq. (3.3), whereas the generators of boosts and rotations are given in eq. (3.4). The translation generators satisfy the communation relations given in 1, whereas the generators of translations and boosts satisfy the commutation relations

$$[L_{\mu\nu}, L_{\rho\sigma}] = i\eta_{\nu\rho} L_{\mu\sigma} - i\eta_{\mu\rho} L_{\nu\sigma} - i\eta_{\nu\sigma} L_{\mu\rho} + i\eta_{\mu\rho} L_{\nu\sigma} \tag{3.7}$$

which are the commutation relations of the $SO(1, 3)$ Lie algebra. The most general representation of the generators of the $SO(1, 3)$ algebra that obeys eq. (3.7) is given by

$$J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}$$

where $S_{\mu\nu}$ obeys eq. (3.7) and commutes with $L_{\mu\nu}$. Additionaly:

$$[J_{\mu\nu}, P_{\rho}] = -i\eta_{\mu\rho} P_{\nu} + i\eta_{\nu\rho} P_{\mu} \tag{3.8}$$

In terms of J_i and K_i the commutation relations become

$$[J_i, P_j] = i\epsilon_{ijk} P_k$$
$$[K_i, P_j] = iH\delta_{ij} \qquad \text{where} \quad \boxed{H = P_0}$$

$$[J_i, H] = 0 \quad , \quad [P_i, H] = 0 \quad , \quad [K_i, H] = iP_i$$

A more elegant form of the commutation relations is obtained by defining the

3.1 Pauli–Lubanski vector:

$$W_{\sigma} = -\frac{1}{2} \epsilon_{\sigma\mu\nu\lambda} J^{\mu\nu} P^{\lambda} \tag{3.9}$$

We note the appearance of the antisymmetric tensor in four dimensions $\epsilon_{\sigma\mu\nu\lambda}$, which generalises the antisymmetric tensor in three dimensions ϵ_{ijk}. Before proceeding to examine the Pauli–Lubasnki vectort, we digress to discuss the generalisation of the antisymmetric tensor in any number of dimensions.

antisymmetric tensor in 3D: ϵ_{ijk} $\quad i, j, k = 1, 2, 3$

$$\begin{aligned} \epsilon_{123} &= +1 & \epsilon_{132} &= -1 \\ \epsilon_{231} &= +1 & \epsilon_{213} &= -1 \\ \epsilon_{312} &= +1 & \epsilon_{321} &= -1 \end{aligned}$$

i.e. $\epsilon_{123} = \epsilon_{\text{even permutations}} = \epsilon_{\text{odd permutations}}$

For a 3×3 matrix $\quad A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$

$$\begin{aligned} \text{Det}A &= \sum_{i,j,k} \epsilon_{ijk} a_i b_j c_k \\ &= \epsilon_{123} a_1 b_2 c_3 + \epsilon_{132} a_1 b_3 c_2 + \epsilon_{213} a_2 b_1 c_3 \\ &\quad + \epsilon_{231} a_2 b_3 c_1 + \epsilon_{312} a_3 b_1 c_2 + \epsilon_{321} a_3 b_2 c_1 \\ &= a_1(b_2 c_3 - b_3 c_2) - a_2(b_1 c_3 - b_3 c_1) + a_3(b_1 c_2 - b_2 c_1) \end{aligned}$$

useful identities : $\epsilon_{ijk}\epsilon_{ilm} = (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})$

$\epsilon_{ijl}\epsilon_{ijk} = 2\delta_{lk}$

facilitates vector calculus calculations in $3D$.

Generalises to nD $\quad \epsilon_{\mu\nu\rho \,\cdots\, \sigma}$: $\quad \epsilon_{123\cdots n} = +1 = \epsilon_{e.p.} = -\epsilon_{o.p.}$

For a $n \times n$ matrix $\quad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$

$$\text{Det}A = \sum_{i,j,k,\cdots} \epsilon_{ijk\,\ldots}\, a_{1i} a_{2j} \,\cdots\, a_{n\rho}$$

in $4D$ $\epsilon_{\mu\rho\sigma\tau}$ $\quad \mu, \rho, \sigma, \tau = 0, 1, 2, 3$

Returning to the Pauli–Lubanski vector eq. (3.9) we have

$$\begin{aligned}
W_\sigma P^\sigma &= -\frac{1}{2}\epsilon_{\sigma\mu\nu\lambda}J^{\mu\nu}P^\lambda P^\sigma = +\frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}J^{\mu\nu}P^\lambda P^\sigma = 0 \\
W_0 &= -\frac{1}{2}\epsilon_{0ijk}J^{ij}P^k = -J_k P^k = -\vec{J}\cdot\vec{P} \\
W_i &= -\frac{1}{2}\epsilon_{ijk0}J^{jk}P^0 - \frac{1}{2}\epsilon_{ij0k}J^{j0}P^k - \frac{1}{2}\epsilon_{i0jk}J^{0j}P^k \\
&= \frac{1}{2}\epsilon_{0ijk}J^{jk}P^0 - \frac{1}{2}\epsilon_{0ijk}J^{j0}P^k - \frac{1}{2}\epsilon_{0ijk}J^{j0}P^k \\
&= P_0 J_i + \epsilon_{ijk}P_j K_k \\
\\
\vec{W} &= P_0\vec{J} + \vec{P}\times\vec{K}
\end{aligned} \tag{3.10}$$

For $\vec{P}=0 \quad , \quad P_0 = m \quad \Rightarrow \quad \vec{W} = +m\vec{J} = +m\vec{S}$

The commutation relations become:

$$\begin{aligned}
&[J_{\mu\nu}, W_\rho] = i(\eta_{\nu\rho}W_\mu - \eta_{\mu\rho}W_\nu) \\
&[W_\mu, P_\nu] = 0 \\
&[W_\mu, W_\nu] = i\epsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma
\end{aligned}$$

3.2 Casimir invariants of the Poincare group

A Casimir operator is one that commutes with all the generators of the group.
Eigenvalues of the Casimir operators are labels of 1–particle states.
The Casimir operator C commutes with the Hamiltonian

$$\begin{aligned}
&H = P_0 \Rightarrow [C, H] = 0 \\
&P_0 = i\hbar\frac{\partial}{\partial t} \leftrightarrow \text{translation in time}
\end{aligned}$$

Hence, the eigenvalues of the Casimir operator are constants in time $\rightarrow$ constants of the motion.

single particle states: $\psi(x) = |\vec{P}, S\rangle$
where $\vec{P}$ represents the momentum vector and S stands for other quantum numbers
$P_\mu P^\mu$ is the first Casimir operator of the Poincare group

$$P_\mu P^\mu|\vec{P}, S\rangle = m_0^2|\vec{P}, S\rangle \quad \rightarrow \quad m_0 \text{ rest mass of the particle}$$

The rest mass of the particle is a Poincare invariant.
The second Casimir operator of the Poincare group is given by $W_\mu W^\mu$.
We saw that W_μ is a Lorentz four vector. Hence, $W_\mu W^\mu$ is a Lorentz scalar, and commutes with $J_{\mu\nu}$, the generators of the Lorentz group.
From the explicit form of W_μ it also follows that

$$[W_\mu, P_\nu] = 0$$

by using the asymmetry of $\epsilon_{\mu\nu\rho\sigma}$ and the symmetry of $P_\nu = i\partial_\nu$. Hence, it follows that

$$[W_\mu W^\mu, P^\nu] = 0$$

Since, $W_\mu W^\mu$ is a Lorentz invariant, we can compute it in a convenient frame.
If $m \neq 0$ it is convenient to choose the rest frame of the particle. In this frame

$$P^\mu = (m, 0, 0, 0).$$

$$W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho 0} J_{\nu\rho} P_0 + \cdots = -\frac{m}{2}\epsilon^{\mu\nu\rho 0} J_{\nu\rho} \cdots = \frac{m}{2}\epsilon^{0\mu\nu\rho} J_{\nu\rho}$$

$$\begin{aligned} \Rightarrow \quad W^0 &= 0 \\ W^i &= \frac{m}{2}\epsilon^{0ijk} J_{jk} = \frac{m}{2}\epsilon^{ijk} J_{jk} = m J^i \end{aligned}$$

Therefore, on a one particle state with mass m and spin j we have $-W_\mu W^\mu = m^2 J_i J^i = m^2 \vec{J}^2$

$$-W_\mu W^\mu |\vec{P}, S\rangle = m^2 j(j+1)|\vec{P}, S\rangle \qquad (m \neq 0)$$

If $m = 0$ there is no rest frame and $|\vec{v}| = c$, *i.e.* the particle is travelling at the speed of light. We choose a frame with

$$P^\mu = (w, 0, 0, w).$$

$$\begin{aligned} \Rightarrow W^0 &= -\frac{1}{2}\epsilon^{0ijk} J_{ij} P_k = w J^3 = W^3 \\ W^1 &= -\frac{1}{2}\epsilon^{10jk} J_{0j} P_k - \frac{1}{2}\epsilon^{1j0k} J_{j0} P_k - \frac{1}{2}\epsilon^{1jk0} J_{jk} P_0 \\ &= w(J^1 - K^2) \\ \text{Similarly,} \qquad W^2 &= w(J^2 + K^1) \end{aligned}$$

$$\text{Therefore,} \qquad -W_\mu W^\mu = w^2\left[(K^2 - J^1)^2 + (K^1 + J^2)^2\right] \qquad (m = 0)$$

The limit $m \to 0$ is nontrivial. We study separately the massive and massless representations.

- Massive representations

$$\begin{aligned} P_\mu P^\mu &= m^2 \\ W_\mu W^\mu &= -m^2 j(j+1) \end{aligned}$$

Take $m > 0 \quad \rightarrow \quad$, massive representations are labeled by their mass m and spin j. For massive states we can choose the frame $P^\mu = (m, 0, 0, 0)$.

$\Rightarrow$ invariant under spatial rotations

The "Little group" of Lorentz transformations is the set of Lorentz transformations that leaves P^μ invariant.

$$P^\mu = \Lambda^\mu{}_\alpha P^\alpha$$

We see from the form of P^μ in the rest frame that for massive representations the "little group" correspond to spatial rotations *i.e.*

$$\Rightarrow \quad \text{the little group for massive representations is } SU(2)$$

$\Rightarrow$ massive representations of mass m are labeled by their spin $j = 0,\ \frac{1}{2},\ 1,\ \cdots$
states within each representations are labelled by $j_z = -j, -j+1m \cdots, j-1, j$

$$\Rightarrow \quad \text{massive particles of spin } j \text{ have } \underline{2j+1} \text{ degrees of freedom}$$

- Massless representations $\quad P_\mu P^\mu = P^2 = m^2 = 0.$

  Two choices that satisfy this condition are $P^\mu = (w, 0, 0, w)$ or $P^\mu = (0, 0, 0, 0)$.

  The second case is unphysical. It is unchanged under Lorentz transformations.

  In the case of massless states there is no rest frame.

  We want to find the little group in the case of massless states, *i.e.* the set of Lorentz transformations $\Lambda^\mu{}_\nu$ that leaves the momentum 4–vector invariant. Our problem is to solve the equation

$$\Lambda^\mu{}_\nu(p)P^\nu = P^\mu$$

where $P^\nu = (w, 0, 0, w)$.

First, we note that rotations in the $x,\ y$ plane leaves this p^ν invariant

$$\lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ o & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.11}$$

This is an $SO(2)$ subgroup of $SU(2)$ generated by J^3, the generator of rotations in the $x-y$ plane. To find the most general transformation that leaves $P^\mu = (w, 0, 0, w)$ invariant, it is sufficient to look at the infinitesimal Lorentz transformations,

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

We look for the most general matrix $\omega^{\mu\nu}$ that satisfies

$$\omega^{\mu\nu} = -\omega^{\nu\mu}$$

and

$$\Lambda^{\mu\nu}P_\nu = (\delta^{\mu\nu} + \omega^{\mu\nu})P_\nu = \delta^{\mu\nu}P_\nu + \omega^{\mu\nu}P_\nu = P^\mu$$

$$\text{Hence,} \omega^{mu\nu}P_\nu = 0$$

For $P^\nu = (w, 0, 0, w) \ \rightarrow \ P_\nu = (w, 0, 0, -w)$

$$\Rightarrow \quad \begin{pmatrix} 0 & w^{01} & w^{02} & w^{03} \\ -w^{01} & 0 & w^{12} & w^{13} \\ -w^{02} & -w^{12} & 0 & w^{23} \\ -w^{03} & -w^{13} & -w^{23} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = 0$$

This gives the constraints

$$\begin{aligned} w^{03} &= 0 \\ w^{01} + w^{13} &= 0 \\ w^{02} + w^{23} &= 0 \end{aligned}$$

Denoting $w^{01} = \alpha;\ w^{02} = \beta;\ w^{12} = \theta$, the most general transformation that leaves P^μ invariant is given by

$$\Lambda = \mathrm{e}^{-i(\alpha A + \beta B + \theta C)}$$

where (with a lower second index)

$$A^\mu{}_\nu = i \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad B^\mu{}_\nu = i \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{3.12}$$

$$C^\mu{}_\nu = i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The matrix C corrspond to rotations in the $x - y$ plane, whereas the A and B matrices are given by combinations of boost and rotation generators.

$$\begin{aligned} C^\mu{}_\nu &= \left(J^3\right)^\mu{}_\nu \\ A^\mu{}_\nu &= \left(K^1 + J^2\right)^\mu{}_\nu \\ B^\mu{}_\nu &= \left(K^2 - J^1\right)^\mu{}_\nu \end{aligned}$$

We showed that for massless states the Poincare invariant $-W_\mu W^\mu$ is given by

$$-W_\mu W^\mu = w^2 \left[(K^2 - J^1)^2 + (K^1 + J^2)^2\right]$$

Hence, we obtained

$$-W_\mu W^\mu = w^2 \left[A^2 + B^2\right]$$

Using the expressions for the matrices A, B and C that we found or the commutation relations of the Lorentz algebra we find that the J^3, A and B generators close an algebra

$$\left[J^3, A\right] = iB \quad ; \quad \left[J^3, B\right] = -iA \quad ; \quad [A, B] = 0$$

This is the same as the algebra generated by

$$P^x,\ P^y,\ \text{and } L^z = (xP^y - yP^x)\ ,$$

i.e. translations and rotations in the Euclidean $x - y$ plane with A and B playing the role of translation operators. The algebra is denoted as $ISO(2)$. It is a non–compact algebra, *i.e.* it is infinite dimensional due to the continuous eigenvalues of the momentum operators P^x and P^y. Since A and B commute their eigenvalues are continuous and non–compact.

We find that for massless particles there exists a continuous degree of freedom that is not realised physically. We demand therefore that the operators A and B annihilate the physical massless states

$$\begin{aligned}\hat{A}|\vec{P},a,b\rangle &= a|\vec{P},a,b\rangle \\ \hat{B}|\vec{P},a,b\rangle &= b|\vec{P},a,b\rangle\end{aligned}$$

with $a = b = 0$ for physical massless states. Therefore,

$$-W_\mu W^\mu = 0$$

for physical massless states. This agrees well with

$$\lim_{m\to 0} W_\mu W^\mu = \lim_{m\to 0} -m^2(j(j+1)) = 0$$

that we found in the massive case. For massless states with $a,\ b\ = 0$ the little group is $SO(2)$ or $U(1)$. The generator of rotations in the $x-y$ plane is J^3. Hence, the representations are labeled by the eigenvalue h of J^3, which is the angular momentum in the direction of propagation.

$\rightarrow$ helicity : projection of the spin on the direction of momentum.

The helicity is quantised. The proof is based on topological properties of the Lorentz group.

$\rightarrow$ massless states are labeled by their helicity.

$$h = \frac{\vec{P}}{|\vec{P}|}\cdot\vec{J}$$

where $\frac{\vec{P}}{|\vec{P}|}$ is a unit vector in the direction of the momentum. Helicity is quantised. For massless states there are only two helicity states.

$$h = 0,\ \pm\frac{1}{2},\ \pm 1,\ \pm\frac{3}{2},\ \pm 2,\ \cdots$$

$$\begin{array}{lcll}\text{Photon } m^2 & = & 0 & \text{two polarisation states } h = \pm 1 \\ \text{Graviton } m^2 & = & 0 & \text{two polarisation states } h = \pm 2\end{array}$$

For massive particles there are $(2j+1)$ helicity states.

For massive states we can go from $-h$ to $+h$ polarisation states by a Lorentz transformation, *i.e.* by a boost.

For massless particles this is not possible as $c = 1$ in all inertial frames.

Helicity is a Lorentz invariant of massless states.

4 Lagrangian & Hamiltonian Mechanics

Newtonian mechanics : specify position and velocity at $t = t_i$

$$m\frac{d^2X}{dt^2} = F(X) \Rightarrow X = X(t=t_f) \ ; \quad \dot{X} = \dot{X}(t=t_f)$$

In Newtonian mechanics the mechanical problem is solved by specifying initial conditions for the position and velocity, and solving the second order differential Newton equation, which then provides a solution for the position and velocity at ensuing times.

Modern particle physics : specify energy and momentum at $t = t_i$

Energy and momentum are constants of the motion. Extract quantities that remain constant in the initial and final time and measure them experimentally. The analysis of the experimental data is performed subject to the tenet that energy and momentum are conserved.

In modern particle physics calculations are done in the framework of quantum field theories. A bridge between the "old" Newtonian mechanics and the "modern" particle physics is provided by the classical Lagrangian & Hamiltonian formulations of classical mechanics.

Newton : $\vec{F} = m\vec{a} \Rightarrow -\vec{\nabla}V(\vec{X}) = m\frac{d^2\vec{X}}{dt^2}$ for conserved forces

$$\vec{v} = \frac{d\vec{X}}{dt} \qquad \vec{a} = \frac{d^2\vec{X}}{dt^2}$$

$$\vec{P} = m\frac{d\vec{X}}{dt} \qquad \vec{F} = \frac{d\vec{P}}{dt}$$

Form dynamical functions of $\vec{X}$ and $\vec{v}$

Examples : $\vec{L} = \vec{X} \times \vec{P}$

$$E = \frac{1}{2}m\vec{v}^2 + V(\vec{X}, t) = T + V(\vec{X})$$

4.1 Constants of the motion

Energy is conserved if $V = V(\vec{X}) \neq V(t)$ and $\vec{F} = -\vec{\nabla}V(\vec{X})$, which follows from

$$\begin{aligned}\frac{dE}{dt} &= \frac{d}{dt}\left(\frac{1}{2}m\vec{v}^2 + V(\vec{X})\right) = m\vec{v}\cdot\frac{d\vec{v}}{dt} + \frac{d}{dt}V(\vec{X}) \\ &= m\vec{v}\cdot\dot{\vec{v}} + \vec{\nabla}V(\vec{X})\frac{d\vec{X}}{dt} = \left(m\dot{\vec{v}} + \vec{\nabla}V\right)\cdot\vec{v} = 0\end{aligned}$$

where the last equality follows from Newton's equation.

4.2 An alternative way to write Newton's equations

Take

$$\begin{aligned} L = T - V &= \frac{1}{2}mv^2 - V(\vec{X}) \\ &= \frac{1}{2}m\dot{\vec{X}}^2 - V(x,y,z) \\ &= \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - V(x,y,z) \end{aligned}$$

$$\text{for } x \quad \frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} \qquad \frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x}$$

$$\text{we get} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = m\ddot{x} + \frac{\partial V}{\partial x} = 0$$

This is a very important result and generalises to many mechanical systems and modern field theories.

For a conserving mechanical system with n–degrees of freedom $q_1, \ q_2, \cdots, q_n$ with potential $V(q_1, q_2, \cdots q_n)$.

The 2^{nd} order Euler–Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

The motion of the physical system is solved by specifying $2n$ boundary conditions:

$$\begin{array}{llll} & q_1, \ \cdots, q_n \ , \ \dot{q}_1, \ \cdots, \dot{q}_n, & \text{at } t = t_0 & \\ \text{or} & q_1, \ \cdots, q_n \ , & \text{at } t = t_0 & t = t_f \end{array}$$

We define the conjugate momentum

$$p_i = \frac{\partial L}{\partial \dot{q}_i}(\vec{q}, \dot{\vec{q}}, t)$$

<u>Cyclic coordinate</u> a coordinate that does not appear explicitly in L

$$L = L(q_1, \ \cdots, q_{n-1}, \dot{q}_1, \ \cdots, \dot{q}_n, t) \ \Rightarrow \ \frac{\partial L}{\partial q_{\ n}} \ = \ 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) = 0 \ \Rightarrow \ \frac{d}{dt}(p_n) = 0 \ \Rightarrow \ p_n = \text{constant} \tag{4.1}$$

$$\Rightarrow \quad \text{Aim} \rightarrow \text{find cyclic coordinates} \rightarrow \text{constants of the motion}$$

In the Lagrange formulation we describe the system in terms of n 2^{nd} order differential equations

An alternative formulation is provided by the Hamiltonian formulation

Hamilton $\rightarrow$ describe the system in terms of 1^{st} order differential equations.

Still need to specify $2n$ boundary conditions

the price is that we have $2n$ first order differential equations, rather than n second order equations.

Hamilton change variables from configuration space to phases space

$$\begin{aligned}
&\underline{\text{Configuration space :}} \quad (q_1, \ \cdots \ , q_n, \dot{q}_1, \ \cdots \ , \dot{q}_n), \\
&\underline{\text{Phase space :}} \qquad\qquad (q_1, \ \cdots \ , q_n, p_1, \ \cdots \ , p_n), \\
&\text{where} \qquad\qquad\qquad p_i = \frac{\partial L}{\partial \dot{q}_i}
\end{aligned}$$

The tranformation is made by a Legendre transformation

$$H = \sum_{k=1}^{n} p_k \dot{q}_k - L(q_1, \ \cdots \ , q_{n-1}, \dot{q}_1, \ \cdots \ , \dot{q}_n, t) = H(q_k, p_k),$$

$$\text{where} \quad p_k = \frac{\partial L}{\partial \dot{q}_k}$$

Example: a particle in one dimension with constant energy E.

$$\begin{aligned}
L &= \frac{1}{2} m \dot{q}^2 - V(q) \\
p &= \frac{\partial L}{\partial \dot{q}} = m\dot{q} \quad \Rightarrow \quad \dot{q} = \frac{p}{m} \\
H &= p\dot{q} - L(q, \dot{q} = p\dot{q}) - \left(\frac{1}{2} m \dot{q}^2 - V(q)\right) \\
&= \frac{p^2}{m} - \frac{m}{2}\frac{p^2}{m^2} + V(q) = \frac{p^2}{2m} + V(q)
\end{aligned}$$

In the Hamiltonian formalism we perform a Legendre transformation from configuration space to phase space that consist of the $2n$ Independent variables (q_i, p_i). The Hamiltionian is a function of these $2n$ variables and possibly of time, $H = H(q_i, p_i, t)$.

Legendre transformation from configuration to phase space $(q_i, p_i) \ \rightarrow 2n$ independent variables $H = H(q_i, p_i, t)$.

The Hamilton equation of motion are obtained by taking

$$dH = \frac{\partial H}{\partial q_k} dq_k + \frac{\partial H}{\partial p_k} dp_k + \frac{\partial H}{\partial t} dt \tag{4.2}$$

Whereas from† $H = p_k \dot{q}_k - L$, we have

$$dH = p_k d\dot{q}_k + \dot{q}_k dp_k - \frac{\partial L}{\partial q_k} dq_k - \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k - \frac{\partial L}{\partial t} dt \tag{4.3}$$

$$\text{with} \quad p_k = \frac{\partial L}{\partial \dot{q}_k} \qquad \text{and with Euler–Lagrange equations} \qquad \frac{dp_k}{dt} - \frac{\partial L}{\partial q_k} = 0$$

we get,

$$dH = \dot{q}_k dp_k - \frac{\partial L}{\partial q_k} dq_k - \frac{\partial L}{\partial t} dt \tag{4.4}$$

†here and in eq. (4.2) summation of k is implied.

Comparing the coefficients of eqs. (4.2) and (4.4) yields the Hamilton equations of motion

$$\begin{aligned}
\frac{\partial H}{\partial p_k} &= \dot{q}_k \\
\frac{\partial H}{\partial q_k} &= -\frac{\partial L}{\partial q_k} = -\frac{dp_k}{dt} = -\dot{p}_k \\
\frac{\partial H}{\partial t} &= -\frac{\partial L}{\partial t}
\end{aligned}$$

We get $2n+1$ 1^{st} order differential equations which are the Hamilton equations of motion.

When L does not depend explicitly on time $\Rightarrow$ $\frac{\partial L}{\partial t} = 0$

$$\begin{aligned}
\Rightarrow \quad \frac{dH}{dt} &= \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial t} \\
&= \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} = 0
\end{aligned}$$

when $L \neq L(t) \quad \Rightarrow \quad H \neq H(t) \quad \Rightarrow$ H is a constant of the motion.

When H does not depend explicitly on t H is identified with the conserved energy $H = E =$ constant.

4.3 Poisson brackets

Using Hamilton equations we can write for any canonical function $G(q_i, p_i, t)$

$$\begin{aligned}
\frac{dG}{dt} &= \frac{\partial G}{\partial q_i}\dot{q}_i + \frac{\partial G}{\partial p_i}\dot{p}_i + \frac{\partial G}{\partial t} \\
&= \frac{\partial G}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial G}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial G}{\partial t}
\end{aligned}$$

the Possion brackets are defined as

$$\{A, B\} = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}$$

$$\underline{\text{we can write}} \quad \frac{dG}{dt} = \{G, H\} + \frac{\partial G}{\partial t}$$

$$\Rightarrow \quad \text{if} \quad \frac{\partial G}{\partial t} = 0 \ \ \& \ \ \{G, H\} = 0 \Rightarrow \frac{dG}{dt} = 0$$

i.e. if G does not depend explicitly on time and the Poisson brackets of G with H vanish then G is conserved in time.

In Quantum mechanics the Poisson brackets are replaced by the commutator of the hermitian operators A and B, *i.e.*

$$\{A, B\} \ \rightarrow \ [A, B] \quad \text{where} \ \ A\ , B \text{ are hemitian operators}$$

$$\Rightarrow \quad \frac{dA}{dt} = [A, H] + \frac{\partial \langle A \rangle}{\partial t} \quad \Rightarrow \quad \text{if} \ \ A \neq A(t) \ \ \& \ \ [A, H] = 0 \quad \Rightarrow \quad \frac{dA}{dt} = 0$$

$$\Rightarrow \quad A \ \ A \text{ is a conserved operator}$$

5 The action principle

The Euler–Lagrange equations of motion can be derived from an action principle. Given the Lagrangian

$$L(q_i, \dot{q}_i) = \sum_i \frac{1}{2} m_i \dot{q}^i - V(q_i)$$

define

$$S = \int dt L(q_i, \dot{q}_i).$$

Action principle for fixed values of $q(t_i) = q_{in}$, $q(t_f) = q_{out}$, then the classical trajectory which satisfies these boundary conditions is an extremum of the action

$$\delta \int_{t_{in}}^{t_{out}} dt L(q_i, \dot{q}_i) = 0$$

$$\begin{aligned}
\delta S &= \int_{t_{in}}^{t_{out}} dt\ \delta L(q_i, \dot{q}_i = \int_{t_{in}}^{t_{out}} dt \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) \\
&= \int_{t_{in}}^{t_{out}} dt \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \frac{dq_i}{dt} \right) \\
&= \int_{t_{in}}^{t_{out}} dt \left(\frac{\partial L}{\partial q_i} \delta q_i - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \right) + \left. \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right|_{t_{in}}^{t_{out}} = 0 \\
&\Rightarrow \int_{t_{in}}^{t_{out}} dt \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right) \delta q_i = 0 \qquad (5.1)
\end{aligned}$$

This must hold for any variation δq_i. Hence we must have

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0 \qquad (5.2)$$

6 Classical field theory

So far we discussed systems with discrete and finite number of particles

Classical field $\Psi(\vec{r}, t)$: function of $\vec{r}$, t, specifying the value of the field at a spacetime point.

A field representation can simplify a many body mechanical problem

Example:

$$\frac{\partial^2 y(x,t)}{\partial t^2} - \frac{1}{v^2} \frac{\partial^2 y(x,t)}{\partial x^2} = 0$$

where $y(x, t)$ represents the motion of molecules along a chain. With a large number of molecules in the chain, say of the order of 10^{23} we can write the deviation from some equilibrium position as a function of the variables x and t.

6.1 Continuous dynamics in the Lagrangian formalism

The Lagrangian for longitudenal motion of a N–particle linear elastic chain.

$$L = \sum_{i=1}^{N} \frac{1}{2} m \dot{y}_i^2 - \sum_i \frac{1}{2} K (y_{i+1} - y_i)^2$$

where

$$\begin{aligned} m_i = m &\quad \rightarrow \quad \text{mass of the } i^{th} \text{ particle} \\ y_i &\quad \rightarrow \quad \text{longitudenal displacement from equilibrium} \\ K_i = K &\quad \rightarrow \quad \text{elastic constant} \\ a &\quad \rightarrow \quad \text{equilibrium position} \\ Na &\quad \rightarrow \quad \text{total length of the chain} \end{aligned}$$

$$L = \frac{1}{2} \sum_{i=1}^{N} a \left[\frac{m}{a} \dot{y}_i^2 - Ka \left(\frac{y_{i+1} - y_i}{a} \right)^2 \right] = \sum_i a L_i$$

$$\lim_{a \to 0} = \sum_{i=1}^{N} a L_i = \int dx \mathcal{L}(x) = L$$

x is a continuous index replacing the discrete index i.

$$\begin{aligned} \lim_{a \to 0} \frac{m}{a} &\rightarrow \frac{dm}{dx} = \mu \rightarrow \text{linear mass density} \\ \lim_{\substack{a \to 0 \\ N \to \infty}} Ka &\rightarrow \tau = \text{elastic tenion; Young's modulus} \end{aligned}$$

$$L = \lim_{a \to 0} \frac{1}{2} \sum a \left\{ \frac{m}{a} \dot{y}_i^2 - Ka \left(\frac{y_{i+1} - y_i}{a} \right)^2 \right\} \rightarrow \frac{1}{2} \int dx \left(\mu \dot{y}^2 - \tau {y'}^2 \right)$$

$$\text{Lagrangian density} \quad \mathcal{L}(x) = \frac{1}{2} \left\{ \mu \dot{y}^2 - \tau {y'}^2 \right\} \quad \text{per unit length}$$

$\mathcal{L}$ is a function of the field velocity $\dot{y}$, the field coordinate $y(x,t)$ and the field gradient ${y'}^2$.

x is an index. Apoint in the field. It replaced the i variable in the discrete case.

The generalisation to three dimensions

$$L = \int \mathcal{L}(\phi, \dot{\phi}, \nabla \phi) dx dy dz \quad \text{where} \quad \phi = \phi(x, y, z, t)$$

the action is given by

$$\begin{aligned} S = \int_{t_1}^{t_2} dt \; L[\phi] &= \int_{t_1}^{t_2} dt dx dy dz \; \mathcal{L}(\phi, \dot{\phi}, \nabla \phi) \\ &= \int d^4x \; \mathcal{L}(\phi, \partial^\mu \phi) \quad \leftarrow \quad \text{relativistic notation} \end{aligned}$$

The equations of motions for the field ϕ are obtained by requiring that the variation of the action vanishes and demanding that $\delta\phi = 0$ at t_1 and t_2.

$$\delta \int \mathcal{L}(\phi, \partial^\mu \phi) d^4x =$$
$$\int \left(\frac{\partial \mathcal{L}(\phi, \partial^\mu \phi)}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}(\phi, \partial^\mu \phi)}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right) d^4x$$

we note that there is no explicit dependence of $\mathcal{L}$ on x^μ. With $\delta(\partial^\mu\phi) = \partial^\mu(\delta\phi)$ we get

$$= \int_\Omega \left(\frac{\partial \mathcal{L}(\phi, \partial^\mu \phi)}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}(\phi, \partial^\mu \phi)}{\partial(\partial_\mu \phi)} \partial_\mu \delta\phi \right) d^4x$$
$$= \int_\Omega \delta\phi \left(\frac{\partial \mathcal{L}(\phi, \partial^\mu \phi)}{\partial \phi} \right) d^4x + \int_\Omega \partial_\mu \left(\frac{\partial \mathcal{L}(\phi, \partial^\mu \phi)}{\partial(\partial_\mu \phi)} \delta\phi \right) d^4x - \int_\Omega \delta\phi \partial_\mu \left(\frac{\partial \mathcal{L}(\phi, \partial^\mu \phi)}{\partial(\partial_\mu \phi)} \right) d^4x$$

where we used integration by parts $\int V dU = VU - \int U dV$.
the second term is a boundary term that vanishes by the divergence theorem (Gauss law).

$$= \int \delta\phi \left(\frac{\partial \mathcal{L}(\phi, \partial^\mu \phi)}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}(\phi, \partial^\mu \phi)}{\partial(\partial_\mu \phi)} \right) \right) d^4x = 0$$

Since this holds for any $\delta\phi$ we have that

$$\frac{\partial \mathcal{L}(\phi, \partial^\mu \phi)}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}(\phi, \partial^\mu \phi)}{\partial(\partial_\mu \phi)} \right) = 0$$

These are the Euler–Lagrange equations of motion for the field ϕ.

Example': in the case of the harmonic chain

$$\mathcal{L} = \frac{1}{2}\mu \dot{y}^2 - \frac{1}{2}\tau y'^2$$
$$\frac{\partial \mathcal{L}}{\partial \dot{y}} = \mu \dot{y}^2 \quad ; \quad \frac{\partial \mathcal{L}}{\partial y'} = \tau \dot{y}^2 \quad ; \quad \frac{\partial \mathcal{L}}{\partial y} = 0$$
$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu y)} - \frac{\partial \mathcal{L}}{\partial y} = \mu \frac{\partial^2 y}{\partial t^2} - \tau \frac{\partial^2 y}{\partial x^2} = 0$$

which is the familiar wave equation. If

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\eta^{\alpha\beta}(\partial_\alpha \phi)(\partial_\beta \phi) - \frac{1}{2}m^2\phi^2$$

$$\begin{aligned}
\frac{\partial \mathcal{L}(\phi, \partial^\mu \phi)}{\partial(\partial_\mu \phi)} &= \eta^{\alpha\beta} \frac{\partial(\partial_\alpha \phi)}{\partial(\partial_\mu \phi)}(\partial_\beta \phi) + \eta^{\alpha\beta}(\partial_\alpha \phi)\frac{\partial(\partial_\beta \phi)}{\partial(\partial_\mu \phi)} \\
&= \eta^{\alpha\beta}\delta_{\alpha\mu}(\partial_\beta \phi) + \eta^{\alpha\beta}\delta_{\beta\mu}(\partial_\alpha \phi) \\
&= \eta^{\mu\beta}(\partial_\beta \phi) + \eta^{\mu\alpha}(\partial_\beta \phi) = 2\eta^{\mu\nu}(\partial_\nu \phi)
\end{aligned}$$

$$\begin{aligned}
\partial_\mu \left(\frac{\partial \mathcal{L}(\phi, \partial^\mu \phi)}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}(\phi, \partial^\mu \phi)}{\partial \phi} &= \frac{1}{2}\partial_\mu(2\eta^{\mu\nu}\partial_\nu \phi) + m^2\phi \\
&= (\partial_\mu \partial^\mu + m^2)\phi(\vec{x}, t) = 0
\end{aligned}$$

This is the Klein–Gordon equation of a reletivistic free scalar field that we will encounter in more detail later on. The Hamiltonian for a field can be written has

$$H = \int h(\phi, \pi) dxdydz$$

where the generalised momentum density is defined by

$$h = \dot{\phi}(\vec{x}, t)\pi(\vec{x}, t) - \mathcal{L}$$

and

$$H = \int d^3x(\dot{\phi}(\vec{x}, t)\pi(\vec{x}, t) - \mathcal{L}) = \int d^3xh$$

For the Klein–Gordon field,

$$\mathcal{L} = \frac{1}{2}(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2)$$

We used here $\partial^\mu = (\partial_t, \vec{\nabla})$, $\partial_\mu = (\partial_t, -\vec{\nabla})$ $\rightarrow$ $\Box = \partial_\mu\partial^\mu = (\partial_t^2 - \vec{\nabla}^2)$

$$\begin{aligned} \Rightarrow \frac{\partial\mathcal{L}}{\partial\dot{\phi}} &= \dot{\phi} \\ h &= \dot{\phi}^2 - \left(\frac{1}{2}(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2)\right) = \frac{1}{2}\left(\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2\right) \end{aligned}$$

the total energy $\int hd^3x$ should be conserved if $H \neq H(t)$, and be the zero component of some four vaector P^μ. To construct it we introduce the energy momentum tensor

$$T^{\mu\nu} = \partial^\mu\phi\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)} - \mathcal{L}\eta^{\mu\nu}$$

$$\begin{aligned} \Rightarrow P^\mu &= \int T^{\mu 0}d^3x \\ \text{and } H = P^0 &= \int T^{00}d^3x \end{aligned}$$

$T^{\mu\nu}$ is conserved as it satisfies the continuity equation

$$\begin{aligned} \partial_\nu T^{\mu\nu} &= \partial_\nu\left(\partial^\mu\phi\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)} - \mathcal{L}\eta^{\mu\nu}\right) \\ &= \partial^\mu\phi\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\right) + \partial_\nu\partial^\mu\phi\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)} - \eta^{\mu\nu}\left(\frac{\partial\mathcal{L}}{\partial\phi}\partial_\nu\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi)}\partial_\nu\partial_\lambda\phi\right) \\ &= \left(\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi}\right)\partial^\mu\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\partial_\nu\partial^\mu\phi - \frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi)}\partial^\mu\partial_\lambda\phi = 0 \end{aligned}$$

The conservation of $T^{\mu\nu}$ implies that $P^\mu = \int T^{\mu 0}d^3x$ transforms as a 4–vector and is time independent. P^μ is the energy–momentum 4–vector.

7 The Klein–Gordon equation

In non–relativistic quantum mechanics

$$\vec{P} \to -i\hbar\vec{\nabla} \quad , \quad E \to i\hbar\frac{\partial}{\partial t} \text{ quantum operators}$$

The Hamiltonian for an energy conserving system is

$$H = \frac{\vec{P}}{2m} + V(\vec{q}) = E$$

leads to the Shrödinger equation by substitution

$$\left(-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{q})\right)\Psi(\vec{q},t) = i\hbar\frac{\partial}{\partial t}\Psi(\vec{q},t)$$

In special relativity the four vector P^μ is given by

$$P^\mu = (\frac{E}{c}, \vec{P})$$

we have $\partial^\mu = (\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}) \quad ; \quad \partial_\mu = (\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla})$ Therefore we can identify $P^\mu = i\hbar\partial^\mu$.

In special relativity $E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \quad , \quad \vec{P} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}$ Therefore, $P_\mu P^\mu = \frac{E^2}{c^2} - \vec{P}^2 = m^2c^2$. Following the example of non–relativistic quantum mechanics we use $P^\mu \to i\hbar\partial^\mu$ and obtain the wave equation

$$-\hbar^2\partial_\mu\partial^\mu\phi = m^2c^2\phi$$

or

$$\left(\partial^2 + \frac{m^2c^2}{\hbar^2}\right)\phi(\vec{X},t) = 0 \qquad \leftarrow \qquad \text{the Klein–Gordon equation}$$

Its interpretation as a single particle is problematic. The equatio describes a scalar field but not in a single state but a multi–state, *i.e.* a quantised field. To find solutions of the Klein–Gordon equation we put it in a box and impose that the wave–function vanishes on the boundaries. We then take the volume to infinity. We assume that the particle is free *i.e.* $V(\vec{X}) = 0$. The solution in the box is a plane wave solution of the form

$$\phi(\vec{X},t) \sim \mathrm{e}^{ik_\mu x^\mu}$$

$$\begin{array}{rcl} x^\mu & = & (t, \vec{x}) \\ k^\mu & = & (w, \vec{k}) \qquad , \qquad k_\mu = (w, -\vec{k}) \quad \leftarrow \text{constant} \\ \phi(\vec{x},t) & \sim & \mathrm{e}^{i(wt - \vec{k}\cdot\vec{x})} \end{array}$$

We want the solution to describe a free particle of mass m. We substitute this solution into the Klein–Gordon equation

$$\Rightarrow \quad w^2 - |\vec{k}|^2 = m^2$$

The particle is confined in a box $\stackrel{\text{assume}}{\Rightarrow}$ $\phi(x,y,z,t) = T(t)X(x)Y(y)(Z(z)$

$$\underline{\text{substituting in the KG equation}} \quad \rightarrow \quad -\frac{\ddot{T}}{T} + \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} \; = \; m^2 \quad \rightarrow \quad \text{const}$$

$$\underline{\text{all constants}} \quad \rightarrow \quad w^2 \; - \; k_x^2 \; - \; k_y^2 \; - \; k_z^2$$

To satisfy the boundary conditions we impose:

$k_x = \frac{n_1 \pi}{L}$, $k_y = \frac{n_2 \pi}{L}$, $k_z = \frac{n_3 \pi}{L}$ where $n_1,\ n_2,\ n_3$ are integers

and obtain $w^2 = m^2 + \frac{\pi^2}{L^2}(n_1^2 + n_2^2 + n_3^2)$ $\leftarrow$ dispersion relations

The solutions of the KG equations are momentum eigenstates

$$\begin{aligned}
\underline{\text{we have:}} \qquad \phi \;&=\; A\mathrm{e}^{-ik\cdot x} = A\mathrm{e}^{-ik_\mu x^\mu} \;\Rightarrow\; \partial^\mu \phi = \frac{\partial \phi}{\partial x_\mu} = -ik^\mu A\mathrm{e}^{-ik\cdot x} = -ik^\mu \phi \\
\partial^2 \phi \;&=\; -k_\mu k^\mu \phi = -k^2 \phi \\
\Rightarrow \qquad & \left(\partial^2 + \frac{m^2c^2}{\hbar^2}\right)\phi = 0 \;\;\Rightarrow\;\; k^2 = \frac{m^2c^2}{\hbar^2} = \frac{1}{\lambda^2}
\end{aligned}$$

$\lambda = \frac{\hbar}{mc}$ has dimensions of length $\rightarrow$ Compton wave length of particle of mass m

$$\underline{\text{we have:}} \qquad P^\mu \phi = i\hbar \partial^\mu \phi = \hbar k^\mu \phi$$

ϕ describes a momentum eigenstate with eigenvale $\hbar k^\mu$

The condition $k^2 = \frac{m^2c^2}{\hbar^2}$ is the same as $P^2 = m^2c^2$ $\rightarrow$ the mass shell condition

$$\underline{\text{we have:}} \qquad P^2 = \frac{E^2}{c^2} - \vec{p}^2 = m^2c^2 \quad \Rightarrow \quad \frac{E}{c} = \pm\sqrt{m^2c^2 + \vec{p}^2}$$

$\Rightarrow$ some solutions of the KGE correspond to negative energy states

$\rightarrow$ interpretation as a single particle state is problematic. Such interpretation is also in conflict with the probability interpretation of the wave function.

Quantum mechanics $\rightarrow$ non–relativistically $|\psi(x)|^2 = \rho(x)$.

$\rho(x)$ $\rightarrow$ probability density $\rightarrow$ $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$

The probability density in quantum mechanics obeys a continuity equation.

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

$$\text{where} \qquad \vec{J} = -\frac{i\hbar}{2m}(\psi^* \vec{\nabla}\psi - \psi \vec{\nabla}\psi^*)$$

Provided that the potential $V(\vec{r})$ is real, $\rho(x)$ is conserved.

We want to construct similar quantities for the KG equation. Furthermore, we want the continuity equation to be covariant *i.e.* to hold in all inertial frames, $\rightarrow$

$$\partial_\mu j^\mu = 0 \quad \leftarrow \quad \text{covariant}$$

find a 4–vector $j^\mu = (j^0, \vec{J})$ which obeys $\partial_\mu j^\mu = 0$.

$$\begin{aligned}
\underline{\text{Start with the KGE}} \qquad & 1.\ \ \phi^* \cdot / \;\; (\partial^2 + m^2)\phi = 0 \\
& 2.\ \ \phi \cdot / \;\; (\partial^2 + m^2)\phi^* = 0 \\
1 \; - \; 2 \quad \rightarrow \qquad & \phi^* \partial^2 \phi - \phi \partial^2 \phi^* = 0
\end{aligned}$$

$$\eta\alpha\beta(\phi^*\partial^\alpha\partial^\beta\phi - \phi\partial^\alpha\partial^\beta\phi^*) = \eta\alpha\beta\partial^\alpha(\phi^*\partial^\beta\phi - \phi\partial^\alpha\phi^*) = \partial_\beta(\phi^*\partial^\beta\phi - \phi\partial^\alpha\phi^*) = 0$$
$$\Rightarrow \quad j^\mu \sim (\phi^*\partial^\beta\phi - \phi\partial^\alpha\phi^*) \quad \Rightarrow \quad \partial_\mu j^\mu = 0$$

In Schrödinger case $\rho = \psi^*\psi$. Here,

$$\rho = \frac{i\hbar}{2mc^2}(\phi^*\frac{\partial}{\partial t}\phi - \phi\frac{\partial}{\partial t}\phi^*)$$

$$\phi \sim \mathrm{e}^{\pm i(Et - \vec{p}\cdot\vec{x})} = \mathrm{e}^{\pm iEt}\mathrm{e}^{\mp\vec{p}\cdot\vec{x}}$$

The case with $+E$ in the exponent corresponds to the antiparticle case. For $\phi \sim \mathrm{e}^{+iEt}(\ )$, with $E \approx mc^2$ we have

$$\rho = \frac{i\hbar}{2mc^2}\left(\frac{imc^2}{\hbar}\phi^*\phi + \frac{imc^2}{\hbar}\phi^*\phi\right) = -\phi^*\phi < 0$$

We get negative probability associated with the antiparticle and positive probability asscociated with the particle. This does not make sense as probability density $\rightarrow$ multiply by charge e.

$$\Rightarrow \quad \text{Charge density} \;\; \rho = \frac{i\hbar e}{2mc^2}\left(\frac{imc^2}{\hbar}\phi^*\phi + \frac{imc^2}{\hbar}\phi^*\phi\right)$$

$$\text{Charge current density} \;\; \rho = \frac{e\hbar}{imc}\left(\phi^*\vec{\nabla}\phi + \phi\vec{\nabla}\phi^*\right)$$

The interpretation of the solution of the KGE makes sense as charge density, not as probability density.

$\rightarrow$ it makes sense as a quantum field $\rightarrow$ creating–annihilating particles

7.1 Quantisation of the KG field (real scalar field).

$$\underline{\text{The Lagrangian density :}} \;\; \mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2$$

$$\underline{\text{leads to the KG equation :}} \;\; (\partial^2 + m^2)\phi = 0$$

$$\underline{\text{we can write :}} \;\; \mathcal{L} = \frac{1}{2}\dot{\phi} - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2$$

$$\text{from which we derive} \;\; \Pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}$$

In ordinary quantum mechanics we impose the relations:

$$[q_i, p_j] = i\hbar\delta_{ij}, \;\; [q_i, q_j] = 0, \;\; [p_i, p_j] = 0$$

In classical field theory the coordinates are replaced by the fields

$$q_i \rightarrow \phi(x) \qquad , \qquad p_i \rightarrow \pi(x)$$

analoguesly, in quantum field theory we impose the equal–time commutation relations

$$[\phi(\vec{x},t),\pi(\vec{x}',t)] = i\hbar\delta(\vec{x}-\vec{x}')$$
$$[\phi(\vec{x},t),\phi(\vec{x}',t)] = [\pi(\vec{x},t),\pi(\vec{x}',t)] = 0$$

where the Dirac δ–function satisfies the properties

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$
$$\int_{\alpha}^{\beta} f(x)\delta[F(x)]dx = \frac{f(a)}{|F'(a)|} \quad \text{where} F(a)=0$$

The commutation relations are equal time commutation relations $\to$ canonical commutation relations

In the Heisenberg picture the equations of motion are given by

$$i\hbar\dot{\alpha} = [\alpha, H]\,, \qquad \text{where } \alpha \to \text{operator, } H\text{–Hamiltonian } \alpha \neq \alpha(t)$$

consider:

$$\begin{aligned}
[\phi(\vec{x},t),H] &= \left[\phi(\vec{x},t), \int\left[\frac{1}{2}\pi(\vec{x}',t)^2 + \frac{1}{2}(\nabla\phi(x',t))^2 + \frac{1}{2}m^2\phi(x',t)^2\right]d^3x'\right] \\
&= \frac{1}{2}\int d^3x'\left\{\left[\phi(\vec{x},t),\pi(\vec{x}',t)^2\right] + \left[\phi(\vec{x},t),(\nabla\phi(x',t))^2\right] + m^2\left[\phi(\vec{x},t),\phi(x',t)^2\right]\right\}
\end{aligned}$$

Recalling that $[\phi(\vec{x},t),\phi(\vec{x}',t)]=0 \Rightarrow [\phi(\vec{x},t),\nabla'\phi(\vec{x}',t)]=0$

$$\begin{aligned}
\Rightarrow \quad [\phi(\vec{x},t),H] &= \frac{1}{2}\int d^3x'\left(\left[\phi(\vec{x},t),\pi(\vec{x}',t)^2\right]\right) \\
&= \frac{1}{2}\int d^3x'\left(\phi(\vec{x},t)\pi(\vec{x}',t)^2 - \pi(\vec{x}',t)^2\phi(\vec{x},t)\right) \\
&= \frac{1}{2}\int d^3x'\,(\phi(\vec{x},t)\pi(\vec{x}',t)\pi(\vec{x}',t) - \pi(\vec{x}',t)\phi(\vec{x},t)\pi(\vec{x}',t) \\
&\qquad + \pi(\vec{x}',t)\phi(\vec{x},t)\pi(\vec{x}',t) - \pi(\vec{x}',t)\pi(\vec{x}',t)\phi(\vec{x},t)) \\
&= \frac{1}{2}\int d^3x'\,([\phi(\vec{x},t),\pi(\vec{x}',t)]\,\pi(\vec{x}',t) + \pi(\vec{x}',t)\,[\phi(\vec{x},t),\pi(\vec{x}',t)]) \\
&= \int d^3x'\left(\pi(\vec{x}',t)\delta^3(\vec{x}-\vec{x}') = i\hbar\pi(\vec{x},t) = i\hbar\dot{\phi}(\vec{x},t)\right)
\end{aligned}$$

we obtained the correct equations of motion.

The connection between the quatised field and its particle interpretation is seen by looking at the Fourier transformed field

$$\begin{aligned}
\phi(x) &= \frac{1}{(2\pi)^4}\int d^4p\;\tilde{\phi}(p)\mathrm{e}^{-ip\cdot x} \\
\tilde{\phi}(p) &= \int d^4x\;\phi(x)\mathrm{e}^{ip\cdot x}
\end{aligned}$$

For $\phi(x)$ to satisfy the KG equation we must have

$$\begin{aligned}
(\partial^2 + m^2)\phi &= \frac{1}{(2\pi)^4}\int d^4p\ (m^2 - p^2)\tilde{\phi}(p)\mathrm{e}^{-ip\cdot x}d^4p \ = \ 0 \\
& \quad i.e.\ (p^2 - m^2) = 0 \\
& \quad i.e.\ \tilde{\phi}(p) \neq 0 \text{ only when } p^2 = m^2 \\
& \qquad\quad \tilde{\phi}(p) = (2\pi)\delta(p^2 - m^2)f(p) \\
p^0 &= \pm\sqrt{\vec{p}^2 + m^2}
\end{aligned}$$

$$\underline{\text{we may set :}}\ \ f(\vec{p}) = \theta(p^0)f_+(\vec{p}) + \theta(-p^0)f_-(\vec{p})$$

where

$$\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

<u>from:</u> the properties of the delta function we have

$$\int f(x)\delta(x-a)dx = f(a)$$

$$\text{using } y = \lambda x\ ,\ \frac{dy}{\lambda} = dx$$

$$\int f(x)\delta(\lambda x - \lambda a)dx = \int f(\frac{y}{\lambda})\delta(y - \lambda a)\frac{dy}{\lambda} = \frac{f(a)}{\lambda}$$

Further recalling

$$\int_\alpha^\beta f(x)\delta(F(x))dx =$$

and using $y = F(x)\ ,\ dy = F'(x)dx\ ,\ x = F^{-1}(y)\ ,\ dx = \dfrac{dy}{F'(F^{-1}(y))}$ we have

$$= \int_{F(\alpha)}^{F(\beta)} f(F^{-1}(y))\delta(y)\frac{dy}{F'(F^{-1}(y))} = \sum_i \frac{f(a_i)}{F'(a_i)}$$

where $y = F(a_i) = 0$. Hence,

$$\begin{aligned}
\delta(p^2 - m^2) &= \delta(p_0^2 - (\vec{p}^2 + m^2)) \\
&= \frac{1}{2p_0}\delta(p_0 - (\vec{p}^2 + m^2)^{\frac{1}{2}}) + \frac{1}{2p_0}\delta(p_0 + (\vec{p}^2 + m^2)^{\frac{1}{2}})
\end{aligned}$$

$$\phi(\vec{x}, t) = \frac{1}{(2\pi)^4}\int dt \int \frac{d^3\vec{p}}{2p_o}\left(\mathrm{e}^{-ip\cdot x}f_+(\vec{p}) + \mathrm{e}^{ip\cdot x}f_-(\vec{p})\right)$$

$$\text{Here } p_0 \ = \ \sqrt{\vec{p}^2 + m^2}$$

$$f_+(\vec{p}) = f(+\sqrt{\vec{p}^2}, +\vec{p})$$

$$f_-(\vec{p}) = f(-\sqrt{\vec{p}^2}, -\vec{p})$$

The f_+ term corresponds to positive energy states. The f_- term corresponds to negative energy states.

So far the Fourier decomposition that we discussed is classical, *i.e.* we didn't yet impose the commutation relations.

If we impose the commutation relations

$$[\pi, \phi] = \delta^3(\vec{x} - \vec{x}') \ , \ \ [\phi, \phi] = [\pi, \pi] = 0$$

Then the aplitude of the Fourier modes become annihilation and creation operators,

$$\begin{aligned} f_-(\vec{p}) &= (f_+(\vec{p}))^\dagger = a^\dagger(\vec{p}) \\ f_+(\vec{p}) &= \qquad\qquad = a(\vec{p}) \end{aligned}$$

It can then be shown that $a(\vec{p})$ and $a^\dagger(\vec{p})$ satisfy the commutation relations

$$\left[a(\vec{p}), a^\dagger(\vec{p}')\right] = 2p^0\delta^3(\vec{p} - \vec{p}')XS(2\pi)^3$$
$$[a(\vec{p}), a(\vec{p}')] = 0 = \left[a^\dagger(\vec{p}), a^\dagger(\vec{p}')\right]$$

i.e. the quantum field $\phi(x)$ creates and annihilates particle states with momentum $\vec{p}$. The vacuum is defined by

$$a(\vec{p})|0\rangle = 0 \qquad \forall \ \vec{p} \qquad \langle 0|0\rangle = 1$$

$$\begin{aligned} 1-\text{particle state} \qquad & |\vec{p}\rangle = a^\dagger(\vec{p})|0\rangle \\ 2-\text{particle state} \qquad & |\vec{p}_1, \vec{p}_2\rangle = a^\dagger(\vec{p}_1)a^\dagger(\vec{p}_2)|0\rangle \end{aligned}$$

and so forth.

So far we discussed the free KG equation.

How do we incorporate interactions?

In Newtonian mechanics we describe interactions by adding a potential.

$$L = \frac{1}{2}\sum_i \left(\frac{dx_i}{dt}\right) - V(\vec{x})$$

$$m\sum_i \ddot{x}_i = -\vec{\nabla}V(\vec{x} = \vec{F}(\vec{x})$$

if we have more than one particle

$$\sum_j m_j \sum_i \ddot{x}_{ji} = \sum_{ij} -\vec{\nabla}V(\vec{x}_i - \vec{x}_j)$$

typically we consider the interactions to be 2–body interactions and we sum over all the interacting particles

For the harmonic oscillator in one dimension

$$\begin{aligned} V(x) &= \frac{1}{2}kx^2 \\ m\ddot{x} &= -\frac{\partial V(x)}{\partial x} = -kx \end{aligned}$$

In the case of the KGE

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2$$

The second term looks like $V(\phi) \sim \frac{1}{2}k\phi^2$. Classically the particle is in a potential well. If it has initial energy E, it will oscillate about the vacuum $V(\phi) = 0$. The lowest point that the field can be in is the vacuum. So far we are describing a free field. Suppose that we want to describe a field which is not free in a potential

$$V(\phi) = A\phi^4 + B\phi^2 \text{ ,with } A > 0, \text{ and } B < 0.$$

In Newtonian mechanics this will correspond to a double well. The point $V(\phi) = 0$ is no longer the vacuum

$$\begin{aligned}
\frac{\partial V(\phi)}{\partial\phi} &= 4A\phi^3 + 2B\phi = 0 \\
\text{or} \qquad & (2A\phi^2 + B)\phi = 0 \\
\Rightarrow \qquad & \phi = 0 \ , \ \phi = \pm\sqrt{\frac{-B}{2A}}
\end{aligned}$$

We can now see how we can use this formalism to describe particles and their interactions.

We developed a diagramatic representation of interactions.

$\rightarrow$ Feynman diagrams

The interactions that we described so far are by using a single scalar field

In nature we are familiar so far with gravity, E&M, weak, and strong interactions.

Force	Gravity	E&M	Weak	Strong
Mediator	Graviton	Photon	$W^\pm$, Z-bosons	Gluons
Spin	+2	+1	+1	+1
mass	0	0	~ 80, ~ 90GeV	0
gauge symmetry	spacetime diff.	$U(1)$	$SU(2)$	$SU(3)$

How can we describe these interactions?

Standard Model $\rightarrow$ $SU(3)_C \times SU(2)_L \times U(1)_Y$ local gauge interactions.

In the modern language of elementary particles, interactions correspond to invariances of the Lagrangian under some local symmetry.

Interaction $\leftrightarrow$ invariance under a local gauge symmetry

Symmetries of the Lagrangian correspond to conserved currents

consider a free scalar field $calL = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$

It is invariant under the discrete symmetry $\phi \rightarrow -\phi$

consider two free scalar fields with equal mass m

$$\mathcal{L} = \frac{1}{2}\left[\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2\right] - \frac{1}{2}m^2(\phi_1^2 + \phi_2^2) \tag{7.1}$$

with the transformations

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \rightarrow \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$$

where α is a global constant.

The Lagrangian in eq. (7.1) is invariant under the rotations in the $\phi_1,\ \phi_2$ plane.

α is a continuous parameter $\rightarrow$ global continuous symmetry, $\alpha \neq \alpha(x)$.

Consider the complex field

$$\begin{aligned}
\Phi &= \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \\
\Phi^* &= \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)
\end{aligned}$$

The lagrangian in terms of Φ, with $\Phi^\dagger\Phi = \frac{1}{2}(\phi_1^2 + \phi_2^2)$

$$\mathcal{L} = (\partial_\mu\Phi)^\dagger(\partial^\mu\Phi) - m^2\Phi^\dagger\Phi \tag{7.2}$$

In terms of Φ the rotation becomes

$$\begin{aligned}
\Phi &= \frac{1}{\sqrt{2}}\left[(\cos\alpha\phi_1 + \sin\alpha\phi_2) + i(-\sin\alpha\phi_1 + \cos\alpha\phi_2)\right] \\
\Phi &= \frac{1}{\sqrt{2}}\left[(\cos\alpha - i\sin\alpha)\phi_1 + i(\cos\alpha - i\sin\alpha)\phi_2\right] \\
&= \mathrm{e}^{-i\alpha}\Phi
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad \Phi &\rightarrow \mathrm{e}^{-i\alpha}\Phi \\
\Phi^* &\rightarrow \mathrm{e}^{i\alpha}\Phi^*
\end{aligned}$$

The continuous symmetry in terms of Φ is a symmetry under a phase transformation. The symmetry is a global $U(1)$ symmetry.

$\rightarrow$ arbitrary choice of the phase $\rightarrow$ continuous global symmetry

What happens if $\alpha = \alpha(x)$?

$$\Phi(x) \rightarrow \mathrm{e}^{-i\alpha(x)}\Phi(x)$$

$$\partial_\mu\Phi \left(-i\partial_\mu\alpha(x)\Phi(x) + \partial_\mu\Phi(x)\right)\mathrm{e}^{-i\alpha(x)}$$

$$\begin{aligned}
\mathcal{L} \rightarrow \quad & \eta^{\mu\nu}\left[\partial_\mu(\mathrm{e}^{-i\alpha(x)}\Phi)\right]^\dagger\left[\partial_\nu(\mathrm{e}^{-i\alpha(x)}\Phi)\right] + m^2\left(\mathrm{e}^{-i\alpha(x)}\Phi\right)^\dagger\left(\mathrm{e}^{-i\alpha(x)}\Phi\right) \\
& \eta^{\mu\nu}\left[\partial_\mu\Phi(x) - i(\partial_\mu\alpha(x))\Phi(x)\right]^\dagger\left[\partial_\nu\Phi(x) - i(\partial_\nu\alpha(x))\Phi(x)\right]^\dagger + m^2\Phi^\dagger\Phi
\end{aligned}$$

if $\alpha \neq \alpha(x)$ the derivative term $\partial\alpha(x)$ drops out.

We have to fix the Lagrangian to get a Lagrangian which is invariant under local phase transformaions $\alpha = \alpha(x)$.

We redefine the derivative as $\partial_\mu \rightarrow \partial_\mu + a_\mu(x)$ where $a_\mu(x)$ is a function of x.

Under local phase transformations

$$a_\mu(x) \rightarrow a'_\mu(x) = a_\mu(x) + i\partial_\mu\alpha(x)$$

where $\alpha(x)$ is a scalar function of x. Then

$$\begin{aligned}
&(\partial_\mu + a_\mu(x))\Phi(x) \;\rightarrow\; (\partial_\mu + a'_\mu(x))(\mathrm{e}^{-i\alpha(x)}\Phi(x)) \\
&= \mathrm{e}^{-i\alpha(x)}(\partial_\mu + a'_\mu(x) + i\partial_\mu\alpha(x) - i\partial_\mu\alpha(x))\Phi(x) \\
&= \mathrm{e}^{-i\alpha(x)}(\partial_\mu + a'_\mu(x))\Phi(x)
\end{aligned}$$

We now get that the Lagrangian is invariant under the local phase transformations.

$$\Phi(x) \;\rightarrow\; \Phi'(x) = \mathrm{e}^{-i\alpha(x)}\Phi(x)$$

$$\begin{array}{lll}
\underline{\text{requiring local phase invariance}} & \rightarrow & \text{introduce } \; a_\mu(x) \\
 & \rightarrow & \text{local gauge field}
\end{array}$$

$\rightarrow$ the electromagnetic field
All interactions in the Standard Model are gauge interactions
$\rightarrow$ invariance under local phase transformations + internal symmetries
electromagnetic interactions $\leftrightarrow$ continuous local symmetry

8 The electromagnetic field

Maxwell's equations ($\epsilon_0 = \mu_0 = c = 1$)

$$\vec{\nabla} \cdot \vec{E} = \rho_{em} \tag{8.1}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{8.2}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{8.3}$$

$$\vec{\nabla} \times \vec{B} = \vec{J}_{em} + \frac{\partial \vec{E}}{\partial t} \tag{8.4}$$

$$\underline{Define} \quad J^\mu_{em} = (\rho_{em}, \vec{j}_{em})$$

In terms of scalar and vector potential V and $\vec{A}$

$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad (B_i = \epsilon_{ijk}\partial_j A_k)$$

$$\begin{aligned}
\text{so} \qquad & (\vec{\nabla} \times \vec{\nabla} \times \vec{A})_i = \epsilon_{ijk}\partial_j(\epsilon_{klm}\partial_l A_m)_k \\
= \;& \epsilon_{ijk}\epsilon_{klm}\partial_j\partial_l A_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j\partial_l Am \\
= \;& \partial_i\partial_j A_j - \partial_j\partial_j A_i = (\vec{\nabla}(\vec{\nabla} \cdot \vec{A}))_i - \vec{\nabla}^2\vec{A}_i
\end{aligned}$$

$$\Rightarrow \qquad \begin{aligned} \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A} &= \vec{j}_{em} - \frac{\partial^2\vec{A}}{\partial t^2} - \frac{\partial}{\partial t}\vec{\nabla}V \\ \vec{\nabla}\cdot\vec{E} = -\vec{\nabla}^2 V - \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} &= \rho_{em} \end{aligned} \qquad (8.5)$$

$$\begin{aligned} \text{or}\ \left(\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}^2\vec{A}\right) + \vec{\nabla}\frac{\partial}{\partial t}V + \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) &= \vec{j}_{em} \\ \left(\frac{\partial^2 V}{\partial t^2} - \vec{\nabla}^2\vec{V}\right) - \frac{\partial}{\partial t}\frac{\partial}{\partial t}V - \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) &= \rho_{em} \end{aligned}$$

we can write this in four vector notation

$$\underline{\text{defining}}\quad 4--\text{vector potential}\quad A^\mu = (V,\vec{A})\ ;\ A_\mu = (V,-\vec{A})$$

$$\begin{aligned} &\partial_\nu\partial^\nu A^\mu - \partial^\mu(\partial_\nu A^\nu) = \partial_\nu(\partial^\nu A^\mu - \partial^\mu A^\nu) \\ &= \partial^\nu F^{\nu\mu} = J^\mu_{em} \end{aligned}$$

where we defined the electromagnetic field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu) = -F_{\nu\mu}$$

$$\begin{aligned} F_{0i} &= \partial_0 A_i - \partial_i A_0 = (-\vec{\nabla}V - \partial_t\vec{A})_i = (\vec{E})_i \\ F_{ij} &= \partial_i A_j - \partial_j A_i = -(\vec{\nabla}\times\vec{A})_k = -\epsilon_{ijk}\partial_i A_j = -(\vec{B})_k \end{aligned}$$

$$\underline{\text{hence}}\quad F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix} \qquad (8.6)$$

The elecromagentic field strength tensor and hence Maxwell's equations are invariant under the gauge transformations

$$A^\mu \quad\rightarrow\quad A'^\mu = A^\mu + \partial^\mu\chi$$

where χ is a scalar function.

$$\begin{aligned} F^{\mu\nu} \quad\rightarrow\quad F'^{\mu\nu} &= \partial^\mu(A^\nu + \partial^\nu\chi) - \partial^\nu(A^\mu + \partial^\mu\chi) \\ &= \partial^\mu A^\nu - \partial^\nu A^\mu + \partial^\nu\chi - \partial^\mu\chi \\ &= \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu} \end{aligned}$$

$\Rightarrow$ we can always choose $\partial_\mu A^\mu = 0$ (Lorentz gauge) (8.7)

If $\partial_\mu A^\mu = f \neq 0 \;\rightarrow\; \partial_\mu(A^\mu + \partial^\mu\chi) = 0 \;\rightarrow\; \partial_\mu\partial^\mu\chi = -f$

$\Rightarrow$ in free space $(J^\mu = 0)$ we have $\partial_\nu\partial^\nu A^\mu = 0$

$\rightarrow$ massless KGE for each component of A^μ
$\rightarrow$ A^μ is the wave function of the photon
$\rightarrow$ A^μ is a four vector $\rightarrow$ photon has spin 1

Plane–wave solutions :

$$A^\mu = \epsilon^\mu \mathrm{e}^{-i(wt-\vec{k}\cdot\vec{r})}$$

where

$$\begin{aligned} \epsilon^\mu &= \text{polarization four vector} \\ k^\mu &= (w, \vec{k}) \quad \rightarrow \quad \text{wave 4–vector} \end{aligned}$$

From the wave equation for A^μ $\rightarrow$ $k{\cdot}k = 0$ $\Rightarrow$ $w^2 = \vec{k}^2$ $\Leftrightarrow$ $E^2 = p^2c^2$ (for $m = 0$)
From the Lorentz gauge condition

$$\partial_\mu A^\mu = 0 \quad \Rightarrow \quad \epsilon \cdot k = 0 \quad \Rightarrow \quad \epsilon^0 = \frac{\vec{\epsilon}\cdot\vec{k}}{w}$$

$\Rightarrow$ $\epsilon'^\mu = \epsilon^\mu + ak^\mu$ is equivalent ϵ^μ for any $a =$ costant
$\Rightarrow$ choose $\epsilon_0 = 0$ $\Rightarrow$ $\epsilon_\mu k^\mu = \vec{\epsilon}\cdot\vec{k} = 0$

$$\Rightarrow \quad \text{for} \quad \vec{k} = (0,0,k_z) \quad \rightarrow \quad \epsilon^\mu_x = (0,1,0,0) \quad , \quad \epsilon^\mu_x = (0,0,1,0)$$

$$2 \text{ polarization states } \; \epsilon^\mu_{R,L} = \frac{(0,1,\pm i,0)}{\sqrt{2}} \quad \leftarrow \quad \text{circular polarization}$$

8.1 Electromagentic interactions

We introduce electromagnetic interactions via the minimal substitution in the equations of motion

$$E \rightarrow E - eV \quad , \vec{p} \rightarrow \vec{p} - e\vec{A}$$

where e is the electric charge.

$$\underline{\text{relativistically :}} \;\; p^\mu \rightarrow p^\mu - eA^\mu \quad , \partial^\mu \rightarrow \partial^\mu + ieA^\mu$$

The KG equation becomes

$$\begin{aligned} &(\partial_\mu + ieA_\mu)(\partial^\mu + ieA^\mu)\phi + m^2\phi \\ &(\partial_\mu\partial^\mu + m^2)\phi = -IE[A_\mu\partial^\mu\phi + \partial_\mu(A^\mu\phi)] + e^2A_\mu A^\mu\phi \end{aligned} \tag{8.8}$$

We saw that the minimal coupling prescription and the gauge condition $A^\mu \rightarrow A^\mu + \partial^\mu\chi$ are the as the local phase invariance

$$\phi(x) \rightarrow \mathrm{e}^{-i\alpha(x)}\phi(x) \rightarrow \text{local } U(1) \text{ symmetry}$$

The conserved current is now

$$J^\mu = i(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) - 2eA^\mu(x)\phi^*(x)\phi(x)$$

The second term provides the coupling of the scalar field to the electromagnetic field

9 The Dirac equation

The Schrödinger equation: $i\hbar\frac{\partial}{\partial t}\psi = \hat{H}\psi$ is linear in $i\hbar\frac{\partial}{\partial t}$ $\rightarrow$ linear in E

since $P_\mu = i\hbar\partial_\mu$ & $P^2 = E^2 - \vec{P}^2 = m^2$, the KGE is quadratic in $\frac{\partial}{\partial t}$.

Dirac wanted to find a relativistically covariant equation which is linear in $i\hbar\frac{\partial}{\partial t}$ *i.e.* linear in energy $\rightarrow$ linear in the Hamiltonian $\rightarrow$ linear in the generator of time translations $\rightarrow$ linear in time + relativistically covariant $\Rightarrow$ linear in $-i\hbar\vec{\nabla}$

$-i\hbar\vec{\nabla}$ $\rightarrow$ the quantum generator of spatial translations

$$\text{relativistically} \quad E^2 = \vec{p}^2c^2 + m^2c^4 \quad \Rightarrow \quad E = \pm\sqrt{\vec{p}^2c^2 + m^2c^4}$$

$$i\hbar\frac{1}{c}\frac{\partial}{\partial t}\psi(\vec{r},t) = \pm\sqrt{\vec{p}^2 + m^2c^2}$$

we have to get rid of the $\sqrt{}$ write

$$\sqrt{\vec{p}^2 + m^2c^2} = \alpha_i p_i + \beta m \tag{9.1}$$

Diraq $\rightarrow$ find $\alpha_i,\ \beta$ such (9.1) holds?

$\Rightarrow$ covariant equation lineas in $i\hbar\frac{\partial}{\partial t}$ and $-i\hbar\vec{\nabla}$

Take the square of eq. (9.1)

$$\begin{aligned} \vec{p}^2 + m^2c^2 = \sum p_ip_i + m^2c^2 &= (\alpha_ip_i + \beta mc)^2 \\ &= (\alpha_ip_i + \beta mc)(\alpha_jp_j + \beta mc) \\ &= \alpha_i\alpha_jp_ip_j + (\alpha_i\beta + \beta\alpha_i)p_imc + \beta^2m^2c^2 \end{aligned}$$

For the euation to hold we must impose the following requirements

1 $\beta^2 = 1$

2 $\alpha_i\beta + \beta\alpha_i = 0 \quad \leftarrow$ no linear term in p_i in the square

these conditions can hold only if α_i and β are matrices with $\alpha_i,\ \beta$ anti–commuting matrices

3 $\alpha_i\alpha_2 j + \alpha_j\alpha_i = 0 \quad i \neq j$

4 $\alpha_i^2 = 1 \quad i = j$

can we find $\alpha_i,\ \beta$ that satisfy these conditions?

1 2 x 2 matrices $\alpha_i = \sigma_i$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad ; \quad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad ;$$

$$\sigma_i^2 = 1 \qquad ; \qquad \sigma_i\sigma_j + \sigma_j\sigma_i = 0 \ \text{ for } \ i \neq j$$

But we lack a 2x2 β matrix that satisfies 1 & 2.

2 3 x 3 matrices

No solutions $\longleftrightarrow$ solution must be even order.

Proof: Assume an odd order solution

Assume: A β matrix which is diagonal. As in the 2x2 we can always diagomalise at least one matrix

$$\beta = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & \lambda_n \end{pmatrix} \qquad \beta^2 = 1 \quad \Rightarrow \quad \lambda_i = \pm 1$$

We don't know how may λ_i are positive or negative? Prove that $\mathrm{Tr}\beta = 0$

In that case: # of +1 eigenvalues = # of -1 eigenvalues

$\Rightarrow$ β must be even

$$\begin{aligned} & \alpha_i\beta + \beta\alpha_i = 0 \qquad / \cdot \alpha \\ & \alpha_i\beta\alpha_i + \beta\alpha_i^2 = 0 \quad \Rightarrow \quad \alpha_i\beta\alpha_i + \beta = 0 \\ \Rightarrow \quad & \mathrm{Tr}\beta = -\mathrm{Tr}(\alpha_i\beta\alpha_i) = -\mathrm{Tr}(\alpha_i\alpha_i\beta) = -\mathrm{Tr}\beta \\ \Rightarrow \quad & \mathrm{tr}\beta = -\mathrm{Tr}\beta \quad \Rightarrow \quad \mathrm{Tr}\beta = 0 \quad \rightarrow \quad \beta_{nxn} \ \text{ with } n\text{–even} \end{aligned}$$

3 at order 4 *i.e.* 4x4 matrices → there is a solution

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} I_{2x2} & 0 \\ 0 & I_{2x2} \end{pmatrix}$$

where σ_i are 2x2 Pauli matrices

9.1 The Dirac equation

$$i\hbar\frac{1}{c}\frac{\partial}{\partial t}\Psi(\vec{x},t) = (-i\hbar\vec{\alpha}\cdot\vec{\nabla} + \beta mc)\Psi(\vec{x},t) \tag{9.2}$$

The wave function $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$ is a 4–component vector (not a four spacetime vector)

2 components are spin ↑ spin ↓ particles
2 components are spin ↑ spin ↓ anti–particles
consequences:

1 The Dirac equation predicts the existence of anti–particles

2 Time and space derivatives are linear

Multiply eq. (9.2) by β, recalling that $\beta^2 = 1$

$$
\begin{aligned}
& i\hbar\beta\frac{1}{c}\frac{\partial}{\partial t}\Psi(\vec{x},t) = (-i\hbar\beta\vec{\alpha}\cdot\vec{\nabla} + \beta^2 mc)\Psi(\vec{x},t) \\
\text{or} \quad & i\hbar(\underbrace{\beta}_{\gamma_0}\frac{1}{c}\frac{\partial}{\partial t} + \underbrace{\beta\vec{\alpha}}_{+\gamma^i = \beta\alpha_i = -\alpha_i\beta}\cdot\vec{\nabla})\Psi(\vec{x},t) = mc\Psi(\vec{x},t) \\
\underline{\text{hence}} \quad & i\hbar(\gamma^0\frac{\partial}{\partial ct} + \gamma^i\cdot\partial_i)\Psi(\vec{x},t) = mc\Psi(\vec{x},t)
\end{aligned}
$$

$$
\begin{aligned}
\text{or} \;\; i\hbar(\eta^{\mu\nu}\gamma_\mu\partial_\nu)\Psi &= i\hbar\gamma^\nu\partial_\nu\Psi = i\hbar\not{\partial}\Psi = mc\Psi \\
\gamma^\mu p_\mu\Psi &= mc\Psi \\
\not{p}\Psi &= mc\Psi \;\to\; (\not{p} - mc)\Psi = 0 \;\leftarrow\; \text{free Dirac equation} \\
\text{Setting } \hbar = c = 1 &\;\Rightarrow\; (i\not{\partial} - m)\Psi = 0
\end{aligned}
$$

Lowest order Dirac γ^μ matrices $\to$ 4x4 $\to$ massive particles
massless particles $\to$ no constraint on $\beta \Rightarrow$ 2x2 solution $\to$ Pauli matrices
The Dirac equation is of the form $i\hbar\frac{\partial}{\partial t}\Psi = H\Psi$

$$H = c\vec{\alpha}\cdot\vec{p} + \beta mc^2$$

$$
\begin{aligned}
\underline{\text{hermiticity}} \;\to\; H = H^\dagger \;&\Leftrightarrow\; \alpha_i = \alpha_i^\dagger \quad \beta = \beta^\dagger \\
\Rightarrow \gamma^{0\dagger} = \;& \gamma^0\alpha_i = \gamma^0\gamma^i(\gamma^0\gamma^i)^\dagger = \gamma^{i\dagger}\gamma^{0\dagger} = \gamma^{i\dagger}\gamma^0 \\
\gamma^0\gamma^{i\dagger}\gamma^0 = \gamma^i & \\
\underline{\text{summarise}} \;& \gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu\dagger} \quad \mu = 0,1,2,3
\end{aligned}
$$

$$\underline{\text{together with}} \;\; \{\gamma^\mu,\gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$$

are the two properties that define the Dirac γ–matrices

$$\underline{\text{representation}} \qquad \gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \qquad i = 1,2,3$$

where $\sigma_i,\; i = 1,2,3$ are the Pauli matrices
Solutions of the Dirac equation $\to$ 4–component objects $\to$ spinors (not 4–vectors)

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

Each component obeys the KGE by construction

$$E^2 = H^2 = \vec{p}^2 + m^2 \;\;\Rightarrow\;\; E^2 I\psi = (\vec{p}^2 + m^2)I\psi$$

Spin of the Dirac Particles

How do we prove that the Dirac equation correspond to spin 1/2 particles?

Show: There exist an operator $\vec{S}$ such that $\vec{J} = \vec{L} + \vec{S}$ is a constant of the motion, and

$$\vec{S}^2|s\rangle = s(s+1)|s\rangle = \frac{3}{4}I|s\rangle \quad (\hbar = 1)$$

Note: $\vec{L} = \vec{r} \times \vec{p}$ is not a constant of the motion:

$$\begin{aligned} H &= \beta m + \vec{\alpha} \cdot \vec{p} = \beta m + \alpha_i p_i \\ \vec{L} = L_x\hat{i} + L_y\hat{j} + L_x\hat{k} & \\ e.g. \quad & L_z = (xp_y - yp_x) \end{aligned}$$

$$[L_z, H] = [x, H]\, p_y - [y, H]\, p_x = i\alpha_x py - i\alpha_y p_x = i(\vec{\alpha} \times \vec{p})_z \qquad (\hbar = 1)$$

in general: $\left[\vec{L}, H\right] = i\vec{\alpha} \times \vec{p} \neq 0$

$\Rightarrow$ we need $\left[\vec{S}, H\right] = -i\vec{\alpha} \times \vec{p}$

This is true if $\vec{S} = \frac{1}{2}\Sigma$ with $\Sigma_j = \begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}$

$$\begin{aligned} \underline{\text{we want}}: \quad & [S_j, \alpha_i p_i + \beta m] = -i(\vec{\alpha} \times \vec{P})_j \\ = \; & [S_j, \alpha_i]\, p_i + [S_j, \beta]\, m \end{aligned}$$

$$\begin{aligned} [S_j, \beta] &= \begin{pmatrix} A & \\ & A \end{pmatrix}\begin{pmatrix} I & \\ & -I \end{pmatrix} - \begin{pmatrix} I & \\ & -I \end{pmatrix}\begin{pmatrix} A & \\ & A \end{pmatrix} = 0 \;\rightarrow\; S_j = \begin{pmatrix} A & \\ & A \end{pmatrix} \\ [S_j, \alpha_i] &= \sim \alpha_k \sim \begin{pmatrix} A & \\ & A \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}\begin{pmatrix} A & \\ & A \end{pmatrix} = \begin{pmatrix} 0 & \sigma_j\sigma_i - \sigma_i\sigma_j \\ \sigma_j\sigma_i - \sigma_i\sigma_j & 0 \end{pmatrix} \\ &= -i\begin{pmatrix} 0 & 2\sigma_k \\ 2\sigma_k & 0 \end{pmatrix} - 2i\alpha_k \end{aligned}$$

Defining $S_j = \frac{1}{2}\begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}$ we get the desired result

$$\underline{\textit{Then}} \;\; \vec{S}^2 \frac{1}{4}\left(\Sigma_x^2 + \Sigma_y^2 + \Sigma_z^2\right) = \frac{3}{4}I \;\Rightarrow\; \text{spin} = \frac{1}{2}$$

$$\underline{\text{ underline}}: [L_i, S_j] = 0$$

Magnetic moment of the Dirac equation

In an electromagnetic field we make the usual minimal substitutions

$$H \rightarrow H - eV \quad , \quad \vec{p} \rightarrow \vec{p} - e\vec{A}$$

where e is the electric charge.

In the Dirac equation we obtain

$$H - eV \;=\; \vec{\alpha}\cdot(\vec{p} - e\vec{A}) + \beta m$$

Note: We no longer get the KG equation when squaring

$$\begin{aligned}(H - eV)^2 &= \sum_{j,k} \alpha_j\alpha_k(p_j - eA_j)(p_k - eA_k) + m^2 \\ &= (\vec{p}^2 - e\vec{A})^2 + m^2 - \sum_{j\neq k}(\alpha_j\alpha_k p_j A_k + \alpha_j\alpha_k A_j p_k)\end{aligned}$$

For $j \neq k$: $\alpha_j\alpha_k = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0\end{pmatrix}\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0\end{pmatrix} = \begin{pmatrix} \sigma_j\sigma_k & 0 \\ 0 & \sigma_j\end{pmatrix} = i\epsilon_{jkl}\begin{pmatrix} \sigma_l & 0 \\ 0 & \sigma_l\end{pmatrix} = i\epsilon_{jkl}\Sigma_l$

where we used

$$\begin{aligned} & [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \\ & \{\sigma_i, \sigma_j\} = 2\delta_{ij} \\ \Rightarrow \quad & \sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k\end{aligned}$$

$$P_j A_k f = (-i\nabla_j A_k)f = A_k(-i\nabla_j f) - i(\nabla_j A_k)f = (A_k P_j - i(\nabla_j A_k))f$$

$$\epsilon_{ijl}\Sigma_l\nabla_j A_k = \Sigma_l\epsilon_{ljk}\nabla_j A_k = \Sigma_l(\vec{\nabla}\times\vec{A})_l = \vec{\Sigma}\cdot\vec{B}$$

we get :

$$\begin{aligned} & -e\sum_{j\neq k}(\alpha_j\alpha_k p_j A_k + \alpha_j\alpha_k A_j p_k) \\ &= -e\sum_{j\neq k}(\alpha_j\alpha_k A_j p_k + \alpha_j\alpha_k A_k p_j) - e\vec{\Sigma}\cdot\vec{B} \\ &= -e\sum_{j\neq k}(\alpha_j\alpha_k + \alpha_k\alpha_j)A_j p_k - e\vec{\Sigma}\cdot\vec{B} = -e\vec{\Sigma}\cdot\vec{B}\end{aligned}$$

Hence :

$$\begin{aligned}(H - eV)^2 &= (\vec{p} - e\vec{A})^2 + m^2 - e\vec{\Sigma}\cdot\vec{B} \\ (H - eV) &= m\left(1 + \frac{(\vec{p} - e\vec{A})^2 - e\vec{\Sigma}\cdot\vec{B}}{m * 2}\right)^{\frac{1}{2}} \\ \text{NR limit} \quad &\simeq m + \frac{1}{2m}(\vec{p} - e\vec{A})^2 - \frac{e}{2m}\vec{\Sigma}\cdot\vec{B}\end{aligned}$$

This correspond to a magnetic moment

$$\mu = \frac{e}{m}\vec{S} = g_e(\frac{e}{2m})\vec{S}$$

where $g_e = 2$ (experiment $\Rightarrow$ 2.0023193...) where the (0.0023193...) are quantum field theory corrections.

Great success of Dirac equation. (g-2) of the muon is of great contemporary interest and substantial experimental effort to measure it.

Dirac density and current

To give a probabilistic interpretation of the Dirac wave function ψ we have to construct a conserved current j^μ.

$$\underline{\text{We have}}: \qquad \begin{aligned} (\ i\gamma^\mu \ \ \partial_\mu \ -m\ )\psi &= 0 \\ \psi^\dagger(-i\gamma^{\mu\dagger}\ \overleftarrow{\partial}_\mu - m\ ) \ &= 0 \end{aligned} \tag{9.3}$$

$$\begin{aligned} \underline{\text{using}} \qquad & \gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \\ \Rightarrow \qquad & \psi^\dagger(-i\gamma^0\gamma^\mu\gamma^0\overleftarrow{\partial}_\mu - m) = 0 \qquad\qquad / \ \cdot\gamma^0 \\ \Rightarrow \qquad & \psi^\dagger\gamma^0(-i\gamma^\mu\overleftarrow{\partial}_\mu - m\gamma^0) = 0 \end{aligned}$$

We define the Dirac adjoint $\bar{\psi} = \psi^\dagger\gamma^0$.

$$\Rightarrow \quad \bar{\psi}(i\gamma^\mu\overleftarrow{\partial}_\mu + m) = 0 \tag{9.4}$$

multiplying (9.3) by $\bar{\psi}$ from the left and
(9.4) by ψ from the right

$$\text{we get } i\bar{\psi}\gamma^\mu\overrightarrow{\partial}_\mu\psi + i\bar{\psi}\gamma^\mu\overleftarrow{\partial}_\mu\psi = i\partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0$$

$$\text{so} \quad j^\mu = \bar{\psi}\gamma^\mu\psi \quad \Rightarrow \quad \partial_\mu j^\mu = 0$$

$\Rightarrow$ j^μ is our conserved current.

$$\begin{aligned} \underline{\text{Then}}: \rho = j^0 \ &= \ \bar{\psi}\gamma^0\psi = \psi^\dagger(\gamma^0)^2\psi = \psi^\dagger\psi = \sum_{\alpha=1}^{4}|\psi_\alpha|^2 \geq 0 \\ j^j \ &= \ \bar{\psi}\gamma^j = \psi^\dagger\gamma^0\gamma^j\psi = \psi^\dagger\alpha^j\psi \end{aligned}$$

$\rightarrow$ ρ is positive definite but we still get negative energy

the negative energy solutions correspond to antiparticles. The Dirac field describes a multi–state solution, *i.e.* it is a quantum field.

9.2 Solutions of the Dirac equation

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

Each component obeys the Klein–Gordon equation.

$$\psi = u(E, \vec{p})\mathrm{e}^{-ip\cdot x} = u(E, \vec{p})\mathrm{e}^{-i(Et - \vec{p}\cdot\vec{x})}$$

$\rightarrow$ positive energy plane wave solution

$$(i\gamma^\mu\partial_\mu - m)\psi = (\gamma^\mu p_\mu - m)u = 0$$

$$\underline{\text{writing}} \quad u = \begin{pmatrix}\phi\\ \chi\end{pmatrix} = \begin{pmatrix}\phi_1\\ \phi_2\\ \chi_1\\ \chi_2\end{pmatrix} \tag{9.5}$$

$$\begin{aligned}(\gamma^0 E - \vec{\gamma}\cdot\vec{p} - m)\begin{pmatrix}\chi\\ \phi\end{pmatrix} &= \left[\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}E - \begin{pmatrix}0 & \sigma_j\\ -\sigma_j & 0\end{pmatrix}p_j - m\begin{pmatrix}I & 0\\ 0 & I\end{pmatrix}\right]\begin{pmatrix}\phi\\ \chi\end{pmatrix}\\ &= \begin{pmatrix}E-m & -\vec{\sigma}\cdot\vec{p}\\ \vec{\sigma}\cdot\vec{p} & -E-m\end{pmatrix}\begin{pmatrix}\phi\\ \chi\end{pmatrix} = 0\end{aligned}$$

$$\begin{aligned}\Rightarrow \quad (E-m)\,\phi &= \vec{\sigma}\cdot\vec{p}\,\chi\\ \vec{\sigma}\cdot\vec{p}\,\phi &= (E+m)\,\chi\end{aligned}$$

$$\Rightarrow \quad \chi = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,\phi$$

$$\underline{\text{Recall that}} \quad \vec{S} = \frac{1}{2}\vec{\Sigma} = \frac{1}{2}\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix} \qquad \Rightarrow \quad S_z = \frac{1}{2}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$$

$$\begin{aligned}\text{Hence} \qquad \phi &= N\begin{pmatrix}1\\ 0\end{pmatrix} \quad \text{for spin up along } z\text{–axis}\\ \phi &= N\begin{pmatrix}0\\ 1\end{pmatrix} \quad \text{for spin down along } z\text{–axis}\end{aligned}$$

$$\underline{\text{also have}}: \quad \vec{\sigma}\cdot\vec{p}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}p_x + \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}p_y + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}p_z = \begin{pmatrix}p_z\\ p_x + ip_y\end{pmatrix}$$

$$\underline{\text{Similarly}}: \qquad \vec{\sigma}\cdot\vec{p}\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}p_x - ip_y\\ p_z\end{pmatrix}$$

$$\Longrightarrow \quad u^\uparrow = N\begin{pmatrix}1\\ 0\\ \frac{p_z}{E+m}\\ \frac{p_x+ip_y}{E+m}\end{pmatrix} \qquad u^\downarrow = N\begin{pmatrix}0\\ 1\\ \frac{p_x-ip_y}{E+m}\\ \frac{p_z}{E+m}\end{pmatrix}$$

Normalisation is calculated from $\rho = \psi^\dagger\psi = u^\dagger u = 2E$
$2E$ is the relativistic particle density per unit volume
This gives $N^2\left[1 + \frac{px^2+p_y^2+p_z^2}{(E+m)^2}\right] = 2E$

$$
\begin{aligned}
\text{Using} \qquad & \vec{p}^{\,2} = E^2 - m^2 \quad \text{gives} \\
& N^2 \left[1 + \frac{(E-m)(E+m)}{(E+m)^2} \right] = 2E \\
\Rightarrow \qquad & N^2 \left(\frac{2E}{E+m} \right) = 2E \\
\Rightarrow \qquad & N = \sqrt{E+m}
\end{aligned}
$$

For a particle in the rest frame $p^m u = (m, 0, 0, 0) \;\; \Rightarrow \;\; \vec{p} = 0$ we get

$$
\Longrightarrow \qquad u^{\uparrow} = N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad\qquad u^{\downarrow} = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}
$$

In the non–relativistic limit $u^{\uparrow} \;\rightarrow\;$ spin up $\quad u^{\downarrow} \;\rightarrow\;$ spin down $\quad$ fields.

For anti–particles of 4–momentum $(E, \vec{p})$ we need a solution with $p^{\mu} \rightarrow (-E, -\vec{p})$

$$
\psi = v(E, \vec{p}) \mathrm{e}^{i p_{\mu} x^{\mu}} = v(E, \vec{p}) \mathrm{e}^{i(Et - \vec{p} \cdot \vec{x})}
$$

from the Dirac Equation :

$$
\begin{aligned}
& (i\gamma^{\mu}\partial_{\mu} - m)\psi = (-\gamma^{\mu} p_{\mu} - m) v(E, \vec{p}) = \\
& \left[\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} E + \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} p_j - m \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \right] v(E, \vec{p}) = \\
& \begin{pmatrix} -E - m & \vec{\sigma} \cdot \vec{p} \\ -\vec{\sigma} \cdot \vec{p} & E - m \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \qquad & \vec{\sigma} \cdot \vec{p} \chi = (E+m)\phi \quad \Rightarrow \quad \phi = \frac{\vec{\sigma} \cdot \vec{p}}{E+m} \chi \\
& \vec{\sigma} \cdot \vec{p} \phi = (E-m)\chi
\end{aligned}
$$

Like the 4–momentum, spin is reversed

$$
\Longrightarrow \qquad v^{\downarrow} = N \begin{pmatrix} \frac{p_x - i p_y}{E+m} \\ \frac{p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \quad v^{\uparrow} = N \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + i p_y}{E+m} \\ 1 \\ 0 \end{pmatrix}
$$

9.3 Charge conjugation

Consider the KGE for a charged particle in an electromagnetic field

$$
\begin{aligned}
\vec{p} \quad &\rightarrow \quad \vec{p} - e\vec{A} \\
E \quad &\rightarrow \quad E - eV
\end{aligned}
$$

$$(E - eV)^2 = (\vec{P} - e\vec{A})^2 + m^2$$

In quantum mechanics $E \ \rightarrow \ i\hbar\partial_t$, $\vec{p} \ \rightarrow \ -i\hbar\vec{\nabla}$

$$\Rightarrow \quad (i\hbar\partial_t - eV)^2\phi(\vec{x},t) = (-i\hbar\vec{\nabla} - e\vec{A})^2\phi(\vec{x},t) + m^2\phi(\vec{x},t)$$

The complexified equation

$$\begin{aligned} & (-i\hbar\partial_t - eV)^2\phi^* = (i\hbar\vec{\nabla} - e\vec{A})^2\phi^* + m^2\phi^* \\ \Rightarrow \quad & (i\hbar\partial_t + eV)^2\phi^* = (-i\hbar\vec{\nabla} + e\vec{A})^2\phi^* + m^2\phi^* \end{aligned}$$

Hence if $\phi = \mathrm{e}^{-i(Et - \vec{p}\cdot\vec{x})} = \mathrm{e}^{-iEt + \vec{p}\cdot\vec{x}}$ is a solution of the KGE with $E > 0$, $\vec{p}$.

$$\phi^* = \mathrm{e}^{i(Et - \vec{p}\cdot\vec{x})} = \mathrm{e}^{-i(E(-t) + \vec{p}\cdot\vec{x})}$$

ϕ^* is a solution of the KGE with $E > 0,\ \vec{p} \rightarrow -\vec{p}$ and $e \rightarrow -e,\ m = m_0$.

$$\text{This operation is :} \quad \begin{array}{ccc} \phi & \rightarrow & \phi^* \\ e & \rightarrow & -e \end{array}$$

called charge conjugation C. The KGE is invariant under charge conjugation.

The Dirac equation is also invariant under charge conjugation. $\psi \ \rightarrow$ negative energy solution. What is the charge conjugation operation that leaves the Dirac equation invariant

$$\psi \ \rightarrow \psi^C = C\psi^* \ \rightarrow \ \text{charge conjugation } \ C$$

Such that ψ^C is a positive energy solution with $e \rightarrow -e$.

To find C write

$$\begin{aligned} & \gamma^\mu(i\partial_\mu - eA_\mu)\psi - m\psi = 0 \quad \leftarrow \quad \text{Dirac eq.} \\ \rightarrow \quad & \gamma^\mu(\partial_\mu + ieA_\mu)\psi + im\psi = 0 \\ \Rightarrow \quad & \gamma^{\mu *}(\partial_\mu - ieA_\mu)\psi^* - im\psi^* = 0 \\ & -C\gamma^{\mu *}C^{-1}(\partial_\mu - ieA_\mu)\psi^C + im\psi^C = 0 \end{aligned}$$

Hence we need

$$\begin{aligned} C\gamma^{\mu *}C^{-1} &= -\gamma^\mu \\ i.e. \qquad \gamma^\mu C &= -C\gamma^{*\mu} \end{aligned}$$

Since all γ^μ are reall, except γ^2 (which is purely imaginary) in our standard representation we can take

$$C = i\gamma^2 = i\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

For free particles we have $v^{\uparrow C} = u^\uparrow$ $v^{\downarrow C} = -u^\downarrow$

9.4 Parity invariance

Similarly to charge conjugation we can also extract the transformation properties of the Dirac wave function under parity transformations

$$\underline{\text{define}}: \qquad \psi(\vec{r},t) \;\; \to \;\; \psi^P(\vec{r},t) = P\psi(-\vec{r},t)$$

$$\underline{\text{invariance}}: \qquad \text{we want to find } P \text{ such that } \psi^P \text{ is also a solution}$$

$$\begin{array}{rcl}
\underline{\text{Dirac equation}}: & \to & (i\gamma^\mu\partial_\mu - m)\psi(\vec{r},t) = 0 \\
& \to & (\gamma^\mu\partial_\mu + im)\psi(\vec{r},t) = 0 \\
& \to & (\gamma^0\partial_0 + \gamma^j\partial_j + im)\psi(\vec{r},t) = 0 \\
\vec{r} \to \; -\vec{r} & \to & (\gamma^0\partial_0 - \gamma^j\partial_j + im)\psi(-\vec{r},t) = 0 \\
\psi \; \to \; \psi^P & \to & (P\gamma^0P^{-1}\partial_0 - P\gamma^jP^{-1}\partial_j + im)\psi^P(\vec{r},t) = 0
\end{array}$$

Hence for invariance to hold we need

$$\begin{aligned}
P\gamma^0P^{-1} &= \gamma^0 \\
P\gamma^jP^{-1} &= -\gamma^j
\end{aligned}$$

$$\Rightarrow \quad P\gamma^0 = \gamma^0 P \quad , \quad P\gamma^j = -P\gamma^j$$

This relations are satisfied by

$$P = \gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{9.6}$$

For a particle at rest $\qquad \psi = u(m,\vec{0})\mathrm{e}^{-imt} = \sqrt{2m}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\mathrm{e}^{-imt} \;\Rightarrow\; \psi^P = P\psi = +\psi$

For an anti–particle at rest $\quad \psi = v(m,\vec{0})\mathrm{e}^{+imt} = \sqrt{2m}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\mathrm{e}^{+imt} \;\Rightarrow\; \psi^P = P\psi = -\psi$

$\Rightarrow$ particle and anti–particles have opporite intrinsic parity.
For KGE the parity transformation is

$$\phi(\vec{r},t) \to \phi^P(\vec{r},t) = \phi(-\vec{r},t)$$

Since $\phi(\vec{r},t)$ is a scalar function under LT

$$\phi'(\vec{r}',t') = \phi(\vec{r},t)$$

$$\text{which follows from the KGE} \qquad \left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2\right)\phi(\vec{r},t) = 0$$

$$P \rightarrow \qquad \left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2\right)\phi'(-\vec{r},t) = 0$$

Hence ϕ and ϕ' are solutions of the same equations.

In the case of the Dirac equation the scalar is

$$\Phi = \bar{\psi}\psi = \psi^\dagger\gamma^0\psi$$

$$\begin{aligned}
\text{check}: \quad \Phi(\vec{r},t) &= \psi^\dagger(\vec{r},t)\gamma^0\psi(\vec{r},t) \\
\Phi^P(\vec{r},t) &= \underbrace{\psi^\dagger(-\vec{r},t)\gamma^{0\dagger}}_{\overline{\psi^P}}\ \gamma^0\ \underbrace{\gamma^0\psi(-\vec{r},t)}_{P\psi(-\vec{r},t)} \\
&= \psi^\dagger(-\vec{r},t)\ \gamma^0\ \psi(-\vec{r},t) \\
&= \Phi(-\vec{r},t)
\end{aligned}$$

$\rightarrow \quad \bar{\psi}\psi$ transforms as a scalar under parity transformations.

Similarly, j^μ is a true vector.

$$\begin{aligned}
j^\mu(\vec{r},t) &= \psi^\dagger\gamma^0\gamma^\mu\psi(\vec{r},t) \\
j^{\mu P}(\vec{r},t) &= \psi^\dagger(-\vec{r},t)\gamma^{0\dagger}\gamma^0\gamma^\mu\gamma^0\psi(-\vec{r},t)
\end{aligned}$$

$$\begin{aligned}
\text{But}\quad \gamma^{0\dagger}\gamma^0\gamma^\mu\gamma^0 = \gamma^\mu\gamma^0 &= \gamma^0\gamma^\mu \quad \text{for} \quad \mu = 0 \\
&= -\gamma^0\gamma^\mu \quad \text{for} \quad \mu = 1,2,3
\end{aligned}$$

$$\underline{\text{Hence}}: \quad j^{P0}(\vec{r},t) = j^0(-\vec{r},t) \qquad , \qquad \vec{j}^P(\vec{r},t) = -\vec{j}(-\vec{r},t)$$

As we would expect from a true 4–vector $\quad (t,\vec{x} \overset{P}{\rightarrow} (t,-\vec{x})$

We define the matrix

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I_{2\times2} \\ I_{2\times2} & 0 \end{pmatrix} \tag{9.7}$$

in our representation.

We will see that weak interactions involve the axial current

$$J_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi = \psi^\dagger\gamma^0\gamma^\mu\gamma^5\psi$$

$$\text{under parity transformations} \qquad \psi \rightarrow \psi^P = \gamma^0\psi(-\vec{r},t)$$

$$J_A^{P\mu} = \psi^\dagger(-\vec{r},t){\gamma^0}^\dagger\gamma^0\gamma^\mu\gamma^5\gamma^0\psi(-\vec{r},t)$$

$$\begin{aligned}
\text{now}\quad \{\gamma^\mu,\gamma^5\} &= 0 \quad \text{for} \mu = 0,1,2,3 \\
(\gamma_5)^2 &= I
\end{aligned}$$

Hence

$$J_A^{P0}(\vec{r},t) = -J_A{}^0(-\vec{r},t) \qquad , \qquad \vec{J}_A^P(\vec{r},t) = \vec{J}_A(-\vec{r},t)$$

As expected for an axial vector.

$$\underline{\text{Similarly}} : \Phi_P = \bar{\psi}\gamma^5\psi = \psi^\dagger\gamma^0\gamma^5\psi \qquad \text{a pseudo scalar}$$

$$\begin{aligned}
\Phi_P^P(\vec{r},t) &= \underbrace{\psi^\dagger(-\vec{r},t)\gamma^{0\dagger}}_{\psi^{P\dagger}}\gamma^0\gamma^5\underbrace{\gamma^0\psi(-\vec{r},t)}_{\psi^P} \\
&= -\bar{\psi}(-\vec{r},t)\gamma^5\psi(-\vec{r},t) \\
&= -\Phi_P(-\vec{r},t)
\end{aligned}$$

9.5 Massless Dirac particles

$$\underline{\text{The Dirac equation}} : \qquad H\psi = i\hbar\frac{\partial\psi}{\partial t} = \left(-i\hbar\vec{\alpha}\cdot\vec{\nabla} + \beta m\right)\psi$$

where

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$$

The positive energy free particle solutions are

$$\psi = u(E,\vec{p})\mathrm{e}^{-i(Et-\vec{p}\cdot\vec{x})}$$

For $m = 0 \quad \Rightarrow \quad E = |\vec{p}|$ and $u = \binom{\phi}{\chi}$ gives

$$EIu = \vec{\alpha}\cdot\vec{p}u = \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix} u$$

$$\underline{\text{hence}} : \quad \begin{pmatrix} |\vec{p}| & -\vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & |\vec{p}| \end{pmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0 \quad \Rightarrow \quad \begin{array}{l} \vec{\sigma}\cdot\vec{p}\chi = |\vec{p}|\phi \\ \vec{\sigma}\cdot\vec{p}\phi = |\vec{p}|\chi \end{array} \tag{9.8}$$

$$\begin{aligned}
&\Rightarrow \qquad \chi = \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\phi \quad \& \quad \phi = \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\chi \quad \Rightarrow \quad \chi = \left(\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\right)^2\chi \\
&\Rightarrow \qquad \left(\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\right)^2 = I \\
&\Rightarrow \qquad \Lambda = \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} \quad \text{is the helicity operator with} \quad \Lambda^2 = I
\end{aligned}$$

$$\text{and eigenvalues} \quad \Lambda = \pm 1 \;\rightarrow\; \text{spin}\frac{\text{along}}{\text{against}}\;\vec{p} \;\rightarrow\; \frac{\text{right}}{\text{left}} \quad \text{handed}$$

Helicity operator: projection of spin on $\frac{\vec{p}}{|\vec{p}|}$.

$\Rightarrow$ for massless particles with $m = 0$ the 2 components spinors χ and ϕ are eigenstates of the helicity operator.

For massless particles helicity is a Lorentz invariant.

Note that if ψ represents a massless particle them

$$\gamma^5\psi = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} \Lambda\phi \\ \phi \end{pmatrix} = \begin{pmatrix} \Lambda\phi \\ \Lambda\chi \end{pmatrix} = \Lambda\psi \qquad (\Lambda^2 = 1)$$

Hence, γ^5 is the helicity operator for massless particles (minus helicity for massless anti–particles).

In the case of massless particles we can decompose the Dirac equation into two equations for the two helicity eigenstates.

We can introduce the basis

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad ; \quad \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \quad ; \quad \gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

This basis is called the chiral basis. In this basis

$$\left(\gamma^0 - \vec{\gamma}\cdot\vec{p}\right)\begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} 0 & |\vec{p}| - \vec{\sigma}\cdot\vec{p} \\ |\vec{p}| + \vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0$$

Hence, in this basis,

$$\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\chi = +1\chi \quad ; \quad \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\phi = -1\phi$$

Therefore, in this basis χ and ϕ are the eigenstates of the helicity operator with eigenvalues $+1$ and -1 respectively. We denote,

$$\chi = \psi_R \quad ; \quad \phi = \psi_L$$

ψ_L and ψ_R are two component spinors that tranform as $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations of the Lorentz group.

A Dirac spinor can be written as

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \qquad \psi_L,\ \psi_R \text{ are called Weyl spinors}$$

We define the operators

$$P_{L,R} = \left(\frac{1 \pm \gamma^5}{2}\right)$$

In the chiral basis we have

$$\begin{aligned} P_L &= \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad , \qquad {P_L}^2 = P_L \\ P_R &= \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad , \qquad {P_R}^2 = P_R \end{aligned} \tag{9.9}$$

and

$$\begin{aligned} &P_R P_L = P_L P_R = 0 \\ &P_L + P_R = \frac{1+\gamma^5}{2} + \frac{1-\gamma^5}{2} = 1 \\ &P_L\gamma^\mu = \gamma^\mu P_R \quad ; \quad P_R\gamma^\mu = \gamma^\mu P_L \end{aligned}$$

$$\text{Hence, we have} \qquad P_L\psi = \begin{pmatrix}\psi_L\\0\end{pmatrix} = \psi_L$$
$$P_R\psi = \begin{pmatrix}0\\\psi_R\end{pmatrix} = \psi_R$$

Furthermore, from $P_RP_L = 0$ we have $P_R\psi_L = 0$, $P_L\psi_R = 0$.

The weak interactions were observed experimentally to have the vector minus axial–vector form, $(V-A)$, *i.e.*

$$(J^\mu_V - J^\mu_A)_{fi} = \bar\psi_f\gamma^\mu(1-\gamma^5)\psi$$

If i is a massless particle then $(1-\gamma^5)\psi_i$ vanishes for helicity $+1$, *i.e.* only left–handed fields interact.

The same applies to particle f since

$$\bar\psi_f\gamma^\mu(1-\gamma^5)\psi_i = \psi_f^\dagger\gamma^0(1+\gamma^5)\gamma^\mu\psi_i = \psi_f^\dagger(1-\gamma^5)\gamma^0\gamma^\mu\psi_i = \left[(1-\gamma^5)\psi_f\right]^\dagger\gamma^0\gamma^\mu\psi_i$$

This is non–vanishing only if the f–particle is a left–handed field.

<u>In the Standard Model only left–handed fields interact via the weak interactions.</u>

The Langrangian density that gives the Dirac equation of motion

$$\begin{aligned}
\mathcal{L}_D &= \bar\psi i\gamma^\mu\partial_\mu\psi - m\bar\psi\psi = \\
&\quad \bar\psi(P_L^2+P_R^2)i\gamma^\mu\partial_\mu\psi - m\bar\psi(P_L^2+P_R^2)\psi = \\
&\quad \psi^\dagger P_R\gamma^0 i\gamma^\mu\partial_\mu P_R\psi + \psi^\dagger P_L\gamma^0 i\gamma^\mu\partial_\mu P_L\psi - m\psi^\dagger P_R\gamma^0P_L\psi - m\psi^\dagger P_L\gamma^0P_R\psi = \\
&\quad \psi_R^\dagger\gamma^0 i\gamma^\mu\partial_\mu\psi_R + \psi_L^\dagger\gamma^0 i\gamma^\mu\partial_\mu\psi_L - m(\psi_R^\dagger\gamma^0\psi_L + \psi_L^\dagger\gamma^0\psi_R) = \\
&\quad \bar\psi_R i\gamma^\mu\partial_\mu\psi_R + \bar\psi_L i\gamma^\mu\partial_\mu\psi_L - m(\bar\psi_R\psi_L + \bar\psi_L\psi_R).
\end{aligned}$$

The first two terms are the kinetic terms, whereas the last two are the Dirac mass terms.

We see that the kinetic terms containing the derivatives involve $L \leftrightarrow L$ and $R \leftrightarrow R$ terms, whereas the mass terms involve $L \leftrightarrow R$ and $R \leftrightarrow L$ terms. This is a crucial result for modern particle physics.

9.6 Majorana fields

So far we encountered Weyl and Dirac spinors.

A Majorana spinor is a Dirac spinor in which ψ_L and ψ_R are not independent.

<u>Rather</u> $\quad \Psi_M = \begin{pmatrix}\psi_L\\ i\sigma^2\psi_L^*\end{pmatrix} \quad \rightarrow$ same # of D.o.f as Weyl spinor

A Majorana spinor is invariant under charge conjugation

$$\Psi_M^C = \Psi_M$$

in a Majorana spinor we have $\psi_R = i\sigma_2/\psi_L^*$.

Majorana spinor $\rightarrow$ neutral field $\rightarrow$ its own anti–particle $\Rightarrow \quad m\bar\Psi_M\Psi_M$.

10 Classification of elementary particles

We assembled some of the ingredients that are needed to classify elementary particles.

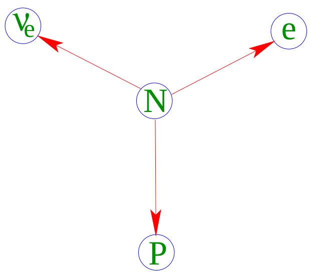
We saw : spin: - 0 scalars; - $\frac{1}{2}$ fermions; - +1 gauge bosons; - +2 graviton;
mass
Spin & mass are the Casimir labels of the Poincare group
charge – electric – weak – strong
where the charges depend on the particle interactions.

Additional properties :		Light Leptons don't interact strongly		heavy Hadrons interact strongly	
$1940's$ fermions	electron e	neutrino ν_e	proton P $\tau \le 10^{32}$years	neutron N $\tau_{\frac{1}{2}} \sim 15$min	
bosons	γ				

The electron neutrino was suggested By Pauli to account for observed spread of energy in nuclear β–decay. It was observed experimentally in 1956.

The lifetime τ depends on global and local conservation rules for different interactions.

The process of beta decay



In Fermi theory $G_F \bar{P}\gamma^\mu N \bar{e}\gamma^\nu \nu_e \quad \leftarrow \quad$ four Fermi interaction, with $G_F \sim 10^{-5}$.
In modern particle physics the interaction is mediated by a heavy vector boson
Isospin symmetry $\to$ classify elementary particles $\to$ broken symmetry

1 Proton and neutron form a doublet of Isospin
   $s = \frac{1}{2} \qquad m = 939\text{MeV};\ 938\text{MeV} \qquad e - \text{charge} + 1,\ 0$
   if we turn off eletromagnetic interactions we cannot distinguish between the proton and the neutron

$SU(2)_I$ – Isospin – global continuous $SU(2)$ symmetry, which is exact if we ignore E&M interactions. Isospin symmetry is approximate in nature versus E&M which is exact.

In the 1950's a slew of particles (resonances) were discovered
All the particles that interacted strongly formed families of Isospin interactions
Examples:

2 pions
$m(\pi^0) \sim 139\text{MeV} \quad ; \quad m(\pi^\pm) \sim 139.5\text{MeV}$
spin $= 0$
Isospin $= +1 \rightarrow$ triplet

3 Kaons
$m(K^+) \sim 439.7\text{MeV} \quad ; \quad m(K^0) \sim 497,8\text{MeV}$
Spin $= 0$
Isospin $= \frac{1}{2} \rightarrow$ doublet

4 P^- and $\bar{N}^0$
$m(P^-) \sim 938\text{MeV} \quad ; \quad m(\bar{N}^0) \sim 939\text{MeV}$
Spin $= \frac{1}{2}$
Isospin $= \frac{1}{2}$

5 Sigmas
$m(\Sigma^+) \sim 1189.36\text{MeV} \quad ; \quad m(\Sigma^0) \sim 1192.46\text{MeV} \quad ; \quad m(\Sigma^-) \sim 1197.34\text{MeV}$
Spin $= \frac{1}{2}$
Isospin $= +1$

6 Λ^0
$m(\Lambda^0) = 1115.6\text{MeV}$
Spin $= \frac{1}{2}$ Isospin $= 0$

7 η
$m(\eta) = 458.8\text{MeV}$
Spin $= 0$ Isospin $= 0$

Classification: states with same spin and comperable mass form Isospin families

Additionally, the resonances are classified by their lifetime and decay products. The observed resonances decay via their strong, electromagnetic, and weak interactions. The decays are typified by the decay rates which are inversely propotional to their lifetime.

$$\begin{array}{rcl} \text{Strong} & \sim & 10^{-24} \text{ sec} \\ \text{E\&M} & \sim & 10^{-18} \text{ sec} \\ \text{Weak} & \sim & 10^{-8} \text{ sec} \end{array}$$

Hadrons – strongly interacting
$$\begin{array}{llll} \text{baryons.} & \text{spin} & = n + \frac{1}{2} & n = 0, 1, ... \\ \text{mesons.} & \text{spin} & = n & n = 0, 1, ... \end{array}$$

Leptons – not strongly interacting
$$\begin{array}{ll} \text{charged} \rightarrow & e, \ \mu, \ , \tau \\ \text{neutral} \rightarrow & \nu_e, \ \nu_\mu, \ \nu_\mu \end{array}$$

gauge bosons. spin 1 $\quad \gamma; \ W^\pm, \ Z; \ G$

All interactions respect the familiar conservation laws, *e.g.* $\rightarrow$ charge, energy, momentum, angular momentum.

In addition: some additional conservation laws must be imposed *e.g.* the process

$$P \;\to\; e^+ \, \pi^0$$

respects conservation of charge, angular momentum and energy, but is not observed in nature, with $\tau_P \leq 10^{32}$years. Proton decay is forbidden in the renormalisable Standard Model but is predicted to occur in many extensions of the Standard Model. Consequently, there are many experimental searches looking for proton decay. Furthermore, some degree of proton instability is necessary to create an excess of matter over anti–matter. Without such excess, all of the protons would have annihilated with anti–protons in the early universe and there would have been non left to form galaxies, stars, planets, and us. On the other had "we know it in our bones" that the proton has to be lond lived. Otherwise, we would have decayed long ago.

$\Rightarrow$ introduce conserved baryon charge $B(P) = +1, \; B(\bar{P}) = -1, \; B(L) = 0 = B(M)$

$$\Rightarrow \quad P \;\not\to\; e^+ \, \pi^0$$

Similarly for leptons

$$\begin{array}{lclcl} \mu^- & \to & e^- \, \bar{\nu}_e \, \nu_\mu & & \text{allowed} \\ \mu^- & \to & e^- \, \gamma & & \text{forbidden} \end{array}$$

Introduce $L_e, \; L_\mu, \; L_\tau$

$$\begin{array}{l} L_e = +1 \;\; \text{for} \;\; e^- \text{ and } \nu_e \\ L_e = -1 \;\; \text{for} \;\; e^+ \text{ and } \bar{\nu}_e \\ L_e = \;\;\; 0 \;\;\; \text{for everyone else} \end{array}$$

Therefore, for the process $\mu^- \;\to\; e^- \, \bar{\nu}_e \, \nu_\mu$

$$\begin{array}{ccccccc} & \mu^- & \to & e^- & \bar{\nu}_e & \nu_\mu & \\ \text{spin} & \frac{1}{2} & \to & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \quad J_f = j_1 + j_2 - j_3 = \frac{1}{2} \\ \text{charge} & -1 & \to & -1 & 0 & 0 & \\ L_\mu & +1 & \to & 0 & 0 & +1 & \\ L_e & 0 & \to & +1 & -1 & 0 & \end{array}$$

Baryon & Lepton numbers are observed to be exactly conserved charges in nature.

Some conservation laws may be approximate, *e.g.*

$$\begin{array}{lclcl} K^+ & \to & \pi^+\pi^0 & \to & \sim \; 20\% \;\; \text{branching ratio} \\ \tau(K^+) & \sim & 10^{-8}\text{sec} & \to & \text{weak decay?} \\ & & & & \text{why not strong } K^+ \text{ decay?} \end{array}$$

Gellmann & Nishijima $\to$ suggested a new additive conserved quantum number.
S – strangeness

$$\begin{array}{l} S(P) = S(\pi) = 0 \\ S(K^+) = S(K^0) = +1 \\ S(\Lambda, \; \Sigma) = -1 \\ S(\Xi) = -2 \end{array}$$

Strong & electromagnetic interactions conserve strangeness
Weak interaction violates strangeness
$\rightarrow$ classification by Isospin is not sufficient to classify hadronic states
$\rightarrow$ need a larger symmetry group S such that

$$SU(2)_I \subset G \leftarrow SU(2)_I \text{ is a subgroup of } G$$

$$G =? \rightarrow G = SU(3)_{flavour} \rightarrow \text{Gellmann \& Neeman} \rightarrow \text{the eightfold way}$$

11 Unitary groups $SU(n)$

simple unitary group of rank n where the rank is the number of mutualy commuting diagonal generators.

$$\begin{aligned} \text{Unitary :} \quad & U^\dagger U \;\Rightarrow\; U^\dagger = U^{-1} \\ \text{Simple :} \quad & \text{Det}U = 1 \end{aligned}$$

Examples : $\quad n = 1 \quad , \quad \psi \rightarrow U\psi \;,\; U^\dagger = U^{-1} \quad \rightarrow \quad U = \mathrm{e}^{i\alpha} \quad \Rightarrow \quad U^\dagger = \mathrm{e}^{-i\alpha} = U^{-1}$

$$n = 2 \quad , \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \rightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad , \quad U^\dagger = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \quad , \quad U^{-1} = \frac{1}{(AD - BC)} \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}$$

$$U^\dagger = U^{-1} \;\Rightarrow\; A^* = \frac{D}{\text{DetU}} \;,\; D^* = \frac{A}{\text{DetU}} \;,\; C^* = -\frac{B}{\text{DetU}} \;,\; A^* = -\frac{C}{\text{DetU}} \tag{11.1}$$

Unitary matrices: $U^\dagger = U^{-1} \quad \Rightarrow \quad U^\dagger U = I$

$$\text{Det}(U^{-1}) = (\text{Det}U)^{-1} \quad ; \quad \text{Det}(U^\dagger) = (\text{Det}U)^*$$
$$U^\dagger = U^{-1} \Rightarrow (\text{Det}U)^* = \frac{1}{\text{Det}U} \Rightarrow |\text{Det}U| = +1$$

Since $|\text{Det}U| = 1$ we get from equations (11.1) that

$$|A| = |D| \quad \& \quad |B| = |C|$$

The most general 2x2 unitary matrix can therefore be written as

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{i\alpha} & 0 \\ 0 & \mathrm{e}^{i\beta} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathrm{e}^{i\gamma} & 0 \\ 0 & \mathrm{e}^{i\delta} \end{pmatrix}$$

which is the most general solution of the equations in (11.1).

# of D.O.F. in U is 4: $\theta +$ three of $\alpha,\ \beta,\ \gamma,\ \delta$.
The phases appearing in U are $\alpha+\gamma$, $\alpha+\delta$, $\beta+\gamma$, $\beta+\delta$.
in a 2x2 unitary matrix there are 4 D.O.F.
generalisation: An NxN unitary matrix has N^2 D.O.F.
Theorem: A unitary matrix U can be written as $U = \mathrm{e}^{iH}$, where H is hermitian ($H^\dagger = H$).
Assume an hermitian NxN matrix: $H = S + iA$,
where S is a real symmetric matrix and A is a real antisymmetric matrix.

$$\text{The \# of D.O.F. in } H: \quad S \to N + \left(\frac{N^2-N}{2}\right) = \frac{N(N+1)}{2}$$

where N is the number of diagonal terms

$$A \to \frac{N^2-N}{2} = \frac{N(N-1)}{2}$$

$$\underline{\text{Hence}}\text{: \# of D.O.F. in } H: \frac{N(N+1)}{2} + \frac{N(N-1)}{2} = N^2$$

A unitary NxN matrix also has N^2 degrees of freedom.
In $U(2)$ write the most general 2×2 unitary matrix
we need 4 independent hermitian matrices:
The space of 2×2 hermitian matrices is spanned by the basis:

$$H = \left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\ ,\ \vec{\tau}\right]$$

Where $\vec{\tau}$ are the Pauli matrices

$$\tau_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad , \quad \tau_2 \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \quad , \quad \tau_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$\text{general } H\text{–matrix} \quad H = \beta\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \vec{\alpha}\cdot\vec{\tau}$$

$$\text{general } U(2) \text{ matrix}: \quad U = \mathrm{e}^{i(\beta I + \vec{\alpha}\cdot\vec{\tau})}$$

The S in an $SU(2)$ matrix stands for Simple, which means that $\mathrm{Det}U = 1$
If $U = \mathrm{e}^{iH}$ then $\mathrm{Det}U = \mathrm{e}^{i\mathrm{Tr}H}$
In general: if $\mathrm{Det}A \neq 0 \Rightarrow \mathrm{Det}A = \mathrm{Det}PAP^{-1} = \mathrm{Det}A_D = \lambda_1\cdots\lambda_n$
If $\mathrm{Det}A \neq 0 \Rightarrow \mathrm{Tr}A = \mathrm{Tr}PAP^{-1} = \mathrm{Tr}A_D = \lambda_1 + \cdots + \lambda_n$

Hence, if $U = \mathrm{e}^A$

$$
\begin{aligned}
\mathrm{Det}U &= \mathrm{Det}\ \mathrm{e}^A = \\
&= \mathrm{Det}(I + A + \frac{A^2}{2} + \cdots) = \\
&= \mathrm{Det}(I + PAP^{-1} + \frac{PAP^{-1}PAP^{-1}}{2} + \cdots) = \\
&= \mathrm{Det}(I + A_D + \frac{A_D^2}{2} + \cdots) = \\
&= \mathrm{Det}\begin{pmatrix} \mathrm{e}^{\lambda_1} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \mathrm{e}^{\lambda_n} \end{pmatrix} = \\
&= \mathrm{e}^{\lambda_1} \cdots \mathrm{e}^{\lambda_n} = \mathrm{e}^{\lambda_1 + \cdots + \lambda_n} = \mathrm{e}^{\mathrm{Tr}A}
\end{aligned}
$$

$$\Rightarrow \quad \mathrm{Det}(\mathrm{e}^A) = \mathrm{e}^{\mathrm{Tr}A}$$

$$\text{Hence, } \det U = \mathrm{e}^{i\mathrm{Tr}H} \quad \Leftrightarrow \quad \mathrm{Det}U = 1 \quad \Rightarrow \quad \mathrm{Tr}H = 0$$

$$\Rightarrow \quad \beta = 0 \quad , \quad \mathrm{Tr}I = 2 \quad , \quad \mathrm{Tr}\tau_j = 0$$

Therefore, the most general $SU(2)$ matrix with $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$.

$$U = \mathrm{e}^{i\vec{\alpha}\cdot\vec{\tau}} \simeq I + i\vec{\alpha}\cdot\vec{\tau}$$

Find the most general hermitian 3x3 matrix $H_{3\times 3}$ $\#(\text{D.O.F.}) = 9$.

Requiring $\mathrm{Det}U = 1 \quad \Rightarrow \quad \mathrm{Tr}H = 0 \quad \Rightarrow \quad \#(\text{D.O.F.}) = 8$

For $SU(2)$ we have 3–matrices & 3 coefficients

For $SU(3)$ we have 8–matrices & 8 coefficients

Such that $\mathrm{Det}U = \mathrm{e}^{i\mathrm{Tr}H} = 1$

$U = \mathrm{e}^{i\vec{\alpha}\cdot\vec{\lambda}} \quad \leftarrow \quad \vec{\lambda}$ are hermitian matrices with $\mathrm{Tr}\lambda_j = 0 \quad \lambda_1, \cdots, \lambda_8$

The number of $SU(2)$ group generators is three

The number of $SU(3)$ group generators is eight

For $SU(2)$ we have only one diagonal hermitian matrix with trace $H_{2\times 2} = 0$

$$\rightarrow \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The other two matrices are $\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

For $SU(2) \quad \rightarrow \quad$ only one diagonal matrix

Find the $\vec{\lambda}$ matrices in $SU(3)$

$$\underline{\#\text{ of diagonal matrices}:}\quad \lambda = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \quad \text{with } \ \mathrm{Tr}\lambda = 0 \Rightarrow \lambda = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -\alpha-\beta \end{pmatrix}$$

$$\rightarrow \quad \underline{\text{Two diagonal matrices}}$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_8 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The importance of the diagonal matrices is that they provide the maximal set of mutually commuting operators whose eigenvalues characterise the elementary particles.

The other $\vec{\lambda}$ matrices are not diagonal

$$\lambda_1 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_2 \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda_4 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_5 \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$
$$\lambda_6 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad \lambda_7 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$(\lambda_1,\ \lambda_2,\ \lambda_3)$ form an $SU(2)$ subgroup of $SU(3)$.

sungroup: a subgroup of generators that satisfy commutation relations among themselves, *e.g,* $[\lambda_1, \lambda_2] = i\lambda_3$, etc.

$$\underline{\text{in } SU(2)}: \qquad \psi \rightarrow U\psi \quad , U \rightarrow 2\times 2 \text{ matrix}, \ \psi - \text{a 2 component vector}$$

$$\text{take} \quad \frac{1}{2}\tau_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \qquad : \qquad \begin{aligned} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= -\frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

The spinors $\binom{1}{0}$ & $\binom{0}{1}$ are eigenvectors with eigenvalues $+\frac{1}{2}$ & $-\frac{1}{2}$

These are the eigenstates of τ_3.

In $SU(2)$ we cannot characterise these states with an additional eigenvalue.

We can characterise the spin exactly only in one direction, say along the z–axis

In $SU(3)$ the analog of τ_3 is $\lambda_3 \quad \rightarrow \quad \frac{1}{2}\lambda_3 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$

The three eigenvectors of $\frac{1}{2}\lambda_3$ are:

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Similarly to $SU(2)$ the physical states can be characterised by the eigenvalues of the operator $(\frac{1}{2}\lambda_3)$

The eigenvectors of $\frac{1}{2}\lambda_3$ are also eigenvectors of λ_8 $\quad\rightarrow\quad$ $\frac{1}{2}\lambda_8 = \begin{pmatrix} \frac{1}{2\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{2\sqrt{3}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{3}} \end{pmatrix}$

where we normalise the generators to have the same normalisation given by

$$\mathrm{Tr}(\frac{1}{2}\lambda_3)^2 = \mathrm{Tr}(\frac{1}{2}\lambda_8)^2 = \frac{1}{2}$$

$$\frac{1}{2} \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{12}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad \frac{1}{2} \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{12}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$
$$\frac{1}{2} \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -\frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

There are no additional diagonal matrices among $\lambda_1, \cdots, \lambda_8$. Therefore, these are eigenvectors of λ_3 and λ_8 only.

$\Rightarrow$ Eigenstates (representations) of $SU(3)$ are characterised by eigenvalues of λ_3 and λ_8.

In $SU(2)$ we classified particles according to the eigenvalues of τ_3.

11.1 Graphical description

$$\text{We define} \qquad T_3 = \frac{1}{2}\lambda_3$$
$$Y = \frac{1}{\sqrt{3}}\lambda_8$$

In $SU(2)$ we had a doublet with 2 eigenvalues $\frac{1}{2}$, $-\frac{1}{2}$

Graph	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$T_3 = \frac{1}{2}\lambda_3$	$T_3 = -\frac{1}{2}$	$T_3 = +\frac{1}{2}$

Using $\frac{1}{2}(\tau_1 \pm i\tau_2)$ we can move from $\binom{1}{0} \leftrightarrow \binom{0}{1}$

$$\tau_+ = \frac{1}{2}(\tau_1 + i\tau_2) = \frac{1}{2}\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\right] = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$$
$$\tau_- = \frac{1}{2}(\tau_1 - i\tau_2) = \frac{1}{2}\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\right] = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$

$$\tau_+\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$
$$\tau_-\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\tau_+ \downarrow = \uparrow \quad , \quad \tau_+ \uparrow = 0 \quad , \quad \tau_+ \uparrow = \downarrow \quad , \quad \tau_+ \downarrow = 0$$

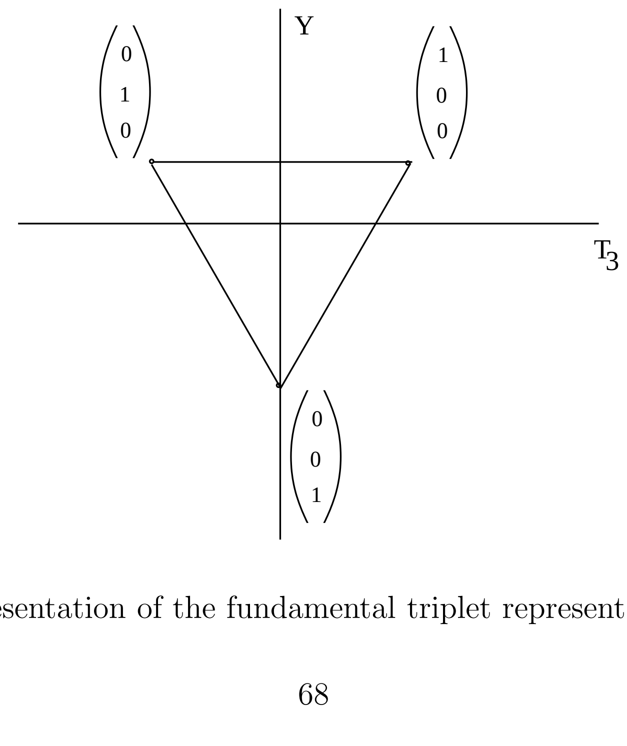
The proton and the neutron form an Isospin doublet

$\rightarrow$ Therefore, $T_3(P) = \frac{1}{2}$, $T_3(N) = -\frac{1}{2}$

For $SU(3)$ we characterise the states by T_3 & Y.

$\rightarrow$ (T_3, Y) plane :

$$(T_3, Y) : \begin{pmatrix}1\\0\\0\end{pmatrix} = (\frac{1}{2}, \frac{1}{3})$$
$$(T_3, Y) : \begin{pmatrix}0\\1\\0\end{pmatrix} = (-\frac{1}{2}, \frac{1}{3})$$
$$(T_3, Y) : \begin{pmatrix}0\\0\\1\end{pmatrix} = (0, -\frac{2}{3})$$



$\rightarrow$ graphical representation of the fundamental triplet representation of $SU(3)$

The physical quantities

T_3 – third component of Isospin (same as for $SU(2)$).

Y – hypercharge

We can exchange the τ_1, τ_2, τ_3 generators of $SU(2)$ with

$$\tau_\pm = \frac{1}{2}(\tau_1 \pm i\tau_2)\&\tau_3$$

Similarly in $SU(3)$ define $\tau_\pm = \frac{1}{2}(\lambda_1 \pm i\lambda_2)$

$$\tau_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \tau_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{ccc} \tau_+ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0 & \tau_+ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & \tau_+ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \\ \tau_- \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & \tau_- \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0 & \tau_- \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \end{array}$$

The three proints on the graphic triangular representation of the triplet of $SU(3)$ form a doublet $\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\}$ and a singlet $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ of $SU(2)$.

If we use only $\lambda_1,\ \lambda_2,\ \lambda_3$ we can only exchange $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ but cannot act on $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\rightarrow$ therefore we have a doublet and singlet of $SU(2)$

$$\begin{array}{ccccc} SU(3) & \rightarrow & SU(2)_I \times U(1)_Y & & \\ 3 & = & 2_{\frac{1}{3}} & + & 1_{-\frac{3}{2}} \\ \text{triplet} & & \text{doublet} & & \text{singlet} \end{array}$$

The representations of $SU(3)$ decompose under $SU(2) \times U(1)$.

In $SU(3)$ we can form generators that exchange $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$(\lambda_4 \pm i\lambda_5) exchanges \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \& \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$(\lambda_6 \pm i\lambda_7) exchanges \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \& \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

we user here the other $SU(3)$ generators that are not in $SU(2)_I$.

For every particle we know both T_3 & Y

Hence $SU(3) \supset SU(2)_I \times U(1)_Y$

in $SU(2)$ we can have higher order representations

For example $\quad 3: \quad \begin{array}{ccc} -1 & 0 & +1 \\ \begin{pmatrix}0\\0\\1\end{pmatrix} & \begin{pmatrix}0\\1\\0\end{pmatrix} & \begin{pmatrix}1\\0\\0\end{pmatrix}\end{array} \qquad \begin{array}{ccc} T_1 & T_2 & T_3 \\ \begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} & \begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix} & \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\end{array}$

Check that the generators of the triplet representation $T_1,\ T_2,\ T_3$ satisfy the $SU(2)$ algebra $[T_i, T_j] = i\epsilon_{ijk}T_k$.

Any 3 $n \times n$ matrices that satisfy $[T_i, T_j] = i\epsilon_{ijk}T_k$ form a representatio of the $SU(2)$ algebra.

The proton & neutron formed a doublet of $SU(2)_I$.

In the fundamental representation of $SU(3)$

There isn't a third particle with $m(?) \sim m(P) \sim m(N)$ that fits

$\Rightarrow \quad P,\ N$ form an Isospin doublet but are not part of an $SU(3)$ triplet

Can it be that $P,\ N$ form an Isospin doublet in a higher order representation of $SU(3)$

The $SU(3)$ generators obey the algebra

$$[T_i, T_j] = i f_{ijk} T_k \tag{11.2}$$

with f_{ijk} totally antisymmetric under exchange of any two indices

$$\text{and} \qquad f_{123} = 1 \quad f_{147} = \frac{1}{2} \quad f_{156} = -\frac{1}{2} \quad f_{246} = \frac{1}{2} \quad f_{257} = \frac{1}{2}$$
$$f_{345} = \frac{1}{2} \quad f_{367} = -\frac{1}{2} \quad f_{458} = \frac{\sqrt{3}}{2} \quad f_{678} = \frac{\sqrt{3}}{2}$$

All others vanish

$\rightarrow$ matrices of higher order representations satisfy the 11.2 algebra

For $SU(2)$ we have a solution at any order with matrices $(2\ell+1)(2\ell+1)$

For $SU(3)$ there isn't a solution at every order (*e.g.* order 2)

To find the higher order representations we use a different method. Similar to the addition of angular momentum for $SU(2)$.

$$\text{For } SU(2)\ \ \psi^\alpha = \{|\uparrow\rangle, |\downarrow\rangle\}, \qquad S^2|\uparrow\rangle = \frac{1}{2}(\frac{1}{2}+1)|\uparrow\rangle \qquad S^2|\downarrow\rangle = \frac{1}{2}(\frac{1}{2}+1)|\downarrow\rangle$$
$$S_z|\uparrow\rangle = +\frac{1}{2}|\uparrow\rangle \qquad S_z|\downarrow\rangle = -\frac{1}{2}|\downarrow\rangle$$

$\psi,\ \phi$ two spin $\frac{1}{2}$ wave functions

$$\psi^\alpha \otimes \phi^\beta: \qquad \text{Triplet} \quad \begin{cases} |\uparrow_\psi\uparrow_\phi\rangle & T=1 \quad T_3 = +1 \\ \frac{1}{\sqrt{2}}(|\uparrow_\psi\downarrow_\phi\rangle + |\downarrow_\psi\uparrow_\phi\rangle) & T=1 \quad T_3 = \ 0 \\ |\downarrow_\psi\downarrow_\phi\rangle & T=1 \quad T_3 = -1 \end{cases}$$

these are the symmetric combinations

$$\text{Singlet}(\tfrac{1}{\sqrt{2}}|\uparrow_\psi\downarrow_\phi\rangle - |\downarrow_\psi\uparrow_\phi\rangle) \qquad T=0 \quad T_3 = \ 0$$

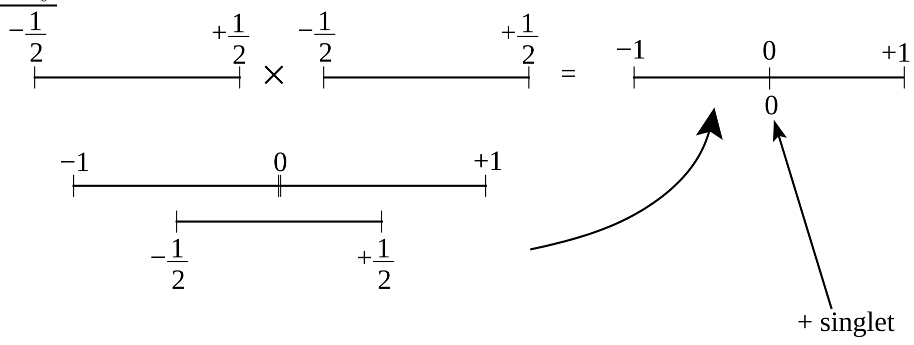
this is the antisymmetric combination

by taking the product of two Isospin doublets we get states with Isospin 1 or 0.

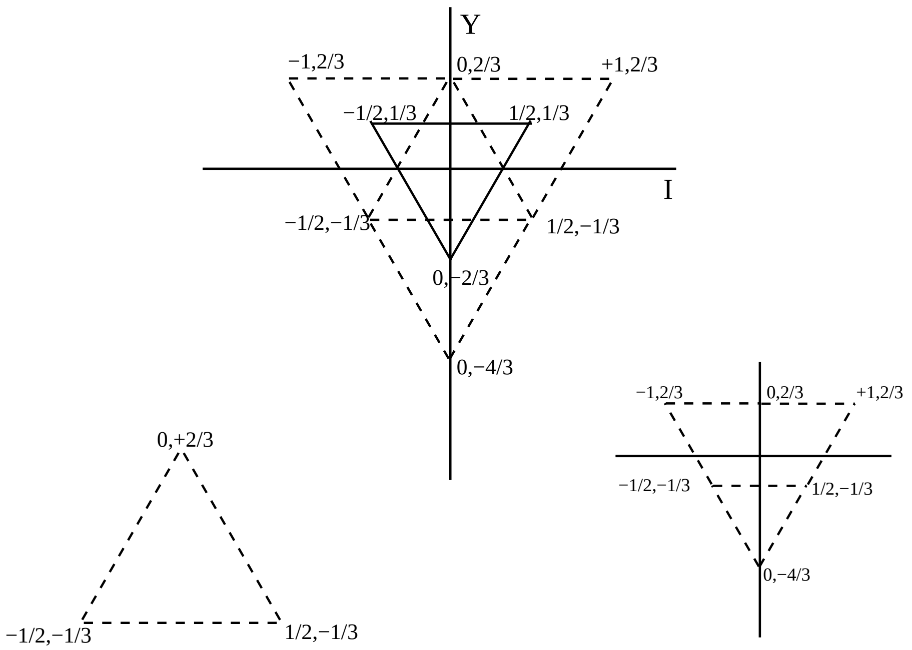
$$\underline{2} \times \underline{2} = \underline{3}_{\ \mathrm{S}} + \underline{1}_{\ \mathrm{A}}$$

We built a higher $\underline{3}$ representation with spin 1 from the $\underline{2}$ representation with spin $\frac{1}{2}$.

<u>Graphically</u>:



We can repeat the graphic analysis for $SU(3)$ with the product $\psi^\alpha \phi^\beta$ where ψ^α and ϕ^β are two triplets of $SU(3)$, and $\alpha,\ \beta = 1, 2, 3$.



The product $3 \times 3 = 6 + \bar{3}$ gives rise to the sextet and $\bar{3}$ representations of $SU(3)$.
Inside the sextet we have

$$6 \;\; = \;\; \begin{cases} (-1, \frac{2}{3} \qquad\qquad (0, \frac{2}{3}) \qquad\qquad (+1, \frac{2}{3}) & \rightarrow \quad SU(2)_I \text{ triplet} \\ \qquad (-\frac{1}{2}, -\frac{1}{3}) \qquad\qquad (\frac{1}{2}, -\frac{1}{3}) & \rightarrow \quad SU(2)_I \text{ doublet} \\ \qquad\qquad\qquad (0, -\frac{4}{3}) & \rightarrow \quad SU(2)_I \text{ singlet} \end{cases}$$

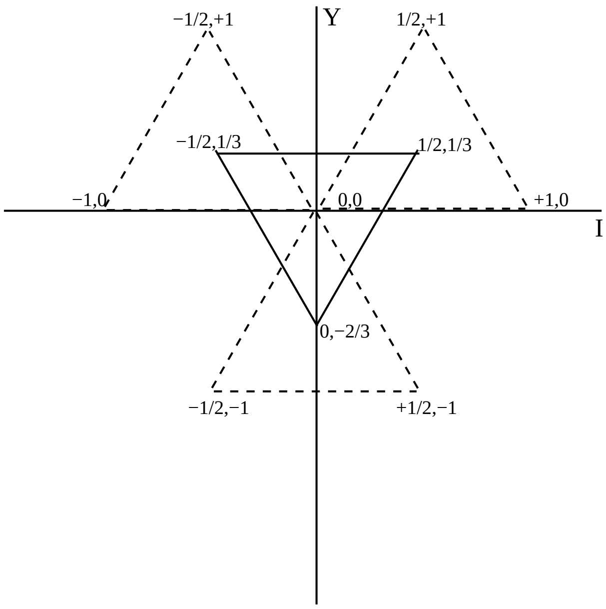
which decomposes under $SU(3) \supset SU(2) \times U(1)$ as

$6 = 3_{\frac{2}{3}} + 2_{-\frac{1}{3}} + 1_{-\frac{4}{3}}$

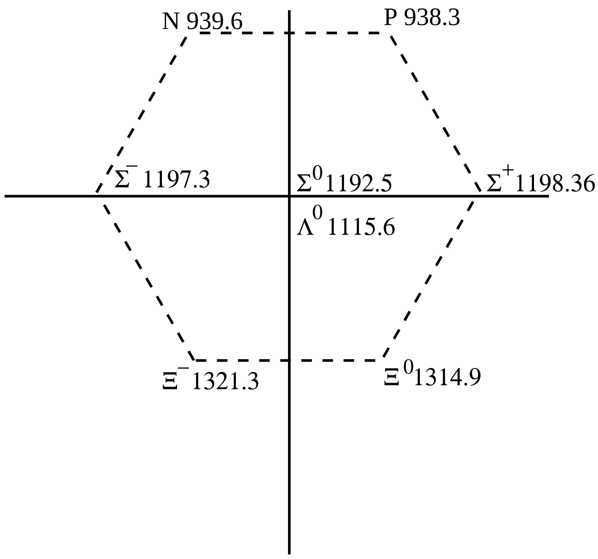
and the $\bar{3}$ decomposes as

$$\bar{3} = 2_{-\frac{1}{3}} + 1_{\frac{2}{3}}$$

we get the $\bar{3}$ representation. Note that $\bar{3} \neq 3$, whereas in $SU(2)$ $\bar{2} = 2$.
Are there physical particle that fit the 6 & $\bar{3}$ together with P, N?
Not yet look at another possibility $3 \times \bar{3}$. Multiply $\bar{3}$ at every point of 3.



$3 \times \bar{3} = 8 + 1 \quad \rightarrow \quad$ octet + singlet
under $SU(2) \times U(1)$ the octet decomposes as $8 = 2_{+1} + 3_0 + 2_{-1} + 1_0$
The octet has a physical assignment

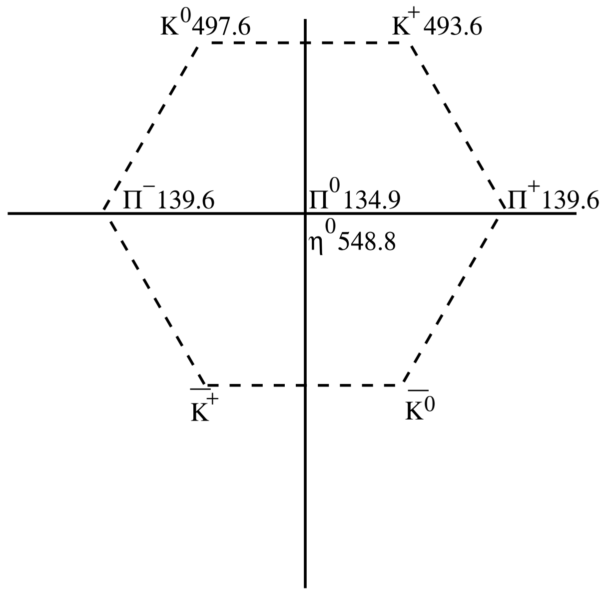


The spin of the particles is $s = \frac{1}{2}$ and baryon number $B = +1$.
We can find a relation between the electric charge and the Isospin T_3 and hypercharge Y

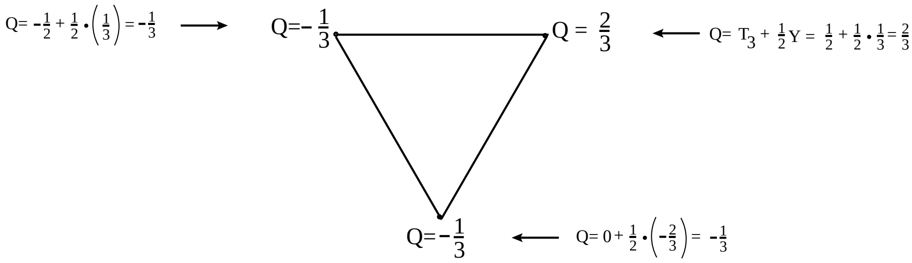
$$Q = \alpha T_3 + \beta Y \qquad \left.\begin{array}{l} Q(P) = +1 = \alpha \cdot (\ \frac{1}{2}) + \beta \cdot 1 \\ Q(N) = \ \ 0 = \alpha \cdot (-\frac{1}{2}) + \beta \cdot 1 \end{array}\right\} \ \Rightarrow \ \ \alpha = 1 \ \ , \ \ \beta = \frac{1}{2}$$

$Q = T_3 + \frac{1}{2} Y \quad \rightarrow \quad$ holds for the other particles in the octet

Second exaple: mesons $spin = 0$; Baryon number $= 0$.



Question: which representations can represent particles and why?
$8 \quad \rightarrow \quad$ Baryons, mesons
$3,\ \bar{3},\ 6,\ \bar{6}$ don't represent integrally charged particles.



Perhaps the physical representations are those that yield integrally charged states.
Another physical representation: 10

$$\begin{array}{ccccccccl}
\Delta^- & & \Delta^0 & & \Delta^+ & & \Delta^{++} & & I=\frac{3}{2} \quad Y=1 \\
& Y^{*-} & & Y^{*0} & & Y^{*+} & & & I=1 \quad Y=0 \\
& & \Xi^{*-} & & \Xi^{*0} & & & & I=\frac{1}{2} \quad Y=-1 \\
& & & \Omega^- & & & & & I=0 \quad Y=-2
\end{array}$$

$S=\frac{3}{2},\ B=\frac{3}{2}$

$$\begin{array}{rcc}
m(\Delta) & \sim & 1230-1234\text{MeV} \\
& & \updownarrow\ 150\text{MeV} \\
m(Y^*) & \sim & 1382\text{MeV} \\
& & \updownarrow\ 150\text{MeV} \\
m(\Xi^*) & \sim & 1531\text{MeV}
\end{array}$$

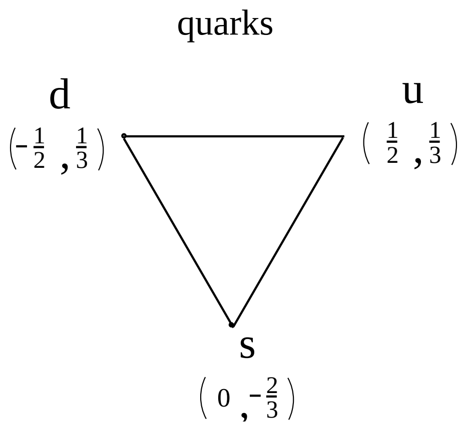
The Ω^- was predicted in Gellman's "The eightfold way" with $m(\Omega^-) \sim 1680$MeV and was discovered in 1964.

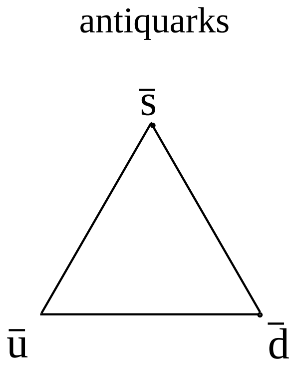
The $SU(3)$ representations are obtained from products of $3,\ \bar{3}$.

Physical	Unphysical
$3 \times \bar{3} = 8 + 1$	$3 \times 3 = 6 + \bar{3}$
$3 \times 3 \times 3 = 10 + 8 + 8 + 1$	
$\bar{3} \times \bar{3} \times \bar{3} = \overline{10} + 8 + 8 + 1$	

Questions: Only some products of 3 and $\bar{3}$ are physical. Why? Is there a physical meaning to the 3, $\bar{3}$?

Gellmann & Zweig: The baryons & mesons are made of more elementary building blocks $\rightarrow$ quarks.





$$
\begin{aligned}
Q(\text{ up }) &= \tfrac{2}{3} & \qquad Q(\text{ antiup }) &= -\tfrac{2}{3}\\
Q(\text{down}) &= -\tfrac{1}{3} & \qquad Q(\text{antidown}) &= \tfrac{1}{3}\\
Q(\text{strange}) &= -\tfrac{1}{3} & \qquad Q(\text{antistrange}) &= \tfrac{1}{3}
\end{aligned}
$$

The spin of the quarks must be $s = \frac{1}{2} \;\Rightarrow\; s(P,\ N) = \frac{1}{2}$.

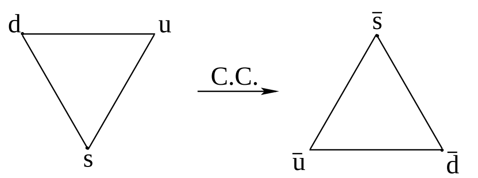
The physical states correspond to bound states of

$$
\begin{aligned}
\text{quark–antiquark} &\rightarrow 3 \times \bar{3}\\
\text{quark–quark–quark} &\rightarrow 3 \times 3 \times 3\\
\text{antiquark–antiquark–antiquark} &\rightarrow \bar{3} \times \bar{3} \times \bar{3}
\end{aligned}
$$

	T_3	Y	$Q = T_3 + \frac{1}{2}Y$
u	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$
d	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{3}$
s	0	$-\frac{2}{3}$	$-\frac{1}{3}$

	T_3	Y	$Q = T_3 + \frac{1}{2}Y$
$\bar{u}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{2}{3}$
$\bar{d}$	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{3}$
$\bar{s}$	0	$\frac{2}{3}$	$\frac{1}{3}$

Under charge conjugation we have



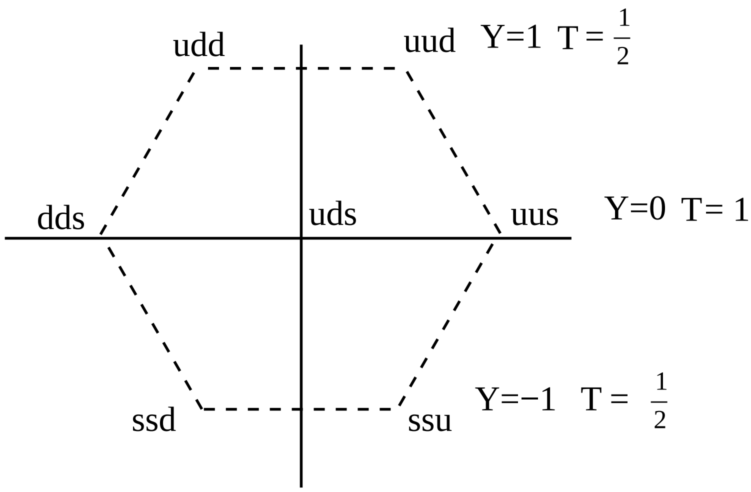
$$\text{In } SU(2) \quad 2 = \bar{2}$$
$$\text{In } SU(3) \quad 3 \neq \bar{2}$$

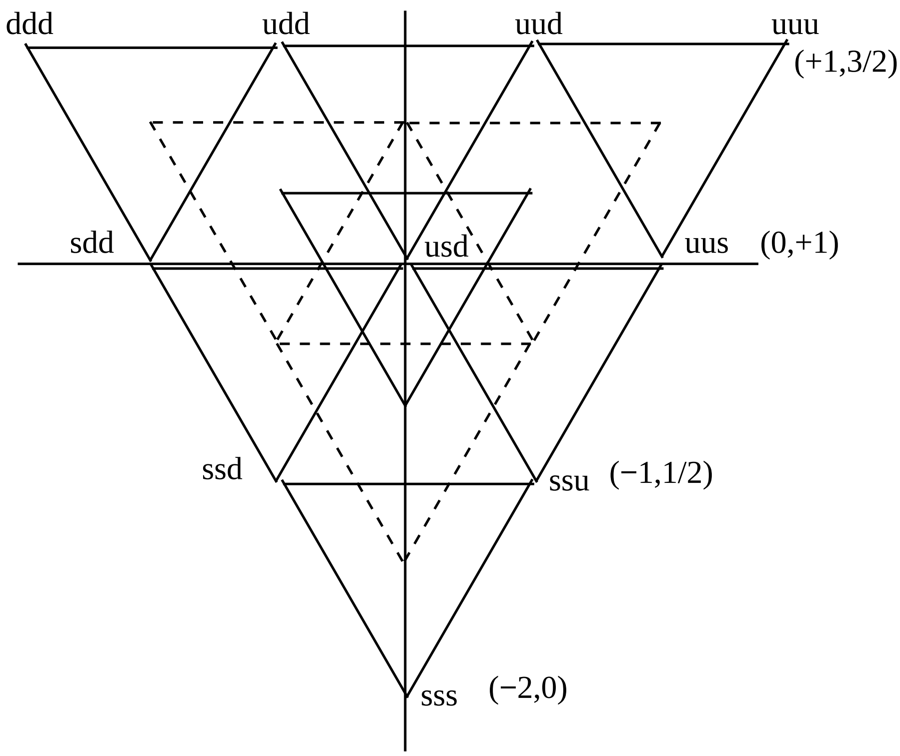
<u>mesons & baryons in the quark model</u>

We can now see how the proton & neutron and all the other slew of hadron resonances fit in the quark model.

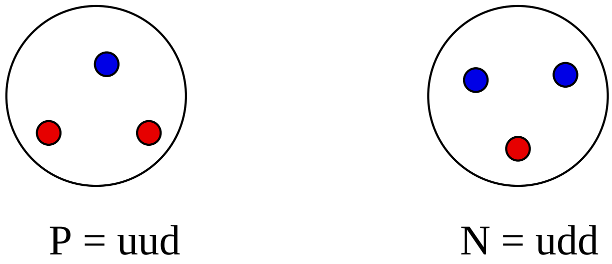
$3 \times 3 \times 3 = (6 + \bar{3}) = 6 \times 3 + \bar{3} \times 3 = 10 + 8 + 8 + 1$

<u>baryons</u>

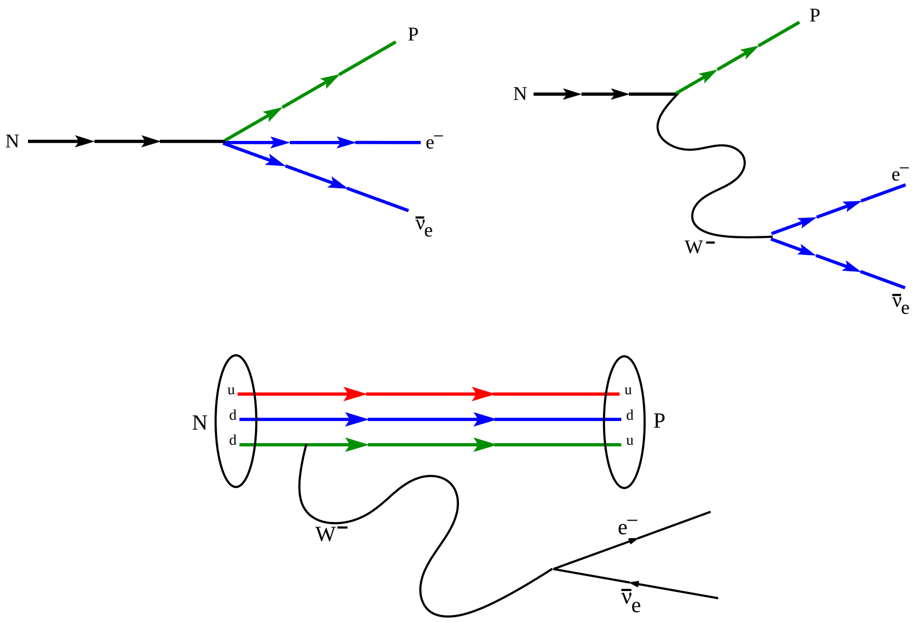




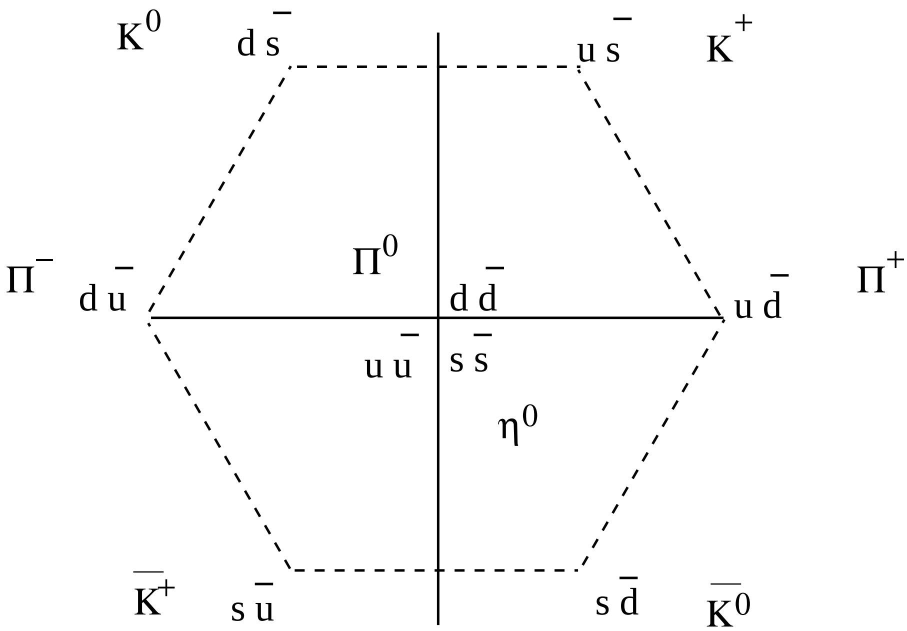
The 10 is the symmetric representation uuu $\rightarrow$ spin $= \frac{3}{2}$



β–beta decay from the quark point of view



<u>mesons</u> $3 \times \bar{3} = 8 + 1$



$$
\begin{aligned}
\pi^0 &= \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) \\
\eta &= \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}) \\
\eta' &= \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})
\end{aligned}
$$

where $\pi^0, \ \eta \ \in 8$ and $\eta' \in 1$.

The decaplet is a fully symmetric representation of $SU(3)$ and spin.

$$
\begin{aligned}
\Delta^{++} &= \quad u^\uparrow u^\uparrow u^\uparrow & \leftarrow \ : \ \frac{1}{\sqrt{2}}(\lambda_1 - i\lambda_2) \\
\Delta^{+} &= \frac{1}{\sqrt{3}}(u^\uparrow u^\uparrow d^\uparrow + u^\uparrow d^\uparrow u^\uparrow + d^\uparrow u^\uparrow u^\uparrow) \\
\Delta^{0} &= \frac{1}{\sqrt{3}}(u^\uparrow d^\uparrow d^\uparrow + d^\uparrow u^\uparrow d^\uparrow + d^\uparrow d^\uparrow u^\uparrow) \\
\Delta^{-} &= \quad d^\uparrow d^\uparrow d^\uparrow
\end{aligned}
$$

to go down the aile we operate with a lowering operator $\swarrow = \frac{1}{\sqrt{2}}(\lambda_4 - i\lambda_5)$, etc.

$$
\begin{aligned}
y^{*+} &= \frac{1}{\sqrt{3}}(u^\uparrow u^\uparrow s^\uparrow + u^\uparrow s^\uparrow u^\uparrow + s^\uparrow u^\uparrow u^\uparrow) \\
Y^{*0} &= \frac{1}{\sqrt{6}}(u^\uparrow d^\uparrow s^\uparrow + u^\uparrow s^\uparrow d^\uparrow + d^\uparrow u^\uparrow s^\uparrow + d^\uparrow s^\uparrow u^\uparrow + s^\uparrow u^\uparrow d^\uparrow + s^\uparrow d^\uparrow u^\uparrow) \\
\Omega^- &= s^\uparrow s^\uparrow s^\uparrow
\end{aligned}
$$

What is the wave function of the Proton?

The Proton is orthogonal to Δ^+. Both are combinations of uud.

$$
\mathrm{P} = (\alpha uud + \beta udu + \gamma duu)
$$

$$
\mathrm{P} \cdot \Delta^+ = 0 \ \Rightarrow \quad \alpha + \beta + \gamma = 0 \Rightarrow \quad 2 \text{ solutions}
$$

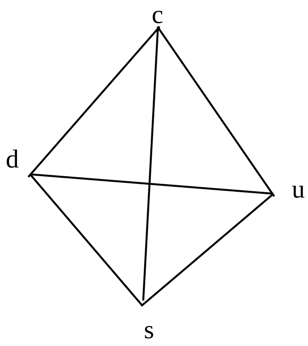
$$
\begin{aligned}
\mathrm{P_A} &= \frac{1}{\sqrt{2}}(ud - du)u & \text{asymmetric under } \ 1 \leftrightarrow 2 \\
\mathrm{P_S} &= \frac{1}{\sqrt{6}}\left[(ud + du)u - uud\right] & \text{symmetric under } \ 1 \leftrightarrow 2
\end{aligned}
$$

Nice ... But ...

Problems:

1) $\Delta^{++} = u^\uparrow u^\uparrow u^\uparrow$ fully symmetric $\rightarrow$ conflict with Pauli's spin–statistics relation

2) charm was discovered $su(3)_{\text{flavour}} \rightarrow$ not enough. $SU(4)_f \quad 4 \ =$



In 1974 J/Ψ particle was discovered with $m \sim 3.1\text{GeV}$. spin $= 0$. $c\bar{c}$.

bottom was discovered in 1978 $m(B-\text{meson}) \approx 10\text{GeV}$. Spin $= 0$. $b\bar{b}$.

top was discovered in 1994 $m_t \sim 175\text{GeV}$.

To resolve the conflict with Pauli's exclusion principle a new quantum number is introduced – colour.

All colour bound states form colour singlets. Hence the Δ^{++} wave function is asymmetric under colour and symmetric under flavour × soin × space quantum numbers.

$$(qqq)_{\text{coloursinglet}} = \frac{1}{\sqrt{6}}(RGB - RBG + BRG - BGR + GBR - GRB)$$

This wave–function is asymmetric under exchange of any two colour.

$\rightarrow$ quarks are in the fundamental representation of $SU(3)_C$.

$$q = \begin{pmatrix} R \\ G \\ B \end{pmatrix} \qquad \bar{q} = \begin{pmatrix} \overline{R} \\ \overline{G} \\ \overline{B} \end{pmatrix}$$

$SU(3)_{\text{colour}}$ is a new degree of freedom, different from $SU(3)_f$.

$SU(3)_{\text{colour}}$ is exact. $SU(3)_f$ is approximate and accidental.

Similarly to $SU(3)_f$ we can introduce $SU(4)_f$. $u,\ d,\ ,s,\ c$.

$SU(3)_f$ violated by mass differences of $O(100\text{MeV})$.

$SU(4)_f$ violated by mass differences of $O(1\text{GeV})$.

The representations would be:

$$\begin{array}{rcll} 4\bar{4} & \rightarrow & \text{mesons} & (3\bar{3})_{\text{colour}} = 8 + \mathbf{1} \\ 4 \cdot 4 \cdot 4 & \rightarrow & \text{baryons} & (3 \cdot 3 \cdot 3)_{\text{colour}} = 10 + 8 + 8 + \mathbf{1} \end{array}$$

quarks have spin $1/2 \rightarrow$ fermions

The problem of flavour is an important open question under experimental and theoretical research.

$$\begin{array}{rcccl} SU(3)_{\text{colour}} & \rightarrow & \text{Exact symmetry} & \rightarrow & \text{strong interactions} \\ U(1)_{\text{E.M.}} & \rightarrow & \text{Exact symmetry} & \rightarrow & \text{E \& M interactions} \end{array}$$

$$\underline{\text{under colour}}: \quad q \rightarrow Uq = e^{i\vec{\alpha}(x)\cdot\vec{\lambda}} q \quad \vec{\lambda} = (\lambda_1, \cdots, \lambda_8); \vec{\alpha} = (\alpha_1, \cdots, \alpha_8)$$

gauge symmetry $\rightarrow$ local phase invariance $\rightarrow$ gauge bosons $\bar{\psi}(\partial_\mu + i\vec{A}_\mu \cdot \vec{\lambda})\psi$, $\vec{A} = (A_1, \cdots, A_8)$.

The strong interactions correspond to local phase invariance under $SU(3)_{\text{colour}}$.

Quarks are observed only as confined states inside hadrons and mesons

$\rightarrow$ not observed as free quarks

observed electric charge is integral $\quad Q(u) = \frac{2}{3} \quad Q(d) = -\frac{1}{3}$

$$Q(c) = \frac{2}{3} \quad Q(s) = -\frac{1}{3}$$

The bound states are (qqq) and $q\bar{q}$ $\Rightarrow$ only integrally charged combinations are observed.

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This wave–function is asymmetric under exchange of any two colour.

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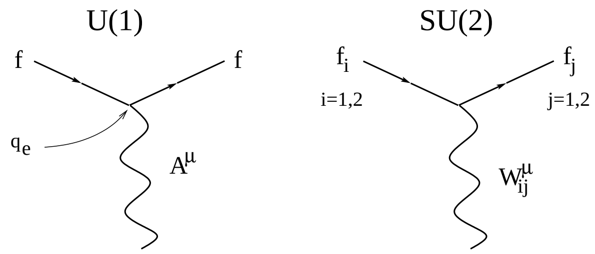
$$Q(c) = \frac{2}{3} \quad Q(s) = -\frac{1}{3}$$

The bound states are (qqq) and $q\bar{q}$ $\Rightarrow$ only integrally charged combinations are observed.
What about the weak interactions?

$$\begin{array}{llll} \text{E\&M} & \rightarrow & U(1) & \text{Abelian local symmetry} \end{array}$$
$$\left.\begin{array}{llll} \text{Weak} & \rightarrow & SU(2) & \text{local symmetry} \\ \text{Strong} & \rightarrow & SU(3) & \text{local symmetry} \end{array}\right\} \rightarrow \text{ non–Abelian}$$

Consider the E&M vs weak interactions

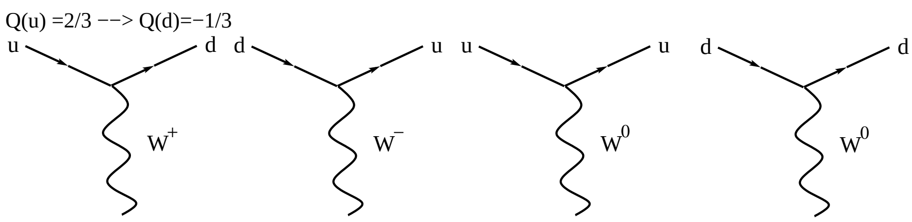


$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \quad \text{doublet}$$

The E&M interactions don't change the identity of the particle
In the weak interactions f_1 may be different from f_2
The gauge bosons W_i^μ are in the adjoint representation $n \times \bar{n} = (n^2 - 1) + 1 \rightarrow$ $SU(n)$ adjoint representation

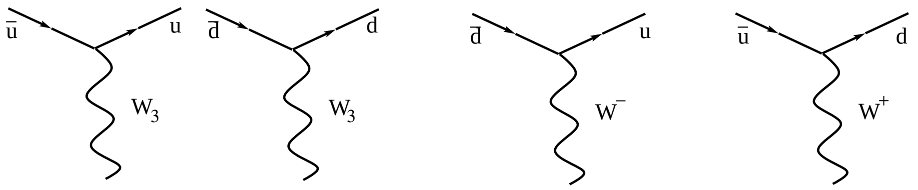
$$\begin{array}{lll} W^\mu_\alpha & \mu = 0, 1, 2, 3 & \text{Lorentz index} \\ & \alpha = 1, 2, 3 & \text{gauge group index} \end{array}$$

Couplings:



The current in the Lagrangian has the form

$$
\begin{aligned}
& \begin{pmatrix}\bar{u} & , & \bar{d}\end{pmatrix} \left(\sum_{i=1}^{3} \tau_i W_i^\mu = \vec{\tau} \cdot \vec{W}^\mu \right) \begin{pmatrix} u \\ d \end{pmatrix} \\
= & \begin{pmatrix}\bar{u} & , & \bar{d}\end{pmatrix} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} W_1^\mu + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} W_2^\mu + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} W_3^\mu \right] \begin{pmatrix} u \\ d \end{pmatrix} \\
= & \begin{pmatrix}\bar{u} & , & \bar{d}\end{pmatrix} \begin{bmatrix} W_3^\mu & W_1^\mu - iW_2^\mu \\ W_1^\mu + iW_2^\mu & -W_3^\mu \end{bmatrix} \begin{pmatrix} u \\ d \end{pmatrix} = \\
& W_3^\mu \left(\bar{u}u - \bar{d}d \right) \qquad + \quad (W_1^\mu + iW_2^\mu)\bar{d}u \quad + \quad (W_1^\mu - iW_2^\mu)\bar{u}d
\end{aligned}
$$

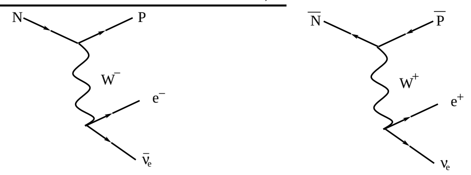


<u>In $SU(3)$</u>:

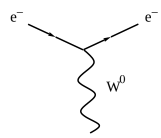
$$
\begin{pmatrix}\bar{u}_1 & , & \bar{u}_2 & , \bar{u}_3\end{pmatrix} \left(\sum_{i=1}^{8} \lambda_j A_j^\mu = \vec{\lambda} \cdot \vec{A}^\mu \right) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}
$$

where λ_j are the Gellmann matrices with $j = 1, ..., 8$

<u>Unification of E&M and weak interactions(Glashow 1961; Weinberg; Salam 1968)</u> <u>Problems</u>:

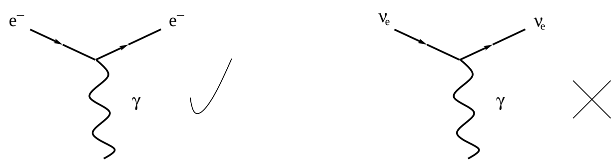


Weak interactions also involve leptons



However, if $W^\pm \in SU(2)$ we must also have $W^0 \in SU(2)$

Do these currents exist in nature?



? indetify $W^0 = \gamma \;\; \rightarrow \;\;$ photon?

ν is neutral $\rightarrow$ does not couple to γ

Must have a new W^0 that couples to ν

The new W^0 must be a mixture of W^3 and γ such that $Q(W^\pm) = \pm 1$.

<u>We saw</u>: Weak interactions only couple to left–handed fields whereas E&M couples to both left & right handed fields

$\rightarrow$ only (e_L, ν_L) and (u_L, d_L) interact weakly

(e_L, e_R) and (u_L, u_R, d_L, d_R) interact E&M
$Q(\nu_L) = 0.$
So far no need for ν_R *i.e.* no strong, weak or E&M interactions for ν_R.
Left–handed fields form doublets of $SU(2)_W$.
Right–handed fields are singlets of $SU(2)_W$.

$$\begin{array}{cccccc}
\binom{\nu_e}{e}_L & \binom{u}{d}_L & e_R & u_R & d_R \\
\binom{\nu_\mu}{\mu}_L & \binom{c}{s}_L & \mu_R & c_R & s_R \\
\binom{\nu_\tau}{\tau}_L & \binom{t}{b}_L & \tau_R & t_R & b_R
\end{array}$$

The quarks are triplets of $SU(3)_{\text{color}}$.
The leptons are singlets of $SU(3)_{\text{color}}$.
The $SU(2)_W$ doublets have T_3^W quantum numbers.
The $SU(2)_W$ singlets have $T_3^W = 0$.
We have to introduce a $U(1)$ symmetry to incorporate E&M charges
$SU(2)_W \times U(1)_Y$
But $U(1)_Y \neq U(1)_{\text{e.m.}}$
All $SU(2)$ representations must have the same $U(1)_Y$ charge,
but $Q(e_L) \neq Q(\nu_L)$
$\Rightarrow \quad U(1)_Y \neq U(1)_{\text{e.m.}}$
Combination : $\quad Q_{\text{e.m.}} = T_{3_W} + \frac{1}{2}Y$
Find values for Y such that $Q_{\text{e.m.}}$ is reproduced for the different particle states.

	T_3	$\frac{1}{2}Y$	$Q_{e.m.}$		T_3	$\frac{1}{2}Y$	$Q_{e.m.}$
ν_L	$\frac{1}{2}$	$-\frac{1}{2}$	0	u_L	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{2}{3}$
e_L	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	d_L	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{3}$
e_R	0	-1	-1	u_R	0	$\frac{2}{3}$	$\frac{2}{3}$
ν_R	0	0	0	d_R	0	$-\frac{1}{3}$	$-\frac{1}{3}$

How can we write a four vector current j^μ that will incorporate both the weak & electromagnetic interactions?

$$\begin{aligned}
J_\mu^+(x) &= \bar{\chi}_L \gamma_\mu \tau_+ \chi_L \qquad , \qquad J_\mu^-(x) = \bar{\chi}_L \gamma_\mu \tau_- \chi_L \\
J_\mu^3(x) &= \bar{\chi}_L \gamma_\mu \frac{1}{2} \tau_3 \chi_L \\
&= (\bar{\nu}_e \bar{e}^-)_L \gamma_\mu \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L = \frac{1}{2} \bar{\nu}_{eL} \gamma^\mu \nu_L - \frac{1}{2} \bar{e}_L \gamma^\mu e_L
\end{aligned}$$

where $\tau_\pm = \frac{1}{2}(\tau_1 \pm i\tau_2)$ and $\chi_L = \binom{\nu_e}{e^-}_L$. We can write this in the form

$$J_\mu^i = \bar{\chi}_L \gamma_\mu \frac{1}{2} \tau_i \chi_L \quad \text{with} \quad i = 1, 2, 3.$$

These currents couple to the vector bosons

$$W^{\mu\pm} = \frac{1}{\sqrt{2}}(W^{\mu 1} \mp i W^{\mu 2})$$

using the identity $\quad \frac{1}{2}(\tau_1 W^1 + \tau_2 W^2) = \frac{1}{\sqrt{2}}(\tau^+ W^+ + \tau^- W^-)$

We can write

$$\underbrace{\frac{1}{\sqrt{2}}(J_\mu^+ W^{+\mu} + J_\mu^- W^{-\mu})}_{\text{charged currents}} + \underbrace{J_\mu^3 W^{3\mu}}_{\text{neutral current}} = \sum_{i=1}^{3} J_\mu^i W^{i\mu}$$

The electromagentic current is given by

$$J_{e.m.}^\mu A_\mu = \underbrace{eQ}_{\text{electric charge coupling}} \bar{\psi}\gamma^\mu \psi A_\mu = \underbrace{e}_{\text{coupling}} \bar{\psi}\gamma^\mu \underbrace{Q}_{\text{charge}} \psi A_\mu$$

where A_μ is the E&M vector boson, *i.e* the photon.

The new $U(1)$ current $\quad \frac{1}{2} \underbrace{g'}_{\text{gauge coupling}} \bar{\psi}\gamma^\mu \underbrace{Y}_{\text{hypercharge}} \psi \underbrace{B_\mu}_{\text{gauge field}}$

The neutral $SU(2)$ current is $\quad g\bar{\psi}\gamma^\mu T_3 \psi W_\mu^3 \quad$ with $T_3 = \frac{\tau_3}{2}$

To get consistency with the charge assignment we should have

$$Q = T_3 + \frac{1}{2}Y \quad \Rightarrow \quad J_\mu^{e.m.} = J_\mu^3 + \frac{1}{2}J_\mu^Y$$

we obtain this by making a rotation on $B_\mu,\ W_\mu^3$.

$$\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} B_\mu \\ W_\mu^3 \end{pmatrix} = \begin{pmatrix} \cos\theta_W B_\mu + \sin\theta_W W_\mu^3 \\ -\sin\theta_W B_\mu + \cos\theta_W W_\mu^3 \end{pmatrix}$$

or inversely $\quad \begin{pmatrix} B_\mu \\ W_\mu^3 \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix}$

$\theta_W \longrightarrow$ Weinberg angle

<u>we get</u> :

$$\begin{aligned} & gJ_\mu^3 W^{3\mu} + \frac{g'}{2} J_\mu^Y B^\mu \\ = \ & gJ_\mu^3(\sin\theta_W A^\mu + \cos\theta_w Z^\mu) + \frac{g'}{2} J_\mu^Y (\cos\theta_W A^\mu - \sin\theta_w Z^\mu) \\ = \ & (g\sin\theta_W J_\mu^3 + g'\cos\theta_W \frac{J_\mu^Y}{2})A^\mu + (g\cos\theta_W J_\mu^3 - g'\sin\theta_W \frac{J_\mu^Y}{2})Z^\mu \end{aligned}$$

The first term is the electromagnetic interaction

$$eJ_\mu^{e.m.} A^\mu = e(J_\mu^3 + \frac{1}{2}J_\mu^Y)A^\mu$$

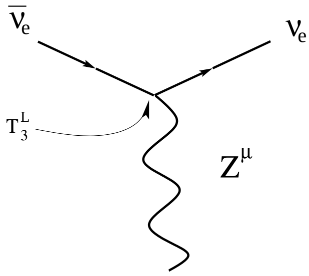
Therefore we have to impose

$$e = g\sin\theta_W = g'\cos\theta_W \quad \Longrightarrow \quad \tan\theta_W = \frac{g'}{g}$$

We can express the neutral current interaction in the form

$$\frac{g}{\cos\theta_W}(J_\mu^3 - \sin^2\theta_W J_\mu^{e.m.})Z^\mu = \frac{g}{\cos\theta_W} J_\mu^{NC} Z^\mu$$

We now have a new neutral current, coupling neutrinos to Z^{μ}



This neutral current was proposed by Glashow in 1961 and observed at CERN in 1970

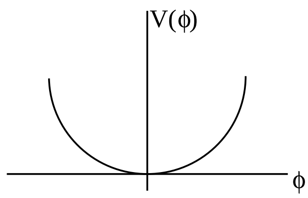
$\rightarrow$ We still have a problem
$\rightarrow$ E&M interactions $\rightarrow$ Long range $\rightarrow$ $m_\gamma = 0$
$\rightarrow$ Weak interactions $\rightarrow$ Short range $\rightarrow$ $m_{W^\pm;Z} \neq 0$
How ? $\rightarrow$ symmetry breaking
Lagrangian is invariant, but vacuum $\rightarrow$ symmetry breaking
The vacuum $\rightarrow$ the states of lowest energy
The Higgs mechanism Consider a real scalar field with the Lagrangian

$$\mathcal{L} = T - V = \frac{1}{2}(\partial_\mu \phi)^2 - (\frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4)$$

$$V(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4 \qquad \text{with} \quad \lambda > 0$$

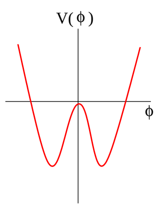
This Lagrangian is invariant under the transformation $\phi \rightarrow -\phi$

For $\mu^2 > 0$ the potential looks like



The Lagrangian describes a self–interacting scalar field with coupling λ and mass μ. The ground state correspond to $\langle\phi\rangle = 0$ and it obeys the reflection symmetry of the Lagrangian.

For $\mu^2 < 0$ the potential looks like



The potential has two minima at $\dfrac{\partial V}{\partial \phi} = \phi(\mu^2 + \lambda\phi^2) = 0$

$\Rightarrow \quad \langle\phi\rangle = \pm v \quad \text{with} \quad v = \sqrt{\dfrac{-\mu^2}{\lambda}}$

The extremum $\phi = 0$ does not correspond to the minimum of the energy.
We perform perturbative calculation around the classical minimum $\phi = v$ or $\phi = -v$.

we write $\qquad \phi(x) = v + \eta(x)$

$\eta(x)$ represents quantum fluctuations about this minimum.
Substituting into the Lagrangian we obtain

$$\begin{aligned}
\mathcal{L}' &= \frac{1}{2}\partial_\mu(v+\eta)\partial^\mu(v+\eta) - \frac{1}{2}\left(\mu^2(v+\eta)^2 + \frac{\lambda}{4}(v+\eta)^4\right) \\
&= \frac{1}{2}(\partial_\mu\eta)^2 - \left(\frac{\mu^2}{2}(v^2 + 2\eta v + \eta^2) + \frac{\lambda}{4}(v^4 + 4v^3\eta + 6v^2\eta^2 + 4v\eta^3 + \eta^4)\right) \\
&= \frac{1}{2}(\partial_\mu\eta)^2 + \frac{\lambda v^2}{2}(v^2 + 2\eta v + \eta^2) - \frac{\lambda}{4}(v^4 + 4v^3\eta + 6v^2\eta^2 + 4v\eta^3 + \eta^4) \\
&= \frac{1}{2}(\partial_\mu\eta)^2 - \lambda v^2\eta^2 - \lambda v\eta^3 - \frac{\lambda\eta^4}{4} + \text{const}
\end{aligned}$$

The field η has a mass term with the correct sign

$$m_\eta = \sqrt{2\lambda v^2} = \sqrt{-2\mu^2}$$

The higher terms in η are self–interaction terms. We do perturbation theory around a stable minimum $\phi = v + \eta$.

η is a massive field

The reflection symmetry is broken by the choice of the vacuum.

Consider now a complex scalar field

$$\phi = \frac{(\phi_1 + i\phi_2)}{\sqrt{2}}.$$

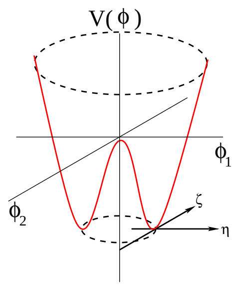
$$\mathcal{L} = (\partial_\mu\phi)^*(\partial^\mu\phi) - \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2$$

$\mathcal{L}$ is invariant under the global $U(1)$ symmetry

$$\phi \to e^{i\alpha}\phi$$

For λ and $\mu^2 < 0$ we rewrite the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left((\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2\right) - \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) - \frac{1}{4}\lambda(\phi_1^2 + \phi_2^2)^2$$



There is now a circle of minima

$$\text{at} \quad \phi_1^2 + \phi_2^2 = v^2 \quad \text{with} \quad v^2 = -\frac{\mu^2}{\lambda}$$

We translate the field ϕ to $\langle\phi_1\rangle = v$, $\langle\phi_2\rangle = 0$.
Expand the Lagrangian around the vacuum with

$$\phi(x) = \frac{1}{\sqrt{2}}(v + \eta(x) + i\zeta(x))$$

$$\begin{aligned}\mathcal{L}' &= \frac{1}{2}\left((\partial_\mu\eta)^2 + (\partial_\mu\zeta)^2\right) + \mu^2\eta^2 + \text{constant} \\ &\qquad\qquad + (\text{cubic \& quartic terms in } \eta \text{ \& } \zeta)\end{aligned}$$

The term $\mu^2\eta^2$ is a mass term for the η field $m_\eta = \sqrt{-2\mu^2}$ as before.
There is no corresponding mass term for ζ $\rightarrow$ massless scalar field
Goldstone theorem $\rightarrow$ spontaneously broken continuous global symmetry
$\rightarrow$ Goldstone boson
Consider now a complex scalar field coupled to a continuous $U(1)$ symmetry

$$\phi(x) \to \mathrm{e}^{i\alpha(x)}\phi(x)$$

As we saw local phase invariance requires that we replace ∂_μ by

$$D_\mu = \partial_\mu - ieA_\mu$$

and the gauge field A_μ transforms as

$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\alpha$$

The gauge invariant Lagrangian is

$$\mathcal{L} = (\partial_\mu + ieA_\mu)\phi^*(\partial^\mu - ieA^\mu)\phi - \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

For $\mu^2 > 0$ $\rightarrow$ Lagrangian for charged self–interacting scalar field with mass μ

For $\mu^2 < 0$ expand $\qquad \phi(x) = \frac{1}{\sqrt{2}}(v + \eta(x) + i\zeta(x))$

$$\begin{aligned}\rightarrow \quad \mathcal{L}' &= \frac{1}{2}\left((\partial_\mu\eta)^2 + (\partial_\mu\zeta)^2\right) - \lambda v^2\eta^2 + \frac{1}{2}e^2v^2A_\mu A^\mu \\ &\quad - evA_\mu\partial^\mu\zeta - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \text{interaction terms}\end{aligned}$$

The particle spectrum appears to be

massless Goldston boson ζ with $m_\zeta = 0$

massive scalar η with $m_\eta = \sqrt{2\lambda v^2}$

massive vector boson A^μ with $m_A = ev$

However, this interpretation should be revised

massless A^μ $\rightarrow$ 2T physical degrees of freedom

massive A^μ $\rightarrow$ 2T + 1L physical degrees of freedom

where did the third degree of freedom come from ? Note that to lowest order

$$\phi = \frac{1}{\sqrt{2}}(v + \eta(x) + i\zeta(x)) \cong \frac{1}{\sqrt{2}}(v + \eta(x))\, \mathrm{e}^{i\frac{\zeta(x)}{v}}$$

$$\rightarrow \text{ use a different set of fields } \; h,\ \theta,\ A_\mu$$

$$\phi \rightarrow \frac{1}{\sqrt{2}}(v + h(x))\mathrm{e}^{i\frac{\theta(x)}{v}}$$

$$A_\mu \rightarrow A_\mu + \frac{1}{ev}\partial_\mu\theta$$

substitute into $\mathcal{L}$. We get

$$\begin{aligned}\mathcal{L}'' &= \frac{1}{2}(\partial_\mu h)^2 - \lambda v^2 h^2 + \frac{1}{2}e^2 v^2 A_\mu A^\mu - \lambda v h^3 - \frac{1}{4}\lambda h^4 \\ &\quad + \frac{1}{2}e^2 A_\mu A^\mu h^2 + v e^2 A_\mu A^\mu h - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\end{aligned}$$

The Goldstone boson disappeared altogether
$\rightarrow$ The Goldstone boson is absorbed as the longitudenal mode of A_μ
$\rightarrow$ only 2 physical fields h and A^μ.
We are ready to see how the Higgs mechanism operates in the Standard Model.

$$\mathcal{L} = (\partial_\mu\Phi)^\dagger(\partial^\mu\Phi) - \mu^2(\Phi^\dagger\Phi) - \lambda(\Phi^\dagger\Phi)^2$$

$$\text{with} \quad \Phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + i\phi_2 \\ \phi_3 + i\phi_4\end{pmatrix}$$

The Lagrangian is invariant under the gauge transformation

$$\begin{aligned}\Phi \;\rightarrow\; \Phi' &= \mathrm{e}^{i\frac{\alpha_a\tau_a}{2}}\Phi \\ \text{The covariant derivative} \qquad D_\mu &= \partial_\mu + ig\frac{\tau_a}{2}W^a_\mu \qquad a = 1,2,3 \\ \text{under} \quad \Phi(x) \rightarrow \Phi'(x) &= (1 + i\frac{\vec{\alpha}\cdot\vec{\tau}}{2})\Phi(x) \\ \vec{W}_\mu \;\rightarrow\; & \vec{W}_\mu - \frac{1}{g}\partial_\mu\vec{\alpha} - \vec{\alpha}\times\vec{W}_\mu\end{aligned}$$

The gauge invariant Lagrangian is

$$\begin{aligned}\mathcal{L} &= (\partial_\mu\Phi + ig\frac{1}{2}\vec{\tau}\cdot\vec{W}_\mu\Phi)^\dagger(\partial^\mu\Phi + ig\frac{1}{2}\vec{\tau}\cdot\vec{W}^\mu\Phi) - V(\Phi) - \frac{1}{4}\vec{W}_{\mu\nu}\cdot\vec{W}^{\mu\nu} \\ &\text{where} \quad \vec{W}_{\mu\nu} = \partial_\mu\vec{W}_\nu - \partial_\nu\vec{W}_\mu - g\vec{W}_\mu\times\vec{W}_\nu\end{aligned}$$

Take $\mu^2 < 0$, $\lambda > 0$
The potential has a minimum at

$$\Phi^\dagger\Phi = \frac{1}{2}(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2) = -\frac{\mu^2}{2\lambda}$$

Expand about a minimum. Choose

$$\phi_1 = \phi_2 = \phi_4 = 0 \qquad , \qquad \phi_3^2 = -\frac{\mu^2}{\lambda} = v^2$$

$$\begin{aligned}
\text{Expand } \Phi(x) \text{ about the vacuum} \langle \Phi_0 \rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \\
\Phi(x) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}
\end{aligned}$$

The three additional degrees of freedom are absorbed as the longitudenal components of W_1^μ, W_2^μ , W_3^μ. The mass term

$$\left| \left(g \frac{\vec{\tau}}{2} \cdot \vec{W}_\mu \right) \Phi \right|^2 = \frac{g^2}{8} \left| \begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \right|^2$$

$$= \frac{g^2 v^2}{8} \left[(W_\mu^1)^2 + (W_\mu^2)^2 + (W_\mu^3)^2 \right] \ \rightarrow \ 3 \text{ massive vector bosons}$$

Now consider the Lagrangian of the $SU(2)_W \times U(1)_Y$ of the Standard Model coupled to Φ

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \qquad \text{with} \qquad \begin{aligned} \phi^+ &= \frac{1}{\sqrt{2}} (\phi_1 + i\phi_2) \\ \phi^0 &= \frac{1}{\sqrt{2}} (\phi_3 + i\phi_4) \end{aligned}$$

The Lagrangian density of the Higgs field is given by:

$$\mathcal{L} = \left| \left(\partial_\mu - ig\vec{T} \cdot \vec{W}_\mu - ig' \frac{Y}{2} B_\mu \right) \Phi \right|^2 - V(\Phi)$$

where $\vec{T} = \frac{\vec{\tau}}{2}$ and

$$V(\Phi) = \mu^2 \Phi^\dagger \Phi + \lambda \left(\Phi^\dagger \Phi \right)^2$$

The relevant term for the gauge boson masses

$$\left| \left(g \tfrac{\vec{\tau}}{2} \cdot \vec{W}_\mu + g' \tfrac{1}{2} B_\mu \right) \Phi \right|^2 =$$

$$\left| \left(\tfrac{g}{2} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} W_\mu^1 + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} W_\mu^2 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} W_\mu^3 \right] + \tfrac{g'}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} B_\mu \right) \Phi \right|^2$$

$$= \tfrac{1}{4} \left| \left[g \begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix} + g' \begin{pmatrix} B_\mu & 0 \\ 0 & B_\mu \end{pmatrix} \right] \tfrac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \right|^2$$

$$= \tfrac{1}{8} \left| \begin{pmatrix} gW_\mu^3 + g'B_\mu & g(W_\mu^1 - iW_\mu^2) \\ g(W_\mu^1 + iW_\mu^2) & -gW_\mu^3 + g'B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \right|^2$$

$$= \tfrac{1}{8} v^2 \left(g \left(W_\mu^1 + iW_\mu^2 \right), \left(-gW_\mu^3 + g'B_\mu \right) \right) \begin{pmatrix} g \left(W^{1\mu} - iW^{2\mu} \right) \\ -gW^{3\mu} + g'B^\mu \end{pmatrix}$$

$$= \tfrac{g^2v^2}{8}\left[(W^1_\mu)^2 + (W^2_\mu)^2\right] + \tfrac{v^2}{8}(g'B_\mu - gW_\mu)(g'B^\mu - gW^\mu)$$

$$= \tfrac{1}{4}g^2v^2W^{+\mu}W^{-\mu} + \tfrac{1}{8}v^2(g^2+g'^2)\frac{\left(-gW^3_\mu+g'B_\mu\right)^2}{\left(\sqrt{g^2+g'^2}\right)^2}$$

$$= \left(\tfrac{1}{2}gv\right)^2 W^+_\mu W^{-\mu} + \tfrac{1}{2}v^2\frac{\left(g^2+g'^2\right)}{4}\left(\frac{-gW^3_\mu+g'B_\mu}{\sqrt{g^2+g'^2}}\right)^2 + 0\left(gW^3_\mu + g'B_\mu\right)^2$$

$$M^2_{W^\pm}\quad W^+_\mu W^{-\mu} \quad + \quad \tfrac{1}{2}M^2_Z Z^2_\mu \quad + \quad \tfrac{1}{2}M_A A^2_\mu$$

The first term is the mass term for $W^+,\ W^-$,

$$M_{W^\pm} = \frac{1}{2}gv$$

Recalling that

$$\tan\theta_W = \frac{g'}{g} \qquad ; \qquad \sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}$$

the normalised A_μ and Z_μ field combinations are

$$A_\mu = \frac{g'W^3_\mu + gB_\mu}{\sqrt{g^2+g'^2}} \quad \text{with} \quad M_A = 0$$

$$Z_\mu = \frac{g'W^3_\mu - gB_\mu}{\sqrt{g^2+g'^2}} \quad \text{with} \quad M_Z = \frac{1}{2}v\sqrt{g^2+g'^2}$$

which gives

$$\frac{M_W}{M_Z} = \frac{g}{\sqrt{g^2+g'^2}} = \cos\theta_W$$

which is verified experimentally to high precision

References