MATH191: Problem Solution Sheet 4

It is a slightly unfortunate fact that the convention is to write "lim $= +\infty$" or "lim $= -\infty$" in cases where the limit does not exist. If the limit exists it has to be a number, not $\pm\infty$. In this question (in contrast to Sheet 3, where I was asking if limits exist) I was asking for a, where a could be any number, or could be $+\infty$ or $-\infty$, such that $\lim_{x\to+\infty} f(x) = a$, and similarly for $\lim_{x\to-\infty} f(x)$.

1. a) $f(x) = x^3 - 3x^2 + x - 1 = x^3(1 - 3x^{-1} + x^{-2} - x^{-3})$ when $x \neq 0$. So

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x^3(1 - 3x^{-1} + x^{-2} - x^{-3}) = \lim_{x\to+\infty} x^3 = +\infty,$$

and similarly

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x^3 = -\infty.$$

b)

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^3 + x^2}{x^2 - 2} = \lim_{x\to\infty} \frac{x+1}{1 - x^{-2}} = \lim_{x\to+\infty} x = +\infty,$$

and similarly

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x = -\infty.$$

c)

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{3x^2 - 2x + 3}{2x^2 - 3}$$

$$= \lim_{x\to\pm\infty} \frac{3 - 2x^{-1} + 3x^{-2}}{2 - 3x^{-2}} = \frac{3}{2}.$$

d)

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{5x^2 - 2x + 3}{x^3 + 4x^2 - 1} = \lim_{x\to\pm\infty} \frac{5x^{-1} - 2x^{-2} + 3x^{-3}}{1 + 4x^{-1} - x^{-2}} = \frac{0}{1} = 0.$$

e) A possible solution is: $f(x) = \cos x$ can be 1 for arbitrarily large positive x and can also be -1 for arbitrarily large positive x. So $\lim_{x\to+\infty} \cos x$ cannot even be evaluated. The limit does not exist and we cannot even say "lim $=$ $+\infty$" or "lim $= -\infty$" because $\cos x$ is always between -1 and $+1$. Similarly, $\lim_{x\to-\infty} \cos x$ does not exist and cannot even be evaluated as $+\infty$ or $-\infty$.

f) Since $-1 \leq \cos x \leq 1$ for all values of x, we have

$$\frac{-1}{|x|} \leq \frac{\cos x}{x} \leq \frac{1}{|x|}.$$

Hence, by the sandwich rule, $\lim_{x\to+\infty} f(x) = 0$.

2. a) We use the fact that $\frac{d}{dx}x^n = nx^{n-1}$. Thus $\frac{d}{dx}x^3 = 3x^2$, $\frac{d}{dx}x^2 = 2x$, $\frac{d}{dx}x = 1$, and $\frac{d}{dx}2 = 0$. So

$$\frac{d}{dx}(x^3 + 2x^2 - 3x + 2) = 3x^2 + 4x - 3.$$

b) Let $f(x) = x^4 \sin x$, and set $u = x^4$ and $v = \sin x$, so $f(x) = uv$. Now $u' = 4x^3$ and $v' = \cos x$, so

$$f'(x) = uv' + u'v = x^4 \cos x + 4x^3 \sin x.$$

c) Let $f(x) = 2\sqrt{x} + \cos x = 2x^{1/2} + \cos x$. Then

$$f'(x) = 2 \cdot \frac{1}{2}x^{-1/2} - \sin x = \frac{1}{\sqrt{x}} - \sin x.$$

d) Let $y = \frac{1}{\sqrt{1+x}} = (1+x)^{-1/2}$. Let $u = 1 + x$, so $y = u^{-1/2}$. Then $\frac{du}{dx} = 1$ and $\frac{dy}{du} = -\frac{1}{2}u^{-3/2}$, so

$$\frac{dy}{dx} = -\frac{1}{2}u^{-3/2} \cdot 1 = -\frac{1}{2}(1+x)^{-3/2}.$$

e) Let $f(x) = \frac{\sin x}{x^2}$, and set $u = \sin x$ and $v = x^2$, so $f(x) = \frac{u}{v}$. Now $u' = \cos x$ and $v' = 2x$, so

$$f'(x) = \frac{vu' - uv'}{v^2} = \frac{x^2 \cos x - 2x \sin x}{x^4} = \frac{x \cos x - 2 \sin x}{x^3}.$$

f) Let $y = \cos(x^2 + 1)$. Write $u = x^2 + 1$, so $y = \cos u$. Now $\frac{dy}{du} = -\sin u$ and $\frac{du}{dx} = 2x$, so

$$\frac{dy}{dx} = -\sin(u) \cdot 2x = -2x \sin(x^2 + 1).$$

g) Let $y = \frac{1}{(2+3x)^2} = (2+3x)^{-2}$. Write $u = 2 + 3x$, so $y = u^{-2}$. Now $\frac{dy}{du} = -2u^{-3}$ and $\frac{du}{dx} = 3$, so

$$\frac{dy}{dx} = -2u^{-3} \cdot 3 = -6(2+3x)^{-3} = \frac{-6}{(2+3x)^3}.$$

3. In each part, we use the formula for the equation of the tangent to the graph of $y = f(x)$ at $(x, y) = (x_0, f(x_0))$, which is:

$$y = f(x_0) + f'(x_0)(x - x_0).$$

a) In this case, $f(x) = x^2$ and $x_0 = 3$, $f(x_0) = 9$. We have $f'(x) = 2x$, so $f'(x_0) = 2 \times 3 = 6$. Thus the equation of the tangent is

$$y = 9 + 6(x - 3),$$

or

$$y = 6x - 9.$$

b) In this case, $f(x) = x^3$ and $x_0 = -1$, $f(x_0) = -1$. We have $f'(x) = 3x^2$, so $f'(x_0) = 3((-1)^2) = 3$. Thus the equation of the tangent is

$$y = -1 + 3(x - (-1)) = -1 + 3(x + 1),$$

or

$$y = 3x + 2.$$

c) In this case $f(x) = x^2 \cos x$ and $x_0 = 0$, $f(x_0) = 0$. To calculate $f'(x)$, set $u = x^2$ and $v = \cos x$, so $f(x) = uv$. Then $u' = 2x$ and $v' = -\sin x$, so

$$f'(x) = uv' + u'v = x^2(-\sin x) + 2x\cos x = 2x\cos x - x^2 \sin x.$$

Hence $f'(x_0) = 0\cos 0 - 0^2 \sin 0 = 0$, and so the equation of the tangent is

$$y = 0 + 0(x - 0) \qquad \text{or} \qquad y = 0.$$

d) In this case $f(x) = \frac{\sin x}{x}$ and $x_0 = \pi$, $f(x_0) = 0$. To calculate $f'(x)$, set $u = \sin x$ and $v = x$, so $f(x) = \frac{u}{v}$. Then $u' = \cos x$ and $v' = 1$, so

$$f'(x) = \frac{vu' - uv'}{v^2} = \frac{x\cos x - \sin x}{x^2}.$$

Hence $f'(x_0) = \frac{\pi\cos\pi - \sin\pi}{\pi^2} = \frac{-\pi - 0}{\pi^2} = \frac{-1}{\pi}$, and so the equation of the tangent is

$$y = 0 - \frac{1}{\pi}(x - \pi),$$

or

$$y = 1 - \frac{x}{\pi}.$$

1. The relevant row of Pascal's triangle is

$$1 \quad 4 \quad 6 \quad 4 \quad 1,$$

so

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

a)

$$(1+x)^4 = 1^4 + 4\cdot 1^3 x + 6\cdot 1^2 x^2 + 4\cdot 1 x^3 + x^4 = 1 + 4x + 6x^2 + 4x^3 + x^4.$$

b)

$$(1+2x)^4 = 1^4+4{\cdot}1^3(2x)+6{\cdot}1^2(2x)^2+4{\cdot}1(2x)^3+(2x)^4 = 1+8x+24x^2+32x^3+16x^4.$$

c)

$$(1-x)^4 = 1^4+4{\cdot}1^3(-x)+6{\cdot}1^2(-x)^2+4{\cdot}1(-x)^3+(-x)^4 = 1-4x+6x^2-4x^3+x^4$$

(using that $(-x)^n$ is x^n if n is even, and $-x^n$ if n is odd).