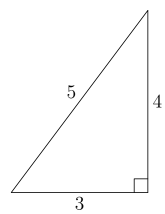
MATH191: Problem Solution Sheet 3

1. Consider the following triangle:




Here $R = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$, and $\phi = \tan^{-1}(4/3) = 0.9273$ to 4 decimal places. Now

$$3 = 5\cos\phi \qquad \text{and} \qquad 4 = 5\sin\phi,$$

so

$$\begin{aligned} 3\cos\theta + 4\sin\theta &= 5\cos\theta\cos\phi + 5\sin\theta\sin\phi \\ &= 5\cos(\theta - \phi) \end{aligned}$$

The equation therefore becomes

$$\cos(\theta - \phi) = \frac{2}{5}.$$

We solve this for $\theta - \phi$: the general solution is

$$\theta - \phi = \pm 1.1593 + 2n\pi \qquad n \in \mathbb{Z}.$$

Since $\phi = 0.9273$, this gives

$$\theta = 0.9273 \pm 1.1593 + 2n\pi = \begin{cases} 2.0866 + 2n\pi & n \in \mathbb{Z} \\ -0.2320 + 2n\pi & n \in \mathbb{Z}. \end{cases}$$

(It's a good idea to check your solutions. For example, for the first type of solution you can work out $\cos(2.0866) = -0.4932$ and $\sin(2.0866) = 0.8699$ on your calculator, and then verify that $(-0.4932 \times 3) + (0.8699 \times 4) = 2$. (You won't necessarily get exactly 2, because of rounding everything to 4 decimal places.)

2. a)

$$x = r\cos\theta = 2\cos\frac{\pi}{4} = \sqrt{2}.$$

$$y = r\sin\theta = 2\sin\frac{\pi}{4} = \sqrt{2}.$$

So $(x, y) = (\sqrt{2}, \sqrt{2})$.

b)

$$x = r\cos\theta = \cos\,\pi = -1.$$

$$y = r\sin\theta = \sin\,\pi = 0.$$

So $(x, y) = (-1, 0)$.

c)

$$r = \sqrt{x^2 + y^2} = \sqrt{1+3} = 2.$$

$$\tan\theta = \frac{y}{x} = \sqrt{3}.$$

Since $x > 0$, $\theta = \tan^{-1}(\sqrt{3}) = \pi/3$. So $(r, \theta) = (2, \pi/3)$.

d)

$$r = \sqrt{x^2 + y^2} = \sqrt{1+1} = \sqrt{2}.$$

$$\tan\theta = \frac{y}{x} = -1.$$

Since $x < 0$,

$$\theta = \tan^{-1}(-1) + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}.$$

So

$$(r, \theta) = (\sqrt{2}, \frac{3\pi}{4}).$$

3.

$$f(0.1) = \frac{\cos(0.1) - 1}{(0.1)^2} = -0.49958347$$

(and $f(-0.1)$ has the same value, since $f(x)$ is even).

$$f(0.01) = \frac{\cos(0.01) - 1}{(0.01)^2} = -0.4999958$$

(and $f(-0.01)$ has the same value).

It seems reasonable to guess that $\lim_{x\to 0} f(x) = -1/2$ (and this is in fact the case).

In degrees: $f(0.1) = f(-0.1) = f(0.01) = f(-0.01) = -0.000152309$ to 9 decimal places. Note that

$$-\frac{\pi^2}{2\times\ 180^2} = -0.0001523087099...$$

4. a) $\lim_{x\to 1} x^2 - 2 = 1 - 2 = -1$.

   1 mark

   b)

$$\lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1}\frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1} x+1 = 2.$$

   2 marks

   c) When $x$ is very close to 1, $x^2-2$ is very close to $-1$ and $x-1$ is close to 0. So $(x^2-2)/(x-1)$ is a large number and the limit does not exist. We only say that the limit exists if the limit is a number. However $x^2-2<0$ for $x$ close to 1, so that $(x^2-2)/(x-1)<0$ for $x-1>0$ and $(x^2-2)/(x-1)>0$ for $x-1<0$. So we can write

$$\lim_{x\to 1+}\frac{x^2-2}{x-1} = -\infty, \quad \lim_{x\to 1-}\frac{x^2-2}{x-1} = +\infty,$$

   d)

$$\lim_{x\to\pm\infty}\frac{x^2-2}{(x-1)^2} = \lim_{xto\pm\infty}\frac{1-(2/x^2)}{(1-(1/x))^2}$$
$$= \lim_{1/x\to 0}\frac{1-(2/x^2)}{(1-(1/x))^2} = 1.$$

   e)

$$\lim_{x\to 0}\frac{\sin 5x}{x} = \lim_{x\to 0}\frac{5\sin 5x}{5x}$$
$$= 5\lim_{5x\to 0}\frac{\sin 5x}{5x} = 5\lim_{y\to 0}\frac{\sin y}{y} = 5.$$

   f) When $x>0$ we have $f(x) = \frac{x}{x} = 1$; while when $x<0$ we have $f(x) = \frac{-x}{x} = -1$. So

$$\lim_{x\to 1+} f(x) = \lim_{x\to 1+} 1 = 1,$$
$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-} -1 = -1.$$

   Since $1 \neq -1$, $\lim_{x\to 0} f(x)$ does not exist.