Chapter 1

Functions and Graphs

1.1 Numbers (1.2.1, 1.2.4)

The most fundamental type of number are those we use to count with: $0, 1, 2, \ldots$. These are called the *natural numbers*: the *set* of all natural numbers is denoted $\mathbb{N}$.

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}.$$

Next we encounter the whole numbers or *integers*: the set of all integers is denoted $\mathbb{Z}$.

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

Clearly every natural number is an integer: that is $\mathbb{N} \subset \mathbb{Z}$.

Third, there are the fractions or *rational numbers*: the set of all rational numbers is denoted $\mathbb{Q}$. The rational numbers are those which can be written in the form p/q, where p and q are integers and $q \neq 0$. In set notation,

$$\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}.$$

Since any integer n can be written as $n/1$, every integer is a rational number: that is $\mathbb{Z} \subseteq \mathbb{Q}$.

Finally there are the *real numbers*: all numbers which can be written with a decimal expansion. The set of all real numbers is denoted $\mathbb{R}$. Not every real number is rational: for example $\sqrt{2}$ and π can't be written in the form p/q. Such real numbers are called *irrational*.

Thus we have

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

Later on we'll come across the *Complex numbers* $\mathbb{C}$.

Interval notation

Interval notation is a very convenient way of denoting sets of real numbers. If a and b are real numbers with $a \leq b$, we write $[a, b]$ for the set of all real numbers x with $a \leq x \leq b$. That is,

$$[a, b] = \{x \in \mathbb{R}\colon a \leq x \leq b\}.$$

Notice that this is really a collection of *real* numbers: thus $[1, 4]$ does not just contain the numbers $1, 2, 3, 4$, but everything between 1 and 4 (for example, π).

Similarly we use the notation (a, b) for the same set excluding the endpoints:

$$(a, b) = \{x \in \mathbb{R}\colon a < x < b\}.$$

We can mix square and round brackets:

$$\begin{aligned} [a, b) &= \{x \in \mathbb{R}\colon a \leq x < b\}. \\ (a, b] &= \{x \in \mathbb{R}\colon a < x \leq b\}. \end{aligned}$$

When we don't want an upper or lower limit, we can use the symbol ∞:

$$\begin{aligned} [a, \infty) &= \{x \in \mathbb{R}\colon a \leq x\} \\ (-\infty, b) &= \{x \in \mathbb{R}\colon x < b\} \end{aligned}$$

You should never put a square bracket next to ∞ or $-\infty$: ∞ is a convenient symbol, but it is *not* a real number.

1.2 Functions, Domain and Range (2.1, 2.2)

We often write expressions like $y = f(x)$. Here f is a *function*: we regard f as a machine, which, when we feed it a real number x, either spits out another real number $f(x)$ or tells us it doesn't like x. For example, if $f(x) = 1/x$, then if we feed f any real number $x \neq 0$, spits out the real number $1/x$: if we accidentally feed it $x = 0$, it complains (remember ∞ is not a real number).

Since we don't want our machine to complain, we have to be careful only to feed it allowable numbers.

The *Maximal Domain* of f is the set of all inputs x which don't make the machine complain (so $f(x)$ is a real number). Thus the maximal domain of $f(x) = 1/x$ is $(-\infty, 0) \cup (0, \infty)$. Sometimes we want to restrict the choice of inputs: a *domain* of f is any set of allowed inputs x: thus $[2, 5]$ is a domain of $f(x) = 1/x$, but $[-2, 2]$ is not (it contains 0, which is disallowed).

The *Range* of f is the set of possible output values y.

The *zeros* of f are all the possible input values x such that the output $f(x) = 0$. Also called *roots*.

1.3 Polynomials (2.4)

Polynomials are a very simple type of function:

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n.$$

The *degree* of the polynomial is the largest power of x that appears.

1. Degree 0: constants $f(x) = c_0$.
2. Degree 1: linear functions $f(x) = c_0 + c_1 x$.
3. Degree 2: quadratics $f(x) = c_0 + c_1 x + c_2 x^2$.
4. Degree 3: cubics $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$.

Examples $f(x) = x^2 - 4$. Draw graph. The maximal domain is $\mathbb{R}$. The range is $[-4, \infty)$. Two zeros, ± 2.

$f(x) = x^3 - 3x$. Draw graph. The maximal domain is $\mathbb{R}$. The range is $\mathbb{R}$. Three zeros, 0 and $\pm\sqrt{3}$.

The maximal domain of a polynomial is always $\mathbb{R}$. Polynomials are also *continuous* (you can draw the graph without taking your pen off the paper) and *smooth* (there are no sharp corners in the graph).

1.4 Rational functions (2.5)

A rational function is one which can be written in the form

$$f(x) = \frac{g(x)}{h(x)},$$

where $g(x)$ and $h(x)$ are polynomials.

Example

$$\frac{x^3 - 3x^2 + 5}{2x^4 + x - 3}.$$

Unlike polynomials, the maximal domain of a rational function may not be $\mathbb{R}$: they explode whenever $h(x) = 0$. The zeros of a rational function are exactly the points where $g(x) = 0$.

Examples $f(x) = 1/x$. Draw graph. The maximal domain is $(-\infty, 0) \cup (0, \infty)$. The range is $(-\infty, 0) \cup (0, \infty)$. The line $x = 0$ is a *vertical asymptote*. f is not continuous: it jumps at $x = 0$. f has no zeros.

$f(x) = (x + 2)/(x - 1)^2$. Don't try to draw graph. The maximal domain is $(-\infty, 1) \cup (1, \infty)$. f has one zero, at $x = -2$.

1.5 Modulus (1.2.4)

The *modulus* $|x|$ of a real number x is just its size: thus $|x| = x$ if $x \geq 0$, and $|x| = -x$ if $x < 0$.

Examples $f(x) = |x|$. Draw graph. The maximal domain in $\mathbb{R}$. The range is $[0, \infty)$. There is one zero, at $x = 0$. f is continuous, but not smooth (there is a sharp corner at $x = 0$).

$f(x) = |x^2 - 4|$. Draw graph. The maximal domain is $\mathbb{R}$. The range is $[0, \infty)$. There are two zeros, at $x = \pm 2$. f is continuous, but not smooth.

$f(x) = |x^2 + 1|$ is just the same as $f(x) = x^2 + 1$.

1.6 Even and Odd Functions (2.2.4)

An even function $f(x)$ is one for which $f(-x) = f(x)$ for all values of x (in the maximal domain). Thus the graph to the left of the y-axis can be obtained from the graph to the right by reflecting in the y-axis.

Examples are x^2, $|x|$, $x^4 + 2x^2 + 3$, any polynomial with only even powers.

An odd function $f(x)$ is one for which $f(-x) = -f(x)$ for all values of x (in the maximal domain). Thus the graph to the left of the y-axis can be obtained from the graph to the right by rotating about the origin.

Examples are x, $1/x$, $x^3 - 3x$, any polynomial with only odd powers.

Unlike numbers, most functions are neither even nor odd. Example $f(x) = x - 3$. Any polynomial with both even and odd powers is neither even nor odd.

To decide whether a function $f(x)$ is even, odd, or neither, work out $f(-x)$ and decide whether it is equal to $f(x)$, to $-f(x)$, or to neither of these.

Examples: $f(x) = \frac{x}{x+2}$, $f(x) = \sin(x^3)$, $f(x) = \sin(|x|)$, $f(x) = \frac{\sin(x)}{x}$. Note last is not defined at $x = 0$.

1.7 Inverse functions (2.2.2)

Suppose f is a function: that is, if we input a real number x, it outputs a real number y. The *inverse function* f^{-1} is the function which takes the output of f and tells us what the input was. Thus if $y = f(x)$ then $x = f^{-1}(y)$.

Example $f(x) = x + 3$. Then $f^{-1}(x) = x - 3$. (If $y = x + 3$, then $x = y - 3$, so $f^{-1}(y) = y - 3$. However a function doesn't care what letter I use to define it, so we can also write $f^{-1}(x) = x - 3$.) Draw graphs.

Thus determining the inverse function boils down to solving the equation $y = f(x)$ for x in terms of y.

Example $f(x) = x/(x-2)$. To find the inverse function, write $y = x/(x-2)$ and solve for x. $y(x-2) = x$, $yx - x = 2y$, $x(y-1) = 2y$, $x = 2y/(y-1)$. Thus $f^{-1}(y) = 2y/(y-1)$, or $f^{-1}(x) = 2x/(x-1)$. Show graphs.

Notice the *reflection rule*: since finding the inverse function is just interchanging the roles of x and y, the graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected in the line $y = x$.

Problem: not every function has an inverse. Consider for example $f(x) = x^2$. Should we say that $f^{-1}(4) = 2$, or -2? We can't decide. Can see this problem on the graphs: draw graph of x^2 and reflection in $y = x$. The problem is that $f(x) = x^2$ is *two to one*: there are two values of x which give rise to each value $f(x)$.

We say that $f(x)$ is *one to one* (or $1-1$) if different values of x always give different values of $f(x)$: that is, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

One to one functions $f(x)$ always have inverses: the maximal domain of $f^{-1}(x)$ is the range of $f(x)$, so may not be the same as the maxiaml domain of $f(x)$.

Examples $f(x) = x^3$ is $1-1$. The maximal domain and the range of $f(x)$ are $\mathbb{R}$, and the same is true of $f^{-1}(x) = \sqrt[3]{x}$.

$f(x) = e^x$ is $1-1$. $f(x)$ has maximal domain $\mathbb{R}$, and range $(0, \infty)$. The inverse $f^{-1}(x) = \ln(x)$ has maximal domain $(0, \infty)$, and range $\mathbb{R}$. (Pretend we know about these functions for now.)

If $f(x)$ is not $1-1$, and we want to talk about its inverse, we have to restrict the domain of $f(x)$ to one where it is $1-1$.

Example $f(x) = x^2$ is not $1-1$ on its maximal domain, but it is $1-1$ on the domain $[0, \infty)$. If we restrict to this domain, then $f(x)$ has an inverse $f^{-1}(x) = +\sqrt{x}$. Draw graphs. (Could just as well have chosen other domain).

Note: if $f(x)$ is continuous, then it is $1-1$ on an interval precisely when it is either strictly increasing or strictly decreasing there. Pictures.

1.8 Trigonometric Functions (2.6)

Draw a circle of radius 1, and pick a point $P = (x, y)$ on the circle, at angle θ to the horizontal.

Then $\sin\theta = y$, $\cos\theta = x$, and $\tan\theta = y/x (= \sin\theta/\cos\theta)$.

Notice that $-1 \leq \sin\theta, \cos\theta \leq 1$ for any value of θ: i.e. the *range* of sin and cos is $[-1, 1]$. On the other hand, $\tan\theta$ can take any real value: the *range* of tan is $\mathbb{R}$.

By pythagoras's theorem, $\sin^2\theta + \cos^2\theta = 1$.

You should *always* express angles in radians rather than degrees: there are very good reasons for this, which will become clear during the course. Remember that a full revolution (360 degrees) is 2π radians, so (give values of 180, 90, 60, 30 degrees).

Special angles: (Use 30-60-90 and 45-45-90 rules)

	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
sin	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
cos	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
tan	0	$1/\sqrt{3}$	1	$\sqrt{3}$	

Graphs of the trigonometric functions

Draw graphs of $\sin\theta$, $\cos\theta$ and $\tan\theta$. Note that cos is even and sin and tan are odd.

A function $f(x)$ is *periodic with period T* or *T-periodic* if $f(x+T) = f(x)$ for all x.

Thus sin and cos are 2π-periodic, tan is π-periodic.

Also define $\cot\theta = 1/\tan\theta = \cos\theta/\sin\theta$, $\operatorname{cosec}\theta = 1/\sin\theta$ and $\sec\theta = 1/\cos\theta$.

Draw graph of $\cot\theta$: odd and π-periodic.

Trigonometric identities

Hand out trigonometric identities and talk about them.

We can derive other identities from these.

Example To get an expression for $\cos(3\theta)$ in terms of $\cos\theta$, write $\cos(3\theta) = \cos(2\theta+\theta)$, and observe

$$
\begin{aligned}
\cos(2\theta+\theta) &= \cos 2\theta\cos\theta - \sin 2\theta \sin\theta \quad (4)\\
&= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta \quad (10, 11)\\
&= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta\\
&= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta \quad (1)\\
&= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta\\
&= 4\cos^3\theta - 3\cos\theta.
\end{aligned}
$$

Check: Try $\theta = 0$. Then $\cos 3\theta = \cos 0 = 1$, and $\cos\theta = 1$, so the right hand side is $4(1)^3 - 3 = 4 - 3 = 1$. So this checks. Try $\theta = \pi/3$. Then $\cos 3\theta = \cos\pi = -1$, while $\cos\theta = \cos\pi/3 = 1/2$. Thus the right hand side is $4(1/2)^3 - 3/2 = 1/2 - 3/2 = -1$. So this checks.

Inverse trigonometric functions

The trigonometric functions are periodic, and so are "∞ to 1". In order to consider their inverse functions, we have to restrict their domain to a *principal domain*, just as we did for $f(x) = x^2$.

The principal domain of $y = \sin x$ (draw graph) is $[-\pi/2, \pi/2]$. Thus the *principal value* of $\sin^{-1} x$ lies in $[-\pi/2, \pi/2]$. (There are infinitely many angles whose sine is x:

we pick the one which lies between $-\pi/2$ and $\pi/2$.) Draw graph. Maximal domain $[-1, 1]$, range $[-\pi/2, \pi/2]$.

The principal domain of $y = \cos x$ (draw graph) is $[0, \pi]$. Thus the *principal value* of $\cos^{-1} x$ lies in $[0, \pi]$. Draw graph. Maximal domain $[-1, 1]$, range $[0, \pi]$.

The principal domain of $y = \tan x$ (draw graph) is $[-\pi/2, \pi/2]$. Thus the *principal value* of $\tan^{-1} x$ lies in $[-\pi/2, \pi/2]$. Draw graph. Maximal domain $\mathbb{R}$, range $[-\pi/2, \pi/2]$.

NB On your calculator, $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$ may be called $\arcsin x$, $\arccos x$, and $\arctan x$. Your calculator should automatically give the principal values of these functions. Make sure it's set on radians.

Trigonometric Equations

Consider solving the equation $\sin\theta = 1/2$ for θ. One solution is $\theta = \sin^{-1}(1/2) = \pi/6$. However there are infinitely many other solutions (draw graph). These are $\pi/6 + 2n\pi$, where n is any integer, and $(\pi - \pi/6) + 2n\pi$, where n is any integer. Thus the *general solution* of $\sin\theta = 1/2$ is

$$\theta = \begin{cases} \pi/6 + 2n\pi & n \in \mathbb{Z} \\ (\pi - \pi/6) + 2n\pi & n \in \mathbb{Z}. \end{cases}$$

We can rewrite the equation as $\sin\theta = \sin\pi/6$. By the same argument, the *general solution* of the equation $\sin\theta = \sin\alpha$ is

$$\theta = \begin{cases} \alpha + 2n\pi & n \in \mathbb{Z} \\ (\pi - \alpha) + 2n\pi & n \in \mathbb{Z}. \end{cases}$$

Example Find the general solution of $\sin\theta = 1/\sqrt{2}$. Write $1/\sqrt{2} = \sin\alpha$: here $\alpha + \sin^{-1}(1/\sqrt{2}) = \pi/4$. Thus the general solution is

$$\theta = \begin{cases} \pi/4 + 2n\pi & n \in \mathbb{Z} \\ (\pi - \pi/4) + 2n\pi & n \in \mathbb{Z}. \end{cases}$$

Analogously, the general solution of the equation $\cos\theta = \cos\alpha$ is

$$\theta = \pm\alpha + 2n\pi \qquad n \in \mathbb{Z},$$

and the general solution of the equation $\tan\theta = \tan\alpha$ is

$$\theta = \alpha + n\pi \qquad n \in \mathbb{Z}.$$

Example Find the general solution of $\tan\theta = 3$. Write $3 = \tan\alpha$: thus $\alpha = \tan^{-1}(3) = 1.2490 = 0.3976\pi$. Thus the general solution is $\theta = 0.3976\pi + n\pi$, or

$\theta = (0.3976 + n)\pi$. That is, the values of θ with $\tan\theta = 3$ are 0.3976π, 1.3976π, 2.3976π, $\ldots$ and -0.6024π, -1.6024π, -2.6024π, etc.

A second type of trigonometric equation can be solved by a trick. These are equations of the form $a\cos\theta + b\sin\theta = c$, where a, b, and c are fixed. To solve this, consider a right-angled triangle with sides a and b, and hypoteneuse $R = \sqrt{a^2+b^2}$: let ϕ be the angle between a and R (so $\phi = \tan^{-1}(b/a)$). Then $a = R\cos\phi$ and $b = R\sin\phi$. Thus

$$a\cos\theta + b\sin\theta = R\cos\phi\cos\theta + R\sin\phi\sin\theta = R\cos(\theta-\phi).$$

Our equation therefore becomes

$$\cos(\theta-\phi) = c/R,$$

which we can solve in the usual way. Writing $c/R = \cos\alpha$, we have

$$\cos(\theta-\phi) = \cos\alpha,$$

which has general solution

$$\theta - \phi = \pm\alpha + 2n\pi,$$

or

$$\theta = \pm\alpha + 2n\pi + \phi.$$

Since $R = \sqrt{a^2+b^2}$ and $\phi = \tan^{-1}(b/a)$, we have $\alpha = \cos^{-1}(c/\sqrt{a^2+b^2}$, so we can write the general solution of $a\cos\theta + b\sin\theta = c$ as

$$\theta = \pm\cos^{-1}(\frac{c}{\sqrt{a^2+b^2}}) + 2n\pi + \tan^{-1}\frac{b}{a}.$$

However, rather than remember this formula, it is better to work through the steps for each example.

Example Find the general solution of the equation $\cos\theta + 2\sin\theta = 1$.

Consider the triangle with sides 1, 2, and $\sqrt{5}$, and let ϕ be the angle $\tan^{-1} 2 = 1.1071$. Then $1 = \sqrt{5}\cos\phi$ and $2 = \sqrt{5}\sin\phi$. Hence $\cos\theta + 2\sin\theta = \sqrt{5}\cos\theta\cos\phi + \sqrt{5}\sin\theta\sin\phi = \sqrt{5}\cos(\theta-\phi)$.

Solving $\cos\theta + 2\sin\theta = 1$ is the same as solving $\sqrt{5}\cos(\theta-\phi) = 1$, or $\cos(\theta-\phi) = 1/\sqrt{5}$. Now $\cos^{-1}(1/\sqrt{5}) = 1.1071$, so the general solution is

$$\theta - \phi = \pm 1.1071 + 2n\pi,$$

or

$$\theta = \phi \pm 1.1071 + 2n\pi = 1.1071 \pm 1.1071 + 2n\pi$$

so $\theta = 2n\pi$ or $\theta = 2.2142 + 2n\pi$.

The first type of solution is easy to check: if $\theta = 2n\pi$ then $\sin\theta = 0$ and $\cos\theta = 1$, so $\cos\theta + 2\sin\theta = 1$. For the second type, we can check that $\cos(2.2142) = -0.6$ and $\sin(2.2142) = 0.8$, so $\cos(2.2142) + 2\sin(2.2142) = 1$.

1.9 Polar Coordinates (2.6.6)

Sometimes, instead of describing a point P by its *Cartesian coordinates* (x, y) (the horizontal and vertical distances from the origin), it's convenient to represent it by its distance r and angle θ from the origin. Draw picture.

To convert from Cartesian to polar coordinates, use the formulae

$$r = \sqrt{x^2 + y^2} \qquad \tan\theta = y/x,$$

and to convert from polar to Cartesian coordinates, use the formulae

$$x = r\cos\theta \qquad y = r\sin\theta.$$

Examples Let P be the point with Cartesian coordinates $(2, 1)$. To find its polar coordinates: $r = \sqrt{2^2 + 1^2} = \sqrt{5}$, and $\tan\theta = 1/2$ so $\theta = \tan^{-1}(1/2) = 0.4636(= 0.1476\pi)$. So the polar coordinates of P are $(r, \theta) = (\sqrt{5}, 0.1476\pi)$.

Let P be the point with polar coordinates $(2, \pi/3)$. To find its Cartesian coordinates: $x = 2\cos\pi/3 = 2(1/2) = 1$ and $y = 2\sin\pi/3 = 2(\sqrt{3}/2) = \sqrt{3}$.

Note:

a) When P is the origin, θ is not defined.

b) Beware when using the formula $\tan\theta = y/x$ to calculate θ: when you calculate $\tan^{-1}(y/x)$ on your calculator, it will return the *principal value* of $\tan^{-1}(y/x)$, which lies between $-\pi/2$ and $\pi/2$ (i.e. $x > 0$). When determining θ, you have to look at the sign of x: if $x > 0$ then $\theta = \tan^{-1}(y/x)$; if $x < 0$ then $\theta = \tan^{-1}(y/x) + \pi$; if $x = 0$ then $\theta = \pi/2$ or $-\pi/2$ depending on whether $y > 0$ or $y < 0$. For example, let P have Cartesian coordinates $(-2, -2)$. Then $r = \sqrt{8}$. If we work out $\tan^{-1}(-2/-2) = \tan^{-1}(1)$, we get $\pi/4$, which is clearly wrong. Since $x < 0$, we have to add π to this to get θ: $\theta = 5\pi/4$.

1.10 Limits (7.8, 7.9)

We look at the behaviour of $f(x)$ as x approaches a certain value. Let's start with some intuitive examples:

Examples

a) $f(x) = x^2$. Clearly when x is very close to 2, $f(x)$ is very close to 4. We say $f(x) \to 4$ as $x \to 2$, or $\lim_{x\to 2} f(x) = 4$. (Boring: this is because x^2 is *continuous* at $x = 2$).

b) $f(x) = 1/x$. When x is negative and very close to 0, $f(x)$ is a very large negative number. When x is postive and very close to 0, $f(x)$ is a very large positive number. We say $\lim_{x\to 0-} f(x) = -\infty$ and $\lim_{x\to 0+} f(x) = \infty$. $\lim_{x\to 0} f(x)$ does not exist, since the limit depends on which side you approach 0 from.

c) Now consider the *Heaviside step function*

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases}$$

We have $\lim_{x\to 0-} f(x) = 0$ and $\lim x \to 0+ f(x) = 1$. Thus $\lim_{x\to 0} f(x)$ doesn't exist. However, if we look close to $x = 2$ then clearly $\lim_{x\to 2} f(x) = 1$.

d) Notice that the limit says nothing at all about $f(a)$ itself: if we defined a function by

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 2 \\ 99 & \text{if } x = 2, \end{cases}$$

Then we still have $\lim_{x\to 2} f(x) = 4$, even though $f(2) = 99$.

This motivates the definition of continuity:
The function $f(x)$ is *continuous* at $x = a$ if

a) a is in the maximal domain of $f(x)$.

b) $\lim_{x\to a} f(x) = f(a)$.

$f(x)$ is *continuous* if it is continuous at all values of x.

Examples $f(x) = x^2$ is continuous. The Heaviside step function is continuous everywhere except at $x = 0$. $f(x) = 1/x$ is continuous everywhere except at $x = 0$.

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 2 \\ 99 & \text{if } x = 2 \end{cases}$$

is not continuous at $x = 2$, since $\lim_{x\to 2} f(x)$ isn't equal to $f(2)$.

Some more examples: Rational functions

Example 1: $f(x) = (x^2 + 3)/(x - 2)$ as $x \to 1$. Clearly when x is very close to 1, $f(x)$ is very close to $f(1) = -4$. Hence $\lim_{x\to 1} f(x) = -4$, and $f(x)$ is continuous at $x = 1$.

Example 2: $f(x) = (x^2+3)/(x-2)$ as $x \to 2$. When x is very close to 2, then x^2+3 is very close to 7. However as x gets closer and closer to 2 from above, $x-2$ becomes a smaller and smaller positive number. Hence $\lim_{x\to 2+} f(x) = \infty$. When x tends to 2 from below, $x-2$ becomes a smaller and smaller negative number: hence $\lim_{x\to 2-} f(x) = -\infty$. Since the limits don't agree, $\lim_{x\to 2} f(x)$ doesn't exist.

Example 3: You should always try to simplify $f(x)$ before calculating the limit.

Consider $f(x) = (x^2-1)/(x-1)$ as $x \to 1$. Simplifies to $f(x) = x+1$ *provided that* $x \neq 1$. Hence $\lim_{x\to 1} f(x) = 2$. Note, however, that $f(x)$ is not continuous at $x = 1$, since $f(1) = (1^2-1)/(1-1)$ is not defined, i.e. 1 isn't in the maximal domain of $f(x)$.

A very important example: sinc$x$

Consider the function $f(x) = \sin x/x$ as $x \to 0$. Show graph. Appears that $\lim_{x\to 0} \sin x/x = 1$. Give geometric interpretation.

Notice that $f(x)$ is *not* continuous at $x = 0$, since $f(0) = \sin 0/0$ is not defined. However, we can define a new function

$$\mathrm{sinc}x = \begin{cases} \sin x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

Then sinc$x$ is continuous at $x = 0$.

Another type of example can be solved using a trick. Consider $f(x) = \sin 2x/x$. Then we can write $f(x) = 2\sin 2x/2x$. As $x \to 0$, $2x \to 0$ also, so $\sin 2x/2x \to 1$ as $x \to 0$. Hence $f(x) \to 2$ as $x \to 0$.

Limits as $x \to \pm\infty$

Sometimes these limits exist: $\lim_{x\to\infty} 1/x = 0$, $\lim_{x\to-\infty} 1/x = 0$. Sometimes they don't: $\lim_{x\to\infty} \sin x$ doesn't exist.

The limit of a non-constant polynomial as $x \to \pm\infty$ is always $\pm\infty$:

a) $f(x) = x^2 + 3x + 3 \sim x^2 \to \infty$ as $x \to \infty$ and as $x \to -\infty$.

b) $f(x) = 2x^3 - 3x^2 - 2 =\sim 2x^3 \to \infty$ as $x \to \infty$ and $\to -\infty$ as $x \to -\infty$.

For rational functions, it depends on the degrees of the polynomials on the top and bottom:

a) If the degree of the numerator is greater than the degree of the denominator, then the limit is $\pm\infty$.

$$f(x) = \frac{x^3+1}{3x^2-2x+1}$$

$$\begin{aligned} &\sim \frac{x^3}{3x^2} \\ &= \frac{x}{3}, \end{aligned}$$

so $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$.

b) If the degree of the numerator is less than the degree of the denominator, then the limit is 0:

$$\begin{aligned} f(x) &= \frac{x^3+1}{2x^4 - x^2 + 2} \\ &\sim \frac{x^3}{2x^4} \\ &= \frac{1}{2x}, \end{aligned}$$

so $f(x) \to 0$ as $x \to \infty$ and as $x \to -\infty$.

c) If the degrees are the same, the limit is a non-zero real number:

$$\begin{aligned} f(x) &= \frac{x^3+1}{2x^3 - 3x + 2} \\ &\sim \frac{x^3}{2x^3} \\ &= \frac{1}{2}, \end{aligned}$$

so $f(x) \to 1/2$ as $x \to \infty$ and as $x \to -\infty$.

The sandwich rule

Suppose that $g(x) \leq f(x) \leq h(x)$ for all large x, and that $g(x) \to 0$ and $h(x) \to 0$ as $x \to \infty$. Then $f(x) \to 0$ as $x \to \infty$.

Example Consider $f(x) = \sin x/x$ as $x \to \infty$. Since $\sin x$ always lies between -1 and 1, we have

$$\frac{-1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

for all $x > 0$, and hence $\sin x/x \to 0$ as $x \to \infty$.

Asymptotes

Recall that we talked about horizontal and vertical asymptotes earlier: for example $f(x) = 1/(x-1)$ has a vertical asymptote $x = 1$ and a horizontal asymptote $y = 0$. We can now define these terms:

The line $x = a$ is a *vertical asymptote* of $f(x)$ if $\lim_{x\to a-} f(x) = \pm\infty$ or $\lim_{x\to a+} f(x) = \pm\infty$ or both.

The line $y = b$ is a *horizontal asymptote* of $f(x)$ if $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$ or both.

Chapter 2

Differentiation (8.1–8.3, 9.5)

2.1 Rate of Change (8.2.1–5)

Recall that the equation of a straight line can be written as $y = mx + c$, where m is the *slope* or *gradient* of the line, and c is the *y-intercept* (i.e. the value of y when $x = 0$).

Example $y = 2x + 1$. Draw it. The slope 2 can also be looked on as the *rate of change* of y with respect to x: when x increases by 1, y increases by 2. For example, if x represents time in seconds, and y represents distance travelled in meters, then the rate of change of y with respect to x is the speed of travel.

If the relationship between y and x is more complicated, for example $y = x^2$, then the rate of change of y wrt x is different for different values of x.

Example What is the rate of change of y wrt x when $x = 1$? When $x = 1$, $y = 1$. If x increases by a small amount δ, then y increases to $(1 + \delta)^2 = 1 + 2\delta + \delta^2$, in other words y increases by $2\delta + \delta^2$. Thus

$$\text{Rate of change} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{2\delta + \delta^2}{\delta} = 2 + \delta.$$

To find the instantaneous rate of change at $x = 1$, we let $\delta \to 0$, to obtain 2. Thus the car is travelling at 2 m/s at time 1.

In general, let $y = f(x)$. The rate of change of y with respect to x at $x = x_0$ is given by

$$\left.\frac{dy}{dx}\right|_{x_0} = \lim_{\delta \to 0} \frac{f(x_0 + \delta) - f(x_0)}{\delta}.$$

Example Return to the example $y = f(x) = x^2$, and let x_0 be any value of x.

Then

$$
\begin{aligned}
\left.\frac{dy}{dx}\right|_{x_0} &= \lim_{\delta\to 0}\frac{f(x_0+\delta)-f(x_0)}{\delta}\\
&= \lim_{\delta\to 0}\frac{(x_0+\delta)^2-x_0^2}{\delta}\\
&= \lim_{\delta\to 0}\frac{x_0^2+2x_0\delta+\delta^2-x_0^2}{\delta}\\
&= \lim_{\delta\to 0}(2x_0+\delta)\\
&= 2x_0.
\end{aligned}
$$

Thus at time x_0, the speed of the car is $2x_0$. Equivalently, at time x the speed of the car is $2x$. We also write

$$\frac{dy}{dx}=2x, \qquad y'=2x, \qquad \frac{df}{dx}=2x, \qquad \text{or } f'(x)=2x.$$

The rate of change is called the *derivative of* y *wrt* x, or the *derivative of* $f(x)$ *wrt* x, or just the *derivative of* $f(x)$.

Geometrically $f'(x_0)$ is the slope of the tangent to $y=f(x)$ at $x=x_0$ (picture). Thus the equation of this tangent is $y=f'(x_0)x+c$, where c is the y-intercept. In order to work out c, we use the fact that the tangent passes through the point $(x_0, f(x_0))$. Putting $x=x_0$ and $y=f(x_0)$ in the equation we get $f(x_0)=f'(x_0)x_0+c$, so $c=f(x_0)-f'(x_0)x_0$, and hence the equation of the tangent is

$$y=f'(x_0)x+f(x_0)-f'(x_0)x_0,$$

or

$$y=f(x_0)+f'(x_0)(x-x_0).$$

Example Find the equation of the tangent to the curve $y=x^2$ at $x_0=3$.

When $x_0=3$ we have $f(x_0)=9$, and $f'(x_0)=2x_0=6$. Hence the equation of the tangent is

$$y=9+6(x-3)$$

or

$$y=6x-9.$$

2.2 Derivatives of common functions: rules of differentiation (8.3.1–7)

Recall that if $f(x) = x^2$, then $f'(x) = 2x$. We found this with our bare hands:

$$f'(x) = \lim_{\delta \to 0} \frac{(x+\delta)^2 - x^2}{\delta} = \lim_{\delta \to 0} 2x + \delta = 2x.$$

We can do the same thing for other common functions.

Example Let $f(x) = x^3$. Then

$$\begin{aligned} f'(x) &= \lim_{\delta \to 0} \frac{(x+\delta)^3 - x^3}{\delta} \\ &= \lim_{\delta \to 0} \frac{x^3 + 3x^2\delta + 3x\delta^2 + \delta^3 - x^3}{\delta} \\ &= \lim_{\delta \to 0} (3x^2 + 3x\delta + \delta^2) \\ &= 3x^2. \end{aligned}$$

Thus

$$\frac{d}{dx} x^3 = 3x^2.$$

To find the derivative of x^n for other values of n, we need to be able to work out $(x+\delta)^n$. To do this, we have the *binomial theorem*: to work out $(a+b)^n$, we don't have to work out

$$(a+b)(a+b)(a+b)\dots(a+b),$$

we can use

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \binom{n}{3} a^{n-3}b^3 + \dots + \binom{n}{n-1} ab^{n-1} + b^n,$$

where

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

Rather than work out the coefficients $\binom{n}{r}$ using this formula, we can use *Pascal's triangle*. Draw it. Thus, for example

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

Example Expand $(1+2x)^5$ using the binomial theorem.

$$
\begin{aligned}
(1+2x)^5 &= 1^5 + 5(1)^4(2x) + 10(1)^3(2x)^2 + 10(1)^2(2x)^3 + 5(1)(2x)^4 + (2x)^5 \\
&= 1 + 5(2x) + 10(4x^2) + 10(8x^3) + 5(16x^4) + (32x^5) \\
&= 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5.
\end{aligned}
$$

We can use this to work out the derivative of x^n for any n. Let $f(x) = x^n$. Then

$$
\begin{aligned}
f'(x) &= \lim_{\delta \to 0} \frac{(x+\delta)^n - x^n}{\delta} \\
&= \lim_{\delta \to 0} \frac{x^n + nx^{n-1}\delta + \text{terms in } \delta^2,\ \delta^3 \text{ etc.} - x^n}{\delta} \\
&= \lim_{\delta \to 0} (nx^{n-1} + \text{terms in } \delta,\ \delta^2 \text{ etc.}) \\
&= nx^{n-1}.
\end{aligned}
$$

Thus

$$\frac{d}{dx} x^n = nx^{n-1}.$$

This gives $\frac{d}{dx}x^2 = 2x$ and $\frac{d}{dx}x^3 = 3x^2$ in agreement with our earlier calculations. We can also now calculate, for example

$$\frac{d}{dx} x^{57} = 57x^{56}.$$

Example Calculate the equation of the tangent to the graph $y = x^{28}$ at $x = 1$.

Write $y = f(x) = x^{28}$. We want to use the formula for the tangent at $x = x_0$:

$$y = f(x_0) + f'(x_0)(x - x_0),$$

so since $x_0 = 1$ the equation is

$$y = f(1) + f'(1)(x - 1).$$

Now $f(1) = 1^{28} = 1$, and $f'(x) = 28x^{27}$, so $f'(1) = 28$. Hence the equation of the tangent is

$$y = 1 + 28(x - 1),$$

or

$$y = 28x - 27.$$

Derivative of $\sin x$ and $\cos x$

Let $f(x) = \sin x$. We can calculate $f'(x)$ using what trigonometric identity (16):

$$\begin{aligned} f'(x) &= \lim_{\delta \to 0} \frac{\sin(x+\delta) - \sin x}{\delta} \\ &= \lim_{\delta \to 0} 2 \frac{\cos\left(\frac{2x+\delta}{2}\right) \sin\left(\frac{\delta}{2}\right)}{\delta} \\ &= \lim_{\delta \to 0} \cos(x + \frac{\delta}{2}) \frac{\sin(\delta/2)}{(\delta/2)} \\ &= \cos x. \end{aligned}$$

Thus $\frac{d}{dx} \sin x = \cos x$.

Similarly $\frac{d}{dx} \cos x = -\sin x$ (exercise).

Example Find the equation of the tangent to the graph $y = \sin x$ at $x = 0$.

Write $f(x) = \sin x$ and $x_0 = 0$. We want to use our formula

$$y = f(x_0) + f'(x_0)(x - x_0)$$

for the equation of the tangent. We have $f(x_0) = \sin 0 = 0$ and $f'(x_0) = \cos 0 = 1$, so the equation is

$$y = 0 + 1(x - 0),$$

or $y = x$.

To find derivatives of other functions, we need some *rules of differentiation*

The constant multiplication rule

If k is a constant, then $\frac{d}{dx} kf(x) = kf'(x)$.

Examples

a) $\frac{d}{dx} 3x^2 = 3(2x) = 6x$.

b) $\frac{d}{dx} 5x^4 = 20x^3$.

c) $\frac{d}{dx} 2 \sin x = 2 \cos x$.

The sum rule

If u and v are functions of x, then $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$. Alternatively, $(u+v)' = u' + v'$.

Examples

a) $\frac{d}{dx}(x^3 + 2x + 1) = 3x^2 + 2$. Similarly, we can work out the derivative of any polynomial.

b) $\frac{d}{dx}(x^2 + 2\sin x - \cos x) = 2x + 2\cos x + \sin x$.

The product rule

If u and v are functions of x, then $(uv)' = uv' + u'v$.

Examples

a) Let $f(x) = x^2 \sin x$. We let $u = x^2$ and $v = \sin x$. Thus $u' = 2x$ and $v' = \cos x$. The product rule says that $f'(x) = x^2 \cos x + 2x \sin x$.

b) Let $f(x) = \cos^2 x = \cos x \cos x$. We let $u = v = \cos x$. Then $u' = v' = -\sin x$. The product rule says that $f'(x) = \cos x(-\sin x) + (-\sin x)\cos x = -2\sin x \cos x$. Note $f'(x) = -\sin(2x)$.

c) Let $f(x) = x^2 \sin x \cos x$. We let $u = x^2 \sin x$ and $v = \cos x$. Thus $u' = x^2 \cos x + 2x \sin x$ (part a)), and $v' = -\sin x$. The product rule says that

$$f'(x) = (x^2 \sin x)(-\sin x) + (x^2 \cos x + 2x \sin x)\cos x = x^2(\cos^2 x - \sin^2 x) + 2x \sin x \cos x.$$

(Note $f'(x) = x^2 \cos 2x + x \sin 2x$.)

The quotient rule

If u and v are functions of x, then

$$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}.$$

Examples

a) Let $f(x) = 1/x$. We let $u = 1$ and $v = x$, so $u' = 0$ and $v' = 1$. The quotient rule says that

$$f'(x) = \frac{x(0) - (1)(1)}{x^2} = -1/x^2.$$

b) Let $f(x) = \tan x = \frac{\sin x}{\cos x}$. We let $u = \sin x$ and $v = \cos x$. Thus $u' = \cos x$ and $v' = -\sin x$. Thus

$$f'(x) = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

c) Let $f(x) = 1/x^n$. We let $u = 1$ and $v = x^n$, so $u' = 0$ and $v' = nx^{n-1}$. The quotient rule says that

$$f'(x) = \frac{x^n(0) - (1)nx^{n-1}}{x^{2n}} = \frac{-n}{x^{n+1}}.$$

Written another way,

$$\frac{d}{dx}x^{-n} = -nx^{-n-1},$$

so we can see that

$$\frac{d}{dx}x^n = nx^{n-1}$$

whether n is positive or negative. In fact, we have $\frac{d}{dx}x^a = ax^{a-1}$ for *any* number a. Some examples:

d) Let $f(x) = \sqrt{x} = x^{1/2}$. Then $f'(x) = (1/2)x^{-1/2} = \frac{1}{2\sqrt{x}}$.

e) Let $f(x) = \frac{1}{\sqrt[3]{x}} = x^{-1/3}$. Then $f'(x) = -(1/3)x^{-4/3} = \frac{-1}{3x\sqrt[3]{x}}$.

The chain rule

Let $f(x) = g(h(x))$. Then $f'(x) = g'(h(x))h'(x)$.

Examples

a) Let $f(x) = (4x-1)^3$. Let $g(x) = x^3$ and $h(x) = 4x - 1$, so $f(x) = g(h(x))$. We have $g'(x) = 3x^2$ and $h'(x) = 4$. Thus

$$f'(x) = g'(h(x))h'(x) = 3(4x-1)^2 \cdot 4 = 12(4x-1)^2.$$

b) Let $f(x) = \sin(3x+2)$. Let $g(x) = \sin x$ and $h(x) = 3x + 2$, so $f(x) = g(h(x))$. We have $g'(x) = \cos x$ and $h'(x) = 3$. Thus

$$f'(x) = g'(h(x))h'(x) = \cos(3x+2) \cdot 3 = 3\cos(3x+2).$$

More generally, $\frac{d}{dx}\sin(ax+b) = a\cos(ax+b)$ and $\frac{d}{dx}\cos(ax+b) = -a\sin(ax+b)$.

c) Let $f(x) = (\sin x + \cos 3x)^3$. Let $g(x) = x^3$ and $h(x) = \sin x + \cos 3x$, so $f(x) = g(h(x))$. We have $g'(x) = 3x^2$ and $h'(x) = \cos x - 3\sin 3x$. Thus

$$f'(x) = g'(h(x))h'(x) = 3(\sin x + \cos 3x)^2(\cos x - 3\sin 3x).$$

d) Let $f(x) = \tan((\sin x + \cos 3x)^3)$. Let $g(x) = \tan x$ and $h(x) = (\sin x + \cos 3x^3)$, so $f(x) = g(h(x))$. We have $g'(x) = \sec^2(x)$ and $h'(x) = 3(\sin x + \cos 3x)^2(\cos x - 3\sin 3x)$, so

$$f'(x) = g'(h(x))h'(x) = \sec^2((\sin x + \cos 3x)^3) \cdot 3(\sin x + \cos 3x)^2(\cos x - 3\sin 3x).$$

The Inverse Function Rule

Let $y = f^{-1}(x)$ (so $x = f(y)$). Then

$$\frac{dy}{dx} = \frac{1}{f'(y)}.$$

Examples

a) Let $y = \sqrt{x}$ (so $x = y^2$, and we have $f(y) = y^2$. Then

$$\frac{dy}{dx} = \frac{1}{2y} = \frac{1}{2\sqrt{x}}.$$

Thus

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

This agrees with our earlier way of calculating this: $\frac{d}{dx}x^{1/2} = frac12x^{-1/2}$.

b) Let $y = \sin^{-1}(x)$ (so $x = \sin y$, and we have $f(y) = \sin y$. Then

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}.$$

Thus

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1 - x^2}}.$$

c) Similarly, it can be shown that

$$\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1 - x^2}}.$$

d) Let $y = \tan^{-1}(x)$ (so $x = \tan y$, and we have $f(y) = \tan y$. Then

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}.$$

Thus

$$\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1 + x^2}.$$

2.3 An application: the Newton-Raphson method (9.5.8)

This is a method for getting an approximate solution to the equation $f(x) = 0$ in cases where we can't get an exact solution. Suppose that, by drawing a graph of $f(x)$ we can see that there is a solution α (so $f(\alpha) = 0$). The aim is to get a good approximation to α. From the graph we can make an initial guess x_0 at α. The idea (draw picture) is that the place x_1 where the tangent to the graph at x_0 hits the x-axis is a better approximation than x_0.

The equation of the tangent is

$$y = f(x_0) + f'(x_0)(x - x_0),$$

which intersects the x-axis when $y = 0$, so

$$f(x_0) + f'(x_0)(x - x_0) = 0,$$

or

$$x - x_0 = \frac{-f(x_0)}{f'(x_0)}.$$

Thus the tangent hits the x-axis when

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Thus

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Now we can take x_1 as our new guess for α, and use the same method to get a better guess

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

We can repeat this as many times as we like to get better and better guesses x_3, x_4, and so on. In general

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Example Consider the equation $x = \cos x$. By drawing the graphs of x and $\cos x$, we can see that there is a solution somewhere between $x = 0$ and $x = \pi/2$. Let's take $x_0 = 1$ as our initial guess at the solution.

We need to write the equation in the form $f(x) = 0$, which we do by setting $f(x) = x - \cos x$. Then $f'(x) = 1 + \sin x$. Thus the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

becomes

$$x_{n+1} = x_n - \frac{x_n - \cos x_n}{1 + \sin x_n}.$$

So

$$x_1 = x_0 - \frac{x_0 - \cos x_0}{1 + \sin x_0} = 1 - \frac{1 - \cos(1)}{1 + \sin(1)} = 0.750364.$$

This should be a better approximation than x_0 to the solution.

For the next approximation

$$x_2 = x_1 - \frac{x_1 - \cos x_1}{1 + \sin x_1} = 0.750364 - \frac{0.750364 - \cos(0.750364)}{1 + \sin(0.750364)} = 0.739113.$$

Then

$$x_3 = x_2 - \frac{x_2 - \cos x_2}{1 + \sin x_2} = 0.739085,$$

and

$$x_4 = x_3 - \frac{x_3 - \cos x_3}{1 + \sin x_3} = 0.739085.$$

Thus the solution is $x = 0.739085$ to six decimal places. Note that we got this on the third step, but we had to go as far as the fourth step to know that it was accurate to six decimal places.

Example Show graphically that the equation $x^3 = \tan^{-1}(x)$ has three solutions, and find an approximation to the positive solution which is correct to four decimal places.

From the graph, it is clear that there are three solutions $x = 0$ and $x = \pm\alpha$. We want to find an approximation to α. In order to be sure that the method finds α and not 0, we'll make sure that our initial guess is bigger than α: let's take $x_0 = 2$.

Write the equation as $f(x) = x^3 - \tan^{-1}(x) = 0$. Then $f'(x) = 3x^2 - \frac{1}{1+x^2}$. So

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - \tan^{-1}(x_n)}{3x_n^2 - \frac{1}{1+x_n^2}}.$$

Thus

$$x_1 = 2 - \frac{2^3 - \tan^{-1}(2)}{3 \cdot 2^2 - \frac{1}{1+2^2}} = 2 - \frac{8 - 1.107149}{12 - \frac{1}{5}} = 2 - \frac{6.892851}{11.8} = 1.415860.$$

Then

$$x_2 = x_1 - \frac{x_1^3 - \tan^{-1}(x_1)}{3x_1^2 - \frac{1}{1+x_1^2}} = 1.084510.$$

$$x_3 = x_2 - \frac{x_2^3 - \tan^{-1}(x_2)}{3x_2^2 - \frac{1}{1+x_2^2}} = 0.937997.$$

$$x_4 = x_3 - \frac{x_3^3 - \tan^{-1}(x_3)}{3x_3^2 - \frac{1}{1+x_3^2}} = 0.903896.$$

$$x_5 = x_4 - \frac{x_4^3 - \tan^{-1}(x_4)}{3x_4^2 - \frac{1}{1+x_4^2}} = 0.902031.$$

$$x_6 = x_5 - \frac{x_5^3 - \tan^{-1}(x_5)}{3x_5^2 - \frac{1}{1+x_5^2}} = 0.902025.$$

Thus the solution is $x = 0.902025$, which is correct to at least 4 decimal places. In fact, $(0.902025)^3 - \tan^{-1}(0.902025) = -0.00000093$.

2.4 Differentiability (8.2.4)

The derivative $f'(a)$ gives the slope of the tangent to the graph $y = f(x)$ at $x = a$. If there is no well-defined tangent at $x = a$, or if $f(x)$ isn't continuous at $x = a$, then we say that $f(x)$ is *not differentiable* at $x = a$. Thus $f(x)$ is *differentiable* at $x = a$ if

a) $f(x)$ is continuous at $x = a$, and

b) The graph of $y = f(x)$ has a well-defined (non-vertical) tangent at $x = a$.

We say that $f(x)$ is *differentiable* if it is differentiable at $x = a$ for every value of a.

Examples $1/x$, $|x|$, $|\sin x|$.

2.5 Higher derivatives (8.3.13)

The derivative $f'(x)$ of a function $f(x)$ is also a function, and may be differentiable itself. Differentiating a function $y = f(x)$ twice yields the *second derivative*, which is written $f''(x)$, $f^{(2)}(x)$, $\frac{d^2f}{dx^2}$, y'' or $\frac{d^2y}{dx^2}$. It tells us the rate of change of $f'(x)$ wrt x: i.e. how the slope of the tangent to $y = f(x)$ is changing as x changes.

Similarly, the second derivative $f''(x)$ may be differentiable, yielding the *third derivative* $f'''(x)$, $f^{(3)}(x)$, or $\frac{d^3f}{dx^3}$. In general, we get the *nth derivative* $f^{(n)}(x)$ or $\frac{d^nf}{dx^n}$ by differentiating $f(x)$ n times in succession.

We say that $f(x)$ is *n times differentiable* if it is possible to differentiate it n times in succession, and that it is *infinitely differentiable* or *smooth* if there is no limit to the number of times it can be differentiated.

Examples

a) Let $f(x) = x^3 + 2x^2 + 3x + 1$. Then $f'(x) = 3x^2 + 4x + 3$, $f''(x) = 6x + 4$, $f'''(x) = 6$, and $f^{(n)}(x) = 0$ for all $n \geq 4$. Thus $f(x)$ is smooth.

b) Let $f(x) = \sin x$. Then $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f^{(4)}(x) = \sin x$, and so on for ever. Thus $f(x)$ is smooth.

c) Let $f(x) = \frac{1}{x} = x^{-1}$. Then $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, $f'''(x) = -6x^{-4}$, $f^{(4)}(x) = 24x^{-25}$, and so on. $f(x)$ isn't differentiable at $x = 0$ (since 0 isn't in its maximal domain), but it is smooth everywhere else.

2.6 Maclaurin Series and Taylor Series (9.5.1–2)

Suppose that $f(x)$ is a smooth function, and *suppose* that it can be written as

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \cdots$$

or equivalently as

$$f(x) = \sum_{r=0}^{\infty} a_r x^r.$$

Then we can work out the coefficients a_r by repeatedly differentiating $f(x)$:

$$f(0) = a_0.$$

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots, \qquad \text{so } f'(0) = a_1.$$

$$f''(x) = 2a_2 + 6a_3x + 12a_4x^2 + \cdots, quad \text{ so } f''(0) = 2a_2 \text{ or } a_2 = f''(0)/2.$$

$$f'''(x) = 6a_3 + 24a_4x + \cdots, \qquad \text{so } f'''(0) = 6a_3 \text{ or } a_3 = f'''(0)/6.$$

$$f^{(4)}(x) = 24a_4 + \cdots, \qquad \text{so } f^{(4)}(0) = 24a_4 \text{ or } a_4 = f^{(4)}(0)/24.$$

In general

$$a_n = f^{(n)}(0)/n!,$$

so

$$f(x) = f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

or more concisely

$$f(x) = \sum_{r=0}^{\infty} \frac{f^{(r)}(x)}{r!}x^r.$$

(where we take 0! to be 1).

This is called the *Maclaurin Series expansion* of $f(x)$. Note that we have simply made the assumption that it is possible to write $f(x)$ in this way: we'll see more later about which functions $f(x)$ this is possible for, and for which values of x it makes sense.

Examples

a) Let $f(x) = x^3 + 2x^2 + 2x + 1$. We have $f(0) = 1$, $f'(x) = 3x^2 + 4x + 2$, so $f'(0) = 2$, $f''(x) = 6x + 4$, so $f''(0) = 4$, and $f'''(x) = 6$, so $f'''(0) = 6$. Then $f^{(n)}(x) = 0$ for all $n \geq 4$, so $f^{(n)}(0) = 0$ for all $n \geq 4$. Thus the Maclaurin series expansion is

$$1 + 2x + \frac{4}{2!}x^2 + \frac{6}{3!}x^3 = x^3 + 2x^2 + 2x + 1.$$

Thus for polynomials, we just recover the original polynomial.

b) Let $f(x) = \sin x$. We have $f(0) = 0$, $f'(x) = \cos x$, so $f'(0) = 1$, $f''(x) = -\sin x$, so $f''(0) = 0$, $f'''(x) = -\cos x$, so $f'''(0) = -1$, $f^{(4)}(x) = \sin x$, so $f^{(4)}(0) = 0$, $f^{(5)}(x) = \cos x$, so $f^{(5)}(0) = 1$, and so on for ever. Thus

$$\sin x = \frac{1}{1!}x + \frac{-1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{-1}{7!}x^7 + \cdots,$$

or

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

c) Similarly, the Maclaurin series expansion of $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

Working out the factorials in the series for $\sin x$ we get

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots,$$

and the denominators get small very quickly. If x is also small, then the terms in the Maclaurin series get small very quickly: for example

$$\begin{aligned}\sin(0.1) &= (0.1) - \frac{(0.1)^3}{6} + \frac{(0.1)^5}{120} - \frac{(0.1)^7}{5040} + \cdots \\ &= (0.1) - \frac{1}{6000} + \frac{1}{12000000} - \frac{1}{50400000000} + \cdots\end{aligned}$$

Thus we can get a good approximation to $\sin(0.1)$ by just taking the first few terms.

The first approximation is $\sin(0.1) = 0.1$. The second is $\sin(0.1) = 0.1 - 1/6000 = 0.09983333\ldots$. The third is $\sin(0.1) = 0.1 - 1/6000 + 1/12000000 = 0.09983341666\ldots$, and so on. In fact, $\sin(0.1) = 0.099833416647\ldots$.

Maclaurin's theorem is very good for getting approximations to $f(x)$ when x is very small, but what happens if, for example, we want to get an approximation to $\sin(10)$? The Maclaurin series tells us that

$$\sin(10) = 10 - \frac{10^3}{3!} + \frac{10^5}{5!} - \frac{10^7}{7!} + \frac{10^9}{9!} - \cdots,$$

or

$$\sin(10) = 10 - \frac{1000}{6} + \frac{100000}{120} - \frac{10000000}{5040} + \frac{1000000000}{362880} - \cdots$$

The terms do eventually get small (for example $\frac{10^{35}}{35!} = 0.00012\ldots$), but it takes a long time.

One way to deal with this is to change variable, setting $y = x - a$ for some a, so that when x is close to a, y is close to 0. This change of variable gives the *Taylor series expansion of $f(x)$ about $x = a$:*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots,$$

or

$$f(x) = \sum_{r=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

This is good for approximating $f(x)$ when x is close to a (so that $x - a$ is small).

Examples

a) Let $f(x) = x^3 + x^2 + x + 1$, and let $a = 1$. We have $f(a) = 4$, $f'(x) = 3x^2 + 2x + 1$, so $f'(a) = 6$, $f''(x) = 6x + 2$, so $f''(a) = 8$, and $f'''(x) = 6$, so $f'''(a) = 6$. Then $f^{(n)}(x) = 0$ for all $n \geq 4$, so $f^{(n)}(a) = 0$ for all $n \geq 4$. Thus the Taylor series expansion of $f(x)$ about $x = 1$ is

$$f(x) = 4 + 6(x-1) + \frac{8}{2!}(x-1)^2 + \frac{6}{3!}(x-1)^3 = 4 + 6(x-1) + 4(x-1)^2 + (x-1)^3.$$

Thus for a polynomial, we are simply rewriting it as a polynomial in $x - a$.

b) Find an approximation for $f(x) = 1/x$ near $x = 1$ by using the first three terms in the Taylor series expansion.

We have $f(1) = 1$. $f'(x) = -1/x^2$, so $f'(1) = -1$. $f''(x) = 2/x^3$, so $f''(x) = 2$. Hence

$$\frac{1}{x} = 1 - (x-1) + \frac{2}{2!}(x-1)^2 = 1 - (x-1) + (x-1)^2 = x^2 - 3x + 3.$$

Tricks

$x \sin x$, $\sin^2 x$, $\cos^2 x$.

2.7 L'Hopital's rule (9.5.3)

What is $\lim_{x\to 0} \frac{\sin x}{x}$? If we write $\sin x$ as its Maclaurin series expansion, then

$$\frac{\sin x}{x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots,$$

and it is obvious that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Similarly, consider $\lim_{x\to 0} \frac{1-\cos x}{x}$. We have

$$\frac{1-\cos x}{x} = \frac{1 - (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots}{x} = \frac{\frac{x^2}{2!} - \frac{x^4}{4!} + \cdots}{x} = \frac{x}{2!} - \frac{x^3}{4!} + \cdots,$$

and it is obvious that $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$.

We can do the same to work out any limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ where $f(a) = g(a) = 0$. Expand $f(x)$ and $g(x)$ as Taylor series about $x = a$ to get

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

$$g(x) = g(a) + g'(a)(x-a) + \frac{g''(a)}{2!}(x-a)^2 + \cdots$$

and so

$$\frac{f(x)}{g(x)} = \frac{f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots}{g'(a)(x-a) + \frac{g''(a)}{2!}(x-a)^2 + \cdots} = \frac{f'(a) + \frac{f''(a)}{2!}(x-a) + \cdots}{g'(a) + \frac{g''(a)}{2!}(x-a) + \cdots},$$

from which it is clear that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

This is *L'Hôpital's rule.* **Note** it only works when $f(a) = g(a) = 0$.

If $f'(a) = g'(a) = 0$, then we can extend this to show that the limit is $\frac{f''(a)}{g''(a)}$: if these are both 0, then it is $\frac{f'''(a)}{g'''(a)}$, etc.

Examples

a) What is $\lim_{x \to 2} \frac{x^2-x-2}{x-2}$? We have $f(x) = x^2 - x - 2$ and $g(x) = x - 2$, so $f'(x) = 2x - 1$ and $g'(x) = 1$. Hence $f'(2) = 3$ and $g'(2) = 1$, so the limit is 3.

b)

$$\begin{aligned} \lim_{x \to 0} \frac{\sin x - x}{x^3} &= \lim_{x \to 0} \frac{\cos x - 1}{3x^2} \\ &= \lim_{x \to 0} \frac{-\sin x}{6x} \\ &= \lim_{x \to 0} \frac{-\cos x}{6} = -\frac{1}{6}. \end{aligned}$$

2.8 The exponential function (2.7.1, 8.3.9)

Functions of the form $f(x) = a^x$, where $a > 1$ is a constant, are called *exponential functions.*

Notice that $a^0 = 1$, $a^1 = a$, a^x is large when x is large, and $a^{-x} = \frac{1}{a^x}$ is small, but positive, when x is large. Thus all of the exponential functions are increasing and have range $(0, \infty)$. Draw graphs of 2^x, 3^x, 4^x.

Exponential functions have the following important properties:

$$\begin{aligned} a^{x_1} a^{x_2} &= a^{x_1 + x_2} \\ \frac{a^{x_1}}{a^{x_2}} &= a^{x_1 - x_2} \\ a^{kx} &= (a^k)^x = b^x \qquad \text{where } b = a^k. \end{aligned}$$

By the last property, we only need to understand one exponential function and we understand them all: for example $4^x = 2^{2x}$, $3^x = 2^{kx}$ where k is the number with $2^k = 3$.

We choose a preferred value of a in such a way that a^x *is its own derivative.*

Define the exponential function $f(x) = \exp(x)$ to be the function with $f'(x) = f(x)$ and $f(0) = 1$.

This is enough to tell us its Maclaurin series expansion: since $f^{(n)}(0) = 1$ for all n, we have

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{r=0}^{\infty} \frac{x^r}{r!}.$$

It can be shown (not hard, but quite a lot of work), that $\exp(x) = e^x$, where

$$e = \exp(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = 2.7182818\ldots$$

.

Thus e^x and $\exp(x)$ are just different ways of writing the same function, which has the crucial property that

$$\frac{d}{dx}e^x = e^x.$$

Draw graph. Note maximal domain is $\mathbf{R}$, range is $(0, \infty)$, increasing, neither even nor odd.

2.9 The logarithmic function (2.7.2, 8.3.9)

The inverse function of $f(x) = a^x$ is called the *logarithm to base a*, written $\log_a$. Thus if $y = a^x$ then $x = \log_a y$.

The inverse of *the* exponential function $y = e^x$ is called the *natural logarithm*, written ln (so ln is just another way of saying $\log_e$). Thus if $y = e^x$ then $x = \ln y$.

We can draw the graph of $y = \ln x$ by using the reflection rule. The maximal domain in $(0, \infty)$, the range is $\mathbf{R}$, and $\ln x$ is increasing.

To differentiate $\ln x$, we use the inverse function rule: if $y = f^{-1}(x)$ (so $x = f(y)$), then $\frac{dy}{dx} = \frac{1}{f'(y)}$. In this case, $y = \ln x$ (so $x = f(y) = e^y$), so

$$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{e^y} = \frac{1}{x}.$$

Thus

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

The properties of the exponential function give corresponding properties of the logarithm:

$e^{x_1}e^{x_2} = e^{x_1+x_2}$ translates to $\ln(x_1x_2) = \ln x_1 + \ln x_2$.
$\frac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$ translates to $\ln\frac{x_1}{x_2} = \ln x_1 - \ln x_2$.
$e^{nx} = (e^x)^n$ translates to $\ln x^n = n\ln x$.
Thus for example

$$\begin{aligned} \ln\left(\frac{\sqrt{10x}}{y^2}\right) &= \ln(\sqrt{10x}) - \ln(y^2) \\ &= \frac{1}{2}\ln(10x) - 2\ln y \\ &= \frac{1}{2}(\ln(10) + \ln x) - 2\ln y. \end{aligned}$$

$\ln x$ doesn't have a Maclaurin series expansion, since $x = 0$ isn't in the maximal domain of $\ln x$. However, it is possible to calculate the Maclaurin series of $\ln(1+x)$ (this comes down to the same thing as finding the Taylor series of $\ln x$ about $x = 1$).

Have $f(x) = \ln(1+x)$, so $f(0) = \ln(1) = 0$.
$f'(x) = \frac{1}{1+x}$, so $f'(0) = \frac{1}{1} = 1$.
$f''(x) = \frac{-1}{(1+x)^2}$, so $f''(0) = -1$.
$f'''(x) = \frac{2}{(1+x)^3}$, so $f'''(0) = 2$.
$f''''(x) = \frac{-6}{(1+x)^4}$, so $f''''(0) = -6$.
Thus

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

This expansion clearly doesn't make sense if $x \leq -1$, since ln is only defined for $x > 0$. We shall see later that it also doesn't make sense if $x > 1$: in that case, the numerators of the terms grow faster than the denominators. It does, however, work for $x = 1$, when we get

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

Because the terms get small very slowly, we need to take very many terms to get an accurate approximation to $\ln 2$.

2.10 Hyperbolic functions (2.7.3, 8.3.9)

The *hyperbolic functions* $\sinh x$, $\cosh x$, and $\tanh x$ are defined in terms of e^x: their relationship with the ordinary trig functions will become clear when we do complex numbers.

We have $\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$, and $\tanh x = \frac{\sinh x}{\cosh x}$. We can also define $\text{sech}x$ etc. by analogy with the trig functions.

Notice that $\sinh(-x) = \frac{e{-x}-e^{-(-x)}}{2} = -\sinh x$, so $\sinh x$ is an odd function. Draw graph. Odd, maximal domain and range are $\mathbf{R}$, increasing.

Similarly $\cosh x$ is even, its maximal domain is $\mathbf{R}$, and its range is $[1, \infty)$.

$\tanh x$ is odd, its maximal domain is $\mathbf{R}$, and its range is $(-1, 1)$.

Notice that $\frac{d}{dx}e^{-x} = -e^{-x}$, so it follows that $\frac{d}{dx}\sinh x = \cosh x$ and $\frac{d}{dx}\cosh x = \sinh x$. By the quotient rule, we have

$$\frac{d}{dx}\tanh x = \frac{d}{dx}\frac{\sinh x}{\cosh x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = 1 - \tanh^2 x.$$

In fact, there's another way to write $\cosh^2 x - \sinh^2 x$. Notice that $\cosh^2 x - \sinh^2 x = (\cosh x + \sinh x)(\cosh x - \sinh x)$. Now $\cosh x + \sinh x = e^x$ and $\cosh x - \sinh x = e^{-x}$, so $\cosh^2 x - \sinh^2 x = e^x e^{-x} = e^0 = 1$. Compare this with the standard trig identity $\cos^2 x + \sin^2 x = 1$.

In fact, every standard trig identity has a corresponding version for hyperbolic trig functions, which can be obtained by *Osborn's rule*: change the sign of term which involves a product (or implied product) of two sines.

For example $\sin(A + B) = \sin A \cos B + \cos A \sin B$ becomes $\sinh(A + B) = \sinh A \cosh B + \cosh A \sinh B$. $\cos(A+B) = \cos A \cos B - \sin A \sin B$ becomes $\cosh(A+B) = \cosh A \cosh B + \sinh A \sinh B$. To understand what is meant by an *implied product* the identity $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ becomes $\tanh(A + B) = \frac{\tanh A + \tanh B}{1 + \tanh A \tanh B}$ since $\tan A \tan B = \frac{\sin A \sin B}{\cos A \cos B}$ involves an implied product of two sines.

Finally, let's work out the Maclaurin series expansions of $\sinh x$ and $\cosh x$. We can either do this directly, by differentiating, or we can note that

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots, \\ e^{-x} &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots. \end{aligned}$$

Thus

$$\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

and

$$\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

Thus the series for $\cosh x$ consists of the even terms of that for e^x, and the series for $\sinh x$ consists of the odd terms. Compare this with the series expansions of $\sin x$ and $\cos x$.

2.11 Implicit Differentiation (8.3.11)

Sometimes it is difficult (or impossible) to put the relationship between x and y in the form $y = f(x)$. In such cases, we have to use *implicit differentiation* to find $\frac{dy}{dx}$.

Examples

a) Consider the circle of radius 1 centred on the origin. The equation of this circle is $x^2 + y^2 = 1$. Find the slopes of the tangents to the circle at the points $(x_0, y_0) = (1/2, \sqrt{3}/2)$ and $(x_1, y_1) = (1/2, -\sqrt{3}/2)$.

   We could write the equation as $y = \sqrt{1 - x^2}$, but it is easier to leave it the way it is. We differentiate both sides of the equation wrt x. This gives $2x + \frac{d}{dx}(y^2) = 0$. What is $\frac{d}{dx}(y^2)$? We can write y^2 as $f(y)$, where $f(y) = y^2$, and then by the chain rule

$$\frac{d}{dx}(f(y)) = f'(y)y' = 2y\frac{dy}{dx}.$$

   Thus

$$2x + 2y\frac{dy}{dx} = 0,$$

   or

$$\frac{dy}{dx} = -\frac{x}{y}.$$

   So the slope of the tangent at $(x_0, y_0) = (1/2, \sqrt{3}/2)$ is

$$-\frac{x_0}{y_0} = -\frac{1/2}{\sqrt{3}/2} = \frac{-1}{\sqrt{3}}.$$

   Similarly, the slope of the tangent at $(x_1, y_1) = (1/2, -\sqrt{3}/2)$ is $1/\sqrt{3}$.

   The general rule which we use in implicit differentiation, is

$$\frac{d}{dx}f(y) = f'(y)\frac{dy}{dx}.$$

b) Find the equation of the tangent to the curve

$$x^3 - 4x^2y + y^3 = 1$$

   at the point $(x_0, y_0) = (1, 2)$.

   Differentiating the equation with respect to x we get

$$3x^2 - 4\frac{d}{dx}(x^2y) + 3y^2\frac{dy}{dx} = 0.$$

What is $\frac{d}{dx}(x^2y)$. We have to use the product rule:

$$\frac{d}{dx}(x^2y) = x^2\frac{dy}{dx} + y\frac{d}{dx}(x^2) = x^2\frac{dy}{dx} + 2xy.$$

Thus we have

$$3x^2 - 4x^2\frac{dy}{dx} - 8xy + 3y^2\frac{dy}{dx} = 0.$$

Collecting the terms in $\frac{dy}{dx}$, we get

$$(4x^2 - 3y^2)\frac{dy}{dx} = 3x^2 - 8xy,$$

so

$$\frac{dy}{dx} = \frac{3x^2 - 8xy}{4x^2 - 3y^2}.$$

Thus when $(x, y) = (x_0, y_0) = (1, 2)$, we have

$$\frac{dy}{dx} = \frac{3 - 16}{4 - 12} = \frac{13}{8}.$$

The equation of the tangent at (x_0, y_0) is

$$y = y_0 + \frac{dy}{dx}(x - x_0),$$

or

$$y = 2 + \frac{13}{8}(x - 1).$$

We can simplify this to

$$8y = 16 + 13(x - 1),$$

or

$$8y = 3 + 13x.$$

2.12 Stationary points (9.2.1)

We know that $f'(a)$ gives the slope of the tangent to the graph of $y = f(x)$ at $x = a$. Thus when $f'(a) > 0$, the graph is *increasing* at a: when $f'(x) < 0$ the graph is *decreasing* at a. (Pictures). When $f'(a) = 0$, the tangent to the graph at $x = a$ is horizontal: such a point a is called a *stationary point*.

There are three types of stationary point a: a can be a local maximum, a local minimum, or a point of inflection (draw pictures).

To decide which of the three types a given stationary point a is, consider what happens to $f'(x)$ for x near a. Draw pictures of $f(x)$ and $f'(x)$.

Thus if $f'(a) > 0$ then

a) If $f''(a) > 0$ then a is a local minimum.

b) If $f''(a) < 0$ then a is a local maximum.

c) If $f''(a) = 0$ then a is a point of inflection.

In fact the last of these is a little lie: if $f'''(a) = 0$ also, then we have to look at $f''''(a)$ to decide which case we're in: in general, we have to keep differentiating until we find an $f^{(n)}(a)$ which isn't 0.

Examples

a) Find and classify the stationary points of $f(x) = x^3 - 6x^2 + 9x - 2$.

   We have $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$. To find the stationary points, we have to solve $f'(x) = 0$: thus we have stationary points at $x = 1$ and at $x = 3$.

   To determine which type they are, we differentiate again: $f''(x) = 6x - 12$. Thus $f''(1) = 6 - 12 = -6 < 0$, so $x = 1$ is a local maximum. $f''(3) = 18 - 12 = 6 > 0$, so $x = 3$ is a local minimum.

b) Find and classify the stationary points of $f(x) = x^2e^{-x}$.

   We have $f'(x) = 2xe^{-x} - x^2e^{-x} = e^{-x}(2x - x^2) = e^{-x}x(2-x)$. Thus the stationary points are at $x = 0$ and $x = 2$. (Note e^{-x} is never 0).

   To classify them, we calculate $f''(x) = -e^{-x}(2x - x^2) + e^{-x}(2 - 2x) = e^{-x}(2 - 4x + x^2)$. Thus $f''(0) = 2$, so 0 is a local minimum; $f''(2) = e^{-2}(2 - 8 + 4) < 0$, so 2 is a local maximum.

2.13 Graph Sketching

Method:

i) Maximal domain.

ii) Where crosses y axis.

iii) Where crosses x-axis (if possible).

iv) Stationary points.

v) Behaviour as $x \to \pm\infty$.

vi) Vertical Asymptotes.

vii) If necessary, see where $f'(x) > 0$ and where $f'(x) < 0$.

viii) Sketch the graph.

Examples

a) $x^2 - 3x + 2$.

b) $x^3 - 12x + 3$.

c) $\frac{x-3}{x-1}$.

Chapter 3

Integration (F.11)

3.1 The area under a curve (F.11.26–32)

Suppose that $f(x)$ is a continuous function between $x = a$ and $x = b$ (where $a \leq b$). We write

$$\int_a^b f(x)dx$$

for the area under the graph $y = f(x)$ between a and b: the 'integral of $f(x)$ between a and b'.

Examples

a) Let $f(x) = 2$. Then $\int_1^4 f(x)dx = 6$.

b) Let $f(x) = x$. Then $\int_0^6 f(x)dx = 18$.

By convention, areas underneath the x-axis are taken to be negative: thus, for example $\int_{-\pi}^{\pi} \sin(x)dx = 0$. Also, if $b < a$ then we let

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

Thus, for example,

$$\int_6^0 xdx = -18.$$

The first aim of integration is to calculate such areas for as many functions $f(x)$ as possible.

Given $f(x)$, let $F(x) = \int_0^x f(x)dx$, the area under the graph between 0 and x. Notice that $F(x)$ is itself a function.

Examples

a) Let $f(x) = 2$. Then $F(x) = \int_0^x 2dx = 2x$.

b) Let $f(x) = x$. Then $F(x) = \int_0^x xdx = x^2/2$.

Notice that in each case, $F'(x) = f(x)$. This is always the case: for

$$\begin{aligned} F'(x) &= \lim_{\delta \to 0} \frac{F(x+\delta) - F(x)}{\delta} \\ &= \lim_{\delta \to 0} \frac{\int_0^{x+\delta} f(x)dx - \int_0^x f(x)dx}{\delta} \\ &= \lim_{\delta \to 0} \frac{\int_x^{x+\delta} f(x)dx}{\delta} \qquad \text{(pic)}. \end{aligned}$$

But the closer δ gets to 0, the closer $\int_x^{x+\delta} f(x)dx$ gets to $f(x)\delta$, and hence the closer

$$\frac{\int_x^{x+\delta} f(x)dx}{\delta}$$

gets to $f(x)$. Hence $F'(x) = f(x)$.

Thus $F(x)$ is a function whose derivative is $f(x)$: this is the *fundamental theorem of calculus.*

Now $\int_a^b f(x)dx = F(b) - F(a)$ (picture): thus we have:

$$\int_a^b f(x)dx = F(b) - F(a),$$

where $F(x)$ is a function whose derivative is $f(x)$. $F(b) - F(a)$ is commonly written $[F(x)]_a^b$.

Examples

a) Let $f(x) = x^2$. A function whose derivative is $f(x)$ is $F(x) = x^3/3$. Hence

$$\int_1^3 x^2 dx = \left[\frac{x^3}{3}\right]_1^3 = \frac{3^3}{3} - \frac{1^3}{3} = 9 - \frac{1}{3} = 8\frac{2}{3}.$$

b) Let $f(x) = \cos x$. A function whose derivative is $f(x)$ is $F(x) = \sin x$. Hence

$$\int_0^{\pi/2} \cos x dx = [\sin x]_0^{\pi/2} = \sin(\pi/2) - \sin(0) = 1 - 0 = 1.$$

3.2 Definite and indefinite integrals

Notice that if the derivative of $F(x)$ is $f(x)$, then so is the derivative of $F(x)+C$ for all constants C. Hence we have a choice of functions $F(x)$ with $F'(x)=f(x)$.

When we're working out $\int_a^b f(x)dx$, this choice doesn't matter, since

$$[F(x)+C]_a^b = F(b)+C-(F(a)+C) = F(b)-F(a) = [F(x)]_a^b$$

.

$\int_a^b f(x)dx$ is called a *definite* integral: it has a definite value (the area under the curve from a to b. That is, the definite integral is a *number*.

Sometimes we don't want to specify a and b: this gives an *indefinite* integral, in which we can choose a constant. For example

$$\int x^2 dx = \frac{x^3}{3} + C,$$

where C is a constant. This is the *indefinite integral of* x^2. The indefinite integral is a *function*, with an *arbitrary constant* C.

3.3 Common integrals (F.11.1–15)

We can integrate a lot of functions just by spotting what they're the derivative of. For example, we know $\frac{d}{dx}x^{a+1} = (a+1)x^a$, so

$$\int x^a dx = \frac{x^{a+1}}{a+1} + C.$$

This formula is fine provided $a \neq -1$. To integrate $x^{-1} = 1/x$, remember that $\frac{d}{dx}\ln x = 1/x$, so

$$\int \frac{1}{x} dx = \ln x + C \qquad (x > 0).$$

If $x < 0$, then $\frac{d}{dx}\ln(-x) = \frac{-1}{-x} = \frac{1}{x}$, so

$$\int \frac{1}{x} dx = \ln(-x) + C \qquad (x < 0).$$

Combining these, we get

$$\int \frac{1}{x} dx = \ln(|x|) + C \qquad (x \neq 0).$$

Similarly we have

$$
\begin{aligned}
\int \sin x dx &= -\cos x + C \\
\int \cos x dx &= \sin x + C \\
\int e^x dx &= e^x + C \\
\int (ax+b)^k dx &= \frac{1}{a(k+1)}(ax+b)^{k+1} \qquad (k \neq -1) \\
\int \sin(ax+b) dx &= \frac{-1}{a}\cos(ax+b) + C \\
\int \cos(ax+b) dx &= \frac{1}{a}\sin(ax+b) + C \\
\int e^{ax+b} &= \frac{1}{a}e^{ax+b} + C.
\end{aligned}
$$

See handout of common integrals. Note: using the rules of differentiation, we can differentiate almost any function. The same is not true of integration: e.g. there is no good expression for $\int e^{x^2} dx$.

Examples

a)

$$\int_1^4 2e^x + \frac{1}{x} dx = [2e^x + \ln(|x|)]_1^4 = 2e^4 + \ln(4) - (2e^1 + \ln(1)) = 105.146\ldots$$

b)

$$\int_0^\pi \cos(2x) + \sin x dx = [\frac{\sin(2x)}{2} - \cos x]_0^\pi = \frac{\sin(2\pi)}{2} - \cos(\pi) - (\frac{\sin(0)}{2} - \cos 0) = 0 - (-1) - (0-1) = 2.$$

c)

$$\int_1^2 \sqrt{2x+1} dx = \int_1^2 (2x+1)^{1/2} dx = [\frac{1}{3}(2x+1)^{3/2}]_1^2 = \frac{1}{3}(5^{3/2} - 3^{3/2}) = 1.9947\ldots.$$

Chapter 4

Vectors (6)

Vectors provide a compact language to describe and work with quantities which have both *magnitude* and *direction* (e.g. Force, velocity, electric field).

4.1 Components of a vector (6.31–38)

We'll work in two dimensions for now (I can draw the pictures).

Since a vector is a quantity with magnitude and direction, we can represent it by a line, starting at the origin, pointing in the direction of the vector, and with length the magnitude of the vector.

Draw picture. Vectors are written with lines under them, and printed with bold letters.

We can thus represent vectors by the endpoint of the line: $\mathbf{a} = (1, 2)$ in the example.

In general, the vector $\mathbf{a} = (a_1, a_2)$ has *magnitude* or *size* $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$ and *direction* given by $\tan\theta = a_2/a_1$ (note connection with polar coordinates).

Example The vector $\mathbf{a} = (1, 2)$ has magnitude $|\mathbf{a}| = |(1, 2)| = \sqrt{1^2 + 2^2} = \sqrt{5}$, and its direction is given by $\theta = \tan^{-1}(2/1) = 1.107\ldots$.

If k is a number (often called a *scalar* to distinguish it from a vector), then $k\mathbf{a}$ is the vector with components $k\mathbf{a} = (ka_1, ka_2)$. If $k > 0$ then this has the same direction as $\mathbf{a}$ but k times the magnitude (pics). If $k < 0$, then $k\mathbf{a}$ has the opposite direction to $\mathbf{a}$. For example, if $k = -1$ then we get $-\mathbf{a} = (-a_1, -a_2)$, which has the same magnitude as $\mathbf{a}$ but opposite direction.

Vectors are often used to represent displacements between points: for example, if $A = (1, 1)$ and $B = (2, 3)$ are *points*, then the displacement between A and B is the *vector* $\vec{AB} = (1, 2)$. (Picture). Notice that if $C = (1, 3)$ and $D = (2, 5)$, then $\vec{CD} = (1, 2) = \vec{AB}$. We represent both points and vectors by pairs of numbers (x, y), but a point has *position* but no size or direction: a vector has size and direction, but no position.

In general, the displacement between the points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ is the vector $\vec{AB} = (x_2 - x_1, y_2, y_1)$. The magnitude of this vector $|\vec{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ is the distance between A and B. Notice $\vec{BA} = -\vec{AB}$.

4.2 Addition of vectors (6.32–35)

If $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$, then $\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2)$.

Geometrically, $\mathbf{a} + \mathbf{b}$ is the vector you obtain by *following* $\mathbf{a}$ by $\mathbf{b}$ (picture). Physically, it is the vector which describes the *combination* of $\mathbf{a}$ and $\mathbf{b}$.

Examples

a) If a spaceship is subject to a force $(2, 0)$ due to the thrust of its engines, and to a force $(0, -1)$ due to gravity, then the total force on it is $(2, 0) + (0, -1) = (2, -1)$.

b) If a boat is driven at a velocity $(3, 1)$ by its motor against a current with velocity $(-2, 0)$, then its total velocity is $(3, 1) + (-2, 0) = (1, 1)$.

We can also subtract vectors: $\mathbf{a} - \mathbf{b}$ is just $\mathbf{a} + (-\mathbf{b})$, or $(a_1 - b_1, a_2 - b_2)$.

Example Let $\mathbf{a} = (1, 2)$, $\mathbf{b} = (-3, 0)$, and $\mathbf{c} = (-1, 1)$. Then $\mathbf{a} + \mathbf{b} = (-2, 2)$, $\mathbf{a} - \mathbf{c} = (2, 1)$, and $\mathbf{a} + 2\mathbf{b} - 3\mathbf{c} = (1, 2) + (-6, 0) + (3, -3) = (-2, -1)$.

4.3 Unit vectors and coordinate vectors (6.31)

A *unit vector* is a vector $\mathbf{a}$ with magnitude 1: thus $(1, 0)$ or $(1/\sqrt{2}, 1/\sqrt{2})$ are unit vectors.

Since $k\mathbf{a}$ (when $k > 0$) is a vector with the same direction as $\mathbf{a}$ but with k times the magnitude, the *unit vector in the direction of* $\mathbf{a}$ is

$$\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}.$$

Examples Let $\mathbf{a} = (1, 1)$. Then $|\mathbf{a}| = \sqrt{2}$, so the unit vector in the direction of $\mathbf{a}$ is $\hat{\mathbf{a}} = (1/\sqrt{2}, 1/\sqrt{2})$.

Let $\mathbf{b} = (2, -1)$. Then $|\mathbf{b}| = \sqrt{5}$, so the unit vector in the direction of $\mathbf{b}$ is $\hat{\mathbf{b}} = (2/\sqrt{5}, -1/\sqrt{5})$.

The *coordinate vectors* are unit vectors in the directions of the coordinate axes: they are written $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$. (Traditional to omit the hats).

Notice that $\mathbf{a} = (a_1, a_2)$ is the same as $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$. For example, $(2, -3) = 2(1, 0) - 3(0, 1) = 2\mathbf{i} - 3\mathbf{j}$.

4.4 The scalar product (6.41–46, 53–58)

The *scalar product* or *dot product* of two vectors is a way of combining two vectors $\mathbf{a}$ and $\mathbf{b}$ to obtain a *number* $\mathbf{a} \cdot \mathbf{b}$. Its magic arises from the fact that there are two equivalent ways to define it: one is easy to calculate, the other is geometric.

Easy to calculate If $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$, then $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2$.

Geometric If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$.

For example, $(1, 2) \cdot (-1, 2) = -1 + 4 = 3$.

We can use the geometric definition to work out the angle between two vectors. Since $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$, we have

$$\theta = \cos^{-1} \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \right).$$

Note this gives θ between 0 and π (the principal domain of $\cos \theta$).

Example What is the angle between $\mathbf{a} = (1, 2)$ and $\mathbf{b} = (-1, 3)$?

It is $\cos^{-1}(3/\sqrt{5}\sqrt{5}) = \cos^{-1}(3/5) = 0.927\ldots$.

In particular, the angle θ between them is $\pi/2$ (90 degrees) if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Two non-zero vectors $\mathbf{a}$ *and* $\mathbf{b}$ *are perpendicular if and only if* $\mathbf{a} \cdot \mathbf{b} = 0$.

Important interpretation of scalar product

.

Suppose $\mathbf{a}$ is any vector, and $\hat{\mathbf{n}}$ is a unit vector (i.e. describing a direction). We can divide $\mathbf{a}$ into a component in the direction of $\hat{\mathbf{n}}$ and a component perpendicular to it (draw picture). The size of the component in the direction of $\hat{\mathbf{n}}$ is $|\mathbf{a}| \cos \theta = |\mathbf{a}||\hat{\mathbf{n}}| \cos \theta = \mathbf{a} \cdot \hat{\mathbf{n}}$.

The component of a vector $\mathbf{a}$ *in the direction of a unit vector* $\hat{\mathbf{n}}$ *is* $\mathbf{a} \cdot \hat{\mathbf{n}}$.

Example A railway line runs in the direction of the unit vector $\hat{\mathbf{n}} = (1/\sqrt{2}, 1/\sqrt{2})$. A force $\mathbf{F} = (2, 3)$ is applied to a train: what is the force acting in the direction of the rails?

The force is $\mathbf{F} \cdot \hat{\mathbf{n}} = 2/\sqrt{2} + 3/\sqrt{2} = 5/\sqrt{2}$.

4.5 Vectors in 3 dimensions

The theory is essentially exactly the same. A vector in 3 dimensions has three components: $\mathbf{a} = (a_1, a_2, a_3)$, and its magnitude is given by $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$. There are three coordinate vectors, $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$. The unit vector $\hat{\mathbf{a}}$ in the direction of $\mathbf{a}$ is $\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$.

The dot product of two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ can be calculated as $a_1b_1 + a_2b_2 + a_3b_3$, and is given geometrically as

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta.$$

The component of $\mathbf{a}$ in the direction of a unit vector $\hat{\mathbf{n}}$ is $\mathbf{a} \cdot \hat{\mathbf{n}}$.

Example A typical exam question. Let $\mathbf{a} = (2, 1, 3)$, $\mathbf{b} = (-1, 0, 1)$, and $\mathbf{c} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. Calculate $\mathbf{a} - 2\mathbf{b}$, $\mathbf{a} + \mathbf{b} + \mathbf{c}$, and find the angle θ between $\mathbf{a}$ and $\mathbf{b}$.

Start by writing $\mathbf{c} = (2, 1, -2)$.

Then $\mathbf{a} - 2\mathbf{b} = (2, 1, 3) - 2(-1, 0, 1) = (4, 1, 1)$. $\mathbf{a} + \mathbf{b} + \mathbf{c} = (2, 1, 3) + (-1, 0, 1) + (2, 1, -2) = (3, 2, 2)$.

We have $|\mathbf{a}| = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$ and $|\mathbf{b}| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$. Finally $\mathbf{a} \cdot \mathbf{b} = -2 + 3 = 1$. Hence

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{14}\sqrt{2}}\right) \simeq 1.381.$$

Chapter 5

Complex numbers ($\mathbb{C}$)

5.1 Historical motivation (3.1)

Consider the equation $z^2 - z + 1 = 0$. Using the formula $(-b \pm \sqrt{b^2 - 4ac})/2a$ we get

$$z = \frac{1 \pm \sqrt{-3}}{2}.$$

No solutions? If we write $j = \sqrt{-1}$, then we have

$$z = \frac{1 \pm \sqrt{(3)(-1)}}{2} = \frac{1 \pm \sqrt{3}\sqrt{-1}}{2} = \frac{1 \pm \sqrt{3}j}{2}.$$

There are two solutions, but they are *complex numbers*: they are of the form $x + yj$, where x and y are real numbers, and $j = \sqrt{-1}$. NB j is often written j.

The real numbers are *not algebraically closed*: you can write down a polynomial equation, using only real numbers, which has no real solutions. The complex numbers are *algebraically closed*: any polynomial equation (even involving complex numbers) has a complex solution. This is one of many mathematical justifications for considering complex numbers.

In fact, the justifications are not just mathematical: in a deep sense, complex numbers are the fundamental number system of the universe, and it is our own limitations which cause us to see them as less natural than real numbers (e.g. quantum mechanics). Even in relatively straightforward physical situations, calculations can become much easier and more natural if we use complex numbers rather than real numbers (e.g. complex impedance, section 3.5).

5.2 Basic definitions and properties (3.2)

A complex number z is one of the form

$$z = x + yj,$$

where x and y are real numbers, and $j^2 = -1$. The *set* of all complex numbers is written $\mathbb{C}$.

A complex number z has a *real part* $x = \Re(z)$, and an imaginary part $y = \Im(z)$. For example, if $z = 2 - 3j$, then $\Re(z) = 2$ and $\Im(z) = -3$ (Note: the imaginary part doesn't include the j). If $\Im(z) = 0$ then z is a real number. If $\Re(z) = 0$ then z is called an *imaginary* number — not a very useful term.

Two complex numbers are equal if their real and imaginary parts are equal: i.e. $a + bj = c + dj$ if $a = c$ and $b = d$.

If $z = a + bj$ is a complex number, its *complex conjugate* $\bar{z}$ is the complex number $\bar{z} = a - bj$ obtained by changing the sign of the imaginary part. Note also written z^*. We have $z = \bar{z}$ if and only if z is a real number.

Since a complex number $z = x + jy$ is described by two real numbers x and y, we can represent it by the point (x, y) in the plane. Draw and show some examples. Real axis and imaginary axis. Note that $\bar{z}$ is the reflection of z in the real axis.

Arithmetic with complex numbers

Addition and subtraction work just as with vectors: if $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$ then $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$ and $z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$. Examples: $(2 + j) + (3 - 2j) = 5 - j$, $(2 - 2j) - (3 - j) = -1 - j$.

Notice that if $z = x + jy$, then $z + \bar{z} = (x + jy) + (x - jy) = 2x$. Thus $z + \bar{z} = 2\Re(z)$. Similarly, $z - \bar{z} = (x + jy) - (x - jy) = 2jy$. Thus $z - \bar{z} = 2j\Im(z)$.

For multiplication, we multiply them out the two expressions and remember that $j^2 = -1$. Thus $(2 + j)(3 - 2j) = 6 - 2j^2 + 3j - 4j = 8 - j$, and $(2 - 2j)(3 - j) = (6 + 2j^2 - 6j - 2j = 4 - 8j$. In general $(x_1 + jy_1)(x_2 + jy_2) = (x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)j$.

Note $j^2 = -1$, $j^3 = -j$, $j^4 = 1$, $j^5 = j$, $j^6 = -1$, $j^7 = -j$, $j^8 = 1$, etc.

An important fact is that, for any complex number z, $z\bar{z}$ is always a real number. For if $z = x + jy$, then $\bar{z} = x - jy$, and

$$z\bar{z} = (x + jy)(x - jy) = x^2 - j^2y^2 + jxy - jxy = x^2 + y^2.$$

Thus for example $(2 + j)(\overline{2 + j}) = 2^2 + 1^2 = 5$. Moreover, $z\bar{z} = x^2 + y^2 = r^2$, where r is the distance of z from the origin in the Argand diagram. We write this distance as $|z|$, the *modulus* of z: thus

$$z\bar{z} = |z|^2 \qquad \text{or } |z| = \sqrt{z\bar{z}}.$$

We can use this to divide two complex numbers z_1 and z_2: if $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$, then we simplify

$$\frac{z_1}{z_2} = \frac{x_1 + jy_1}{x_2 + jy_2}$$

by multiplying top and bottom by $\bar{z}_2 = x_2 - jy_2$: this makes the bottom into a real number, which we can just divide by.

Examples

a)

$$\frac{1}{j} = \frac{-j}{j(-j)} = -j \qquad \text{(important)}.$$

b)

$$\frac{1+j}{1-j} = \frac{(1+j)(1+j)}{(1-j)(1+j)} = \frac{2j}{1^2+1^2} = j.$$

(You can check $j(1-j) = j - j^2 = 1 + j$).

c)

$$\frac{2+j}{3-2j} = \frac{(2+j)(3+2j)}{(3-2j)(3+2j)} = \frac{4+7j}{3^2+2^2} = \frac{4}{13} + \frac{7}{13}j.$$

5.3 The polar form of complex numbers (3.2.5,3.2.6)

Just as with points (x, y), complex numbers can be represented in polar coordinates: we can describe a complex number $z = x + jy$ by its distance r from the origin, and its angle θ with the origin.

We've already seen that $r = |z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$, the *modulus* of z. θ is given by $\tan\theta = y/x$, and is called the *argument* of z, written $\arg(z)$.

Remember (from polar coordinates) that $\tan^{-1}(y/x)$ is an angle between $-\pi/2$ and $\pi/2$. Thus we can't just say $\arg(z) = \tan^{-1}(y/x)$: we have to look at where z is in the argand diagram. In fact, it is traditional to give $\arg(z)$ as an angle between 0 and 2π: thus we have

$$\arg(z) = \begin{cases} \tan^{-1}(y/x) & \text{if } x > 0, y \geq 0, \\ \frac{\pi}{2} & \text{if } x = 0, y > 0, \\ \tan^{-1}(y/x) + \pi & \text{if } x < 0, \\ -\frac{\pi}{2} & \text{if } x = 0, y < 0, \\ \tan^{-1}(y/x) + 2\pi & \text{if } x > 0, y < 0. \end{cases}$$

Examples $\arg(1) = 0$, $\arg(1 + j) = \pi/4$, $\arg(j) = \pi/2$, $\arg(-1 + j) = 3\pi/4$, $\arg(-1) = \pi$, $\arg(-1 - j) = 5\pi/4$, $\arg(-j) = 3\pi/2$, $\arg(1 - j) = 7\pi/4$.

If $z = x + jy$ and $|z| = r$, $\arg(z) = \theta$ then (draw picture) $x = r\cos\theta$ and $y = r\sin\theta$. Thus $z = r\cos\theta + jr\sin\theta$, or

$$z = r(\cos\theta + j\sin\theta),$$

In fact, we can write this in a better way. Recall

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

Thus

$$e^{j\theta} = 1 + (j\theta) + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \cdots$$

Now $j^2 = -1$, $j^3 = -j$, $j^4 = 1$, $j^5 = j$ etc., so

$$\begin{aligned} e^{j\theta} &= 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} - \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + j\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= \cos\theta + j\sin\theta. \end{aligned}$$

This is *Euler's formula*: $e^{j\theta} = \cos\theta + j\sin\theta$.
Thus

$$z = re^{j\theta},$$

the *polar form* of z.

Examples Express the following complex numbers in polar form:

a) $z = 1 + j$.

We have $|z| = \sqrt{2}$ and $\arg(z) = \tan^{-1}(1) = \pi/4$, so

$$1 + j = \sqrt{2}e^{\frac{j\pi}{4}}.$$

b) $z = -1$.

We have $|z| = 1$ and $\arg(z) = \pi$, so

$$-1 = e^{j\pi}.$$

c) $z = -2 - 3j$.

We have $|z| = \sqrt{2^2 + 3^2} = \sqrt{13}$, and $\arg(z) = \tan^{-1}(-3/-2) + \pi \simeq 4.646$, so

$$-2 - 3j = \sqrt{13}e^{4.646j}.$$

In polar coordinates, multiplication and division of complex numbers becomes very easy (while addition and subtraction become harder).

If $z_1 = r_1e^{j\theta_1}$ and $z_2 = r_2e^{j\theta_2}$, then

$$z_1 z_2 = r_1 e^{j\theta_1} r_2 e^{j\theta_2} = r_1 r_2 e^{j(\theta_1+\theta_2)}.$$

(multiply the moduli and add the arguments).

$$\frac{z_1}{z_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1-\theta_2)}.$$

(divide the moduli and subtract the arguments).

Remarks on Euler's formula $e^{j\theta} = \cos\theta + j\sin\theta$

Fjrst, it is important to recognize $e^{j\theta}$ as the complex number which is distance 1 from the origin at angle θ (draw picture with circle of radius 1 and a few examples on it).

Since $\cos\theta$ and $\sin\theta$ are unchanged when you add 2π to θ, so is $e^{j\theta}$. Thus

$$1 = e^{0j} = e^{2\pi j} = e^{4\pi j} = e^{6\pi j} = e^{-2\pi j} = e^{-4\pi j} = \cdots$$

$$j = e^{\pi j/2} = e^{5\pi j/2} = e^{9\pi j/2} = e^{-3\pi j/2} = \cdots$$

and so on.

$$e^{-j\theta} = \cos(-\theta) + j\sin(-\theta) = \cos\theta - j\sin\theta = \overline{\cos\theta + j\sin\theta}.$$

Thus the complex conjugate of $e^{j\theta}$ is

$$\overline{e^{j\theta}} = e^{-j\theta}.$$

(Picture).

Adding the equations

$$\begin{aligned} e^{j\theta} &= \cos\theta + j\sin\theta \\ e^{-j\theta} &= \cos\theta - j\sin\theta \end{aligned}$$

we get

$$e^{j\theta} + e^{-j\theta} = 2\cos\theta$$

or

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}.$$

Subtracting them gives

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}.$$

This shows the relationship between the trigonometric functions cos and sin, and the hyperbolic functions cosh and sinh. In fact,

$$\cos\theta = \cosh(j\theta),$$

$$\sin\theta = \frac{1}{j}\sinh(j\theta) = -j\sinh(j\theta).$$

So, from the point of view of complex numbers cos and cosh are essentially the same function, as are sin and sinh. This also explains Osborn's rule: every time we have a product of two sines, we get $(-j)^2 = -1$.

5.4 de Moivre's theorem (3.3.1, 3.3.2)

We have $e^{j\theta} = \cos\theta + j\sin\theta$. Hence

$$(\cos\theta + j\sin\theta)^n = (e^{j\theta})^n = e^{jn\theta} = \cos(n\theta) + j\sin(n\theta).$$

That is

$$(\cos\theta + j\sin\theta)^n = \cos(n\theta) + j\sin(n\theta).$$

This is *de Moivre's theorem*. One of its uses is to obtain trigonometric identites easily. These are of two main types:

First, we can write $\cos(n\theta)$ and $\sin(n\theta)$ in terms of $\cos\theta$ and $\sin\theta$. The method is to write

$$\cos(n\theta) + j\sin(n\theta) = (\cos\theta + j\sin\theta)^n,$$

and expand the right hand side using the binomial theorem.

Examples

a)

$$\begin{aligned}\cos(2\theta) + j\sin(2\theta) &= (\cos\theta + j\sin\theta)^2 \\ &= \cos^2\theta - \sin^2\theta + 2j\sin\theta\cos\theta.\end{aligned}$$

Equating the real parts gives $\cos(2\theta) = \cos^2\theta - \sin^2\theta$, equating imaginary parts gives $\sin(2\theta) = 2\sin\theta\cos\theta$.

b) With more complicated examples, it's usual to abbreviate $c = \cos\theta$ and $s = \sin\theta$.

$$\begin{aligned}\cos(3\theta) + j\sin(3\theta) = (c + js)^3 & \\ &= c^3 + 3c^2(js) + 3c(js)^2 + (js)^3 \\ &= c^3 + 3jc^2s - 3cs^2 - js^3 \\ &= (c^3 - 3cs^2) + j(3c^2s - s^3).\end{aligned}$$

Thus $\cos(3\theta) = \cos^3\theta - 3\cos\theta\sin^2\theta$, and $\sin(3\theta) = 3\cos^2\theta\sin\theta - \sin^3\theta$.

c)

$$\begin{aligned}\cos(5\theta) + j\sin(5\theta) = (c+js)^5 & \\ &= c^5 + 5c^4(js) + 10c^3(js)^2 + 10c^2(js)^3 + 5c(js)^4 + (js)^5 \\ &= (c^5 - 10c^3s^2 + 5cs^4) + j(5c^4s - 10c^2s^3 + s^5).\end{aligned}$$

Thus $\cos(5\theta) = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$, and $\sin(5\theta) = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$.

Second we can write powers of $\cos\theta$ and $\sin\theta$ in terms of $\cos(n\theta)$ and $\sin(n\theta)$. To do this, write $z = \cos\theta + j\sin\theta$, so

$$z^n = \cos n\theta + j\sin n\theta \tag{5.1}$$

$$z^{-n} = \cos n\theta - j\sin n\theta. \tag{5.2}$$

Adding gives

$$2\cos n\theta = z^n + z^{-n},$$

and subtracting gives

$$2j\sin n\theta = z^n - z^{-n}.$$

To get an expression for $\cos^k\theta$, we write $2\cos\theta = z + z^{-1}$, so

$$2^n\cos^n\theta = \left(z + z^{-1}\right)^n.$$

Expand this using the binomial theorem, and use (1) and (2) to simplify. Similarly for $\sin^k\theta$, using $2j\sin\theta = z - z^{-1}$.

Examples

a)

$$\begin{aligned}2^3\cos^3\theta &= (z+z^{-1})^3 \\ &= z^3 + 3z^2z^{-1} + 3zz^{-2} + z^{-3} \\ &= (z^3 + z^{-3}) + 3(z + z^{-1}) \\ &= 2\cos 3\theta + 6\cos\theta.\end{aligned}$$

So

$$4\cos^3\theta = \cos 3\theta + 3\cos\theta.$$

b)

$$\begin{aligned}(2j)^3\sin^3\theta &= (z - z^{-1})^4 \\ &= z^3 - 3z + 3z^{-1} - z^{-3} \\ &= (z^3 - z^{-3}) - 3(z - z^{-1}) \\ &= 2j\sin 3\theta - 6j\sin\theta.\end{aligned}$$

Thus

$$8j^3 \sin^3 \theta = 2j \sin 3\theta - 6j \sin \theta,$$

or

$$4 \sin^3 \theta = -\sin 3\theta + 3 \sin \theta.$$

c)

$$\begin{aligned}
(2j)^4 \sin^4 \theta &= (z - z^{-1})^4 \\
&= z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4} \\
&= 2 \cos 4\theta - 8 \cos 2\theta + 6.
\end{aligned}$$

Thus

$$8 \sin^4 \theta = \cos 4\theta - 4 \cos 2\theta + 3.$$

Chapter 6

Series

Adding up (infinitely many) different things: e.g. Maclaurin series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Sometimes this makes sense (the series *converges*): sometimes it doesn't (the series *diverges*).

6.1 Convergence and Divergence (7.6.1)

Recall the notation

$$\sum_{r=0}^{n} a_r = a_0 + a_1 + a_2 + \cdots + a_n,$$

$$\sum_{r=0}^{\infty} a_r = a_0 + a_1 + a_2 + \cdots$$

Examples

$$\sum_{r=0}^{3} r^2 = 0^2 + 1^2 + 2^2 + 3^2 = 14.$$

$$\sum_{r=1}^{5} \frac{1}{r} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} (= \frac{137}{60}).$$

$$\sum_{r=0}^{\infty} \frac{x^r}{r!} = e^x.$$

$$\sum_{r=1}^{\infty} (-1)^{r+1} \frac{x}{r} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \ln(1+x).$$

(Note often see $(-1)^r$ or $(-1)^{r+1}$ in series. $(-1)^r$ is $+1$ when r is even, -1 when r is odd. $(-1)^{r+1}$ is the other way round. These give us *alternating* series.)

Finite series always make sense, but infinite ones may or may not.

Given an infinite series $\sum_{r=0}^{\infty} a_r$, define its *partial sums* S_n by cutting it off after a_n

$$S_n = \sum_{r=0}^{n} a_r$$

(these all make sense).

Say that the series *converges* if the partial sums get closer and closer to some *finite* value L, i.e. if $S_n \to L$ as $n \to \infty$. We write

$$\sum_{r=0}^{\infty} a_n = L.$$

We say that the series *diverges* otherwise.

Examples

a) Consider

$$\sum_{r=0}^{\infty} \frac{1}{2^r} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

The partial sums are 1, 3/2, 7/4, 15/8, etc., which clearly get closer and closer to 2. Thus the series is *convergent*, and

$$\sum_{r=0}^{\infty} \frac{1}{2^r} = 2.$$

b) Consider

$$\sum_{r=0}^{\infty} (-1)^r = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

The partial sums are 1, 0, 1, 0, 1, 0, etc., which clearly don't approach any particular value. Thus the series is *divergent.*

These examples illustrate an important fact: if the terms a_r don't get closer and closer to zero, then $\sum_{r=0}^{\infty} a_r$ must diverge.

However (equally important), the opposite is not true. Just because the terms get closer and closer to zero, it doesn't mean the series must converge. For example

$$\sum_{r=1}^{\infty} \frac{1}{r} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

diverges.

6.2 Geometric series (7.3.2, 7.6.1)

One of the few examples when we can actually calculate the value of an infinite series.

A geometric series is one in which each term is a multiple of the previous one, i.e.

$$\sum_{r=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots$$

a is called the *first term* and r is the *common ratio.*

The partial sums are given by

$$S_n = a + ar + ar^2 + \cdots + ar^n.$$

We can work these out with a trick:

$$rS_n = ar + ar^2 + \cdots + ar^n + ar^{n+1},$$

so

$$S_n - rS_n = a - ar^{n+1},$$

or

$$S_n(1-r) = a(1-r^{n+1}),$$

or

$$S_n = a\left(\frac{1-r^{n+1}}{1-r}\right).$$

What happens as $n \to \infty$. If $-1 < r < 1$ then $r^{n+1} \to 0$, and $S_n \to \frac{a}{1-r}$. If $r \leq -1$ or $r \geq 1$, then the terms aren't getting smaller, and the series diverges.

A geometric series $\sum_{r=0} ar^n$ *converges to* $\frac{a}{1-r}$ *if* $-1 < r < 1$*, and diverges otherwise.*

6.3 Convergence tests (7.6.2, 7.6.3)

In most examples, it is impossible to work out the partial sums S_n, and we have to make do with deciding whether the series converges or diverges. There are a number of tests which help to do this.

The comparison test

We suppose *all terms are* ≥ 0.

If $\sum_{r=0}^{\infty} a_r$ converges, and $0 \leq b_r \leq a_r$ for all r, then $\sum_{r=0}^{\infty} b_r$ also converges. If $\sum_{r=0}^{\infty} a_r$ diverges, and $0 \leq a_r \leq b_r$ for all r, then $\sum_{r=0}^{\infty} b_r$ also diverges.

Intuitively obvious.

Examples

a) Consider the factorial series

$$\sum_{r=0}^{\infty} \frac{1}{r!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Each term is less than or equal to the corresponding term in

$$1 + \sum_{r=0}^{\infty} \frac{1}{2^r} = 1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8},$$

which is 1 plus a convergent geometric series. Hence the factorial series converges (in fact, to e).

b) Consider the harmonic series

$$\sum_{r=1}^{\infty} \frac{1}{r} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

We group together the terms as follows:

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots$$

We can then see that each bracket is $\geq 1/2$, so the series is bigger than

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

which is divergent. Hence the harmonic series diverges.

c) Consider the series

$$\sum_{r=1}^{\infty} \frac{1}{r^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

We compare this with the series $\sum_{r=1}^{\infty} a_r$, where $a_1 = 1$ and

$$a_r = \int_{r-1}^{r} \frac{1}{x^2}\, dx$$

for $r \geq 2$. This is a convergent series, since the partial sums are given by

$$S_n = 1 + \int_1^2 \frac{1}{x^2}\, dx + \int_2^3 \frac{1}{x^2}\, dx + \cdots + \int_{n-1}^{n} \frac{1}{x^2}\, dx$$

$$
\begin{aligned}
&= 1 + \int_1^n \frac{1}{x^2}\,dx \\
&= 1 + \left[\frac{-1}{x}\right]_1^n \\
&= 1 + (-\frac{1}{n} + 1) \\
&= 2 - \frac{1}{n} \to 2 \text{ as } n \to \infty.
\end{aligned}
$$

Now $a_r \geq \frac{1}{r^2}$ for all r, so $\sum_{r=1}^{\infty} \frac{1}{r^2}$ converges by the comparison test.

The ratio test

Let

$$l = \lim_{r\to\infty} \left| \frac{a_{r+1}}{a_r} \right|.$$

If $l < 1$ then $\sum_{r=0}^{\infty} a_r$ converges.
If $l > 1$ then $\sum_{r=0}^{\infty} a_r$ diverges.
If $l = 1$ then the ratio test tells you nothing.

Idea: if $l < 1$, choose r with $l < r < 1$: then the series is smaller than a geometric series with common ratio r, so must converge.

Examples

a)

$$\sum_{r=0}^{\infty} \frac{r^2}{3^r}.$$

We have $a_r = \frac{r^2}{3^r}$, so

$$\frac{a_{r+1}}{a_r} = \frac{(r+1)^2 3^r}{r^2 3^{r+1}} = \frac{1}{3}\frac{(r+1)^2}{r^2} \to \frac{1}{3}$$

as $r \to \infty$. Hence the series converges.

b)

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

We have $a_n = \frac{1}{n^2}$, so

$$\frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2} \to 1$$

as $n \to \infty$. Hence the ratio test does not tell us whether this series converges or diverges.

The alternating series test

Suppose each $a_r \geq 0$, $a_{r+1} \leq a_r$ for all r, and $a_r \to 0$ as $r \to \infty$. Then

$$\sum_{r=0}^{\infty} (-1)^r a_r$$

converges.

Examples The series

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges. The series

$$\sum_{r=0}^{\infty} (-1)^r e^{-r} = 1 - e^{-1} + e^{-2} - e^{-3} + \cdots$$

converges.

6.4 Power series (7.7)

Power series involve a variable x:

$$\sum_{r=0}^{\infty} a_r x^r.$$

Whether they converge or diverge can depend on the value of x.

Maclaurin series are examples of power series.

Let

$$l = \lim_{r\to\infty} \left| \frac{a_{r+1}x^{r+1}}{a_r x^r} \right| = \lim_{r\to\infty} |x| \left| \frac{a_{r+1}}{a_r} \right|.$$

By the ratio test, the power series converges if $l < 1$ and diverges if $l > 1$. That is, it converges if

$$|x| < \lim_{r\to\infty} \left| \frac{a_r}{a_{r+1}} \right|,$$

and diverges if

$$|x| > \lim_{r\to\infty} \left| \frac{a_r}{a_{r+1}} \right|.$$

Let

$$R = \lim_{r\to\infty} \left| \frac{a_r}{a_{r+1}} \right|,$$

the *radius of convergence* of the power series.

The power series converges if $-R < x < R$, and diverges if $x > R$ or $x < -R$. If $x = R$ or $x = -R$ the series may converge or diverge: we have to consider these cases separately.

Examples

a) Consider the power series

$$\sum_{r=0}^{\infty} \frac{x^r}{r!}.$$

We have $a_r = \frac{1}{r!}$, so $\frac{a_r}{a_{r+1}} = \frac{(r+1)!}{r!} = (r+1)$, so $R = \infty$. Hence the power series converges for all values of x.

b) Consider the power series

$$\sum_{r=1}^{\infty} \frac{x^r}{r}.$$

We have $a_r = \frac{1}{r}$, so $\frac{a_r}{a_{r+1}} = \frac{r+1}{r} \to 1$ as $r \to \infty$. Hence $R = 1$. The power series converges for $-1 < x < 1$, and diverges for $x > 1$ or $x < -1$. We have to check the cases $x = 1$, $x = -1$ separately.

If $x = 1$, the power series is

$$\sum_{r=1}^{\infty} \frac{1}{r},$$

which diverges. If $x = -1$, the power series is

$$\sum_{r=1}^{\infty} \frac{(-1)^r}{r},$$

which converges by the alternating series test.

Hence the power series converges if $-1 \le x < 1$, and diverges otherwise.

c) Consider the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n2^n}.$$

We have $a_n = \frac{(-1)^n}{n2^n}$, so

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{(n+1)2^{n+1}}{n2^n} = 2\frac{n+1}{n} \to 2$$

as $n \to \infty$. Hence $R = 2$, so the power series converges if $-2 < x < 2$, and diverges if $x < -2$ or $x > 2$.

When $x = 2$ we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which converges by the alternating series test. When $x = -2$, we have

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

which diverges.

Hence the power series converges if $-2 < x \leq 2$, and diverges otherwise.