math 191.8.6

There are no solutions in terms of real numbers.

Introduce: $j = \sqrt{-1}$

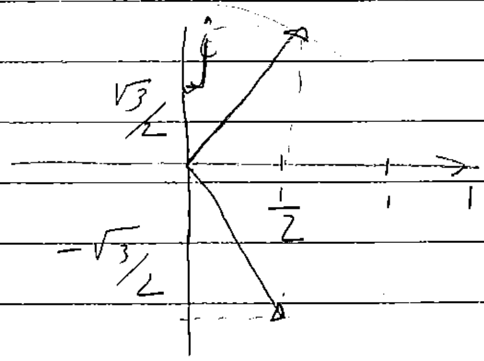
$\Rightarrow$ $j^2 = -1$ $\quad j^3 = -j \quad j^4 = 1$

complex numbers are written in the form: $z = x + jy$

where $x, y \in \mathbb{R}$ and $j^2 = -1$, $j = \sqrt{-1}$

Then $z^2 - z + 1 = 0 \Rightarrow z = \frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

$\rightarrow$ The complex plane



complex number are made of two components along real and imaginary axes.

real numbers $\rightarrow$ not algebraically closed.

e.g. $z^2 - z + 1 = 0$ $\quad a_0, a_1, a_2$ are real

but there are no real solutions.

complex numbers $\Rightarrow$ algebraically closed

i.e. $C_n z^n + C_{n-1} z^{n-1} + \cdots + C_0 = 0$

with $C_n, \ldots, C_0 \rightarrow$ complex constants.

$\Rightarrow$ n - complex solutions $\rightarrow$ always.

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Basic definitions and properties

$z = x + jy \qquad x, y \in \mathbb{R}. \quad$ and $\; j^2 = -1$

$x \rightarrow$ Real part $\qquad y \rightarrow$ imaginary part, $z \in \mathbb{C}$

$z_1 = a + jb \qquad z_2 = c + jd.$

$z_1 = z_2 \iff a = c \;\&\; b = d.$

The complex conjugate:

if $\quad z = x + jy \Rightarrow \bar{z} = x - jy.$

also $\quad z^* = x - jy \quad \leftarrow$ another notation.

Arithmetics with complex numbers

Given: $\quad z_1 = x_1 + jy_1 \qquad z_2 = x_2 + jy_2.$

Then:

Addition: $\quad z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2).$

subtraction: $\quad z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$

multiplication. $\quad z_1 \cdot z_2 = (x_1 + jy_1)(x_2 + jy_2) =$

$$= (x_1 x_2 + j^2 y_1 y_2) + j(x_1 y_2 + y_1 x_2)$$

$$= (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + y_1 x_2)$$

Example: $\quad z_1 = 2 + j \qquad z_2 = 3 - 2j$

$z_1 + z_2 = 5 - j$

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$(2+j)-(3-2j)$

$z_1 - z_2 = -1+3j$

$z_1 \cdot z_2 = (2+j)\cdot(3-2j) = (2\cdot 3-(1)\cdot(-2)) + j(3-4) = 8-j$

Note that if $z = x+jy$ $\quad \bar{z} = x-jy$

then | $z+\bar{z} = 2x \in \mathbb{R}$; $z+\bar{z} = 2\,\mathrm{Re}(z)$

$z-\bar{z} = j2y$; $z-\bar{z} = 2(\mathrm{Im}(z))j$

$$z\bar{z} = (x+jy)(x-jy) = x^2 - j^2y^2 + j(yx-xy)$$
$$= x^2+y^2$$

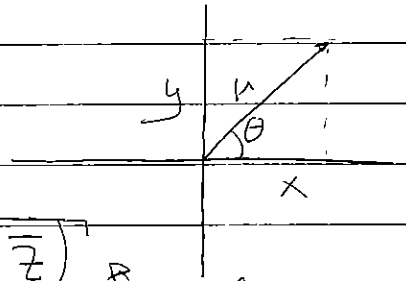
$\Rightarrow z\cdot\bar{z} \in \mathbb{R}$ i.e. it is a real number.

e.g. | $z = 2+j$ $\quad \bar{z} = 2-j$ $\quad z\bar{z} = 2^2+1^2 = 5$

we can write complex numbers using polar coordinates

$z = x+jy$ $\quad x = r\cos\theta$ $\quad y = r\sin\theta$

$\Rightarrow z = r(\cos\theta + j\sin\theta)$



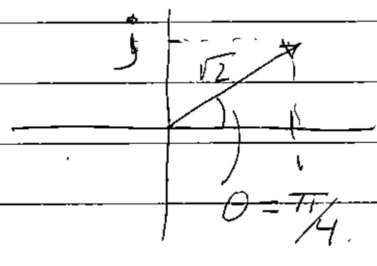
$r \geq 0$ $\quad 0 \leq \theta \leq 2\pi$

$\Rightarrow r = |z| = \mathrm{mod}(z) = \sqrt{x^2+y^2} = \sqrt{(z\bar{z})}$ 'modulus of z'

$\theta = \arg(z) = \tan^{-1}(y/x)$ or 'argument of z'

Example | $z = 1+j \Rightarrow r = \sqrt{1^2+1^2} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$



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we can use the fact that $z\bar{z}$ is Real to do divisions of complex numbers. : eg. if $z = x + jy$

Then $\Rightarrow$ $\frac{1}{z} = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{z\bar{z}} = \frac{x - jy}{x^2+y^2} = \frac{x}{x^2+y^2} - j\frac{y}{x^2+y^2}$

Then if $z_1 = x_1 + jy_1$ $\quad z_2 = x_2 + jy_2$

$\longrightarrow$ $\frac{z_2}{z_1} = z_2 \cdot \frac{1}{z_1} = z_2 \cdot \frac{\bar{z}_1}{z_1\bar{z}_1} = \frac{(x_2+jy_2)(x_1-jy_1)}{x_1^2+y_1^2} =$

$$= \frac{(x_2x_1 + y_2y_1)}{x_1^2+y_1^2} + j\left(\frac{y_2x_1 - y_1x_2}{x_1^2+y_1^2}\right)$$

Examples $z_1 = 2 + j \qquad z_2 = 3 - 2j$

$$\frac{z_1}{z_2} = \frac{2+j}{3-2j} = \frac{(2+j)(3+2j)}{(3-2j)(3+2j)} = \frac{(6-2)+(3+4)j}{13} = \frac{4}{13} + \frac{7}{13}j$$

Properties of conjugation (check!)

$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

$\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

$\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

if $z_2 \neq 0$ $\quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

Property of modulus

$|z_1 z_2| = |z_1|\,|z_2|$

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Note that complex numbers satisfy the usual rules:

i $z_1 + z_2 = z_2 + z_1$

ii $z_1 \cdot z_2 = z_2 \cdot z_1$

iii $z_1(z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$

iv. $z_1 \cdot (z_2 z_3) = (z_1 \cdot z_2) z_3$

Examples let $z_1 = 2-5j \quad z_2 = -3+4j \quad \bar{z}_1 = 2+5j \quad \bar{z}_2 = -3-4j$

$z_1 \cdot z_2 = (2-5j)(-3+4j) = (-6+20) + j(15+8) = -14+23j$

$\bar{z}_1 \cdot \bar{z}_2 = (2+5j)(-3-4j) = (-6+20) + j(-15-8) = -14-23j = \overline{z_1 \cdot z_2}$

$$\left(\frac{z_1}{z_2}\right) = \frac{2-5j}{-3+4j} = \frac{(2-5j)(-3-4j)}{(-3+4j)(-3-4j)} = \frac{1}{25}\cdot(-6-20) + j(15-8) = -\frac{26}{25} + \frac{7}{25}j$$

$$\frac{\bar{z}_1}{\bar{z}_2} = \frac{2+5j}{-3-4j} = \frac{(2+5j)(-3+4j)}{(-3-4j)(-3+4j)} = \frac{1}{25}(-6-20) + j(-15+8) = -\frac{26}{25} - \frac{7}{25}j = \overline{\left(\frac{z_1}{z_2}\right)}$$

$|z_1| = \sqrt{4+25} = \sqrt{29} \qquad |z_2| = \sqrt{9+16} = 5$

$|z_1 \cdot z_2| = \sqrt{14^2 + 23^2} = \sqrt{196+529} = \sqrt{725} = \sqrt{25 \cdot 29} = 5\sqrt{29} = |z_1||z_2|$

EULER'S Formula and de Moivre's theorem.

Recall in polar coordinates: $z = r(\cos\theta + j\sin\theta)$.

We can write this in a better way!

Recall $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$

SO $e^{j\theta} = 1 + (j\theta) + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \cdots$

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NOW: $j^2=-1$, $j^3=-j$, $j^4=j^2j^2=1$, $j^5=j$, $j^6=j^2=-1$, etc...

So. $$e^{j\theta}=1+j\theta-\frac{\theta^2}{2!}-j\frac{\theta^3}{3!}+\frac{\theta^4}{4!}+j\frac{\theta^5}{5!}-\frac{\theta^6}{6!}-j\frac{\theta^7}{7!}+\cdots$$

$$=\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\frac{\theta^6}{6!}+\cdots\right)+j\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}+\cdots\right)$$

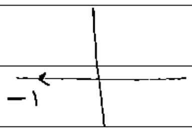
$$=\cos\theta+j\sin\theta.$$

This is Euler's formula: $e^{j\theta}=\cos\theta+j\sin\theta$.

$\Longrightarrow$ $z=re^{j\theta}$ $\longleftarrow$ the Polar form of z.

Examples. Express the following complex numbers in polar form.

1. $z=-1$



$r\cos\theta=-1$

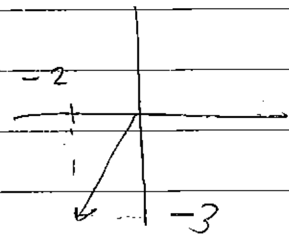
$r\sin\theta=0$

$|z|=1=r$

$$\arg(z)=\tan^{-1}\left(\frac{0}{-1}\right)=\pi$$

$\Rightarrow -1=e^{j\pi}$

2. $z=-2-3j$



$|z|=\sqrt{(-2)^2+(-3)^2}=\sqrt{13}$

$\arg(z)=\tan^{-1}\left(\frac{-3}{-2}\right)+\pi\cong 4.646$

$\Rightarrow -2-3j=\sqrt{13}\,e^{j4.646}$

In polar coordinates multiplication and division math 191. 92 |

of complex numbers takes the form,

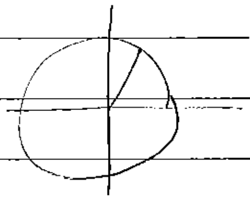
if $z_1 = r_1 e^{j\theta_1}$ $\quad z_2 = r_2 e^{j\theta_2}$

Then $z_1 \cdot z_2 = r_1 e^{j\theta_1} r_2 e^{j\theta_2} = (r_1 r_2) \cdot e^{j(\theta_1 + \theta_2)}$

and $\dfrac{z_1}{z_2} = \dfrac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \left(\dfrac{r_1}{r_2}\right) \cdot e^{j(\theta_1 - \theta_2)}$.

<u>Remarks on Euler's formula</u>

$e^{j\theta}$ is a complex number of magnitude 1.



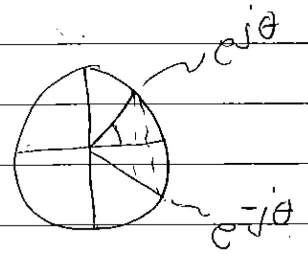
since $\cos(\theta + 2\pi) = \cos(\theta)$ & $\sin(\theta + 2\pi) = \sin(\theta)$.

$\Rightarrow e^{j\theta} = e^{j(\theta + 2\pi n)} = e^{j\theta + j 2\pi n}$

$e^{-j\theta} = \cos(-\theta) + j \sin(-\theta) = \cos\theta - j \sin\theta$

$\Rightarrow$ The complex <u>conjugate</u> of $e^{j\theta}$ is

$\overline{e^{j\theta}} = e^{-j\theta}$



<u>Adding</u>: $e^{j\theta} = \cos\theta + j \sin\theta$

$e^{-j\theta} = \cos\theta - j \sin\theta$

$\Rightarrow e^{j\theta} + e^{-j\theta} = 2\cos\theta$

or $\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$

subtracting $e^{j\theta} - e^{-j\theta}$ gives.

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}.$$

This shows the relation between trigonometric and hyperbolic functions.

$$\cos\theta = \cosh(j\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}.$$

$$\sin\theta = \frac{1}{j}\sinh(j\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}.$$

de Moivre's theorem

we have.

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\Rightarrow (\cos\theta + j\sin\theta)^n = (e^{j\theta})^n = e^{jn\theta} = \cos(n\theta) + j\sin(n\theta).$$

i.e. $(\cos\theta + j\sin\theta)^n = \cos(n\theta) + j\sin(n\theta)$

This is de Moivre's theorem.

→ useful for deriving trigonometric identities.

e.g. express $\cos(n\theta)$ and $\sin(n\theta)$ in terms of $\cos\theta$ and $\sin\theta$ by expanding the RHS of

$$\cos n\theta + j\sin n\theta = (\cos\theta + j\sin\theta)^n$$

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Examples

1. $\cos(2\theta) + j\sin(2\theta) = (\cos\theta + j\sin\theta)^2 =$

$= \cos^2\theta - \sin^2\theta + j(2\cos\theta\sin\theta)$.

equating the real and imaginary part on both sides

$\Rightarrow \cos(2\theta) = \cos^2\theta - \sin^2\theta$.

$\sin(2\theta) = 2\sin\theta\cos\theta$.

2. $\cos 3\theta + j\sin 3\theta = (\cos\theta + j\sin\theta)^3 =$

$= \cos^3\theta + 3\cos^2\theta(j\sin\theta) + 3\cos\theta(j\sin\theta)^2 + (j\sin\theta)^3$

$= \cos^3\theta - 3\cos\theta\sin^2\theta + j(3\cos^2\theta\sin\theta - \sin^3\theta)$

$\Rightarrow \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$.

$\sin(3\theta) = 3\cos^2\theta\sin\theta - \sin^3\theta$.

3. $\cos 5\theta + j\sin 5\theta = (\cos\theta + j\sin\theta)^5 =$

$= \cos^5\theta + 5\cos^4\theta(j\sin\theta) + 10\cos^3\theta(j\sin\theta)^2 + 10\cos^2\theta(j\sin\theta)^3 +$

$+ 5\cos\theta(j\sin\theta)^4 + (j\sin\theta)^5$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$.

$+ j(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$

$\Rightarrow \cos(5\theta) = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

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$\sin(5\theta) = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta.$

we can also write powers of $\cos\theta$ and $\sin\theta$

in terms of $\cos(n\theta)$ and $\sin(n\theta)$.

Write $\quad z = \cos\theta + j\sin\theta = e^{j\theta}$

then $\quad z^n = \cos(n\theta) + j\sin(n\theta) \qquad (*)$

$$z^{-n} = \cos(n\theta) - j\sin(n\theta). \qquad (**)$$

Adding gives

$$2\cos(n\theta) = z^n + z^{-n}$$

And subtracting

$$2j\sin(n\theta) = z^n - z^{-n}$$

To get an expression for $\cos^k\theta$ we write

$$2\cos\theta = z + z^{-1}$$

$$\Rightarrow 2^n\cos^n\theta = (z + z^{-1})^n$$

and expand the RHS using the binomial theorem.

and use $(*)$ and $(**)$ to simplify.

Similarly for $\sin^k\theta$ using $2j\sin\theta = z - z^{-1}$

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Examples

1. $2^3\cos^3\theta = (z+z^{-1})^3 = z^3 + 3z^2z^{-1} + 3zz^{-2} + z^{-3}$

$= (z^3+z^{-3}) + 3(z+z^{-1})$

$= 2\cos 3\theta + 3\cdot 2\cos\theta$

$\Rightarrow 4\cos^3\theta = \cos(3\theta) + 3\cos\theta.$

2. $(2j)^3\sin^3\theta = (z-z^{-1})^3 = z^3 - 3z^2z^{-1} + 3zz^{-2} - z^{-3}$

$= (z^3 - z^{-3}) - 3(z - z^{-1})$

$= 2j\sin 3\theta - 3\cdot 2j\sin\theta$

$\Rightarrow 8j^3\sin^3\theta = 2j\sin 3\theta - 6j\sin\theta$

OR. $4\sin^3\theta = -\sin 3\theta + 3\sin\theta$

3. $(2j)^4\sin^4\theta = (z - z^{-1})^4$

$= z^4 - 4z^3z^{-1} + 6z^2z^{-2} - 4zz^{-3} + z^{-4}$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$= (z^4+z^{-4}) - 4(z+z^{-2}) + 6$

$= 2\cos(4\theta) - 8\cos(2\theta) + 6.$

$\Rightarrow 8\sin^4\theta = \cos(4\theta) - 4\cos(2\theta) + 3$

complex Roots

To solve $w^n = z = re^{j\theta}$

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Write $w = |w| e^{j\alpha}$

$$w^{k} = |w|^{k} e^{jk\alpha} = r e^{j\theta + j2\pi n}.$$

$$\Rightarrow |w| = (r)^{\frac{1}{k}}$$

$$\arg(w) = \alpha = \frac{\theta + 2\pi n}{k} = \frac{\theta}{k} + \frac{2\pi n}{k}.$$

The distinct roots are: $r^{1/k} e^{j\left(\frac{\theta + 2\pi n}{k}\right)}$

with $0 \leq n < k$.

Example 1. $w^3 = 8 + 8j$

$$\Rightarrow |w| = \sqrt{8^2 + 8^2} = 8 \cdot \sqrt{2}$$

$$\theta = \tan^{-1} 8/8 = \tan^{-1} 1 = \pi/4$$

$$\Rightarrow w^3 = (8\sqrt{2}) \cdot e^{j\pi/4} = 8 \cdot 2^{1/2} e^{j\pi/4} = r e^{j\pi/4 + j2n\pi}$$

$$\Rightarrow w = (8 \cdot 2^{1/2})^{1/3} = 2 \cdot 2^{1/6}$$

$$\alpha = (\pi/4 + 2\pi n) \cdot 1/3 = \frac{\pi}{12} + \frac{2\pi}{3} n \qquad n = 0, 1, 2.$$

$$\alpha_0 = \pi/12, \quad \alpha_1 = \frac{\pi}{12} + \frac{2\pi}{3} = \frac{9\pi}{12} = \frac{3\pi}{4} \quad \alpha_2 = \frac{\pi}{12} + \frac{4\pi}{3} = \frac{17\pi}{12}$$

Example 2. $w^2 = -4j = 4e^{j3\pi/2}$

$$\Rightarrow |w| = 4^{1/2} = 2$$

$$\alpha = \left(\frac{3\pi}{2} + 2\pi n\right) \cdot \frac{1}{2} = \frac{3\pi}{4} + \pi n \qquad n = 0, 1$$

$$\alpha_0 = \frac{3\pi}{4} \quad \alpha_1 = \frac{3\pi}{4} + \pi = \frac{7\pi}{4} \quad w = 2e^{j\frac{3\pi}{4}} \text{ or } w = 2e^{j\frac{7\pi}{4}}.$$

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Series.

Adding up infinitely many different things.

e.g. Maclaurin series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

i.e. $$e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \cong 2.718$$

sometimes this makes sense.

$\Rightarrow$ the series converges.

sometimes it doesn't.

$\Rightarrow$ the series diverges.

we want to develop criteria that tells us whether a series converges or diverges

e.g. $$1 + 2 + 2^2 + 2^3 + \cdots + 2^n \xrightarrow[n\to\infty]{} \infty$$

but $$1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} \xrightarrow[n\to\infty]{} \frac{1}{1-\frac{1}{2}} = 2.$$

So the first series diverges whereas the second converges.

Convergence and divergence.

Recall $$\sum_{r=0}^{n} a_r = a_0 + a_1 + \cdots + a_n.$$

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$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots$$

Examples $\sum_{n=0}^{3} n^2 = 0^2 + 1^2 + 2^2 + 3^2 = 14$

$$\sum_{n=1}^{5} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{137}{60}$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$ Maclaurin series of e^x.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \ln(1+x)$$

Maclaurin series of $\ln(1+x)$.

Note: When $(-1)^n$ or $(-1)^{n+1}$ appear in the sum.

The series is an alternating series.

i.e. the signs between the terms alternate.

→ important for convergence.

Finite series $\sum_{n=1}^{n} a_n$ always make sense.

infinite series may or may not make sense.

Given an infinite series

$$\sum_{n=0}^{\infty} a_n$$

define its partial sums by cutting it off after a_n

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$$S_n = \sum_{r=0}^{n} a_r$$

we say: The series converges if the partial sums S_n get closer to some finite value L as $n \to \infty$.

i.e. $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{r=0}^{n} a_r = L$

Otherwise we say that the series diverges.

Examples

1. consider $\sum_{r=0}^{\infty} \frac{1}{2^r} = 1 + 1/2 + 1/4 + 1/8 + \cdots$

$S_0 = 1$; $S_1 = 3/2$; $S_2 = \frac{7}{4}$; $S_3 = \frac{15}{8}$; $S_4 = 31/16$ $S_5 = \frac{63}{32}$ ...

we note: $\lim_{n\to\infty} S_n = \sum_{r=0}^{\infty} \frac{1}{2^r} = 2.$

2. consider. $\sum_{r=0}^{\infty} (-1)^r = 1 - 1 + 1 - 1 + 1 \cdots$

The partial sums are: $1, 0, 1, 0, 1, 0 \cdots$

$\Rightarrow$ the series is divergent.

note: if $\left(\lim_{n\to 0} \sum_{r=0}^{n} a_n\right)$ is convergent

$\Rightarrow \lim_{r\to\infty} a_r = 0$

However, the opposite is not true

i.e. $\lim_{r\to\infty} a_r = 0$ does not imply that $\lim_{n\to\infty} S_n = L$

i.e. it does not guarantee that the series is convergent.

Example $\sum_{r=1}^{\infty} \frac{1}{r} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \longrightarrow$ diverges.

Geometric Series

A geometric series is a series in which each term is a multiple of the privious term.

i.e. $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots$

$a \rightarrow$ first term

$r \rightarrow$ the common ratio.

The sum of a geometric series is calculable.

Take the partial sums.

$$S_n = a + ar + ar^2 + \cdots + ar^n$$

multiply by r: $rS_n = ar + ar^2 + ar^3 + \cdots + ar^{n+1}$

Take $S_n - rS_n = a - ar^{n+1}$

Hence $S_n(1-r) = a(1-r^{n+1})$.

OR $S_n = a\left(\frac{1-r^{n+1}}{1-r}\right)$.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} a\left(\frac{1-r^{n+1}}{1-r}\right)$$

$\Longrightarrow$ if $|r|<1$ then $\lim_{n\to\infty} r^{n+1} = 0$.

Hence $\lim_{n\to\infty} S_n = \frac{a}{1-r}$ for $-1<r<1$

and the series diverges if $|r| \geq 1$.

<u>convergence tests</u>

In most cases it is impossible to calculate the partial sum.

All we can do is apply tests to decide if the series converges or diverges.

<u>The comparison test</u>

we suppose that all terms $a_n \geq 0$.

if $\sum_{n=0}^{\infty} a_n$ diverges and

$0 \leq a_n \leq b_n$ for all n.

then $\sum_{n=0}^{\infty} b_n$ diverges as well.

if $\sum_{n=0}^{\infty} a_n$ converges and

$0 \leq b_n \leq a_n$ for all n.

then $\sum_{n=0}^{\infty} b_n$ converges as well.

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Examples

1. consider the factorial series

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

Each term is smaller or equal to the corresponding

term in: $1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + 1 + \frac{1}{2} + \frac{1}{2\cdot2} + \frac{1}{2\cdot2\cdot2} + \cdots$

The second series is a geometric series with.

$$1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{1-\frac{1}{2}} = 3$$

Hence the first series converges as well.

2. Consider the harmonic series.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

we group the terms in the following way.

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots \frac{1}{16}\right) + \cdots$$

we see that the term in each bracket $\geq \frac{1}{2}$.

So the series is bigger than.

$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$ which diverges.

$\Rightarrow$ the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges as well.

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consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

$$1 + \underbrace{\frac{1}{2^2} + \frac{1}{3^2}}_{< 2\cdot\frac{1}{2^2} = \frac{1}{2}} + \underbrace{\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}}_{< 4\cdot\frac{1}{4^2} = \frac{1}{2^2}} + \underbrace{\frac{1}{8^2} + \cdots + \frac{1}{15^2}}_{< 8\cdot\frac{1}{8^2} = \frac{1}{2^3}}$$

$$\Rightarrow 1 + \qquad + \frac{1}{(2^{n+1}-1)^2} < 1 + \frac{1}{2} + \cdots + \frac{1}{2^n} = 2$$

So $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

An alternative way: compare with the series $\sum_{n=1}^{\infty} a_n$, where $a_1 = 1$ and

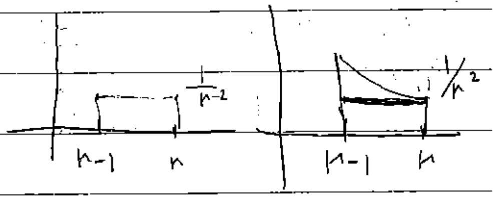
$$a_n = \int_{n-1}^{n} \frac{1}{x^2}\,dx \qquad \text{for } n \geq 2$$

consider: Partial sums

$$S_n = 1 + \int_1^2 \frac{1}{x^2}\,dx + \int_2^3 \frac{1}{x^2}\,dx + \quad + \int_{n-1}^{n} \frac{1}{x^2}\,dx =$$

$$= 1 + \int_1^n \frac{1}{x^2}\,dx = 1 + \left[-\frac{1}{x}\right]_1^n = 1 + \left(-\frac{1}{n} + 1\right) = 2 - \frac{1}{n}$$

So $\lim_{n\to\infty} S_n = 2$.



Now $a_n \geq \frac{1}{n^2}$ for all n.

So $\frac{1}{n^2}$ converges by the comparison test.

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The ratio test.

let $l = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$

if $l < 1$ then $\sum_{n=0}^{\infty} a_n$ converges.

if $l > 1$ then $\sum_{n=0}^{\infty} a_n$ diverges.

if $l = 1$ then the ratio test is not conclusive.

Proof: if $l < 1$ choose r with $l < r < 1$

then the series is smaller than $\sum_{n=0}^{\infty} a r^n$

which is a geometric series and converges.

Examples 1. $\sum_{n=0}^{\infty} \frac{n^2}{3^n} \Rightarrow a_n = \frac{n^2}{3^n}$

$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{(n+1)^2 / 3^{(n+1)}}{n^2 / 3^n} = \frac{1}{3} \frac{(n+1)^2}{n^2}$

$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{3}$

2. $\sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow a_n = \frac{1}{n^2}$

$\frac{a_{n+1}}{a_n} = \left(\frac{n+1}{n^2} \right)^2 \xrightarrow[n \to \infty]{} 1$

$\Rightarrow$ the ratio test is inconclusive.

The alternating series test.

Suppose that $a_n \geq 0$; $a_{n+1} < a_n$; $a_n \xrightarrow[n \to \infty]{} 0$

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Then $\sum_{n=0}^{\infty}(-1)^n a_n$ converges.

Examples The series: $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1-\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$

converges.

similarly the series $\sum_{n=0}^{\infty}(-1)^n e^{-n} = 1-\frac{1}{e}+\frac{1}{e^2}-\frac{1}{e^3}+\cdots$

converges.

Power series

Power series involve a variable x:

$$\sum_{n=0}^{\infty} a_n x^n$$

whether they converge or diverge depend on the value of x

Maclaurin series are examples of power series.

Let $l = \lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim_{n\to\infty}|x|\left|\frac{a_{n+1}}{a_n}\right|$

by the ratio test the power series converges if $l<1$

$$\Rightarrow |x| < \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

and diverges if $l>1 \Rightarrow |x| > \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

is called the Radius of convergence of the power series.

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Examples:

1. Consider the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

we have $a_n = \frac{1}{n!} \Rightarrow \frac{a_n}{a_{n+1}} = \frac{(n+1)!}{n!} = (n+1)$

$\Rightarrow R = \infty \rightarrow$ the series converges for all values of x.

2. consider the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$

we have $a_n = \frac{1}{n} \Rightarrow \frac{a_n}{a_{n+1}} = \frac{n+1}{n} \xrightarrow[n\to\infty]{} 1$

$\Rightarrow R = 1 \rightarrow$ The power series converges for $-1 < x < 1$

and diverges for $x > 1$ OR $x < -1$

For $x = \pm 1$ we have to check seperately

if $x = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} \longrightarrow$ diverges.

$x = -1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \longrightarrow$ converges by the alternating test.

$\Rightarrow$ the power series converges for $-1 \leq x < 1$

and diverges otherwise.

3. Consider the power series: $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n 2^n}$

we have $a_n = \frac{(-1)^n}{n 2^n}$

$\left|\frac{a_n}{a_{n+1}}\right| = \frac{(n+1) 2^{n+1}}{n 2^n} = \frac{n+1}{n} 2 \xrightarrow[n\to\infty]{} 2$

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$\Rightarrow R = 2$

$\Rightarrow$ the power series converges if $-2 < x < 2$

and diverges if $x > 2$ or $x < -2$

For $x = 2 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

which converges by the alternating series test.

For $x = -2 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n \cdot (-1)^n \cdot \cancel{2^n}}{n \cancel{2^n}} = \sum_{n=1}^{\infty} \frac{1}{n}$.

which diverges.

<u>Hence:</u> The power series converges for $-2 < x \leq 2$

and diverges otherwise.