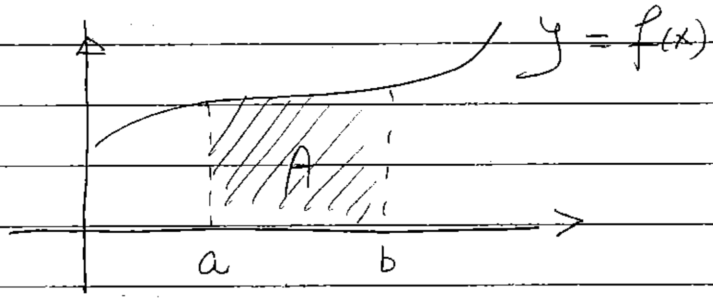
math191.70

Integration.

integration as area under a curve - definite integral



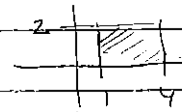
i) $f(x)$ is continuous
ii) $f(x) \geq 0$
for $a \leq x \leq b$.

Then $\int_a^b f(x)\,dx$ = The area under the graph and the x-axis from $a \to b$.

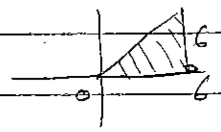
if $f(x) \leq 0$ on $[a,b]$ then

$\int_a^b f(x)\,dx$ = $-$ the area between the graph and the x-axis from $a \to b$.

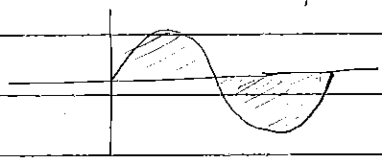
Examples: $f(x) = 2 \Rightarrow \int_1^4 f(x)\,dx = 3 \cdot 2 = 6.$



$f(x) = x = \int_0^6 x\,dx = \frac{6 \cdot 6}{2} = 18$



$f(x) = \sin x \quad \int_0^{2\pi} \sin x\,dx = 0$



if $a \geq b$ then $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$

e.g. $\int_6^0 x\,dx = -18.$

similarly: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ with $a \leq c \leq b$.

Linearity: $\int_a^b (\lambda f(x) + \mu g(x))\,dx = \lambda \int_a^b f(x)\,dx + \mu \int_a^b g(x)\,dx$

where λ, μ are constants.

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Positivity: if $f(x) \geq 0$ on $[a, b] \Rightarrow \int_a^b f(x)dx \geq 0$.

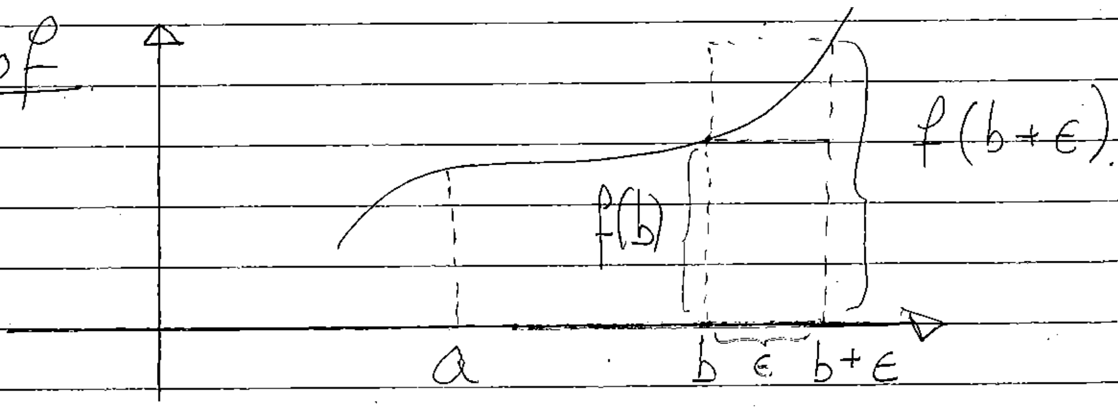
Fundamental theorem of calculus:

show that $\int_a^b f(x)dx = F(b) - F(a)$.

where $\frac{dF(x)}{dx} = f(x)$.

$F(x)$ is an indefinite integral of $f(x)$.

Proof



Let $G(b)$ = area from $x=a$ to $x=b$.

$G(b+\epsilon)$ = area from $x=a$ to $x=b+\epsilon$.

$\epsilon \cdot f(b) \leq G(b+\epsilon) - G(b) \leq \epsilon \cdot f(b+\epsilon)$.

$\Rightarrow f(b) \leq \frac{G(b+\epsilon) - G(b)}{\epsilon} \leq f(b+\epsilon)$.

let $\epsilon \to 0 \Rightarrow f(b) = G'(b)$.

let $F(x)$ be an indefinite integral of $f(x) = F'(x)$.

Then $F(x) = G(x) + C$ and $F'(x) = G'(x)$.

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$G(a) = 0 \implies F(a) = C.$

$\implies G(X) = F(X) - C = F(X) - F(a).$

and $G(b) = F(b) - F(a) = \Big[F(X) \Big|_a^b$ notation.

hence: $F(b) - F(a) = \int_a^b F'(x)\,dx = \int_a^b f(x)\,dx.$

Indefinite integrals

Any function $F(x)$ such that $F'(x) = f(x)$.

is the indefinite integral of $f(x)$ and is

written as $\int^x f(t)\,dt$ or $\int f(x)\,dx$.

Examples

Q. what function has $\frac{d\,f(x)}{dx} = 2.$?

A. $2X$, $2X + C$ for C = constant.

$\implies \int 2\,dx = 2X + C.$

Q. What about $f(x) = X$

A. $\frac{X^2}{2}$ or $\frac{X^2}{2} + C$

$\implies \int x\,dx = \frac{X^2}{2} + C.$

More generally $\int X^n\,dX = \frac{X^{n+1}}{n+1} + C$

for $n \neq -1$.

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For $n=-1$ we have.

$$\int \frac{1}{x}\,dx = \ln x + C \quad (x>0)$$

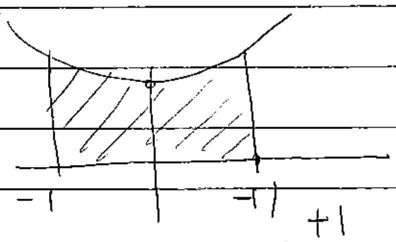
For $x<0$ $\quad \frac{d\ln(-x)}{dx} = \frac{1}{-x}\cdot -1 = \frac{1}{x}$

$$\Rightarrow \int \frac{1}{x}\,dx = \ln(-x) + C \quad (x<0)$$

Hence $\quad \int \frac{1}{x}\,dx = \ln|x| + C. \quad (x \neq 0)$

Example 1. Find the area under $f(x) = x^2+1$ with $-1 \leq x \leq 1$.

A.

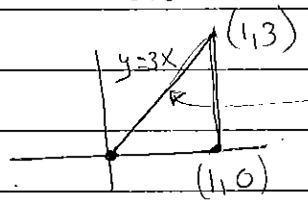


$$\text{Area} = \int_{-1}^{+1} (x^2+1)\,dx = \left[\frac{x^3}{3} + x\right]_{-1}^{1} = \frac{1}{3} + 1 - \left(\frac{-1}{3} - 1\right)$$

$$= 8/3.$$

Example 2. Find the area of the triangle with verticies at $(0,0)$, $(1,0)$, $(1,3)$.

A.



$y = 3x$.

$$\text{Area} = \int_0^1 3x\,dx = \frac{3x^2}{2}\Big|_0^1 = \frac{3}{2} = \frac{\text{base}\cdot\text{height}}{2}.$$

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other integrals → Recognize derivative or bring to a familiar form.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1 \quad n \in \mathbb{R}$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int e^x\,dx = e^x + C$$

$$\int \sin(ax+b) = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \sinh(x)\,dx = \cosh x + C$$

$$\int \cosh(x)\,dx = \sinh x + C$$

$$\int \sec^2 x\,dx = \int \frac{1}{\cos^2 x}\,dx = \tan x + C$$

$$\int dx \frac{1}{(1-x^2)^{1/2}} = \sin^{-1}(x) + C = \arcsin(x) + C$$

Examples 1. $$\int \sin^2 x\,dx = \int \left(\frac{1-\cos 2x}{2}\right)dx = \frac{1}{2}x - \frac{\sin 2x}{2} + C$$

2. $$\int \left(\sqrt{x} + e^{2x} - \cos^2 x\right)dx = \frac{x^{3/2}}{3/2} + \frac{e^{2x}}{2} - \int\left(\frac{1+\cos 2x}{2}\right)dx$$

$$= \frac{2x^{3/2}}{3} + \frac{e^{2x}}{2} - \frac{1}{2}x - \frac{\sin 2x}{4} + C$$

Recall! $$\left.\begin{array}{l}\cos^2 x - \sin^2 x = \cos 2x \\ \cos^2 x + \sin^2 x = 1\end{array}\right\} \Rightarrow \begin{array}{l}\cos^2 x = \frac{1+\cos 2x}{2} \\ \sin^2 x = \frac{1-\cos 2x}{2}\end{array}$$

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3. $\int_1^2 \left(\frac{1}{x} + x^2\right) dx = \left[\ln x + \frac{x^3}{3}\right]_1^2 = \ln 2 + \frac{8}{3} - \left(\ln 1 + \frac{1}{3}\right)$

$= \ln 2 + \frac{7}{3}$

4. $\int_0^{\pi} (\cos(2x) + \sin x)\, dx = +\frac{\sin(2x)}{2} - \cos x \Big|_0^{\pi} =$

$= \frac{\sin(2\pi)}{2} - \cos(\pi) - \left(\frac{\sin(0)}{2} - \cos(0)\right) = 0 + 1 - 0 + 1 = 2.$

5. $\int_1^2 \sqrt{2x+1} = \int_1^2 (2x+1)^{1/2} dx = \frac{1}{3/2} \cdot \frac{1}{2} (2x+1)^{3/2} \Big|_1^2 = \frac{1}{3}\left(5^{3/2} - 3^{3/2}\right)$

Example | integration by parts:

Recall: $\frac{d}{dx}(u \cdot v) = u'v + v'u \Rightarrow u\frac{dv}{dx} = \frac{d(u \cdot v)}{dx} - v\frac{du}{dx}$

$\Rightarrow \int u \cdot \frac{dv}{dx} dx = u(x)v(x) - \int v \frac{du}{dx} dx$.

Example $\int \ln x \cdot 1 dx$

$\ln x = u(x) \quad \frac{dv(x)}{dx} = 1. \Rightarrow v(x) = x. \quad \frac{du}{dx} = \frac{1}{x}.$

Hence $\int \ln x\, dx = x \ln x - \int \frac{1}{x} \cdot x \cdot dx = x \ln x - x + C.$

Example $\int x e^x dx$

$x = u(x) \quad e^x = \frac{dv}{dx} \Rightarrow v(x) = e^x \quad \frac{du}{dx} = 1.$

$\Rightarrow \int x e^x = x e^x - \int e^x dx = x e^x - e^x + C$

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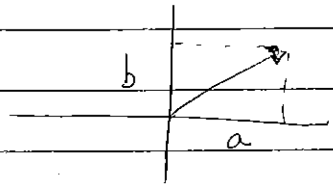
vectors

we will consider vectors in the plane or in space.

Scalar: → magnitude → money in bank account, distance to Manchester.

Vector → magnitude & direction → head east 40 miles to Manchester.

A vector in the plane has two components:

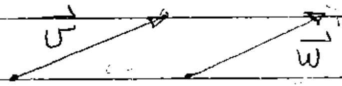


Two vectors are equal if:

1. They have the same direction.
2. They have the same magnitude.

Vectors are denoted by a line with an arrow.

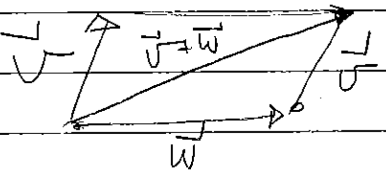
no fixed origin →



$\vec{v} = \vec{w}$ if 1. same direction
2. same magnitude.

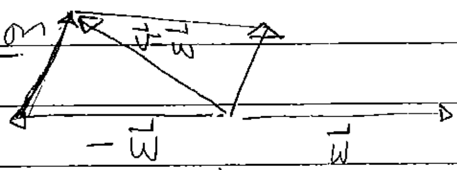
I will denote vectors by an arrow above the letter.

vector addition



$\vec{v} + \vec{w}$

vector subtraction



$\vec{v} + (-\vec{w})$.

we can denote vectors by their coordinates in a cartesian coordinate system $\vec{v} = (a, b)$.

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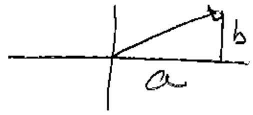
then $|\vec{v}| = \sqrt{a^2+b^2}$



and the direction of $\vec{v}$ is given by $\tan\theta = b/a$.

<u>Manipulations of vectors</u>.

if $\vec{v} = (a, b)$.

$\Rightarrow \lambda \cdot \vec{v} = (\lambda a, \lambda b)$ where λ = constant.

if $\vec{v} = (a_1, b_1)$ $\vec{w} = (a_2, b_2)$

then, $\vec{v} + \vec{w} = (a_1 + a_2, b_1 + b_2)$.

$\vec{v} - \vec{w} = (a_1 - a_2, b_1 - b_2)$

<u>Examples</u> $\vec{v} = (1, -2)$. $\vec{w} = (-3, 1)$ $\vec{u} = (2, 3)$.

1. $\vec{v} + \vec{w} + \vec{u} = (1-3+2, -2+1+3) = (0, 2)$.
2. $\vec{v} - 2\vec{w} + 3\vec{u} = (1+6+6, -2-2+9) = (13, 5)$
3. $|\vec{v}| = \sqrt{1+(-2)} = \sqrt{5}$
4. $|\vec{w}| = \sqrt{(-3)^2+1} = \sqrt{10}$.

Zero vector $= (0,0)$ (or $(0,0,0)$ in 3D).

<u>Unit vectors and coordinate vectors</u>

A unit vector is a vector with magnitude 1.

<u>Example</u> $\vec{v} = (1, 0)$ $\vec{v} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

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we denote unit vector by a hat, $\hat{v}$.

A unit vector in the direction of $\vec{v}$ is

$$\hat{v} = \frac{1}{|\vec{v}|}\vec{v} = \frac{\vec{v}}{|\vec{v}|}$$

Example 1. let $\vec{v} = (1,1)$ then $|\vec{v}| = \sqrt{1+1} = \sqrt{2}$.

and $\hat{v} = \frac{1}{\sqrt{2}}(1,1) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

2. let $\vec{w} = (1,-2)$ then $|\vec{w}| = \sqrt{1^2+(-2)^2} = \sqrt{5}$

and $\hat{w} = \frac{1}{\sqrt{5}}(1,-2) = \left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right)$.

<u>Coordinate vectors:</u> are unit vectors in the direction of the coordinate axes.

They are written as: $\hat{i} = (1,0)$ $\hat{j} = (0,1)$.

Note 1 $\vec{a} = (a_1, a_2) = a_1\hat{i} + a_2\hat{j}$.

Example 1 $(2,-3) = 2(1,0) - 3(0,1) = 2\hat{i} - 3\hat{j}$.

Example 1 $\vec{a} = (-3,1)$ $\vec{b} = (1,-2)$ $\vec{c} = (1,1)$.

$3\vec{a} + 2\vec{b} - \vec{c} = 3(-3\hat{i} + \hat{j}) + 2(\hat{i} - 2\hat{j}) + (\hat{i} + \hat{j})$

$= (3\cdot(-3) + 2 + 1)\,\hat{i} + (3 + 2\cdot(-2) + 1)\,\hat{j}$.

$= -6\hat{i} + 0\hat{j} = -6\hat{i}$.

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The scalar Product

if $\vec{v} = (v_1, v_2)$ $\quad \vec{w} = (w_1, w_2)$.

Then the scalar, OR dot product of $\vec{v}$ and $\vec{w}$.

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2$$

Note: The outcome of a scalar product is a scalar (number) not a vector, i.e. it has magnitude only, no direction

Geometrical Picture of the scalar product.

using Polar coordinates

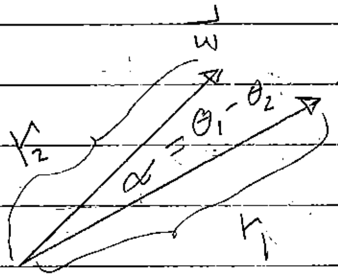
$$\vec{v} = (r_1 \cos\theta_1, r_1 \sin\theta_1) \quad \vec{w} = (r_2 \cos\theta_2, r_2 \sin\theta_2)$$

Then. $\vec{v} \cdot \vec{w} = r_1 \cos\theta_1 r_2 \cos\theta_2 + r_1 \sin\theta_1 r_2 \sin\theta_2 =$

$$= r_1 r_2 (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2)$$

$$= r_1 r_2 \cos(\theta_1 - \theta_2) = |\vec{v}||\vec{w}| \cos(\alpha)$$

where



α is the angle between $\vec{v}$ and $\vec{w}$.

$$\alpha \in [0, \pi]$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|}\right)$$

Example: $\vec{v} = (1, -2)$ $\quad \vec{w} = (3, 1)$. Find α.

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$\vec{v} \cdot \vec{w} = 1 \cdot (-3) + (-2) \cdot 1 = -5$

$|\vec{v}| = \sqrt{1 + (-2)^2} = \sqrt{5}$ $|\vec{w}| = \sqrt{(-3)^2 + 1^2} = \sqrt{10}$.

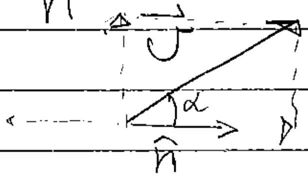
$\alpha = \cos^{-1}\left(\frac{-5}{\sqrt{5} \cdot \sqrt{10}}\right) = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = 3\pi/4$.

Note: if $\alpha = \pi/2$ then $\vec{v} \cdot \vec{w} = 0$.

→ The dot product of two perpendicular vectors is zero.

Note: suppose $\vec{v}$ is a vector and $\hat{n}$ is a unit vector in some direction.

We can decompose the vector $\vec{v}$ into a component which is along $\hat{n}$ and a component orthogonal to $\hat{n}$



The size of the component of $\vec{v}$ along $\hat{n}$ is given by $(\vec{v})_{\parallel} = |\vec{v}| \cos\alpha = \vec{v} \cdot \hat{n}$

Since $\vec{v} = (\vec{v})_{\parallel} + (\vec{v})_{\perp}$ we have

$(\vec{v})_{\perp} = \vec{v} - (\vec{v} \cdot \hat{n})\, \hat{n}$

Example: A railway runs in the direction of $\hat{n} = (1/\sqrt{2}, 1/\sqrt{2})$. A force $\vec{F} = (2,3)$ is applied to the train. What is the component of $\vec{F}$ in the direction of the rails? → $\vec{F} \cdot \hat{n} = \frac{2}{\sqrt{2}} + \frac{3}{\sqrt{2}} = 5/\sqrt{2}$.

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Vectors in three dimensions $\rightarrow$ simple extension.

A vector in three dimension has three components

$\vec{v} = (v_1, v_2, v_3)$ $\quad$ $\vec{w} = (w_1, w_2, w_3)$.

$\vec{v} + \vec{w} = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$.

$\lambda \vec{v} = (\lambda v_1, \lambda v_2, \lambda v_3)$

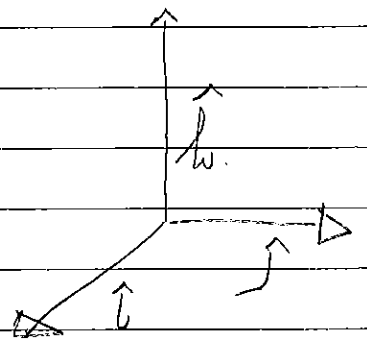
$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 = |\vec{v}| |\vec{w}| \cos\alpha$

where α is the angle between $\vec{v}$ and $\vec{w}$.

The three unit vectors are:

$\hat{i} = (1, 0, 0)$ $\quad$ $\hat{j} = (0, 1, 0)$ $\quad$ $\hat{k} = (0, 0, 1)$



The unit vector in the direction of $\vec{v}$ is

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}}{(v_1^2 + v_2^2 + v_3^2)^{1/2}}$$

Example:

$\vec{u} = (2, 1, 3)$ $\quad$ $\vec{v} = (-1, 0, 1)$ $\quad$ $\vec{w} = 2\hat{i} + \hat{j} - 2\hat{k}$.

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calculate:

$\vec{u}-2\vec{u} = (2,1,3)-2(-1,0,1) = (4,1,1)$

$\vec{u}+\vec{v}+\vec{w} = (2,1,3)+(-1,0,1)+(2,1,-2) = (3,2,2)$

$|\vec{u}| = \sqrt{2^2+1^2+3^2} = \sqrt{14}$

$|\vec{v}| = \sqrt{(-1)^2+0^2+1^2} = \sqrt{2}.$

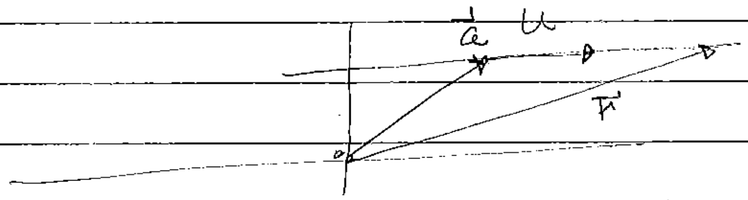
$\vec{u}\circ\vec{v} = (2,1,3)\circ(-1,0,1) = -2+3 = 1$

$\Rightarrow \alpha = \cos^{-1}\left(\frac{\vec{u}\circ\vec{v}}{|\vec{u}||\vec{v}|}\right) = \cos^{-1}\left(\frac{1}{\sqrt{28}}\right) \simeq 1.381$

Application of Vectors

lines: A line has direction

Parametric eq. of a line:



If $\vec{a}$ is a point on the line and $\vec{u}$

is a vector in the direction of the line then,

any point $\vec{r}$ on the line is given by:

$\vec{r} = \vec{a} + \lambda\vec{u}$

where λ is a real constant $\lambda \in \mathbb{R}$.

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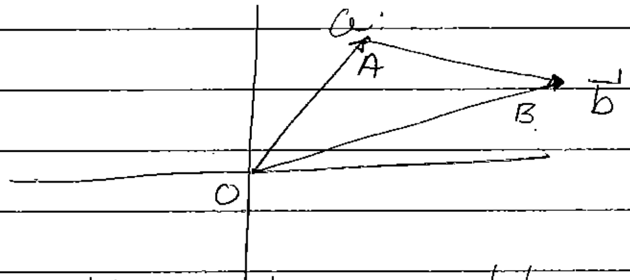
any two points in space are on one line

OR are connected by a vector.



$\vec{a} = \overrightarrow{OA}$ $\vec{b} = \overrightarrow{OB}$ what is $\overrightarrow{AB}$.

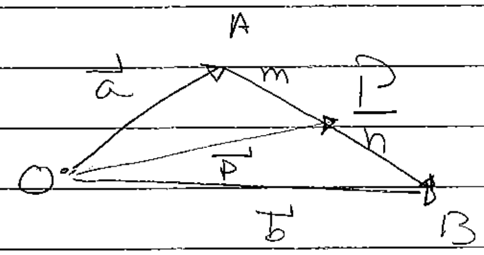
we have $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

hence $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$

$\longrightarrow$ head minus tail rule.

Example The point P divides the segment AB in the Ratio m/n.

Express $\vec{p}$ in terms of the vectors $\vec{a}$, $\vec{b}$ and the scalars m, n.



Answer

$\vec{b} = \overrightarrow{OA} + \overrightarrow{AB} = \vec{a} + \overrightarrow{AB}$

$\Rightarrow \overrightarrow{AB} = \vec{b} - \vec{a}$; $\overrightarrow{AP} = \dfrac{m}{m+n} \cdot \overrightarrow{AB}$

since $\overrightarrow{AB}$ is divided by $\vec{p}$ in the Ratio m/n.

$\Rightarrow \vec{P} = \vec{a} + \overrightarrow{AP} = a + m/(m+n)\ (b - \vec{a}) =$

$= \dfrac{(m+n)\vec{a} + m\vec{b} - m\vec{a}}{(m+n)} = \dfrac{n\vec{a} + m\vec{b}}{(m+n)}$

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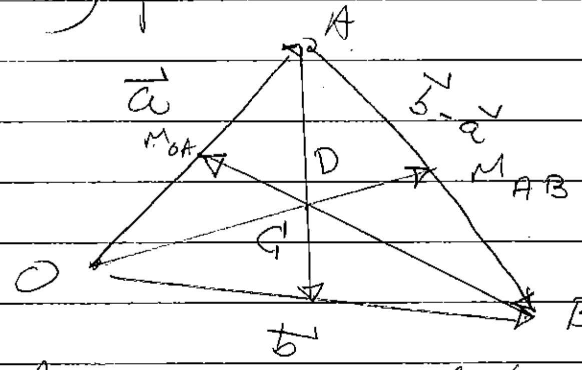
EX.2 Prove that the medians of every triangle intersect at a point which is 2/3 of the way from each vertex to the opposite side,



Proof

The sides of the triangle are specified by $\vec{a}$, $\vec{b}$ and $\vec{b}-\vec{a}$.

The vector median to the side $\vec{b}-\vec{a}$ divides the line segment, $\vec{b}-\vec{a}$ in the ratio $\frac{m}{n}=\frac{1}{1}$

$\Rightarrow$ The median vector to the side $\vec{b}-\vec{a}$ is given by

$$\vec{m}_{AB} = \frac{\vec{a}+\vec{b}}{2}$$

The point G two third of the way from O to the side $\vec{b}-\vec{a}$ along this median is given by

(✱) $$\vec{OG} = \frac{2}{3}\cdot\left(\frac{\vec{a}+\vec{b}}{2}\right) = \frac{\vec{a}+\vec{b}}{3} \qquad (*)$$

The vector median to the $\vec{a}$ side of the triangle:

$$\frac{\vec{a}}{2} = \vec{b} + \vec{m}_{\overline{OA}} \Rightarrow \vec{m}_{\overline{OA}} = \frac{\vec{a}}{2} - \vec{b}$$

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if D is a point two thirds of the way from the vertex B to the side $\vec{a}$ along this median

Then: $\overrightarrow{OD} = \vec{b} + \frac{2}{3}\left(\frac{\vec{a}}{2} - \vec{b}\right) = \frac{\vec{a} + \vec{b}}{3}$

comparison with (*) shows that the two points G and D coincide.

That the third median also intersects at the same point follows by exchanging the roles of $\vec{a}$ and $\vec{b}$.

<u>complex numbers</u>

So far we have → real numbers:

N	Natural numbers	$n = 1, 2, 3 \ldots$	
Z	Integers	$n \in Z$	$= -\infty, \ldots, -1, 0, 1, \ldots, \infty$
Q	Rational numbers	p/q	$p, q \in Z$
r.	Irrational numbers	e.g. $\sqrt{2} \neq p/q$	

<u>consider</u> $z^2 - z + 1 = 0$.

There are no solution in terms of real numbers.

$$z = \frac{1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 1}}{2} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} \cdot \sqrt{-1}$$