math191.1

1. Functions & Graphs.

Numbers: $N = \{0, 1, 2, 3 \dots\}$ → natural numbers

$Z = \{\dots -3, -2, -1, 0, 1, 2, 3 \dots\}$ → integers $N \subset Z$

$Q = \{p/q : p, q \in Z, q \neq 0\}$ → Rational numbers

since $n = n/1 \Rightarrow Z \subseteq Q$

→ every integer is a Rational number.

Finally: $R = \{\text{Real numbers}\}$

→ All numbers which can be written with a decimal expansion. 10.123...

Not every real number is Rational

ie. not every real number can be written as $x = p/q$ where x is real and $p, q \in Z$.

For example $x = \sqrt{2}$. or $x = \pi$.

more generally real numbers → Real solutions of

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = 0$$

with $a_n, a_{n-1} \dots a_0$ = constants.

e.g. $x^2 - 2 = 0$ but not $x^2 + 2 = 0$

complex → later

Note: $N \subset Z \subset Q \subset R$. $(N Z Q R C)$

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Interval notation

→ convenient way of denoting sets of real numbers.

if $a, b \in \mathbb{R}$ with $a \leq b$

then $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$.

that is $[a,b]$ is the set of real numbers with $a \leq x \leq b$.

similarly $(a, b) = \{x \in \mathbb{R} : a < x < b\}$.

$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$.

$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$.

if there is no upper or lower limit.

then $[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$

$(-\infty, b) = \{x \in \mathbb{R} : x < b\}$

→ ∞ is not a number.

Examples:

$1 \notin (0, 1)$ but $1 \in (0, 1]$

$1 \notin (-\infty, 1)$ but $1 \in (-\infty, \infty)$.

Functions, Domain and Range

we usually specify a function by a formula.

$y = f(x)$ take $x \xrightarrow{f} y$.

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Given X we use the rule provided by f to calculate

Examples
i) $y = 2X+1$
ii) $y = (X+2)/(X+1)$
iii) $y = +\sqrt{(X+2)}$

X - is the input y - output

A function has a domain and a Range.

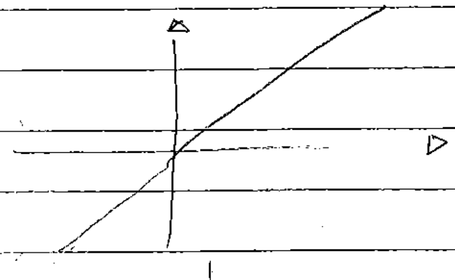
Domain: Allowed inputs. Allowed set of real numbers such that the output is an allowed real number.

Range: of $f(x)$ is the set of possible output values y.

The zeros of $f(x)$ are all possible input values X

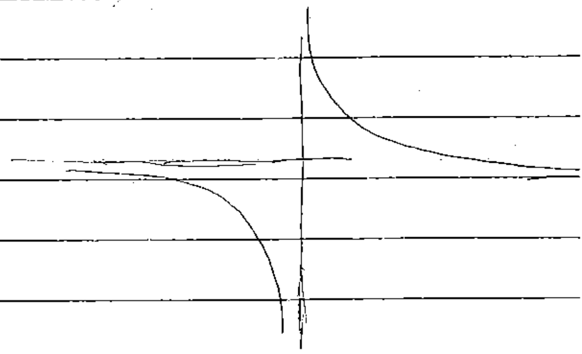
such that $f(x) = 0 \longrightarrow$ Roots

Examples $y = f(x) = X$



i
Domain $= (-\infty, \infty)$
Range $= (-\infty, \infty)$

ii $y = f(X) = 1/X$



Domain $= (-\infty, 0) \cup (0, \infty)$
Range $= (-\infty, 0) \cup (0, \infty)$

iii $y = f(X) = +\sqrt{(X+2)}$

Domain $= \{X \in R \mid X \geq -2\}$
Range $= R^{+} = \{y \in R \mid y \geq 0\}$

Exclude points where x and y are not Real numbers.

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iv] $y = f(x) = x^2$

Domain $= \{ x \in \mathbb{R} \mid -\infty < x < \infty \}$
Range $= \{ y \in \mathbb{R}^+ \mid y \in [0, \infty) \}$

Polynomials

Examples: x, $x+1$, x^2, x^2-3x+2, x^3+4x^2+3x-3

In general: $y = f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$ are real constants.

The degree of the polynomial is the largest power of x

1: degree 0 : constants $f(x) = a_0$

2: degree 1 : linear functions $f(x) = a_1 x + a_0$

3: degree 2 : quadratics : $f(x) = a_2 x^2 + a_1 x + a_0$

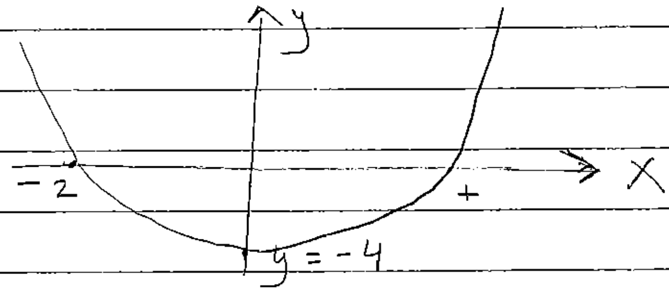
4: degree 3 : cubics : $f(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$

Domain: The domain of a polynomial is always $\mathbb{R}$.

Range: The range can be harder to compute.

Zeros: The points where the polynomial $f(x)$ has real solutions

Examples 1. $f(x) = x^2 - 4$



Domain $= \{ x \in \mathbb{R} \}$

Range $= \{ y \in [-4, \infty) \}$

Zeros: $x = \pm 2$.

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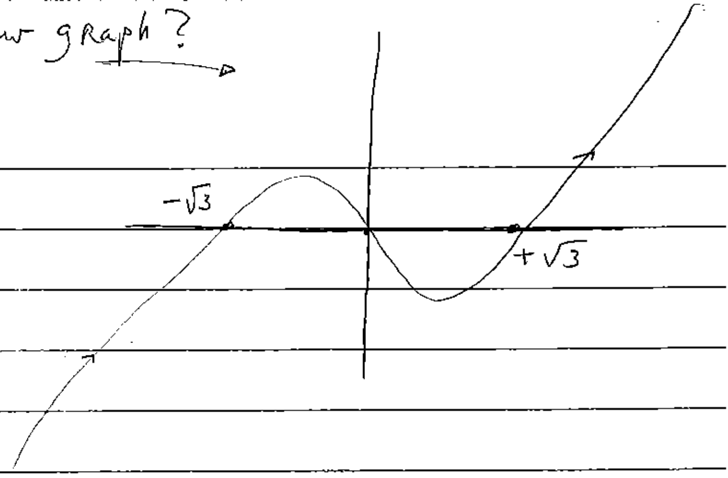
2. $f(x) = x^3 - 3x$

Draw graph?



$f(x) = 0$?

$x(x^2-3) = 0$

$\Rightarrow x = 0$; $x = \pm\sqrt{3}$

$f(x)$ is continuous. For $x \to \infty$ $f(x) \to +\infty$.

For $x \to -\infty$ $f(x) \to -\infty$

Domain $x \in \mathbb{R}$.

Range $y \in \mathbb{R}$.

Rational functions

A rational function has the form.

$$f(x) = \frac{g(x)}{h(x)}$$

where $g(x)$ and $h(x)$ are polynomials

Example $f(x) = \dfrac{x^3 - 3x^2 + 5}{2x^4 + x - 3}$

Domain: $\{ x \in \mathbb{R} \mid h(x) \neq 0 \}$.

set of all $x \in \mathbb{R}$ such that $h(x) \neq 0$

Zeros: Points where $g(x) = 0$ & $h(x) \neq 0$

Examples! $f(x) = 1/x$

Domain: $x = (-\infty, 0) \cup (0, \infty)$.

$x = 0 \to$ vertical asymptote.

$x \to \pm\infty \to$ horizontal asymptotes.

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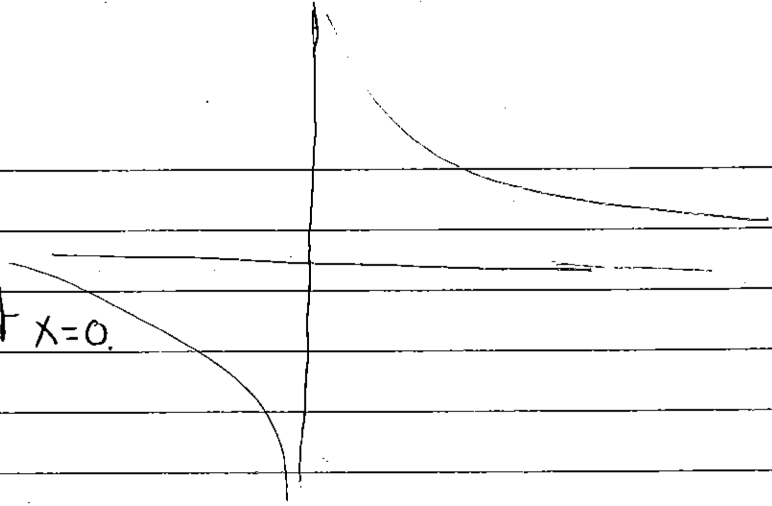


no zeros.

f is not continuous at $x=0$.

Example 2 $f(x) = \dfrac{x+2}{(x-1)^2}$

Domain $x \in (-\infty, 1) \cup (1, \infty)$
Range → we need more tools.
Zeros $x = -2$

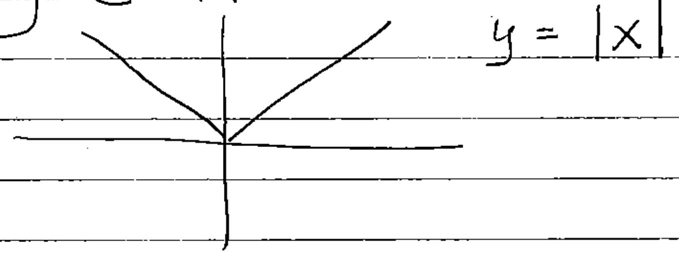
Graphs → need more analysis. minimum, maximum?

<u>modulus</u>

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

modulus size of the number without the sign.

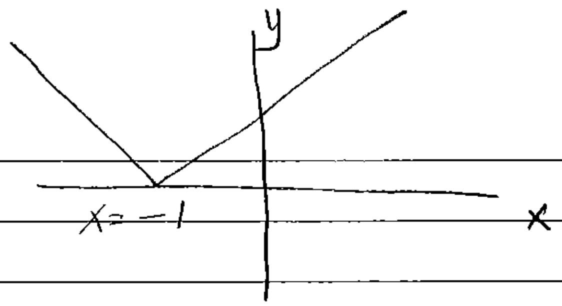
Domain $x \in R$
Range $y \in R^+$
Zero at $x=0$



Example $f(x) = |x+1|$

Domain $x \in R$.
Range $y \in [0, \infty)$
zeros $x = -1$

graph

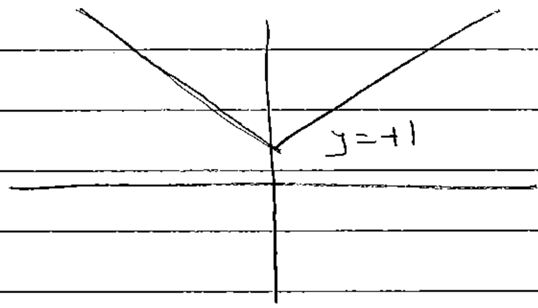


Example $f(x) = |x| + 1$

Domain $x \in \mathbb{R}$

Range $y \in [+1, \infty)$

zeros: none.



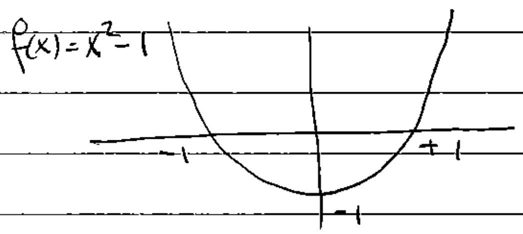
Example $f(x) = |x^2 - 1|$

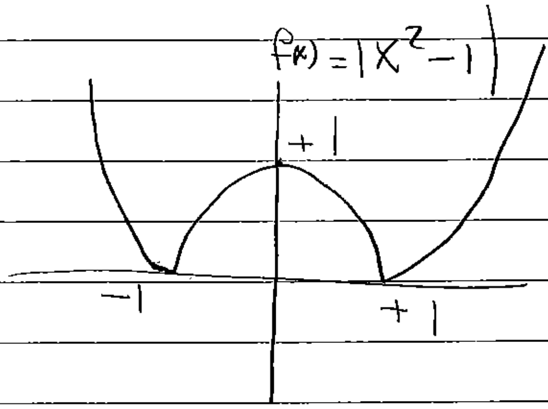
Domain $x \in \mathbb{R}$.

Range $y \in [0, \infty)$

zeros $x = \pm 1$.

graph





Even and odd functions

Even function:

$f(-x) = f(+x)$ for all values of x in the domain.

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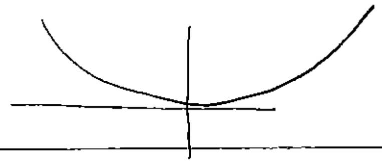


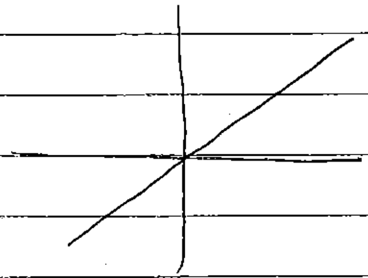
$f(x) = x^2$

Examples: $f(x) = x^2$

Example $f(x) = x^4 + 2x^2 + 3$

odd function

$f(-x) = -f(+x)$ for all values of x in the domain



$f(x) = x$

Example $f(x) = x$

Example $f(x) = x^3$

$\Rightarrow$ A polynomial with only even powers $\rightarrow$ even function.

A polynomial with only odd powers $\rightarrow$ odd function

Note A general function is neither even nor odd.

to find out work out $f(-x)$ and decide

whether $f(-x) = +f(x)$ or $f(-x) = -f(+x)$ or neither.

Examples 1. $f(x) = \dfrac{x}{x+2}$ $\quad f(-x) = \dfrac{-x}{-x+2} = \dfrac{+x}{+x-2} \neq \dfrac{+x}{+x+2}$;

$\neq -\dfrac{x}{(x+2)}$

$\Rightarrow$ neither.

2. $f(x) = \sin(x^2)$; $f(-x) = \sin((-x)^2) = \sin(x^2) \Rightarrow$ even

3. $f(x) = \sin(x^3)$; $f(-x) = \sin((-x)^3) = \sin(-x^3) = -\sin(x^3) \Rightarrow$ odd.

increasing functions and decreasing functions

$f(x)$ is increasing on $[a,b]$ if $f(x) \leq f(y)$ whenever $a \leq x \leq y \leq b$

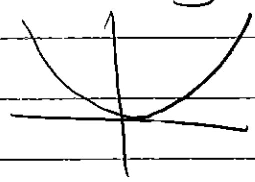
$f(x)$ is strictly increasing on $[a,b]$ if $f(x) < f(y)$ whenever $a \leq x < y \leq b$

$f(x)$ is decreasing on $[a,b]$ if $f(x) \geq f(y)$ whenever $a \leq x \leq y \leq b$

$f(x)$ is strictly decreasing on $[a,b]$ if $f(x) > f(y)$ whenever $a \leq x < y \leq b$

Example $f(x) = x^2$ is strictly decreasing for $x \in (-\infty, 0]$

and is strictly increasing for $x \in [0, +\infty)$



but is neither increasing nor decreasing on $x \in [-1, +1]$

inverse functions

function $f(x)$ $\quad f : x \longrightarrow y$.

inverse function $f^{-1}(y)$ $\quad f^{-1} : y \longrightarrow x$

similarly: $f^{-1}(f(x)) = f^{-1}(y) = x$

hence $f^{-1} \circ f : x \longrightarrow x \implies f^{-1} \circ f = 1$

$f^{-1} \circ f$ is the identity function

inverse functions may or may not exist.

Note $f^{-1}(x)$ does not mean $\dfrac{1}{f(x)}$

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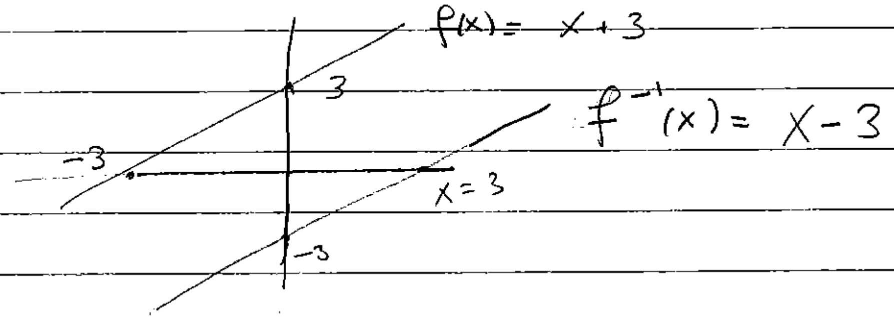
Example: $f(x) = x+3 = y \longrightarrow x = y-3$

So $f^{-1}(y) = y-3$

The name of the variable does not matter.

So $f^{-1}(x) = x-3$

check $f^{-1} \circ f(x) = f^{-1}(f(x)) = f^{-1}(x+3) = x+3-3 = x.$



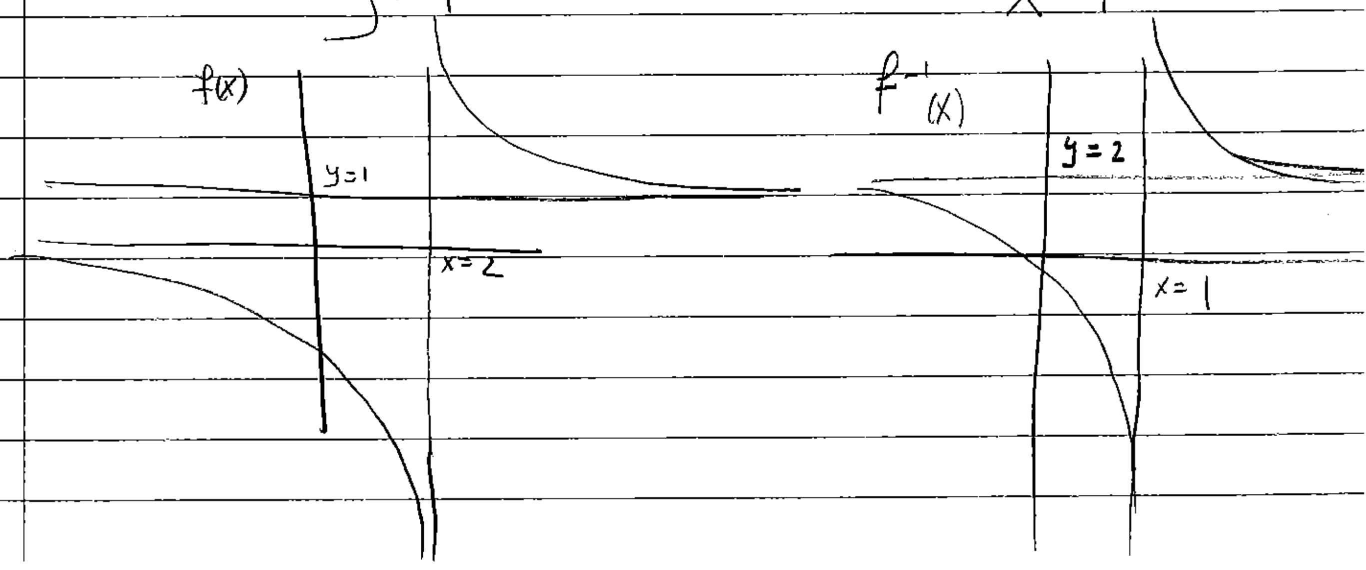
Thus determining the inverse function $\rightarrow$

$\rightarrow$ solving $y = f(x)$ for x in terms of y.

Example $f(x) = \dfrac{x}{x-2} = y \Rightarrow$

$\Rightarrow x = y(x-2) \Rightarrow x = yx - 2y \Rightarrow yx - x = 2y \Rightarrow x = \dfrac{2y}{y-1}$

$\Rightarrow f^{-1}(y) = \dfrac{2y}{y-1}$ or $f^{-1}(x) = \dfrac{2x}{x-1}$



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Notice the reflection rule:

since finding the inverse interchanges the roles of x and y the graph $f^{-1}(x)$ is the graph $f(x)$ reflected in the line $y = x$.

Problem: not every function has an inverse

Example: $f(x) = x^2 \rightarrow$ is $f^{-1}(4) = +2$ or -2.

$\Rightarrow$ $f(x)$ gives one value of y for two values of x.

$\Rightarrow$ we say that $f(x)$ is one to one if different values of x give different values of $f(x)$

that is: if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

one to one functions $f(x)$ always have an inverse.

The maximal domain of $f^{-1}(x)$ is the range of $f(x)$ and may not be the same as the maximal domain of $f(x)$.

e.g. For $f(x) = x^2$ on $\mathbb{R}^+$ i.e. $x \in [0, \infty)$.

then $f^{-1}(y) = +\sqrt{y}$ or $f^{-1}(x) = +\sqrt{x}$

So in the Domain $x \in [0, \infty)$ $f(x)$ is one to one and $f^{-1}(x)$ exists.

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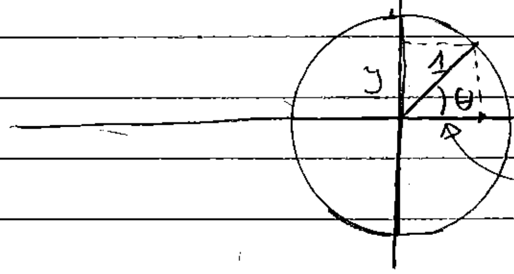
Trigonometric functions

Draw A circle of radius 1 centred on the origin.



$x^2 + y^2 = 1$

$\cos\theta = x$
$\sin\theta = y.$

then $x^2 + y^2 = \cos^2\theta + \sin^2\theta = 1$

and $x = 1 \cdot \cos\theta$
$y = 1 \cdot \sin\theta$ $\Rightarrow \dfrac{y}{x} = \dfrac{\sin\theta}{\cos\theta} = \tan\theta.$

Note $-1 \leq \sin\theta, \cos\theta \leq 1$ for any θ.

$\Rightarrow$ the range of $\sin\theta, \cos\theta$ is $[-1, 1]$

the range of $\tan\theta$ is $(-\infty, +\infty)$.

Express angles in radians.

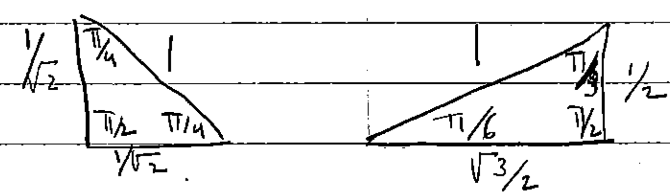
Remember: 360 degrees $= 2\pi$ radians.

then 90° degrees $= \dfrac{\pi}{4}$ radians, etc.

special values

	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
sin	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
cos	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
tan	0	$1/\sqrt{3}$	1	$\sqrt{3}$	undefined

Draw triangles



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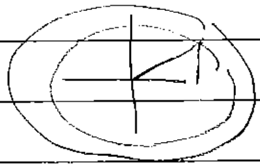
Periodic Functions

A function is periodic if we can find a constant K.

Such that: $f(X+K) = f(X) \quad \forall\, X \in$ domain of f

The smallest positive number K is called the period of f

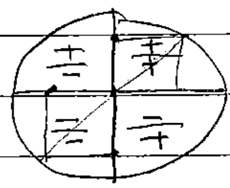
Example: $\cos(X + 2\pi) = \cos(X)$



in fact $\cos(X + 2\pi \cdot n) = \cos(X)$ with $n \in \mathbb{Z}$

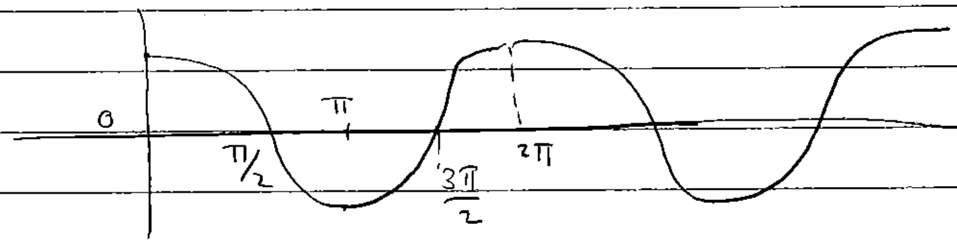
similarly: $\sin(X + 2\pi \cdot n) = \sin(X)$ with $n \in \mathbb{Z}$

but $\tan(X + n\pi) = \tan X$ because

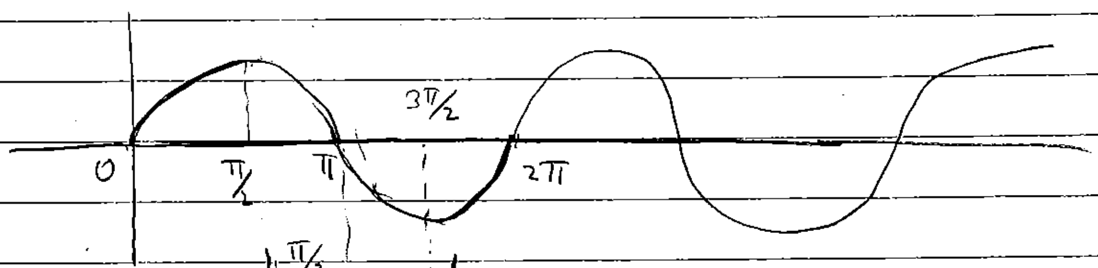


hence $\dfrac{\sin(X+\pi)}{\cos(X+\pi)} = \dfrac{-\sin(x)}{-\cos(X)} = \dfrac{\sin(X)}{\cos(X)} = \tan(x)$.

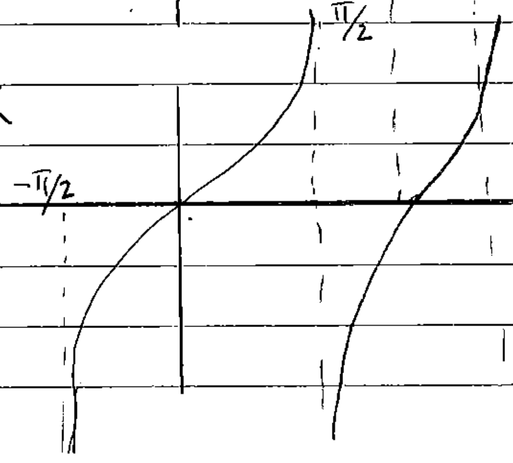
graphically: $\cos X$



$\sin X$



$\tan X$



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Trigonometric identities

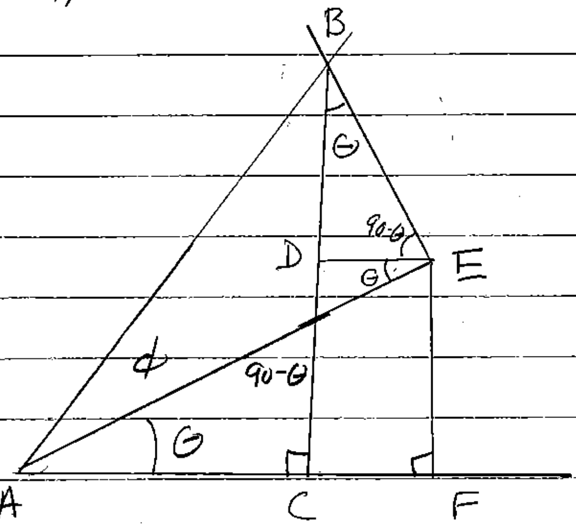
Pythagoras Theorem $\cos^2\theta + \sin^2\theta = 1$

Then from: $\tan\theta = \dfrac{\sin\theta}{\cos\theta} \Rightarrow 1 + \tan^2\theta = \dfrac{1}{\cos^2\theta} = \sec^2\theta$

compound angle formula

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

Proof



$$\cos(\theta+\phi) = \frac{AC}{AB} = \frac{AF - CF}{AB}$$

$$\sin\theta = \frac{DE}{BE} \Rightarrow CF = DE = BE\sin\theta$$

$$\cos\theta = \frac{AF}{AE} \Rightarrow AF = AE\cos\theta$$

$$\Rightarrow \cos(\theta+\phi) = \frac{AE\cos\theta}{AB} - \frac{BE\sin\theta}{AB} = \cos\phi\cos\theta - \sin\phi\sin\theta$$

$$\Rightarrow \cos(2\theta) = \cos^2\theta - \sin^2\theta.$$

similarly $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$ (exercise)

$$\Rightarrow \sin(2\theta) = 2\sin\theta\cos\theta.$$

$$\cancel{\sin(\theta-\phi) = \sin\theta\cos\phi - \cos\theta\sin\phi}$$

$$\Rightarrow \sin\theta\cos\phi = \frac{1}{2}\left(\sin(\theta+\phi) + \sin(\theta-\phi)\right)$$

We can derive other identities from these.

e.g. $\cos(3\theta) = \cos(2\theta+\theta) = \cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta$

$= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta$

$= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$

$= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta$

$= 4\cos^3\theta - 3\cos\theta$

<u>checks</u> $\theta = 0 \Rightarrow \cos 3\theta = 1$; $4\cos^3\theta - 3\cos\theta = 4-3 = 1$ ✓

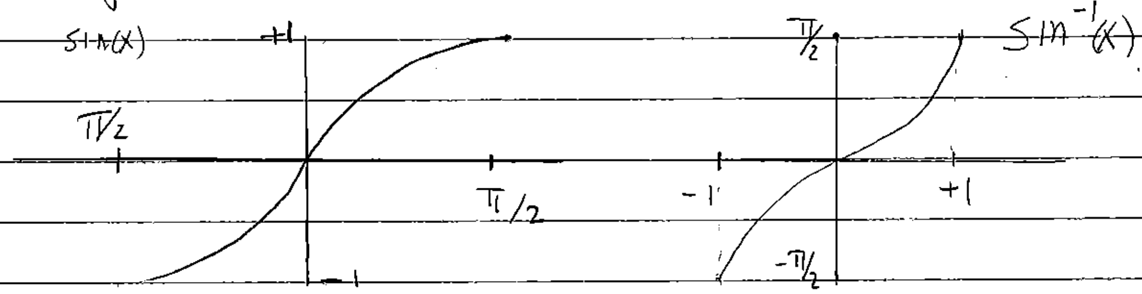
$\theta = \pi/3 \Rightarrow \cos 3\theta = -1$; $\cos\theta = \frac{1}{2} \Rightarrow 4\cdot\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right) = \frac{1}{2} - \frac{3}{2} = -1$ ✓

<u>Inverse trigonometric functions</u>

Trigonometric functions $\rightarrow$ periodic $\rightarrow$ ∞ to 1.

$\rightarrow$ Restrict to the principal domain. (as for x^2)

Principal domain of $y = \sin x$ is $x \in [-\pi/2, \pi/2]$.



$\Rightarrow$ <u>Principal Range</u> of $\sin^{-1}x$ is $x \in [-\pi/2, \pi/2]$

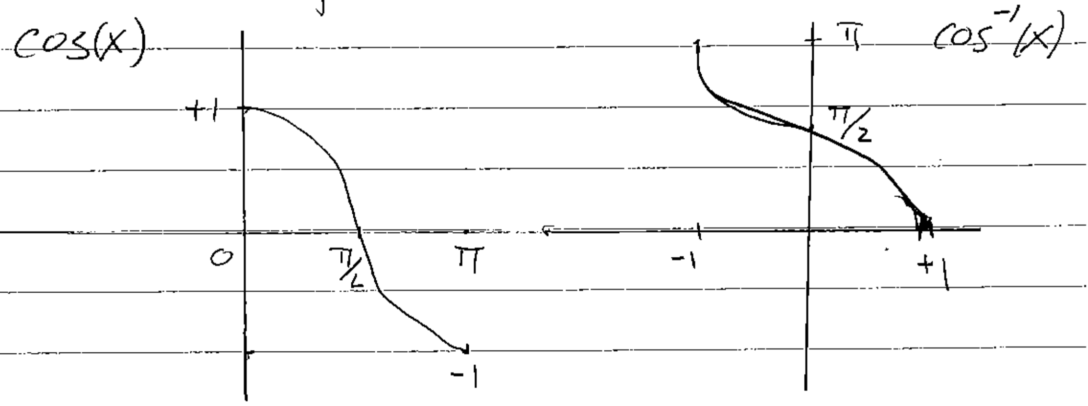
Maximal domain of $\sin^{-1}x$ is $x \in [-1, 1]$

The principal domain of $y = \cos x$ is

$$x \in [0, \pi]$$

$\Rightarrow$ Principal Range of $y = \cos^{-1}(x)$ is $y \in [0, \pi]$

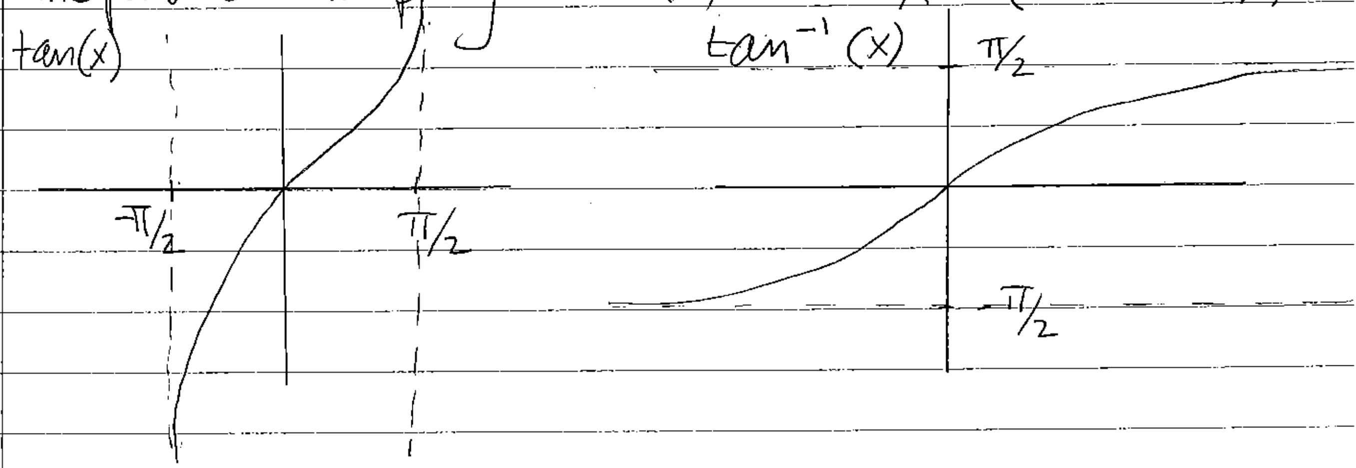
The Maximal domain of $\cos^{-1}(x)$ is $x \in [-1, \pi]$



The principal domain of $y = \tan(x)$ is $x \in [-\pi/2, \pi/2]$

$\Rightarrow$ Principal Range of $y = \tan^{-1}(x)$ is $y \in [-\pi/2, \pi/2]$

Pincipal domain of $y = \tan^{-1}(x)$ is $x \in (-\infty, \infty)$



<u>Note:</u> sometimes $\sin^{-1}(x)$, $\cos^{-1}(x)$, $\tan^{-1}(x)$, etc...

are denoted by $\arcsin(x)$, $\arccos(x)$, $\arctan(x)$, etc.

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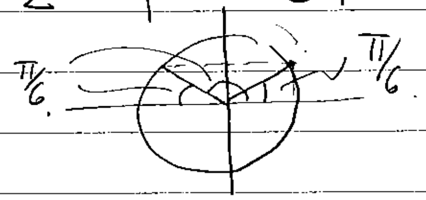
Trigonometric equations

Consider solving $\sin\theta = \frac{1}{2}$ for θ.

$\arcsin(\sin\theta) = \arcsin(\frac{1}{2}) = \theta.$



One solution is $\theta = \pi/6$

However, there are infinitely many solutions.

$$\theta = \frac{\pi}{6} + 2\pi \cdot n. \qquad n \in \mathbb{Z}.$$

and $$\theta = \left(\pi - \pi/6\right) + 2\pi n \qquad n \in \mathbb{Z},$$

by the same argument the general solution of

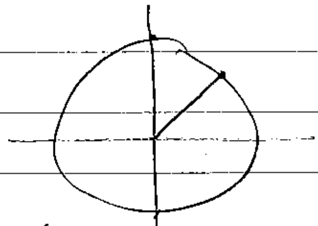
$$\sin\theta = \sin\alpha$$

is $$\theta = \begin{cases} \alpha + 2\pi n & n \in \mathbb{Z} \\ (\pi - \alpha) + 2\pi n & n \in \mathbb{Z}. \end{cases}$$

Example

Find the general solution of $\sin\theta = \frac{1}{\sqrt{2}}$.

Solution

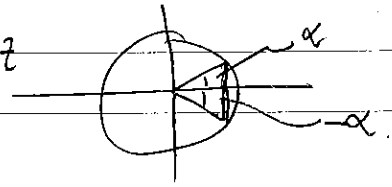


in the principal domain $[-\pi/2, \pi/2]$ $\theta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \pi/4$

or in $[\pi/2, 3\pi/2]$ $\qquad \theta = \sin^{-1}(1/\sqrt{2}) = \pi - \pi/4 = \frac{3\pi}{4}$

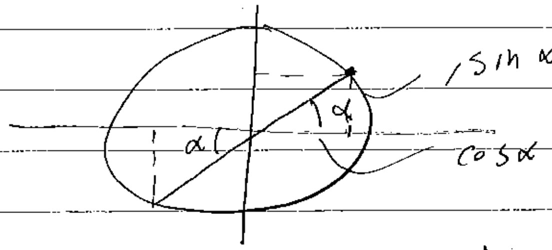
similarly the general solution of $\cos\theta = \cos\alpha$ is

$$\theta = \pm\alpha + 2\pi n \qquad n \in \mathbb{Z}$$



The general solution of $\tan\theta = \tan\alpha$.

is $\theta = \alpha + n\pi \quad n \in \mathbb{Z}$



$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$.

Example Find the general solution of $\tan\theta = 3$.

$\theta = \tan^{-1}(3) = 1.2490$

$\Rightarrow$ general solution $\theta = 1.2490 + n\pi \quad n \in \mathbb{Z}$.

consider the equation

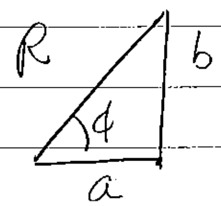
$$a\cos\theta + b\sin\theta = C$$

where a, b, C are constants.

we solve this equation by using a trick.

consider a right-handed triangle with sides a, b and R with

$$R = \sqrt{a^2+b^2} \qquad \tan\phi = \frac{b}{a}$$



then $a = R\cos\phi$ and $b = R\sin\phi$.

Thus,

$a\cos\theta + b\sin\theta = R\cos\phi\cos\theta + R\sin\phi\sin\theta =$

$= R(\cos\phi\cos\theta + \sin\phi\sin\theta) = R\cos(\theta - \phi).$

$\Rightarrow \quad \cos(\theta - \phi) = C/R = \cos(\alpha)$

$\Rightarrow$ general solution $\theta - \phi = \pm\alpha + 2\pi n.$

OR $\quad \theta = \pm\alpha + 2\pi n + \phi$

since $R = \sqrt{a^2+b^2}$ and $\phi = \tan^{-1}\left(\frac{b}{a}\right)$

we have: $\theta = \pm\cos^{-1}\left(\frac{C}{\sqrt{a^2+b^2}}\right) + 2\pi n + \tan^{-1}\left(\frac{b}{a}\right) \quad n \in \mathbb{Z}$

Example Find the general solution of the equation.

$$\cos\theta + 2\sin\theta = 1$$

So $a = 1 \quad b = 2 \quad C = 1$

$R = \sqrt{1^2 + 2^2} = \sqrt{5}$

$\phi = \tan^{-1}(2/1) = 1.1071.$

$\Rightarrow \quad 1 = \sqrt{5}\cos\phi \qquad 2 = \sqrt{5}\sin\phi$

$a = R\cos\phi \qquad b = R\sin\phi.$

and $\quad \sqrt{5}(\cos\phi\cos\theta + \sin\phi\sin\theta) = \sqrt{5}\cos(\theta - \phi) = 1$

so $\quad \theta - \phi = \cos^{-1}(1/\sqrt{5}) = \pm 1.1071 + 2\pi n \quad n \in \mathbb{Z}$

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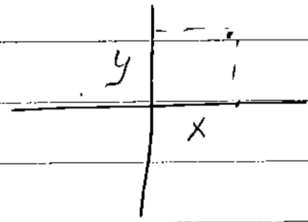
OR $\theta = \pm 1.1071 \pm 2\pi n + 1.1071$

SO $\theta = 2\pi n$ OR $\theta = 2.2142 \pm 2\pi n, \; n \in \mathbb{Z}$.

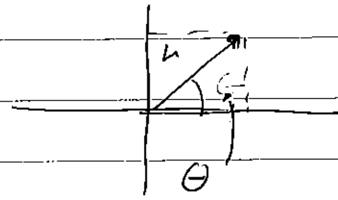
check $\theta = 0 \Rightarrow \sin\theta = 0 \quad \cos\theta = 1 \Rightarrow 1 + 2\cdot 0 = 1$ ✓

$\theta = 2.2142 \Rightarrow \cos(2.2142) = -0.6 \quad \sin(2.2142) = 0.8 \Rightarrow -0.6 + 2\cdot 0.8 = 1$ ✓

Polar coordinates



cartesian coordinates $P(x, y)$

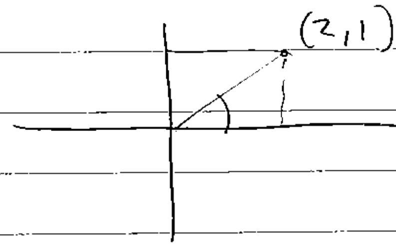


Polar coordinates $P(r, \theta)$

$x = r\cos\theta \qquad y = r\sin\theta$

$r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

Example



$P(x, y) = (2, 1)$

$\Rightarrow r = \sqrt{2^2 + 1^2} = \sqrt{5}$

$\theta = \tan^{-1}\left(\frac{1}{2}\right) = 0.4636 = 0.1476\pi$

$\Rightarrow P(r, \theta) = (\sqrt{5}, 0.1476\pi)$.

Example

$P(r, \theta) = (2, \pi/3)$

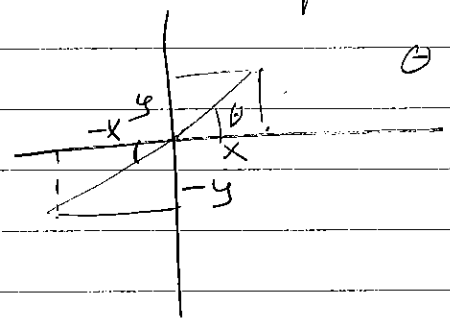
$\Rightarrow x = 2\cos\pi/3 = 2\cdot\frac{1}{2} = 1$

$y = 2\sin\pi/3 = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$

Note

a) When $P(x,y) = (0,0) \longrightarrow \theta$ is not defined.

b) when calculating θ from $\tan\left(\frac{y}{x}\right)$
we have to note the quadrant.

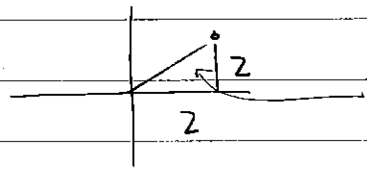


$$\theta = \tan\left(\frac{y}{x}\right) = \tan\left(\frac{-y}{-x}\right)$$

For example $(x,y) = (2,2)$.

$$r = \sqrt{2^2+2^2} = 2\sqrt{2}.$$

$$\theta = \tan^{-1}\left(\frac{2}{2}\right) = \tan 1 = \pi/4.$$

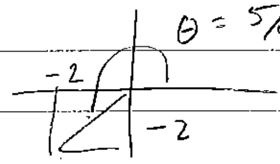


$\theta = \pi/4$

but $(x,y) = (-2,-2)$.

$$r = \sqrt{(-2)^2+(-2)^2} = 2\sqrt{2}.$$

$$\theta = \tan^{-1}\left(\frac{-2}{-2}\right) = \tan 1 = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$$



$\Rightarrow$ For point in the third quadrant we have to take $\theta + \pi$. similarly for $(-x,+y)$ versus $(x,-y)$

math 19 | 1.22 | Limits

Limits $f(x)$ as $x \to x_0$ some value.

$f(x)$ may be undefined at x_0. i.e. $f(x_0)$ may not exist. But the limit may or may not exist.

For example: $\frac{\sin x}{x}$ is not defined at $x=0$

but (as we will see) $\lim_{x\to 0} \frac{\sin x}{x} = 1$

Examples: a) $f(x) = x^2$

$\lim_{x\to 2} x^2 = 4$ as expected because $2^2=4$

b) $f(x) = \frac{1}{x}$ what is $\lim_{x\to 0} (1/x) = ?$

$\lim_{x\to 0^+} 1/x = +\infty$

$\lim_{x\to 0^-} 1/x = -\infty$

$\Rightarrow$ The limit $\lim_{x\to 0}(1/x)$ does not exist because we do not get the same value as we approach zero from the positive or negative sides.

FROM this example we can extract a general rule.

The limit $\lim_{x \to a} f(x) = l$ exists if and only if.

$\lim_{x \to a^+} f(x) = l$ if $f(x)$ tends to l as $x \to a$ $x > a$

and $\lim_{x \to a^-} f(x) = l$ if $f(x)$ tends to l as $x \to a$ $x < a$.

c) Now consider the Heaviside step function.

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

we have $\lim_{x \to 0^-} f(x) = 0$.

and $\lim_{x \to 0^+} f(x) = 1$

$\Rightarrow \lim_{x \to 0} f(x)$ does not exist.

d) Note that the limit say nothing about $f(a)$ itself.

suppose we defined:

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 2 \\ 99 & \text{if } x = 2 \end{cases}$$

then:

$\lim_{x \to 2} f(x) = 4$ but $f(2) = 99$.

This motivates the definition of continuity.

definition | The function $f(x)$ is continuous at $x = a$ if

a) a is in the domain of $f(x)$

b) $\lim\limits_{x \to a} f(x) = f(a)$

$f(x)$ is continuous if it is continuous at all values of x.

Examples.
1. $f(x) = x^2$ is continuous.
2. the Heaviside step function is continuous everywhere except at $x=0$.
3. $f(x) = 1/x$ is continuous everywhere except at $x = 0$
4. $f(x) = \begin{cases} x^2 & \text{if } x \neq 2 \\ 99 & \text{if } x = 2. \end{cases}$

   is not continuous at $x=2$ since $\lim\limits_{x \to 2} f(x) \neq f(2)$.

more Examples: Rational functions

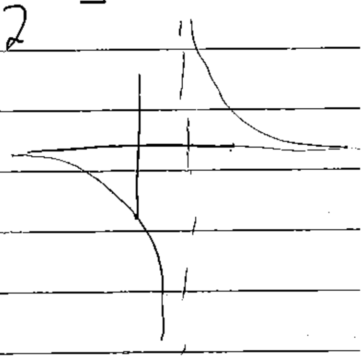
Ex. 1 | $f(x) = \dfrac{x^2+3}{(x-2)}$

<u>clearly:</u> $\lim\limits_{x \to 1} \frac{x^2+3}{(x-2)} = f(1) = -4.$

$\Rightarrow \lim\limits_{x \to 1} f(x) = -4$ and $f(x)$ is continuous at $x=1$

But $\lim\limits_{x \to 2} \frac{x^2+3}{x-2}$ we have

$\lim\limits_{x \to 2^+} \frac{x^2+3}{x-2} = +\infty$. but $\lim\limits_{x \to 2^-} \frac{x^2+3}{x-2} = -\infty$.



$\Rightarrow \lim\limits_{x \to 2} f(x)$ does not exist

<u>Example 2)</u> always simplify $f(x)$
before calculating the limit.

E.g. $f(x) = \frac{(x^2-1)}{x-1}$;

$$\lim_{x \to 1} \frac{(x^2-1)}{(x-1)} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} = \lim_{x \to 1} (x+1) = 2.$$

<u>however)</u> $f(x)$ is not continuous at $x=1$

since $f(1) = \frac{1^2-1}{1-1}$ is not defined

i.e. 1 is not in the domain of $f(x)$.

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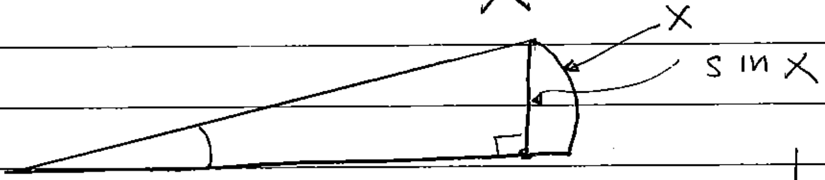
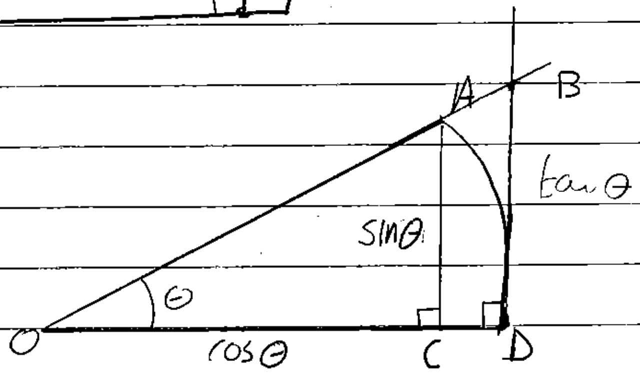
Example. $f(x) = \frac{\sin x}{x}$



$\lim_{x \to 0} \frac{\sin x}{x} = 1.$



Proof: Geometrical

$OA = OD = 1 = R$

Area of $\triangle OBD$ > area of $\triangleleft OAD$ > area of $\triangle OAC$.

(*) Area of $\triangle OBD$:

we have $\frac{OD}{OB} = \cos\theta$ $\quad \frac{BD}{OB} = \sin\theta$

$\Rightarrow \frac{BD/\cancel{OB}}{OD/OB} = \frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{BD}{OD} = BD$

so Area of $\triangle OBD = \frac{1}{2}\tan\theta \cdot 1$

Area of $\triangleleft OAD = \pi R^2 \cdot \frac{\theta}{2\pi} = \frac{\theta}{2}$ (since $R=1$)

Area of $\triangle OAC = \frac{1}{2}\sin\theta\cos\theta.$

Hence: $\frac{1}{2}\cos\theta\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$

Divide by $\frac{1}{2}\sin\theta \Rightarrow \cos\theta < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$

OR $\frac{1}{\cos\theta} > \frac{\sin\theta}{\theta} > \cos\theta$

in the limit $\lim_{\theta \to 0}\cos\theta = 1 \Rightarrow \lim_{\theta \to 0}\frac{\sin\theta}{\theta} = 1.$

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However $f(x) = \frac{\sin}{x}$ is not continuous at $x=0$

since $f(0) = \frac{\sin 0}{0}$ is not defined.

However, <u>Define</u>.

$$f(x) = \operatorname{sinc}(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 1. \end{cases}$$

Then $\operatorname{sinc}(x)$ is continuous at $x = 0$.

<u>Example</u>. $f(x) = \frac{\sin 2x}{x}$

write $f(x) = 2\frac{\sin 2x}{2x} = 2\frac{\sin y}{y}$ where $y = 2x$

clearly as $x \to 0$ $y \to 0$ and $\lim_{y\to 0} 2\frac{\sin y}{y} = 2$

Hence $\lim_{x\to 0} \frac{\sin 2x}{x} = 2$.

<u>Limits as $x \to \pm\infty$.</u>

<u>Example</u> $\lim_{x\to\infty} \frac{1}{x} = 0$ and $\lim_{x\to -\infty} \frac{1}{x} = 0$

$\Rightarrow$ the limit as $x \to \pm\infty$ exists.

But $\lim_{x\to\infty} \sin x$ does not exist

$\lim_{x\to +\infty} a_n x^n + \quad + a_0 = +\infty$

if the degree of the polynomial is larger than 0.

$$\lim_{x \to -\infty} a_n x^n + \cdots + a_1 x + a_0 = 0.$$

if n is even $\lim\limits_{x \to -\infty} = +\infty$

if n is odd $\lim\limits_{x \to -\infty} = -\infty$

The highest power "wins" as $x \to -\infty$.

<u>For rational functions:</u>

a) if the degree of the numerator is larger than the degree of the demonitor the limit is $\pm\infty$

e.g. $f(x) = \dfrac{x^3+1}{3x^2-2x+1}$

$$\lim_{x \to \pm\infty} f(x) \approx \frac{x^3}{3x^2} = \frac{x}{3} = \pm\infty.$$

b) if the degree of the numerator is smaller than the degree of the denominator then the limit is 0.

e.g. $f(x) = \dfrac{x^3+1}{2x^4-x^2+2}$

$$\lim_{x \to \pm\infty} f(x) \approx \frac{x^3}{2x^4} = \frac{1}{2x} = 0$$

c) if the degrees are the same the the limit is a non-zero real number

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e.g. $f(x) = \dfrac{x^3+1}{2x^3-3x+2}$

$$\lim_{x\to\pm\infty} \frac{x^3+1}{2x^3-3x+2} \simeq \frac{x^3}{2x^3} = \frac{1}{2}$$

The sandwich rule.

suppose that $g(x) \leq f(x) \leq h(x)$ for all large x

and that $\lim_{x\to\infty} g(x) = 0$ $\lim_{x\to\infty} h(x) = 0$.

then $\lim_{x\to\infty} f(x) = 0$.

Example consider $f(x) = \dfrac{\sin x}{x}$ as $x \to \infty$

since $-1 \leq \sin x \leq +1 \Rightarrow \dfrac{-1}{x} \leq \dfrac{\sin x}{x} \leq \dfrac{1}{x}$

for all $x > 0 \Rightarrow \lim_{x\to\infty} \dfrac{\sin x}{x} = 0$.

Asymptotes

Recall: we talked about horizontal and vertical asymptotes.

e.g. $f(x) = \dfrac{1}{x-1}$ has vertical asymptote at $x=1$
and horizontal asymptote at $y=0$.

Define 1 The line $x=a$ is a vertical asymptote of $f(x)$ if
$\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$ or both.

The line $y=b$ is a horizontal asymptote of $f(x)$ if
$\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to -\infty} f(x) = b$ or both.