math 191.49

Examples.

1. $f(x) = x^3 + 2x^2 + 2x + 1$

$f(0) = 1 \Rightarrow a_0 = 1$

$f'(x) = 3x^2 + 4x + 2 \Rightarrow f'(0) = 2 \Rightarrow a_1 = 2$

$f''(x) = 6x + 4 \Rightarrow f''(0) = 4 \Rightarrow a_2 = \frac{1}{2} \cdot 4 = 2$

$f'''(x) = 6 \Rightarrow f'''(0) = 6 \quad a_3 = \frac{1}{3!} 6 = 1$

$f^{(n)}(x) = 0$ for $n \geq 4$. $a_n = 0$ for $n \geq 4$.

$\Rightarrow f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 = 1 + 2x + 2x^2 + x^3 = f(x)$.

$\rightarrow$ we recovered the original polynomial.

2. $f(x) = \sin x$.

$f(0) = 0 \Rightarrow a_0 = 0$

$f'(x) = \cos x \Rightarrow f'(0) = 1 \Rightarrow a_1 = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = 0 \Rightarrow a_2 = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -1 \Rightarrow a_3 = -\frac{1}{3!}$

$f^{(4)}(x) = \sin x \Rightarrow f^{(4)}(0) = 0 \Rightarrow a_4 = \frac{1}{4!} \cdot 0 = 0$.

$f^{(5)}(x) = \cos x \Rightarrow f^{(5)}(0) = 1 \Rightarrow a_5 = \frac{1}{5!} \cdot 1 = \frac{1}{120}$.

$\vdots$

$f^{(2n+1)} = (-1)^n \cos x$. $\quad f^{(2n+1)}(0) = (-1)^n$

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$f^{(2n)}(x) = (-1)^n \sin x \qquad f^{(2n)}(0) = 0$

$\Rightarrow \sin x = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} (-1)^n \, x^{2n+1}$

$= x - \frac{x^3}{3!} + \frac{1}{5!} \cdot x^5 - \frac{1}{7!} \cdot x^7 + \dots$

checks: $\sin(-x) = -\sin(x) \qquad \sin(0) = 0.$

similarly: $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

checks $\cos(-x) = +\cos(x) \qquad \cos(0) = 1.$

calculate $\sin(0.1)$.

$\sin(0.1) = (0.1) - \frac{(0.1)^3}{6} + \frac{(0.1)^5}{120} - \frac{(0.1)^7}{5040}$

$= (0.1) - \frac{1}{6000} + \frac{1}{12000000} - \frac{1}{5040000000} + \dots$

we see that successive terms quickly become smaller and smaller.

Thus: $\sin(0.1) \approx 0.09983341666 \rightarrow$ to third order

Versus: $\sin(0.1) = 0.0998334166647$

$\rightarrow$ Very good approximation for $x \sim 0.1$

What about $\sin(10)$?

$\sin(10) = 10 - \frac{10^3}{3!} + \frac{10^5}{5!} - \frac{10^7}{7!} + \frac{10^9}{9!} + \dots$

→ successive terms do become smaller for $2n+1 > 10$.

e.g. $\frac{10^{35}}{35!} = 0.00012$.

→ The series converges but not rapidly.

→ change variable to $y = x - a$.

for some a.

→ Taylor series expansion of $f(x)$ about a.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

OR $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

→ A good approximation of $f(x)$ when $x-a$ is small.

Example: $f(x) = x^3 + x^2 + x + 1$ with $a = 1$.

$f(a) = 4$

$f'(x) = 3x^2 + 2x + 1$ $\quad f'(a) = 3+2+1 = 6$.

$f''(x) = 6x + 2$ $\quad f''(a) = 8$.

$f'''(x) = 6$ $\quad f'''(a) = 6$ $\quad f^{(n)}(x) = 0, n \geq 4$

$$f(x) = 4 + 6(x-1) + \frac{8}{2}(x-1)^2 + \frac{6}{6}(x-1)^3$$

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Example: find an approximation for $f(x) = 1/x$ near $x=1$ by using the first three terms in the Taylor series expansion.

Answer: $f(1) = 1$

$f'(x) = \frac{-1}{x^2}$ $f'(1) = -1$

$f''(x) = \frac{2}{x^3}$ $f''(1) = 2.$

$\frac{1}{x} \cong 1 - (x-1) + \frac{2}{2!}(x-1)^2$

$= x^2 - 3x + 3$

L'Hopital's rule

what is $\lim_{x \to 0} \frac{\sin x}{x} = ?$

write $\sin x$ as a Maclaurin series.

$\frac{\sin x}{x} = \frac{x - x^3/3! + x^5/5! + \cdots}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots$

$\Rightarrow \lim_{x \to 0} \frac{\sin x}{x} = 1$

Similarly: $\frac{1-\cos x}{x} = \frac{1 - (1 - x^2/2 + x^4/4! - \cdots)}{x} = \frac{x}{2} - \frac{x^3}{4!} + \cdots$

$\lim_{x \to 0} \frac{1-\cos x}{x} = 0.$

Do the same to work out the limit $\lim_{x \to a} \frac{f(x)}{g(x)}$

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with $f(a) = g(a) = 0$.

Expand $f(a)$ and $g(a)$ as a Taylor series

about $x = a$.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

$$g(x) = g(a) + g'(a)(x-a) + \frac{g''(a)}{2!}(x-a)^2 + \cdots$$

$$\Longrightarrow \frac{f(x)}{g(x)} = \frac{f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots}{g(a) + g'(a)(x-a) + \frac{1}{2!}g''(a)(x-a)^2 + \cdots}$$

Hence $\displaystyle\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$

This is L'Hopital's rule.

Note: it works only when $f(a) = g(a) = 0$.

if $f'(a) = g'(a) = 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f''(a)}{g''(a)}$$

and so on.

Examples 1. $\displaystyle\lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = ?$ $\quad f(x) = x^2 - x - 2$

$g(x) = x - 2$

$f'(x) = 2x - 1 \qquad g'(x) = 1$

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$$\lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = \frac{f'(2)}{g'(2)} = \frac{3}{1} = 3.$$

2. $$\lim_{x \to 0} \frac{\sin x - x}{x^3} = \lim_{x \to 0} \frac{\cos x - 1}{3x^2} = \lim_{x \to 0} \frac{-\sin x}{3 \cdot 2 \cdot x} =$$

$$= \lim_{x \to 0} \frac{-\cos x}{6} = -\frac{1}{6}.$$

The exponential function

Take any $a > 0$ e.g. $a = 2, 3, 2.718$

Properties $a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n\text{-times}}$ if $n \in \mathbb{Z}$ $n > 0$,

$a^0 = 1$ $\qquad a^{-n} = \frac{1}{a^n}$ $n \in \mathbb{Z}$ $n > 0$.

$(a^{1/n})^n = a$ $\quad n \in \mathbb{Z}$ $n > 0$.

$(a^{p/q})^q = a^p$ $\quad p, q \in \mathbb{Z}$.

similarly For any rational numbers x, y

$$a^{x+y} = a^x a^y$$

1) $a^0 = 1$

2) $a^{-x} = \frac{1}{a^x}$

3) $(a^x)^y = a^{x \cdot y}$

4) $1^x = 1$

if $a > 1$ and $x < y$ then $a^x < a^y$

if $a < 1$ and $x < y$ then $a^x > a^y$.

can extend these properties to define a^x for all $x \in R$.

if $a>1$ then $f(x)=a^x$ is a strictly increasing function.

$\lim_{x\to\infty} a^x = +\infty$ $\quad$ $\lim_{x\to-\infty} a^x = 0$.

if $a<1$ then $f(x)=a^x$ is a strictly decreasing function.

$\lim_{x\to\infty} a^x = 0$ $\quad$ $\lim_{x\to-\infty} a^x = \infty$

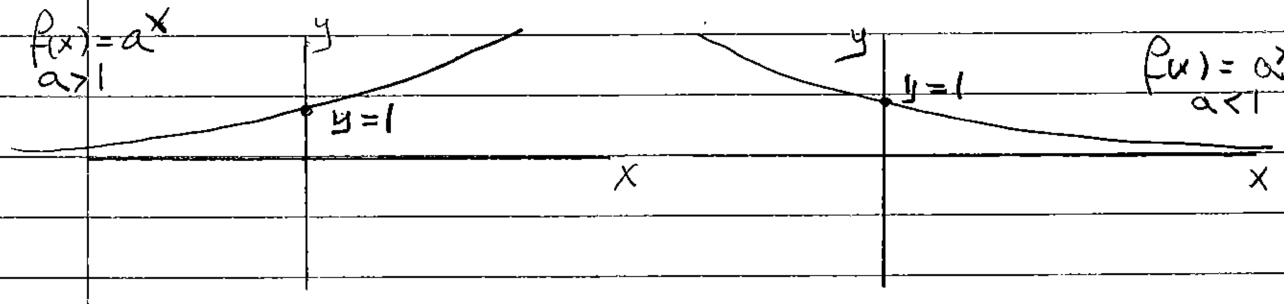
For all $a \neq 1$ $(a>0)$

The domain of a^x is $x \in R$

The Range of a^x is $x \in (0,\infty)$.

$y=0$ is a horizonal asymptote.

$f(x)=a^x$ is continuous i.e. $\lim_{x\to b} a^x = a^b$ for all x.



For $a\neq 1$, since $f(x)=a^x$ is either

strictly increasing or strictly decreasing

$\Rightarrow$ it is one to one $\Rightarrow$ it has an inverse

with domain $(0,\infty)$ and Range R.

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This inverse function is called $\log_a(x)$.

$f(x) = a^x = y \iff \log_a y = X$.

Properties (*) $\operatorname{Log}_a a^x = X$.

(*) $\operatorname{Log}_a \left(a^{X_1+X_2}\right) = X_1 + X_2 = \operatorname{Log}_a a^{X_1} + \operatorname{Log}_a a^{X_2}$

for all $X_1, X_2 \in \mathbb{R} \to \forall X_1, X_2 \in \mathbb{R}$

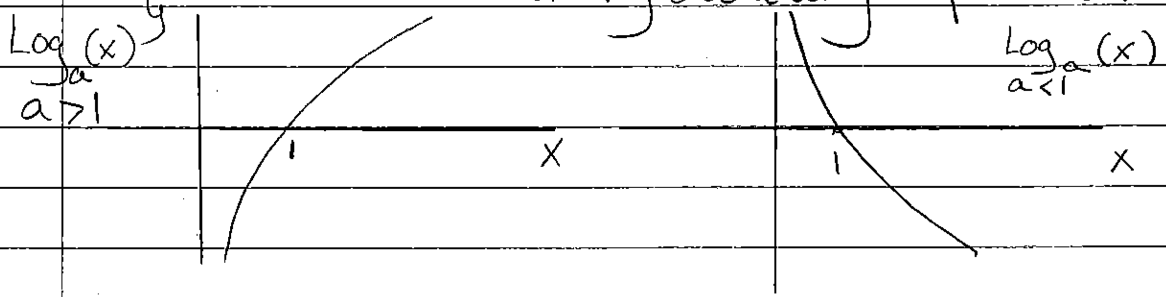
(*) $\operatorname{Log}_a(X_1 X_2) = \operatorname{Log}_a(X_1) + \operatorname{Log}_a(X_2) \quad \forall X_1, X_2 \in (0,\infty)$

(*) $\operatorname{Log}_a(1) = 0$.

(*) $\operatorname{Log}_a\left(\frac{1}{x}\right) = -\operatorname{Log}_a X \quad \forall X \in (0,\infty)$

$\operatorname{Log}_a(X)$ is strictly increasing if $a > 1$

and is strictly decreasing if $a < 1$,



Differentiation

$$f(x) = a^x \Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{a^{x+h} - a^x}{h}$$

$$= a^x \left(\frac{a^h - 1}{h}\right)$$

hence $\lim\limits_{h\to 0} \frac{a^{x+h} - a^x}{h} = a^x \lim\limits_{h\to 0} \frac{a^h - 1}{h}$

Does the limit exist?

we have $f'(x) = a^x f'(0) = f(x) f'(0)$

for all x.

what is $\lim_{h \to 0} \frac{a^h - 1}{h} = f'(0)$?

if $a > 1$ $\frac{a^h - 1}{h} > 0$

if $0 < a < 1$ $\frac{a^h - 1}{h} < 0$.

There is a unique number e such that

$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$.

So for $y = e^x$.

$\frac{dy}{dx} = \lim_{h \to 0} \frac{e^h - 1}{h} \cdot e^x = e^x$.

Maclaurin series of e^x:

$f(x) = e^x$ $f'(x) = e^x$ ... $f^{(n)}(x) = e^x$ $\forall n$.

$\Rightarrow f^{(n)}(0) = 1$.

so $e^x = 1 + x + \frac{x^2}{2} + \dots \frac{x^n}{n!} + \dots\dots$

$e = e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} + \dots = 2.7182818\dots$

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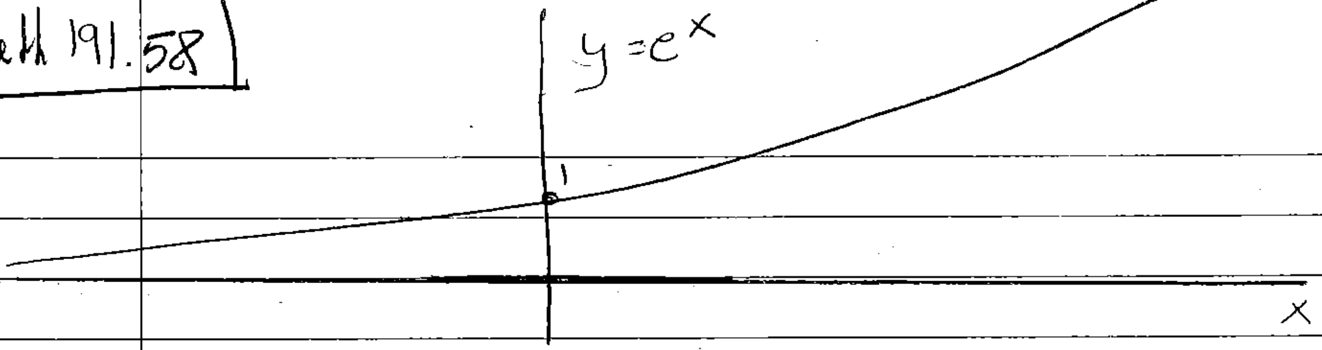


Domain $x \in (-\infty, \infty)$

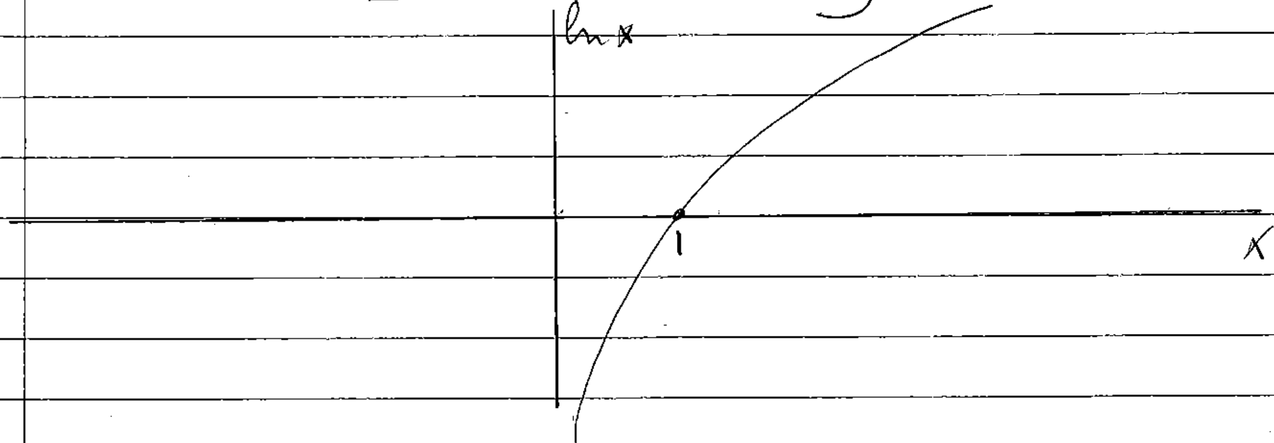
Range $y \in (0, \infty)$

e^x is strictly increasing and is neither even nor odd

<u>The natural logarithm</u>

$\ln(x)$ is the inverse function of e^x.

hence if $y = e^x \Rightarrow \ln y = \log_e e^x = x$



Domain $x \in (0, \infty)$

Range $y \in (-\infty, \infty)$

what is $\dfrac{d\ln x}{dx}$? $\rightarrow$ use the inverse rule.

if $y = \ln x = f(x) \longrightarrow \quad x = f^{-1}(y) = e^y = f(y)$.

$$\frac{dy}{dx} = \frac{1}{\frac{df}{dy}} = \frac{1}{e^y} = \frac{1}{x} \Rightarrow \frac{d\ln x}{dx} = \frac{1}{x}.$$

<u>Properties</u> → from the properties of exponentials

1. $\ln(e^{x}) = f^{-1}(f(x)) = x = x \ln_e e$

2. $e^{x_1 + x_2} = e^{x_1} e^{x_2}$.

$y_1 = e^{x_1} \quad y_2 = e^{x_2} \Rightarrow x_1 = \ln y_1 \quad x_2 = \ln y_2$

So $\ln(y_1 \cdot y_2) = \ln(e^{x_1} \cdot e^{x_2}) = \ln(e^{(x_1+x_2)}) = x_1 + x_2 = \ln y_1 + \ln y_2$.

$\Rightarrow \ln(y_1 \cdot y_2) = \ln y_1 + \ln y_2$.

<u>similarly</u>

3. $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2} \Rightarrow \ln\left(\dfrac{x_1}{x_2}\right) = \ln x_1 - \ln x_2$

4. $e^{nx} = (e^{x})^{n} \Rightarrow \ln x^{n} = n \ln x$,

<u>Example</u>, $\ln\left(\dfrac{\sqrt{10x}}{y^2}\right) = \dfrac{1}{2}(\ln 10 + \ln x) - 2 \ln y$.

<u>note:</u> $\ln x$ does not have a Maclaurin series

since $x = 0$ is not in the domain of $\ln x$.

→ <u>Maclaurin series of $\ln(1+x)$</u>

$f(x) = \ln(1+x) \qquad f(0) = \ln(1) = 0$.

$f'(x) = \dfrac{1}{1+x} \qquad f'(0) = 1$

$f''(x) = -\dfrac{1}{(1+x)^2} \qquad f''(0) = -1$

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$f'''(x) = (-1)(-2)(1+x)^{-3} \Rightarrow f'''(0) = 2.$

$f^{(4)}(x) = (-1)(-2)(-3)(1+x)^{-4} \Rightarrow f^{(4)}(0) = -6.$

$\vdots$

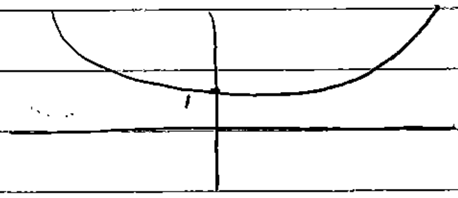
$f^{(n)}(x) = (-1)^{n+1} \cdot (n-1)! \cdot (1+x)^{-n} \Rightarrow f^{(n)}(0) = (-1)^{n+1} (n-1)!$

$\Rightarrow \ln x = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \dots$

OR $\ln x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(n+1)}$

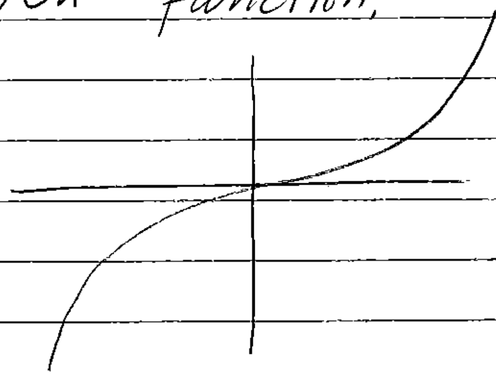
Hyperbolic functions

$\cosh(x) = \frac{e^x + e^{-x}}{2}$



$\cosh(-x) = \cosh(x) \rightarrow$ even function.

$\sinh(x) = \frac{e^x - e^{-x}}{2}$



$\sinh(-x) = -\sinh(x)$

$\rightarrow$ odd function.

$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} \rightarrow$ etc...

$\cosh(x)$: Domain $x \in \mathbb{R}$ Range: $[1, \infty)$

$\sinh(x)$: Domain $x \in \mathbb{R}$ Range: $(-\infty, \infty)$.

$\tanh(x)$: Domain $x \in \mathbb{R}$ Range: $[-1, 1]$

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Note: $\frac{d}{dx}(e^{-x}) = -e^{-x}$

$\Rightarrow \frac{d}{dx}\sinh(x) = \cosh(x)$,

$\frac{d}{dx}\cosh(x) = \sinh(x)$,

$\frac{d}{dx}\tanh(x) = \frac{d}{dx}\left(\frac{\sinh(x)}{\cosh(x)}\right) = \frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)}$,

$$= \frac{\left(\frac{e^{x}+e^{-x}}{2}\right)^2 - \left(\frac{e^{x}-e^{-x}}{2}\right)^2}{\cosh^2 x} = \frac{\frac{1}{4}(2+2)}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

$= 1 - \tanh^2 x$.

(*) every trig identity can be translated to an hyper function identity

osborn rule: product $\sin(x)\cdot\sin(y) \to -\sinh(x)\sinh(y)$,

Thus: $\sin(A+B) = \sin(A)\cos(B) + \sin(B)\cos(A)$

$\to \sinh(A+B) = \sinh(A)\cosh(B) + \sinh(B)\cosh(A)$.

But. $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

$\to \cosh(A+B) = \cosh(A)\cosh(B) + \sinh(A)\sinh(B)$.

similarly.

$\tan(A+B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$

$\to \tanh(A+B) = \frac{\tanh(A) + \tanh(B)}{1 + \tanh(A)\tanh(B)}$.

$\to$ follows from the properties of complex numbers.

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Maclaurin series of sinh(x) & cosh(x).

Note: $e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \dots$$

$$\Rightarrow \cosh(x) = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$$

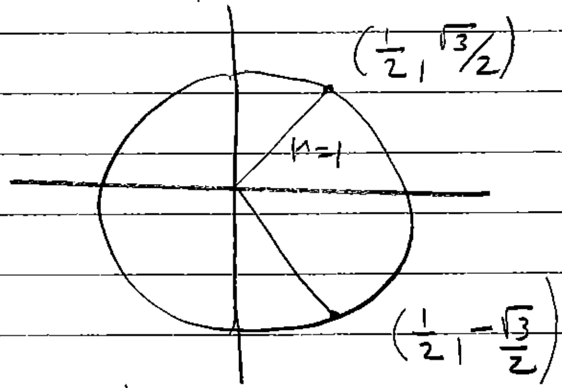
$\rightarrow$ cosh(x) $\rightarrow$ even powers $\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ compare with
sinh(x) $\rightarrow$ odd powers $\quad$ cos(x) & sin(x).

Implicit Differentiation

y is an implicit function of x but not in the for $y = f(x)$ $\rightarrow$ use implicit differentiation.

Example $x^2 + y^2 = 1$

Find the slope of the tangent at the points $(x_0, y_0) = (1/2, \sqrt{3}/2)$ and $(x_1, y_1) = (1/2, -\sqrt{3}/2)$.



can do it directly i.e. $y = \pm\sqrt{1-x^2}$

OR indirectly by implicit differentiation.

$$\frac{d}{dx}\left(x^2 + (y(x))^2\right) = \frac{d}{dx}1, = 0.$$

$$2x + 2y \cdot \frac{dy}{dx} = 0.$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}.$$

So at $(x_0, y_0) = (1/2, \sqrt{3}/2)$

$$\left.\frac{dy}{dx}\right|_{x_0} = -\frac{1/2}{\sqrt{3}/2} = -\frac{1}{\sqrt{3}} \to \text{slope at } (1/2, \sqrt{3}/2).$$

at $(x_1, y_1) = (1/2, -\sqrt{3}/2)$.

$$\left.\frac{dy}{dx}\right|_{x_1} = -\frac{1/2}{-\sqrt{3}/2} = \frac{1}{\sqrt{3}} \to \text{slope at } (1/2, -\sqrt{3}/2).$$

The general rule of implicit differentiation follows from the chain rule.

$$\frac{d f(y(x))}{dx} = \frac{df}{dy} \cdot \frac{dy}{dx}.$$

Example: Find the equation of the tangent to the curve.

$$x^3 - 4x^2 y + y^3 = 1$$

at $(x_0, y_0) = (1, 2)$.

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Recall: eq. of tangent $(y-y_0) = y'(x_0)(x-x_0)$

$\Rightarrow$ we need $y'(x_0)$.

$$\frac{d}{dx}\left(x^3 - 4x^2y + y^3\right) = \frac{d}{dx}(1) = 0.$$

$$\Rightarrow 3x^2 - 8xy - 4x^2\frac{dy}{dy}\cdot\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0.$$

$$\Rightarrow \frac{dy}{dx} = \frac{8xy - 3x^2}{3y^2 - 4x^2}.$$

At $(x_0, y_0) = (1, 2)$.

$$\left.\frac{dy}{dx}\right|_{x_0} = \frac{8\cdot 1\cdot 2 - 3\cdot 1^2}{3\cdot 2^2 - 4\cdot 1^2} = \frac{13}{8}$$

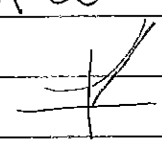
The equation for the tangent is.

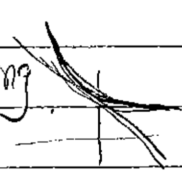
$$y = \frac{13}{8}(x-1) + 2$$

OR $\quad 8y = 13x + 3.$

Stationary points

$f'(a) \rightarrow$ slope of the tangent to the graph of $y = f(x)$ at $x = a$

Thus: when $f'(a) > 0 \Rightarrow$ graph increasing. 

$f'(a) < 0 \Rightarrow$ graph decreasing. 

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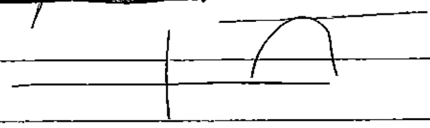
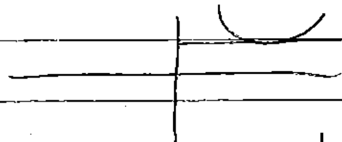
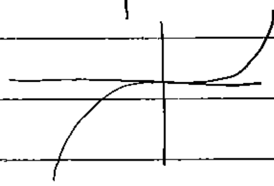
When $f'(a) = 0 \Rightarrow$ tangent to the graph at $x=a$ is horizontal.

$\Rightarrow$ $x = a$ with $f'(a)=0 \rightarrow$ stationary point.

<u>Three types of stationary points</u>

1. local Maximum. 
2. local Minimum. 
3. point of inflection. 

1. $f(a)$ is a local maximum of $f(x)$ if $f(x) \leq f(a)$ for all x near a.

$f(a)$ is a local maximum if $f'(x) > 0$ $x < a$
and $f'(x) < 0$ $x > a$.

so $f'(x)$ is decreasing near a local maximum.

hence $\underline{f''(a) < 0} \Rightarrow$ local Maximum.

<u>similarly</u> 2. $f(a)$ is a local minimum of $f(x)$.
if $f(x) \geq f(a)$ for all x near a.

$f(a)$ is a local minimum if $f'(x) < 0$ $x < a$.
and $f'(x) > 0$ $x > a$.

So $f'(x)$ is increasing near a local minimum.

$\Rightarrow$ $f''(a) > 0$ $\Rightarrow$ local minimum.

if $f''(a) = 0$ and $f'''(a) \neq 0$.

$\Rightarrow$ point of inflection.

Examples

1. Find and classify the stationary point of:

$f(x) = x^3 - 6x^2 + 9x + 2$

$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$

$f'(x) = 0 \Rightarrow x = 1;\ x = +3 \rightarrow$ stationary points.

to find the type we need $f''(x)$.

$f''(x) = 6x - 12$

$\Rightarrow f''(1) = 6 \cdot 1 - 12 = -6 < 0 \Rightarrow$ maximum.

$f''(3) = 18 - 12 = 6 > 0 \Rightarrow$ mimum.

Exercise: 1. Find $f(x) = 0$ using Newton-Rophson method.

2. Sketch the function $f(x)$ $-\infty \leq x \leq \infty$

2. Find and classify the stationary points of $f(x) = x^2 e^{-x}$

$f'(x) = 2x e^{-x} - x^2 e^{-x} = e^{-x}(2x - x^2)$

$\Rightarrow f'(x) = 0 \Rightarrow 2x - x^2 = (2-x)x = 0 \Rightarrow x = 0$ & $x = 2$,

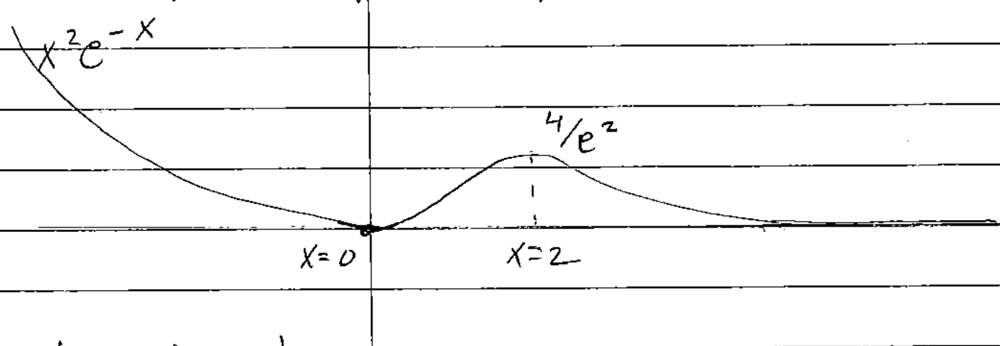
So $x=0$ and $x=2$ are stationary points.

To classify the stationary points calculate $f''(x)$

$$f''(x) = 2 \cdot e^{-x} - 2xe^{-x} - 2xe^{-x} + x^2 e^{-x}$$
$$= (2 - 4x + x^2)\, e^{-x}$$

$\Rightarrow$ At $x=0$ $f''(0) = 2 > 0 \Rightarrow$ minimum.

At $x=2$ $f''(2) = (2-8+4)e^{-2} = (-2)e^{-2} < 0 \Rightarrow$ maximum.



Graph sketching

method:

1. Maximal domain.
2. y - crosses
3. x - crosses
4. stationary points.
5. behaviour as $x \to \pm\infty$.
6. Vertical asymptotes.

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7. check where $f'(x) > 0$ and where $f'(x) < 0$.

8. sketch the graph.

Examples 1. $f(x) = x + \dfrac{1}{x}$.

$$f'(x) = 1 - \frac{1}{x^2}$$

$$f'(x) = 0 \Rightarrow 1 - \frac{1}{x^2} = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1.$$

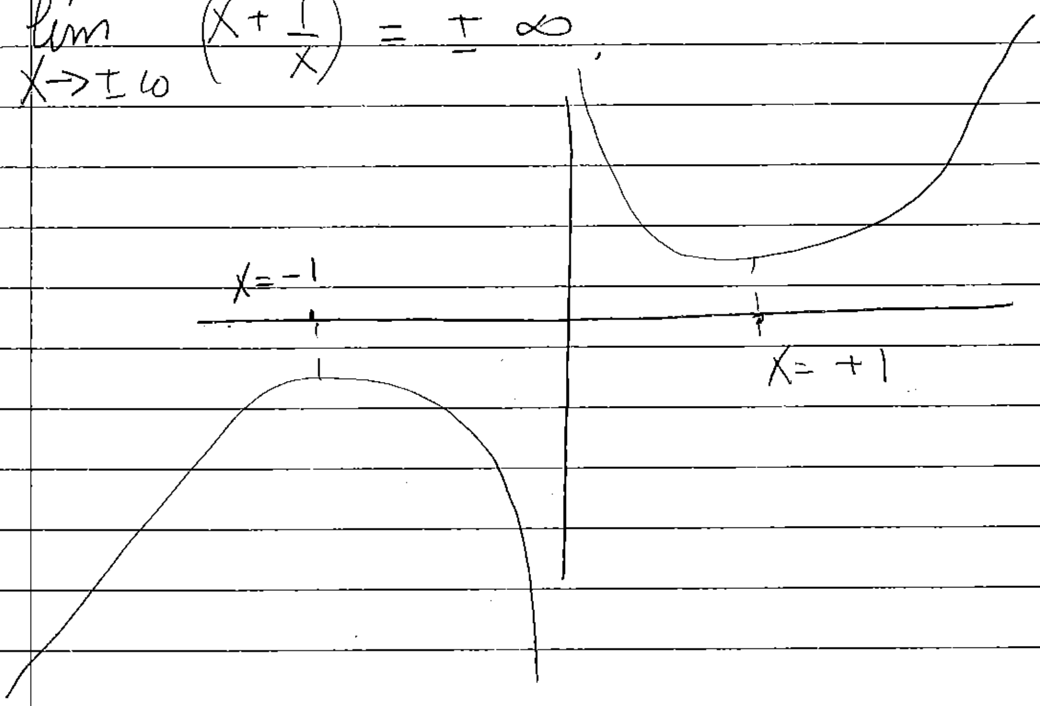
$$f''(x) = \frac{2}{x^3}$$

At $x = (-1) \Rightarrow f''(-1) = \dfrac{2}{(-1)^3} = -2 < 0 \Rightarrow$ maximum.

At $x = +1 \Rightarrow f''(+1) = 2/1^3 = 2 > 0 \Rightarrow$ minimum.

$x = 0$ is a vertical asymptote.

$$\lim_{x \to \pm\infty} \left(x + \frac{1}{x}\right) = \pm\infty.$$



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2. $f(x) = \begin{cases} \frac{4}{x-2} & x < 0 \\ x^2 - 2x - 2 & x \geq 0 \end{cases}$

Domain $x \in (-\infty, \infty)$.

No vertical asymptotes, $x=2$ is in the domain.

horizontal asymptote $y=0$ because $\lim_{x \to -\infty} f(x) = 0$.

$\lim_{x \to +\infty} f(x) = +\infty$.

$f'(x) = \begin{cases} -\frac{4}{(x-2)^2} & x < 0. \\ 2x-2 & x \geq 0. \end{cases}$

$f'(0^-) = -1 \qquad f'(0^+) = -2.$

$-1 \neq -2 \Rightarrow$ not differentiable at $x=0$

$f'(x) = 0 \Rightarrow x = 1$

$f''(x)\Big|_{x=1} = 2 > 0 \Rightarrow$ local minimum.

$f(1) = 1^2 - 2 - 2 = -3$

$f(x) = 0 \Rightarrow x^2 - 2x - 2 = 0 \Rightarrow x = \frac{2 \pm \sqrt{4+4\cdot 2}}{2} = 1 \pm \sqrt{3}$

so at $x = 1+\sqrt{3}$ $f(x)$ crosses x-axis.

at $x=0$ $y=-2$

$f(x)$ crosses y-axis