math 191.30

Differentiation.

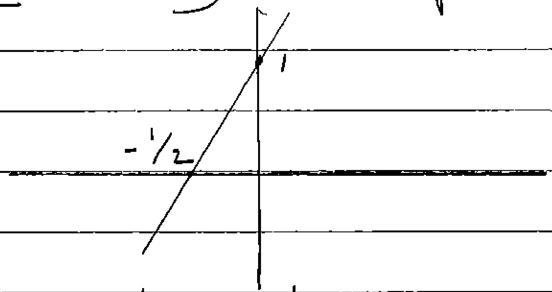
An intuitive meaning of a derivative is as slope or gradient.

Recall: eq for a straight line: $y = mx + c$

where $m \to$ slope

$c \to$ y intercept i.e. $y(\text{at } x=0) = c$

Example. $y = 2x+1$



Slope $= 2 \to$ Rate of change of y with respect to x.

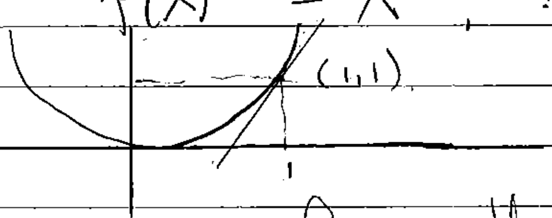
i.e. $$\frac{y(x=2) - y(x=1)}{2-1} = \frac{(2(2)+1) - (2\cdot(1)+1)}{1} = 2.$$

what happens if we take $x+\delta$ with δ any real number

$$\frac{y(x+\delta) - y(x)}{x+\delta - x} = \frac{2(x+\delta)+1-(2x+1)}{\delta} = \frac{2\delta}{\delta} = 2.$$

$\rightarrow$ same.

what about $f(x) = x^2$, ?



in this case we see from the graph

that the slope is not a constant $\Rightarrow$ it changes) math 191.3 | $f(x) = x^2$

$$\text{slope} = \frac{f(x+\delta) - f(x)}{(x+\delta) - x} = \frac{(x+\delta)^2 - x^2}{\delta} =$$

$$= \frac{\cancel{x^2} + 2\delta x + \delta^2 - \cancel{x^2}}{\delta} = 2x + \delta$$

to find the slope at any point x.

take $$\lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{(x+\delta) - x} = \lim_{\delta \to 0} 2x + \delta = 2x.$$

so at $x = 1$ $m = 2$

then we have $1 = 2 \cdot 1 + C \Rightarrow C = -1$

and the equation for the tangent is $y = 2x - 1$

in general | let $y = f(x)$ the rate of change of y with respect to x at $x = x_0$ is given by.

$$\left.\frac{dy}{dx}\right|_{x_0} = \lim_{\delta \to 0} \frac{f(x_0+\delta) - f(x_0)}{(x_0+\delta) - x_0}$$

Example $f(x) = x^3$ $(x+\delta)^3 = x^3 + 3x^2\delta + 3x\delta^2 + \delta^3$

$$\frac{f(x+\delta) - f(x)}{(x+\delta) - x} = \frac{\cancel{x^3} + 3x^2\delta + 3x\delta^2 + \delta^3 - \cancel{x^3}}{\delta} = 3x^2 + 3x\delta + \delta^2$$

$$\lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{(x+\delta) - x} = \lim_{\delta \to 0} (3x^2 + 3x\delta + \delta^2) = 3x^2$$

The rate of change is called the derivative of y WRT x or the derivative of $f(x)$ WRT x.

Geometrically $f'(x_0) = \left.\frac{df(x)}{dx}\right|_{x_0}$ is the slope of the tangent to $y = f(x)$ at $x = x_0$.

$\Rightarrow$ The equation for the tangent is $y = f'(x_0)x + c$, where c is the y-intercept.

we have $y_0 = f'(x_0)\, x_0 + c = f(x_0)$

$\Rightarrow c = f(x_0) - f'(x_0)\cdot x_0$

$\Rightarrow y = f'(x_0)\, x + f(x_0) - f'(x_0)\, x_0$

$= f'(x_0)(x - x_0) + f(x_0)$

Example Find the equation for the tangent to the curve $y = x^2$ at $x_0 = 3$.

$f'(x) = \frac{dy}{dx} = 2x$

$f'(x_0) = 2\cdot 3 = 6$, $\quad f(x_0) = 3^2 = 9$.

$\Rightarrow y = 6\cdot(x-3) + 9 = 6x - 9$.

check $9 = 18 - 9 = 9$ ✓

Example tangent to the line $y = x^3$ at $x = 0$. $f = 0$.

$f'(x) = \frac{dy}{dx} = 3x^2$ $\quad f'(0) = 0$.

$\Rightarrow y = 0\cdot x + c$ and $0 = 0 + c \Rightarrow c = 0$

hence the tangent is the line $y = 0$.

math 191.33

To find the derivative of X^n we use the :

Binomial Theorem

$$(X+y)^n = X^n + \binom{n}{1} X^{n-1} y + \binom{n}{2} X^{n-2} y^2 + \dots + \binom{n}{n-1} X y^{n-1} + \binom{n}{n} y^n$$

$$= \sum_{r=0}^{n} \binom{n}{r} X^{n-r} y^r$$

where $\binom{n}{r} = {}^nC_r = \frac{n(n-1)\cdots(n-r+1)}{1\cdot 2 \cdots r} = \frac{n!}{r!\,(n-r)!}$

we can also use Pascal triangle

1	$(X+y)^0$
1 1	$(X+y)^1$
1 2 1	$(X+y)^2$
1 3 3 1	$(X+y)^3$
1 4 6 4 1	$(X+y)^4$
1 5 10 10 5 1	$(X+y)^5$
1 6 15 20 15 6 1	$(X+y)^6$

Example $(X+y)^4 = X^4 + 4X^3y + 6X^2y^2 + 4Xy^3 + y^4$

Example Expand $(1+2X)^5$ using the binomial theorem.

$$(1+2X)^5 = 1^5 + 5\cdot 1^4\cdot(2X) + 10\cdot 1^3(2X)^2 + 10\cdot 1^2\cdot(2X)^3 + 5\cdot 1\cdot(2X)^4 + (2X)^5$$

$$= 1 + 10X + 40X^2 + 80X^3 + 80X^4 + 32X^5$$

Derivative of $f(X) = X^n$

$$(X+\delta)^n = X^n + nX^{n-1}\delta + \frac{n(n-1)}{2}X^{n-2}\delta^2 + \dots + \delta^n$$

$$\Rightarrow \frac{f(X+\delta) - f(X)}{(X+\delta) - X} = \frac{\cancel{X^n} + nX^{n-1}\delta + \frac{n(n-1)}{2}X^{n-2}\delta^2 + \dots + \delta^n - \cancel{X^n}}{\delta}$$

$$= n x^{n-1} + \delta\left(\frac{n(n-1)}{2} x^{n-2} + \cdots + \delta^{n-1}\right).$$

$$\Rightarrow \lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{x+\delta - x} = n x^{n-1}$$

$$\Rightarrow \frac{d}{dx}(x^n) = n x^{n-1}$$

Example: calculate the equation of the tangent to the graph $y = x^{28}$ at $x = 1$

$y = f(x) = x^{28}$

Eq. for the tangent $y = f(x_0) + f'(x_0)(x - x_0)$

$f'(x) = \frac{d}{dx}(x^{28}) = 28 x^{27}$

$f'(x_0) = f'(1) = 28 \cdot 1^{27}$ $\quad f(x_0) = f(1) = 1^{28} = 1$

$\Rightarrow y = 1 + 28(x - 1) = 28x - 27$

the formula is valid also when the power is not an integer, i.e. $\frac{d}{dx} x^r = r x^{r-1}$

Derivative of $\sin x$ and $\cos x$

$f(x) = \sin x.$

$$\sin(x+\delta) - \sin(x) = 2\cos\left(\frac{2x+\delta}{2}\right)\sin\left(\frac{\delta}{2}\right).$$

using trig. identity $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

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$$\Rightarrow f'(x) = \lim_{\delta \to 0} \frac{\sin(x+\delta) - \sin x}{(x+\delta) - x} = \frac{2\cos(x+\delta/2)\sin(\delta/2)}{\delta} =$$

$$= \lim_{\delta \to 0} \cos(x+\delta/2) \frac{\sin(\delta/2)}{(\delta/2)} = \cos x.$$

$$\Rightarrow \frac{d}{dx}(\sin x) = \cos x$$

Similarly $\frac{d}{dx}(\cos x) = -\sin x$ (exercise)

Example Find the equation of the tangent to the graph $y = \sin x$. at $x = 0$

$y = f(x_0) + f'(x_0)(x - x_0)$ $\to$ Eq. of the tangent

$f'(x) = \frac{d}{dx}\sin x = \cos x \Rightarrow f'(x_0) = \cos(0) = \underline{1}$

$f(x_0) = \sin(0) = 0.$

$$\Rightarrow y = 0 + 1 \cdot (x - 0) = x.$$

Rules of differentiation

if k is a constant then.

$$\frac{d}{dx}(k \cdot f(x)) = k \frac{d}{dx}(f(x)) = k \cdot f'(x).$$

e.g. $\frac{d}{dx}(3x^2) = 3 \cdot \frac{d}{dx}(x^2) = 3(2x) = 6x$

$$\frac{d}{dx}(2\sin x) = 2\frac{d}{dx}\sin x = 2\cos x.$$

The sum rule if $u = u(x)$ & $v = v(x)$

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}.$$

e.g. $\frac{d}{dx}(x^2+x^3) = 2x + 3x^2$

$$\frac{d}{dx}(x^2+2\sin x-\cos x) = 2x + 2\cos x + \sin x$$

The Product rule if $u = u(x)$ & $v = v(x)$

$$\frac{d}{dx}(u \cdot v) = \frac{du}{dx} \cdot v + u\frac{dv}{dx}.$$

OR $(u \cdot v)' = u'v + uv'$

Examples

1) Let $f(x) = x^2 \sin x$ $\quad u = x^2 \quad v = \sin x.$

$u' = 2x \quad v' = \cos x.$

$\Rightarrow f' = 2x\sin x + x^2\cos x$

2) Let $f(x) = \cos^2 x = \cos x \cos x$ $\quad u = \cos x \quad v = \cos x.$

$u' = -\sin x \quad v' = -\sin x.$

$\Rightarrow f'(x) = -\sin x\cos x - \sin x\cos x = -2\sin x\cos x = -\sin 2x$

3) Let $f(x) = x^2\sin x\cos x$ with $u = x^2\sin x$ $\quad v = \cos x$

Then $u' = 2x\sin x + x^2\cos x \qquad v' = -\sin x.$

$$(uv)' = u'v + uv' = (2x\sin x + x^2\cos x)\cos x + x^2\sin x(-\sin x)$$

$$= 2x\sin x\cos x + x^2\cos^2 x - x^2\sin^2 x =$$

$$= x\sin 2x + x^2(\cos^2 x - \sin^2 x) = x\sin 2x + x^2\cos 2x.$$

The quotient rule if $u = u(x)$ & $v = v(x)$.

Then $$\frac{d}{dx}\left(\frac{u}{v}\right) = \left(\frac{u}{v}\right)' = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2} = \frac{u'v - uv'}{v^2}$$

Examples 1. $f(x) = 1/x \qquad u = 1 \qquad v = x$

$u' = 0 \qquad v' = 1$

$$\Rightarrow (1/x)' = \frac{0-1}{x^2} = -1/x^2.$$

2. $f(x) = \frac{\sin x}{\cos x} \qquad u = \sin x \qquad v = \cos x$

$u' = \cos x \qquad v' = -\sin x.$

$$\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x\cdot\cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$= \sec^2 x.$$

3. $f(x) = \frac{1}{x^n} \qquad u = 1 \qquad v = x^n$

$u' = 0 \qquad v' = nx^{n-1}$

$$\Rightarrow (1/x^n)' = \frac{0 - nx^{n-1}}{x^{2n}} = -nx^{n-1-2n} = -nx^{-n-1}$$

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OR $(x^{-n})' = -n x^{-n-1}$

in general For any $r \in \mathbb{R}$ $(x^{r})' = r x^{r-1}$

4. $f(x) = \sqrt{x} = x^{1/2} \Rightarrow f'(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$

5. $f(x) = \frac{1}{\sqrt[3]{x}} = x^{-1/3} \Rightarrow f'(x) = \frac{-1}{3} x^{-1/3-1} = -\frac{1}{3} x^{-4/3}$

The chain rule

Let $f(x) = g(h(x))$

then $f'(x) = g'(h(x)) \cdot h'(x)$

or $\frac{df}{dx} = \frac{dg}{dh} \cdot \frac{dh}{dx}$

Examples 1. $f(x) = (4x-1)^3$ $\quad g(h) = h^3 \quad h = 4x-1$

$\frac{dg}{dh} = 3h^2 \quad \frac{dh}{dx} = 4$

$\Rightarrow \frac{df}{dx} = \frac{dg}{dh} \cdot \frac{dh}{x} = 3h^2 \cdot 4 = 12 \cdot (4x-1)^2$

2. $f(x) = \sin(3x+2)$ $\quad g(h) = \sin h \quad h = 3x+2$

$\frac{dg}{dh} = \cos h \quad \frac{dh}{dx} = 3$

$\frac{df}{dx} = \frac{dg}{dh} \frac{dh}{dx} = \cos h \cdot 3 = 3\cos(3x+2)$

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More generally. $\frac{d}{dx}\left(\sin(ax+b)\right) = a\cos(ax+b)$.

$$\frac{d}{dx}\left(\cos(ax+b)\right) = -a\sin(ax+b).$$

3. $f(x) = (\sin x + \cos 3x)^3 \qquad g(h) = h^3 \qquad h(x) = \sin x + \cos 3x$.

$$\frac{dg}{dh} = 3h^2 \qquad \frac{dh}{dx} = \cos x - 3\sin 3x$$

$$\Rightarrow \frac{df}{dx} = \frac{dg}{dh}\frac{dh}{dx} = 3h^2 \cdot (\cos x - 3\sin 3x) =$$

$$= 3(\sin x + \cos 3x)^2 (\cos x - 3\sin 3x).$$

4. $f(x) = \tan\left((\sin x + \cos 3x)^3\right). = f(v(u(x)))$

Let $f = \tan v \qquad v = u^3 \qquad u = \sin x + \cos 3x$.

$$\frac{df}{dv} = \sec^2 v \qquad \frac{dv}{du} = 3u^2 \qquad \frac{du}{dx} = (\cos x - 3\sin 3x)$$

$$\frac{df}{dx} = \frac{df}{dv}\frac{dv}{du}\frac{du}{dx} =$$

$$= \sec^2 v \cdot 3u^2 \cdot (\cos x - 3\sin 3x) =$$

$$= \sec^2\left((\sin x + \cos 3x)^3\right) \cdot 3(\sin x + \cos 3x)^2 \cdot (\cos x - 3\sin 3x)$$

The inverse function rule

Let $y = f^{-1}(x)$ such that $x = f(y)$.

Then $\frac{dy}{dx} = \frac{1}{f'(y)}$.

math 191.401

Follows from the chain rule.

Recall: $f(f^{-1}(x)) = x$.

OR $f(y(x)) = x$.

then $\frac{df}{dx} = \frac{df}{dy}\frac{dy}{dx} = 1$

OR $\frac{dy}{dx} = \frac{1}{\left(\frac{df}{dy}\right)} = \frac{1}{f'(y)}$

Examples 1. $y = +\sqrt{x}$ $\quad x = y^2 = f(y) \Rightarrow \frac{df}{dy} = 2y$.
$y \in [0, \infty)$

$\frac{dy}{dx} = \frac{1}{2y} = \frac{1}{2\sqrt{x}}$. $\rightarrow$ agrees with what we got before.

2. $y = \sin^{-1}(x)$ $\quad x = \sin y = f(y) \Rightarrow \frac{df}{dy} = \cos y$.

$\frac{dy}{dx} = \frac{1}{(df/dy)} = \frac{1}{\cos y} = \frac{1}{(1-\sin^2 y)^{1/2}} = \frac{1}{(1-x^2)^{1/2}}$

$\Rightarrow \frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$

3. similarly: $\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$ (exercise)

4. $y = \tan^{-1} x$ $\quad x = \tan y = f(y) \Rightarrow \frac{df}{dy} = \sec^2 y$.

$\Rightarrow \frac{dy}{dx} = \frac{1}{(df/dy)} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$

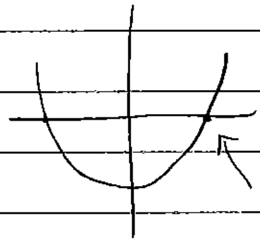
$\Rightarrow \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

math 191.4

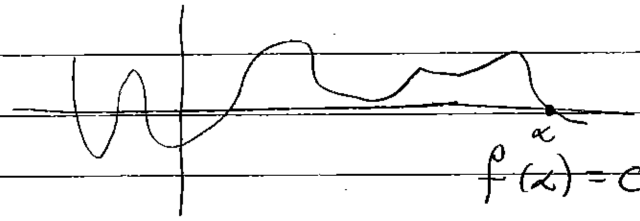
An Application: The Newton-Raphson method.

→ Find approximate solutions for the roots of $f(x)=0$.

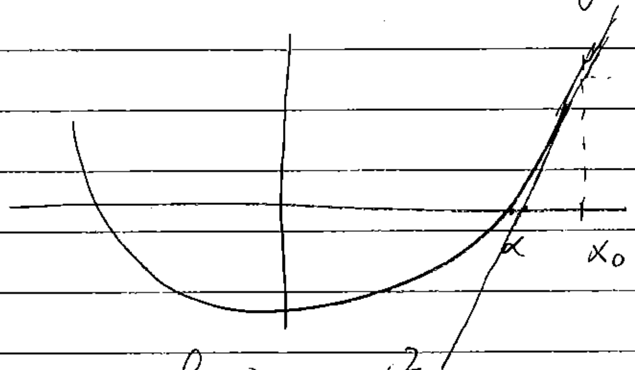
Example



OR



→ start with an initial guess $x_0 \in \mathbb{R}$.



e.g. for $f(x) = x^2 - 2$

start with $x_0 = 1$ or $x_0 = 2$. $(1<\sqrt{2}<2)$

The equation for the tangent is:

$$y = f'(x_0)(x - x_0) + f(x_0).$$

The tangent crosses the x-axis when $y=0$.

$$0 = f'(x_0)(x_1 - x_0) + f(x_0)$$

SO $\quad x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$.

NOW Take x_1 as the new guess for α.

$$\Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

we can repeat as many times as we like. $x_3, x_4 \ldots$

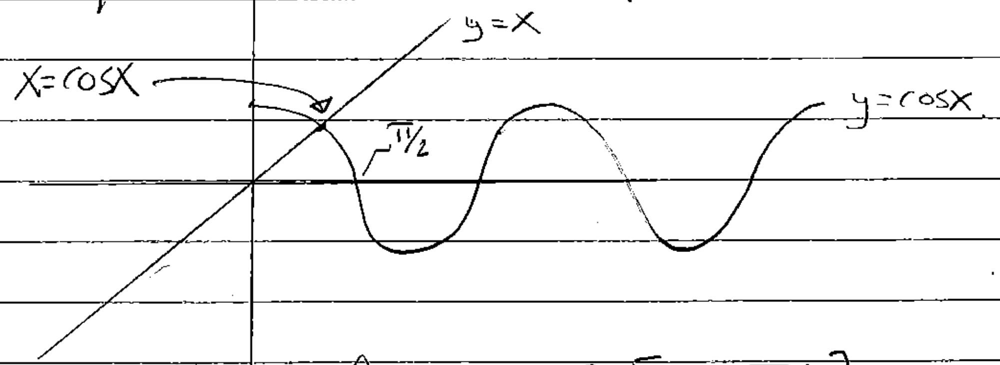
in general $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

The values of x_n then $x_n \longrightarrow \alpha$,

Example

consider the equation $x = \cos x$.

Graph



There is a solution for $x \in [0, \pi/2]$

Take $x_0 = 1$

Write $f(x) = x - \cos(x)$

Then $f'(x) = 1 + \sin(x)$,

$$\Rightarrow \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n - \cos(x_n)}{1 + \sin(x_n)}$$

So $$x_1 = x_0 - \frac{x_0 - \cos(x_0)}{1 + \sin(x_0)} =$$

$$= 1 - \frac{1 - \cos(1)}{1 + \sin(1)} = 0.750364.$$

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This should be a better approximation to the solution than X_0.

Next: $X_2 = X_1 - \dfrac{X_1 - \cos X_1}{1+\sin X_1} = 0.750364 - \dfrac{0.750364 - \cos(0.750364)}{1+\sin(0.750364)} = 0.739113$

$X_3 = X_2 - \dfrac{X_2 - \cos(X_2)}{1+\sin(X_2)} = 0.739085$

$X_4 = X_3 - \dfrac{X_3 - \cos(X_3)}{1+\sin(X_3)} = 0.739085$

$\Rightarrow$ the solution is $X = 0.739085$ to six decimal places.

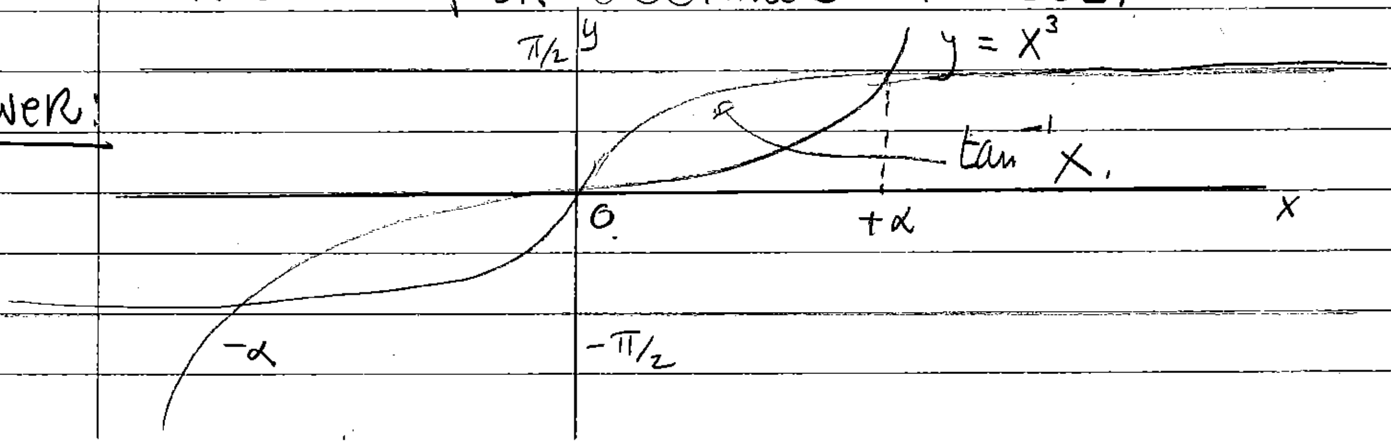
note that we got this for X_3 but had to calculate X_4 to know that it is accurate to six decimal places

Example: Show graphically that the equation

$X^3 = \tan^{-1}(X)$

has three solutions, and find an approximation to the positive solution, correct to four decimal places.

Answer:



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From the graph it is noted that there are three solutions : $X=0$ and $X=\pm\alpha$

take $x_0 = 2$.

$$f(x) = x^3 - \tan^{-1}(x) = 0.$$

$$f'(x) = 3x^2 - \frac{1}{1+x^2}$$

$$\Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - \tan^{-1}(x_n)}{3x_n^2 - \frac{1}{1+x_n^2}}$$

$$\Rightarrow x_1 = 2 - \frac{2^3 - \tan^{-1}(2)}{3\cdot 2^2 - \frac{1}{1+2^2}} = 2 - \frac{8 - 1.107149}{12 - 1/5} = 2 - \frac{6.892851}{11.8} = 1.415860$$

$$\Rightarrow x_2 = x_1 - \frac{x_1^3 - \tan^{-1}(x_1)}{3x_1^2 - \frac{1}{1+x_1^2}} = 1.084510$$

$$\Rightarrow x_3 = x_2 - \frac{x_2^3 - \tan^{-1}(x_2)}{3x_2^2 - \frac{1}{1+x_2^2}} = 0.937997$$

$$x_4 = x_3 - \frac{x_3^3 - \tan^{-1}(x_3)}{3x_3^2 - \frac{1}{1+x_3^2}} = 0.903896$$

$$x_5 = x_4 - \frac{x_4^3 - \tan^{-1}(x_4)}{3x_4^2 - \frac{1}{1+x_4^2}} = 0.902031$$

$$x_6 = x_5 - \frac{x_5^3 - \tan^{-1}(x_5)}{3x_5^2 - \frac{1}{1+x_5^2}} = 0.902025$$

$\Rightarrow$ solution is $x = 0.902025 \rightarrow$ correct to 4 decimals.

math 191.45

Differentiability

$f'(a)$ at a point $x=a$

$\rightarrow$ slope or tangent to the graph $y=f(x)$ at $x=a$.

if the tangent is not well defined at $x=a$.

OR if $f(x)$ is not continuous at $x=a$.

then $\Rightarrow$ $f(x)$ is not differentiable at $x=a$.

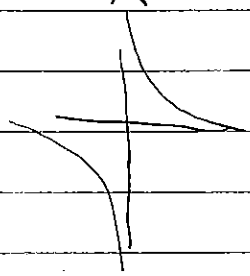
$\Rightarrow$ $f(x)$ is differentiable at $x=a$. if:

a) $f(x)$ is continuous at $x=a$,

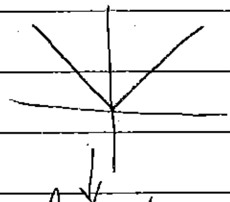
b) the graph $y=f(x)$ has a well defined tangent at $x=a$ (i.e. not vertical).

$f(x)$ is differentiable if it is differential at $x=a$ for every value of a.

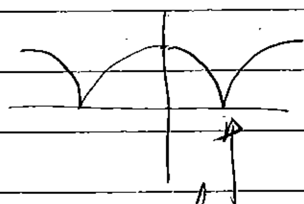
Examples: $\frac{1}{x}$, $|x|$, $|\sin x|$.



vertical slope discontinuous



slope discontinous at $x=0$.



slope discontinuous.

math 191. H6

Higher derivatives

$f'(x) = g(x) \rightarrow$ function of x.

$$g'(x) = \frac{d}{dx}(g(x)) = \frac{d}{dx}\left(\frac{d}{dx}f(x)\right) = \frac{d^2}{dx^2}(f(x)) = f''(x)$$

→ second derivative → Rate of change of $f'(x)$ wrt x.

we can Repeat: $f''(x) = f^{(2)}(x) = h(x)$.

$$h'(x) = \frac{d}{dx}\left(h(x)\right) = \frac{d}{dx}\left(f''(x)\right) = \frac{d}{dx}\left(\frac{d^2 f(x)}{dx^2}\right) = \frac{d^3 f(x)}{dx^3} =$$

$$= f'''(x) = f^{(3)}(x)$$

in general the n^{th} derivative is denoted by: $f^{(n)}(x)$ or $\frac{d^n f}{dx^n}$

$f^{(n)}(x)$ is obtain by differentiating $f(x)$ n-times

we say that $f(x)$ is n-times differentiable if it is possible to differentiate it n-times in succession.

→ $f(x)$ is infinitely differentiable OR smooth if there is no limit to the number of times that it can be differentiated.

Examples 1. $f(x) = x^3 + 2x^2 + 3x + 1$

$f'(x) = 3x^2 + 4x + 3$

$f''(x) = 6x + 4$

$f'''(x) = 6$

$f^{(n)}(x) = 0$ for all $n \geq 4 \Rightarrow f(x)$ is smooth.

2. $f(x) = \sin x$

$f'(x) = \cos x \quad f''(x) = -\sin x \quad f'''(x) = -\cos x \quad f^{(4)}(x) = \sin x$

$\Rightarrow f(x)$ is smooth.

3. $f(x) = \frac{1}{x} = x^{-1}$

$f'(x) = -x^{-2}; \quad f''(x) = +2x^{-3} \quad f'''(x) = -6x^{-4}$

$f^{(4)}(x) = 24x^{-5}; \rightarrow f^{(n)}(x) = (-1)^n n!\, x^{-n-1}$

$\rightarrow f(x)$ isn't differentiable at $x = 0$.

but is smooth everywhere else.

Maclaurin Series and Taylor series

Basic idea $\rightarrow$ Represent functions by infinite power series.

suppose that $f(x)$ is a smooth function

and suppose that $f(x)$ can be written as

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

OR $$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

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Then $\rightarrow$ we can find the coefficients a_k by repeatedly differentiating $f(x)$

$$f(0) = a_0$$

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots \Rightarrow f'(0) = a_1$$

$$f''(x) = 2a_2 + 3\cdot 2\cdot a_3 x + 4\cdot 3\cdot a_4 x^2 + \cdots \Rightarrow f''(0) = 2a_2$$

OR $$a_2 = \frac{f''(0)}{2}$$

$$f'''(x) = 6a_3 + 4\cdot 3\cdot 2a_4 x + \cdots \Rightarrow f'''(0) = 6a_3 \quad \text{OR} \quad a_3 = \frac{f'''(0)}{6}$$

$$f^{(4)}(x) = 24a_4 + \cdots \Rightarrow f^{(4)}(0) = 24a_4 \quad \text{OR} \quad a_4 = \frac{f^{(4)}(0)}{24}$$

in general $$a_n = \frac{f^{(n)}(0)}{n!}$$

so $$f(x) = f(0) + f^{(1)}(0)\,x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

OR $$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\,x^k \qquad \text{where } 0! = 1$$

This is the Maclaurin series expansion of $f(x)$.

Note: we simply assumed that the expansion exist. we still need to determine when it is valid.