MATH181 Solution Sheet 8

1. *Calculate the gradient of*

$$F(x,y) = 3x^2 + 3y^2 - y^3$$

and find its stationary points. Classify these points by using the second derivatives, $F_{xx}, F_{yy}, F_{x,y}$, *and* F_{yx}.

$$\begin{aligned} F_x = \frac{\partial F}{\partial x} &= 6x \\ F_y = \frac{\partial F}{\partial y} &= 6y - 3y^2 = 3y(2-y) \end{aligned}$$

The gradient is the vector (F_x, F_y):

$$\nabla F = (6x,\ 6y - 3y^2) \quad \text{or} \quad 6x\,\mathbf{i} + (6y - 3y^2)\,\mathbf{j}$$

The stationary points are the points where $\nabla F = (0,0)$, i.e. $6x = 0$ and $3y(2-y) = 0$,

$$(0,0) \quad \text{and} \quad (0,2)$$

Classify using the second derivatives

$$F_{xx} = 6,\ F_{xy} = F_{yx} = 0,\ F_{yy} = 6 - 6y$$

At $(0,0)$, $F_{xx} = 6, F_{xy} = 0, F_{yy} = 6$ so $D = F_{xx}F_{yy} - F_{xy}^2 = 36 > 0$. Because $D > 0$ we don't have a saddle point, must be a max or min. F_{xx} and F_{yy} are both greater than 0, so

$(0,0)$ is a local minimum .

At $(0,2)$, $F_{xx} = 6, F_{xy} = 0, F_{yy} = -6$ so $D = F_{xx}F_{yy} - F_{xy}^2 = -36 < 0$. Because $D < 0$

$(0,2)$ is a saddle point.

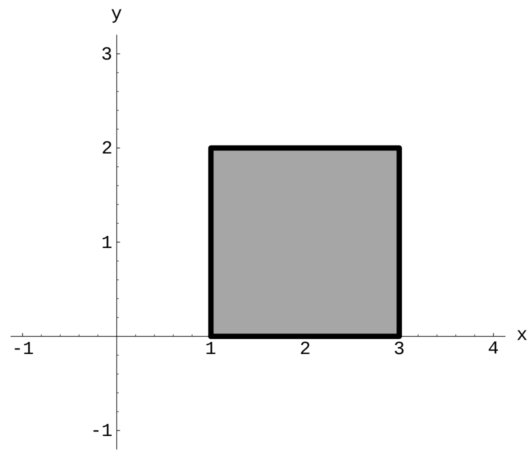
Remember to do the saddle point test from the sign of D *first*, and do the max/min test only if the point is not a saddle point.

2. *Evaluate the repeated integral*

$$\int_1^3 dx \int_0^2 dy\ (3y^2 - 2xy)$$

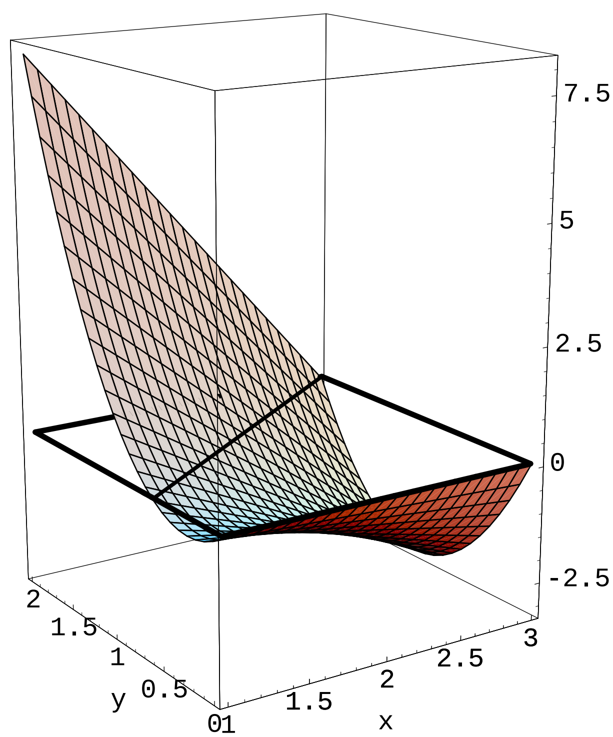
On a sketch of the $x - y$ plane, shade in the region this integral covers. What shape is it?

We integrated over the shaded region below, which is a square. $1 < x < 3,\ 0 < y < 2$.



$$\int_1^3 dx \int_0^2 dy\, (3y^2 - 2xy) = \int_1^3 dx \left[y^3 - xy^2\right]_{y=0}^{y=2}$$
$$= \int_1^3 dx\, (8 - 4x)$$
$$= \left[8x - 2x^2\right]_1^3 = 6 - 6 = \underline{\underline{0}}$$

The answer 0 surprises some people, but it is perfectly sensible. The function $(3y^2 - 2xy)$ is negative in part of the region, positive in another part — when we integrate, the contributions from the two parts happen to cancel exactly — the average value of the function in the shaded square is 0.



The surface $z = (3y^2 - 2xy)$.

Another way of putting this: if we took a piece of flat land, and landscaped it to have a height $3y^2 - 2xy$, the amount of earth we would dig out to make the "valley" in the south part of our region is exactly the same as the amount of earth needed to make the "hill" in the north part; we wouldn't have to bring in any earth, we wouldn't have any earth left over — so the volume is zero.

3. *A rectangular swimming pool covers the area* $0 < x < 10, \quad 0 < y < 20$ *and has a depth given by* $d(x,y) = 1 + \frac{1}{10}y$. *By doing a repeated integral, find out how much water is needed to fill the pool. [All measurements are in metres.]*

To find the volume of water we need to do the integral

$$V = \int_0^{10} dx \int_0^{20} dy \; d(x,y).$$

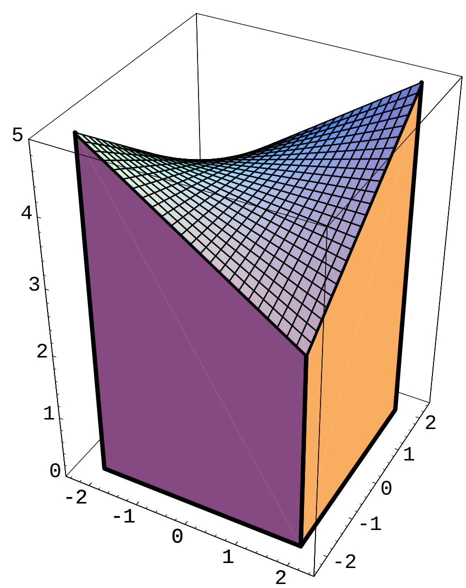
$$\begin{aligned} V = \int_0^{10} dx \int_0^{20} dy \left(1 + \frac{1}{10}y\right) &= \int_0^{10} dx \left[y + \frac{1}{20}y^2\right]_0^{20} \\ &= \int_0^{10} 40 \; dx = \Big[\; 40 \; x \; \Big]_0^{10} = \underline{\underline{400\text{m}^3}} \end{aligned}$$

The answer makes sense - the pool is 1 m deep at the shallow end, 3 m at the deep end, and since the bottom slopes evenly, the average depth is 2 m.

4. *The tea room in theoretical physics has a curved ceiling with height*

$$h(x,y) = 4 + \frac{xy}{4}.$$

The room is a square, with $-2 < x < 2, \quad -2 < y < 2$. *What is the volume of the room? What is the average height of the ceiling? [All measurements are in metres.]*



We need to work out

$$V = \int_{-2}^{2} dx \int_{-2}^{2} dy\, h(x,y) = \int_{-2}^{2} dx \int_{-2}^{2} dy \left(4 + \frac{xy}{4}\right)$$
$$= \int_{-2}^{2} dx \left[4y + \frac{xy^2}{8}\right]_{-2}^{2}$$
$$= \int_{-2}^{2} dx \left(\left(8 + \frac{x}{2}\right) - \left(-8 + \frac{x}{2}\right)\right)$$
$$= \int_{-2}^{2} 16\, dx = \left[\ 16x\ \right]_{-2}^{2} = \underline{\underline{64 \text{ m}^3}}$$

for the volume. For the average height we just need to divide the volume of the room by the floor area, giving

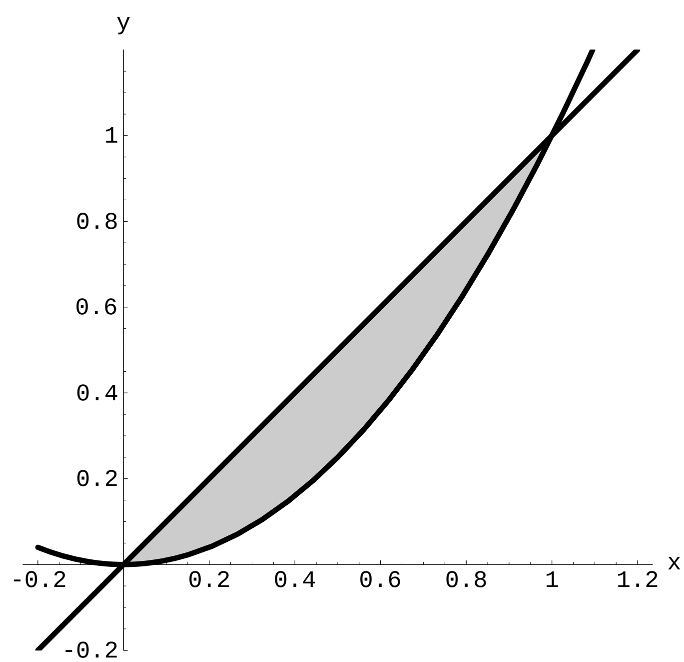
$$\underline{\underline{\text{Average height} = \frac{64}{4^2} \text{ m} = 4 \text{ m}}}$$

5. *Integrate the function*

$$g(x,y) = 2x^2 + y$$

over the region A bounded by the curves $y = x$ and $y = x^2$.

Up till now we've integrated over rectangles, but in this question the region we need to integrate over is more complicated - it's shaded in this diagram.



From the diagram we can see that the equation for the upper edge of the region is $y = x$, the lower edge is $y = x^2$. These boundaries give the limits on the inner integral (the y integral).

$$\int_0^1 dx \int_{x^2}^{x} dy\, g(x,y).$$

The limits of the y integration depend on x, now that our region is not simply a rectangle.

$$\begin{aligned}
\int_0^1 dx \int_{x^2}^{x} dy \left(2x^2 + y\right) &= \int_0^1 dx \left[2x^2 y + \frac{1}{2} y^2\right]_{x^2}^{x} \\
&= \int_0^1 dx \left(2x^3 + \frac{1}{2}x^2 - 2x^4 - \frac{1}{2}x^4\right) \\
&= \left[\frac{1}{2}x^4 + \frac{1}{6}x^3 - \frac{1}{2}x^5\right]_0^1 = \underline{\underline{\frac{1}{6}}}
\end{aligned}$$

6. *The temperature in a metal cube* $0 < x < L, \quad 0 < y < L, \quad 0 < z < L,$ *is*

$$T(x, y, z) = 50 + 20 \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi y}{L}\right) \sin\left(\frac{\pi z}{L}\right) .$$

What is the average temperature in the cube?

To find the average temperature, we integrate temperature over the whole cube, and then divide by the volume (L^3).

$$\begin{aligned}
\text{Average } T &= \frac{1}{V} \int_0^L dx \int_0^L dy \int_0^L dz \left(50 + 20 \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi y}{L}\right) \sin\left(\frac{\pi z}{L}\right)\right) \\
&= \frac{1}{L^3} \int_0^L dx \int_0^L dy \left[50z - 20 \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi y}{L}\right) \frac{L}{\pi} \cos\left(\frac{\pi z}{L}\right)\right]_{z=0}^{z=L} \\
&= \frac{1}{L^3} \int_0^L dx \int_0^L dy \left(50L + 20 \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi y}{L}\right) \frac{2L}{\pi}\right) \\
&= \frac{1}{L^3} \int_0^L dx \left[50Ly - 20 \sin\left(\frac{\pi x}{L}\right) \frac{L}{\pi} \cos\left(\frac{\pi y}{L}\right) \frac{2L}{\pi}\right]_{y=0}^{y=L} \\
&= \frac{1}{L^3} \int_0^L dx \left(50L^2 + 20 \sin\left(\frac{\pi x}{L}\right) \left(\frac{2L}{\pi}\right)^2\right) \\
&= \left[50L^2 x - 20 \frac{L}{\pi} \cos\left(\frac{\pi x}{L}\right) \left(\frac{2L}{\pi}\right)^2\right]_{x=0}^{x=L} \\
&= \frac{1}{L^3} \left(50L^3 + 20 \left(\frac{2L}{\pi}\right)^3\right) = \underline{\underline{50 + \frac{160}{\pi^3} = 55.16}}
\end{aligned}$$

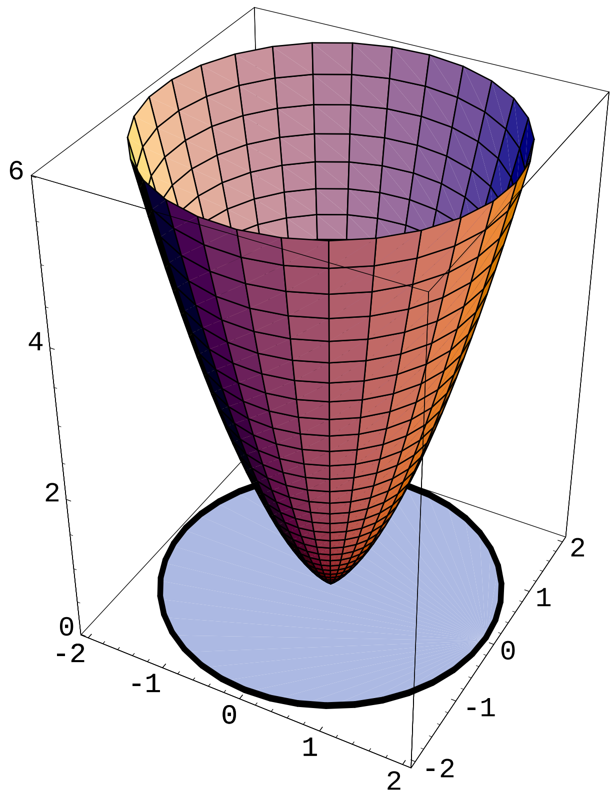
This is a sensible answer. The temperature distribution gives the cube a temperature of 70 in the centre, and 50 on the outside, so 55.16 is a plausible average.

7. *Use polar coordinates to integrate*

$$F(x, y) = x^2 + y^2 + \sqrt{x^2 + y^2}$$

over the area A *enclosed by the curve* $x^2 + y^2 = 4$

The integral corresponds to finding the volume enclosed between the surface $z = F(x, y)$ and the surface $z = 0$, in the circular region shown in the figure, the disk of radius 2. You can see from the figure that the region we integrate over, and the function F, are both rotationally symmetric, so the calculation will probably be easier in polar coordinates.



In polar coordinates

$$F(x,y) = x^2 + y^2 + \sqrt{(x^2+y^2)} \Leftrightarrow F(r,\theta) = r^2 + r.$$

The boundary of the circle is $r = 2$, the interior of the circle is $0 \leq r < 2$ (r, the distance from the origin, can't be less than 0). So the region we integrate over becomes $0 < r < 2,\ 0 < \theta < 2\pi$.

As explained in the lectures, we have to replace $dxdy$ by $rdrd\theta$, (**important!**).

So our integral becomes

$$\begin{aligned} I = \int_0^2 dr \int_0^{2\pi} d\theta\, rF(r,\theta) &= \int_0^2 dr \int_0^{2\pi} d\theta\, r(r^2+r) \\ &= \int_0^2 dr\, 2\pi(r^3+r^2) = 2\pi\left[\frac{1}{4}r^4 + \frac{1}{3}r^3\right]_0^2 = \underline{\underline{\frac{40}{3}\,\pi}} \end{aligned}$$

The answer looks about right. The area of the region we integrate over (the circle of radius 2) is 4π. So a volume of $\frac{40}{3}\,\pi$ corresponds to F having an average value of $\frac{10}{3} = 3.33333\cdots$. From the picture, this is reasonable.