MATH181 Solution Sheet 7

See Stroud, Chapter 10, Chapter 11.

1. *Given that* $f(x,y) = y\sin(xy)$ *find the partial derivatives*

$$\frac{\partial f}{\partial x},\quad \frac{\partial f}{\partial y},\quad \frac{\partial^2 f}{\partial x^2},\quad \frac{\partial^2 f}{\partial y \partial x},\quad \frac{\partial^2 f}{\partial x \partial y},\quad \text{and} \quad \frac{\partial^2 f}{\partial y^2}$$

Do the single derivatives first:

$$\begin{aligned} \frac{\partial f}{\partial x} &= y^2\cos(xy) \\ \frac{\partial f}{\partial y} &= \sin(xy) + xy\cos(xy) && \text{product rule} \end{aligned}$$

Now we can do all the second derivatives:

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(y^2\cos(xy)\right) = -y^3\sin(xy) \\ \frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(y^2\cos(xy)\right) = 2y\cos(xy) - xy^2\sin(xy) \\ \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\sin(xy) + xy\cos(xy)\right) = 2y\cos(xy) - xy^2\sin(xy) \\ \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left(\sin(xy) + xy\cos(xy)\right) = 2x\cos(xy) - x^2y\sin(xy) \end{aligned}$$

A useful piece of shorthand is to write f_x for $\frac{\partial f}{\partial x}$, f_{xx} for $\frac{\partial^2 f}{\partial x^2}$, f_{xy} for $\frac{\partial^2 f}{\partial x \partial y}$, and so on.

2. *A hill has a height given by*

$$h(x,y) = Ae^{-x^2-2y^2}\ .$$

Find the gradient of h.

The gradient (∇h) is the vector $h_x\,\mathbf{i} + h_y\,\mathbf{j}$.

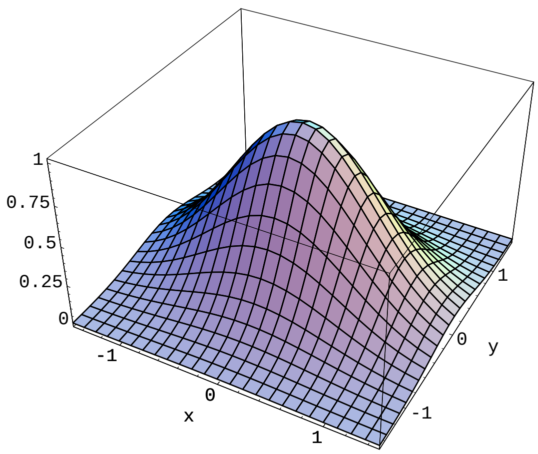
$$\begin{aligned} h_x &= -2xAe^{-x^2-2y^2} \\ h_y &= -4yAe^{-x^2-2y^2} \\ \nabla h &= -(2x\,\mathbf{i} + 4y\,\mathbf{j})Ae^{-x^2-2y^2} \end{aligned}$$

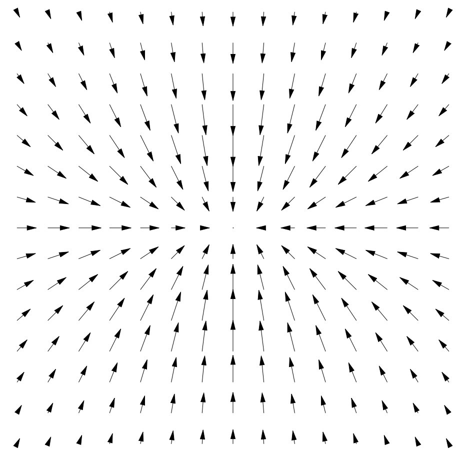
A child at the point $(2,1)$ *drops a ball. Which direction will the ball roll initially?*

The ball will roll in the steepest downhill direction. The gradient points in the steepest up-hill direction, so the ball will roll in the direction opposite to ∇h.

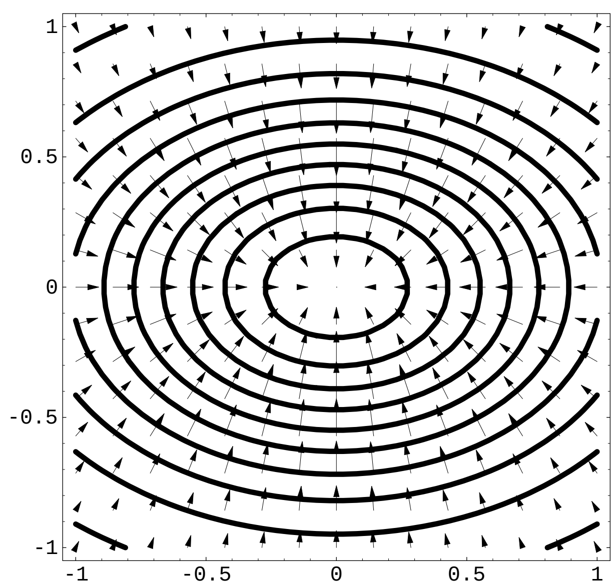
At $x = 2, y = 1$ the gradient is $(-4\mathbf{i} - 4\mathbf{j})Ae^{-6}$, so the ball will roll in the direction ($\mathbf{i}$ + $\mathbf{j}$), ie NE (if we take x as east, and y as north).

To help you see the relation between the shape of the hill and the gradient vector, here are some pictures of the hill, its gradient, and the contours on a map.





The summit of the hill is at $(0,0)$, the gradient vectors point uphill from any point, they are largest where the hill is steepest. The gradient vector is zero at the summit.



The contours join points of equal height. They are closest together where the hill is steepest. The gradient vector is always at right angles to the contour lines.

3. *Show that if* $V(x,y) = e^{2x}\cos 2y$*, then*

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0\ .$$

Work out all the derivatives we need, and substitute them into the differential equation.

$$\begin{aligned} V_x &= 2e^{2x}\cos 2y & \qquad V_y &= -2e^{2x}\sin 2y \\ V_{xx} &= 4e^{2x}\cos 2y & \qquad V_{yy} &= -4e^{2x}\cos 2y \end{aligned}$$

so

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 4e^{2x}\cos 2y - 4e^{2x}\cos 2y = 0$$

so V satisfies this differential equation (which is called the Laplace equation).

4. *Show that* $F(x,t) = a\sin^3(x - ct)$*, is a solution of the wave equation:*

$$\frac{\partial^2}{\partial t^2}F = c^2\frac{\partial^2}{\partial x^2}F\ .$$

$$\begin{aligned} F &= a\sin^3(x-ct) \\ F_x &= 3a\cos(x-ct)\sin^2(x-ct) \\ F_{xx} &= 6a\cos^2(x-ct)\sin(x-ct) - 3a\sin^3(x-ct) \\ F_t &= -3ac\cos(x-ct)\sin^2(x-ct) \\ F_{tt} &= 6ac^2\cos^2(x-ct)\sin(x-ct) - 3ac^2\sin^3(x-ct) \end{aligned}$$

So

$$\frac{\partial^2}{\partial t^2}F = c^2\frac{\partial^2}{\partial x^2}F\ .$$

is true for this F.

5. *A cylinder of ice is slowly melting. At a given time the cylinder has a radius of 100 cm and a height of 200 cm. What is its volume? Because of melting, its radius is decreasing by 1 cm hr*$^{-1}$ *and its height is decreasing by 2 cm hr*$^{-1}$*. At what rate is its volume changing?*

The volume of a cylinder is $V = \pi r^2 h = 6.283 \times 10^6\ \text{cm}^3 = 6.283\ \text{m}^3$.

The chain rule for partial derivatives tells us that the volume changes at the rate

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$

The question tells us dr/dt and dh/dt, we need to work out the partial derivatives,

$$\frac{\partial V}{\partial r} = 2\pi r h, \qquad \frac{\partial V}{\partial h} = \pi r^2$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = 2\pi r h(-1) + \pi r^2(-2) = \underline{\underline{-1.88 \times 10^5 \text{ cm}^2 \text{ hr}^{-1} = -0.188 \text{ m}^3 \text{ hr}^{-1}}}$$

(Change rate is negative, the ice cylinder is becoming smaller.)

6. *The air temperature at height z over the point (x, y) is given by*

$$T(x, y, z) = (40 + x^2 - 3y + xy)e^{-z} - 10$$

A migrating bird is at the point (0,0,1), flying with velocity $(1, 3, \frac{1}{10})$. Use the partial derivative chain rule to find the rate at which the bird feels the temperature changing.

The chain rule says

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt} = \frac{\partial T}{\partial x}v_x + \frac{\partial T}{\partial y}v_y + \frac{\partial T}{\partial z}v_z$$

The derivatives we need are

$$\begin{aligned} \frac{\partial T}{\partial x} &= (2x + y)e^{-z} \\ \frac{\partial T}{\partial y} &= (-3 + x)e^{-z} \\ \frac{\partial T}{\partial z} &= -(40 + x^2 - 3y + xy)e^{-z} \end{aligned}$$

We need the values at the point (0,0,1), which are

$$(T_x, T_y, T_z) = (0, -3\ e^{-1}, -40\ e^{-1})$$

Combining this with the velocity $(v_x, v_y, v_z) = (1, 3, \frac{1}{10})$ gives

$$\underline{\underline{\frac{dT}{dt} = (0, -3\ e^{-1}, -40\ e^{-1}) \cdot (1, 3, \frac{1}{10}) = -13\ e^{-1} = -4.782}}$$