Solution to Problem Set 6, M181

Solution to problem 1

a.

$$\begin{aligned} I_n &= \int_0^{\frac{\pi}{2}} \cos^n(x)\, dx \;= \int_0^{\frac{\pi}{2}} \cos(x)\cos^{n-1}(x)\, dx \\ &= [\sin(x)\cos^{n-1}(x)|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \sin^2(x)(n-1)\cos^{n-2}(x)\, dx \\ &= 0 + (n-1)\int_0^{\frac{\pi}{2}} (1-\cos^2(x))\cos^{n-2}(x)\, dx \\ &= (n-1)\left[\int_0^{\frac{\pi}{2}} \cos^{n-2}(x)\, dx - \int_0^{\frac{\pi}{2}} \cos^n(x)\, dx\right] \\ &= (n-1)\,[I_{n-2} - I_n] \end{aligned}$$

Solving for I_n we get

$$I_n = \frac{n-1}{n} I_{n-2}\ .$$

b)

$$I_{10} = \int_0^{\frac{\pi}{2}} \cos^{10}(x)\, dx$$

Using the reduction formula found in part a:

$$I_{10} = \frac{9}{10} I_8 = \frac{9}{10}\frac{7}{8} I_6 = \cdots = \frac{9}{10}\frac{7}{8}\frac{5}{6}\frac{3}{4}\frac{1}{2} I_0$$

where for I_0 we have

$$I_0 = \int_0^{\frac{\pi}{2}} \cos^0(x)\, dx = \frac{\pi}{2}\ .$$

Hence

$$I_{10} = \frac{63\pi}{512}\ .$$

Solution to problem 2

$$I = \int_0^{\frac{\pi}{2}} \sin^5(x) dx \tag{1}$$

Using the substitution $u = \cos(x)$, we find $du = -\sin(x)dx$. We also need to change the limits. When x= 0 then u = 1. When $x = \frac{\pi}{2}$ then $u = 0$.

$$I \;=\; \int_0^{\frac{\pi}{2}} (1-\cos^2(x))(1-\cos^2(x))\sin(x)dx$$

$$
\begin{aligned}
&= -\int_1^0 (1-u^2)(1-u^2)du \\
&= \int_0^1 (1-2u^2+u^4)du \\
&= [u - \frac{2}{3}u^3 + \frac{1}{5}u^5]_0^1 \\
&= \frac{8}{15}
\end{aligned} \tag{2}
$$

Solution to problem 3

This integral can be done by substitution $u = (x^5 + 3)$

$$
\begin{aligned}
I &= \int \frac{x^4}{x^5+3}dx \\
&= \frac{1}{5}\int \frac{du}{u} \\
&= \frac{1}{5}\log(u) \\
&= \frac{1}{5}\log(x^5+3) + C
\end{aligned}
$$

Solution to problem 4

$$
I = \int \frac{x^3 + 2x^2 + x + 1}{x^2 + 2} \tag{3}
$$

When integrating rational functions, if the order of the polynomial in the numerator is greater than in the denominator, then you divide through until this condition is satisfied.

$$
\begin{aligned}
\frac{x^3 + 2x^2 + x + 1}{x^2 + 2} = \\
& \frac{x(x^2+2) - 2x + 2x^2 + x + 1}{x^2 + 2} \\
& x + \frac{2x^2 - x + 1}{x^2 + 2} \\
& x + \frac{2(x^2+2) - 4 - x + 1}{x^2 + 2} \\
& x + 2 + \frac{-x - 3}{x^2 + 2}
\end{aligned} \tag{4}
$$

The above manipulations are equivalent to using the long devision notation. The denominator is an irreducible quadratic factor, so there is no further expansion in partial fractions.

$$\begin{aligned} I &= \int (x+2+\frac{-x-3}{x^2+2})dx \\ &= \frac{1}{2}x^2+2x-\int \frac{x}{x^2+2}dx - 3\int \frac{1}{x^2+2}dx \qquad (5) \\ &= \frac{1}{2}x^2+2x-\frac{1}{2}\log(x^2+2)-\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+C \qquad (6) \\ & \qquad (7) \end{aligned}$$

The two integrals in equation 6 are standard. To do the integral below the substitution $x = \sqrt{2}\tan\theta$ is useful. The $\sqrt{2}$ helps the trigonometric functions in the numerator and denominator cancel.

$$\begin{aligned} I &= \int \frac{1}{x^2+2})dx \\ &= \int \frac{\sqrt{2}\sec^2(\theta)}{2\tan^2(\theta)+2}d\theta \\ &= \int \frac{\sqrt{2}}{2}d\theta \\ &= \frac{\sqrt{2}}{2}\theta + C \\ &= \frac{\sqrt{2}}{2}\tan^{-1}(\frac{x}{\sqrt{2}}) + C \end{aligned}$$

Solution to problem 5

$$I = \int \frac{x+1}{x^3(x-2)^2} \qquad (8)$$

This question requires is a more complicated example of the partial fraction expansion. We need the constants A, B, C, D, and E, in the expansion below.

$$\frac{x+1}{x^3(x-2)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + + \frac{D}{x-2} + \frac{E}{(x-2)^2} \qquad (9)$$

First we create a common denominator on the right hand side. Now the numerators on both sides of the equation are equal.

$$x+1 = Ax^2(x-2)^2 + Bx(x-2)^2 + C(x-2)^2 + +Dx^3(x-2) + Ex^3 \qquad (10)$$

The equation is true for all x. If we choose $x = 0$

$$1 = 4C \qquad (11)$$

so $C = \frac{1}{4}$. Now consider $x = 2$, hence

$$1 + 2 = 8E \qquad (12)$$

and $E = \frac{3}{8}$. Now the right hand side of equation 10 is expanded into a polynomial in x.

$$\begin{aligned} x+1 &= (A+D)\,x^4 + (-4\,A + B - 2\,D + E)\,x^3 + (4\,A - 4\,B + C)\,x^2 \\ &+ (4\,B - 4\,C)\,x + 4\,C \end{aligned}$$

$$\begin{aligned} x \text{ coeff. } 1 &= 4B - 4C && (13) \\ x^2 \text{ coeff. } 0 &= 4A - 4B + C && (14) \\ x^4 \text{ coeff. } 0 &= A + D && (15) \end{aligned}$$

Equation 13 gives $B = \frac{1}{2}$. Equation 14 gives $A = \frac{7}{16}$. Equation 15 gives $D = -\frac{7}{16}$.

The integral becomes:

$$I = \int \left(\frac{7}{16x} + \frac{1}{2x^2} + \frac{1}{4x^3} - \frac{7}{16(x-2)} + \frac{3}{8(x-2)} \right) dx \tag{16}$$

Now we can integrate each of the terms in the integrand.

$$I = \frac{7}{16}\ln(x) - \frac{1}{2x} - \frac{1}{8x^2} - \frac{7}{16}\ln(x-2) - \frac{3}{8(x-2)^2} + C \tag{17}$$

Solution to problem 6

The equation for y is given by

$$y = \sqrt{a^2 - x^2}.$$

From the surface integral formula we have

$$\text{surface area} = \int_{-a}^{a} 2\pi y \sqrt{1 + (y')^2}\; dx$$

The derivative of y gives

$$y' = \frac{x}{\sqrt{a^2 - x^2}}$$

Hence

$$y\sqrt{1 + (y')^2} = a$$

and the integral reduces to

$$\int_{-a}^{a} dx = 2a$$

and finally

$$\text{surface area} = 4\pi a^2\ .$$

Solution to problem 6