Solution Sheet 5, M181

1. *Use partial fractions to sketch the function*

$$f(x) = \frac{x^2 - 7x + 10}{x - 1} \ .$$

Find the stationary points of $f(x)$.

Partial Fractions

$$\frac{x^2 - 7x + 10}{x - 1} = Ax + B + \frac{C}{x-1}$$

we need to find A, B, C.

$$x^2 - 7x + 10 = (Ax + B)(x - 1) + C \qquad \text{for all } x$$

Choose x values to give easy equations

$$x = 1 \quad \Rightarrow \quad \underline{\underline{C = 4}}$$

$$x = 0 \quad \Rightarrow \quad 10 = -B + C \ \Rightarrow \ B = C - 10 \ \Rightarrow \ \underline{\underline{B = -6}}$$

$$x \to \infty \quad \Rightarrow \quad x^2 = Ax^2 \ \Rightarrow \ \underline{\underline{A = 1}}$$

so

$$\underline{\underline{f(x) = \frac{x^2 - 7x + 10}{x - 1} = x - 6 + \frac{4}{x-1}}}$$

We see that $f(x)$ has a singularity at $x = 1$, and that at large x it tends asymptotically to the line $y = x - 6$.

Zeroes

We can find the zeroes of $f(x)$ more easily from the original form: it has zeroes at

$$x^2 - 7x + 10 = 0 \ \Rightarrow \ (x-5)(x-2) = 0 \ \Rightarrow \ \underline{\underline{x = 2 \text{ and } x = 5}}$$

Stationary Points

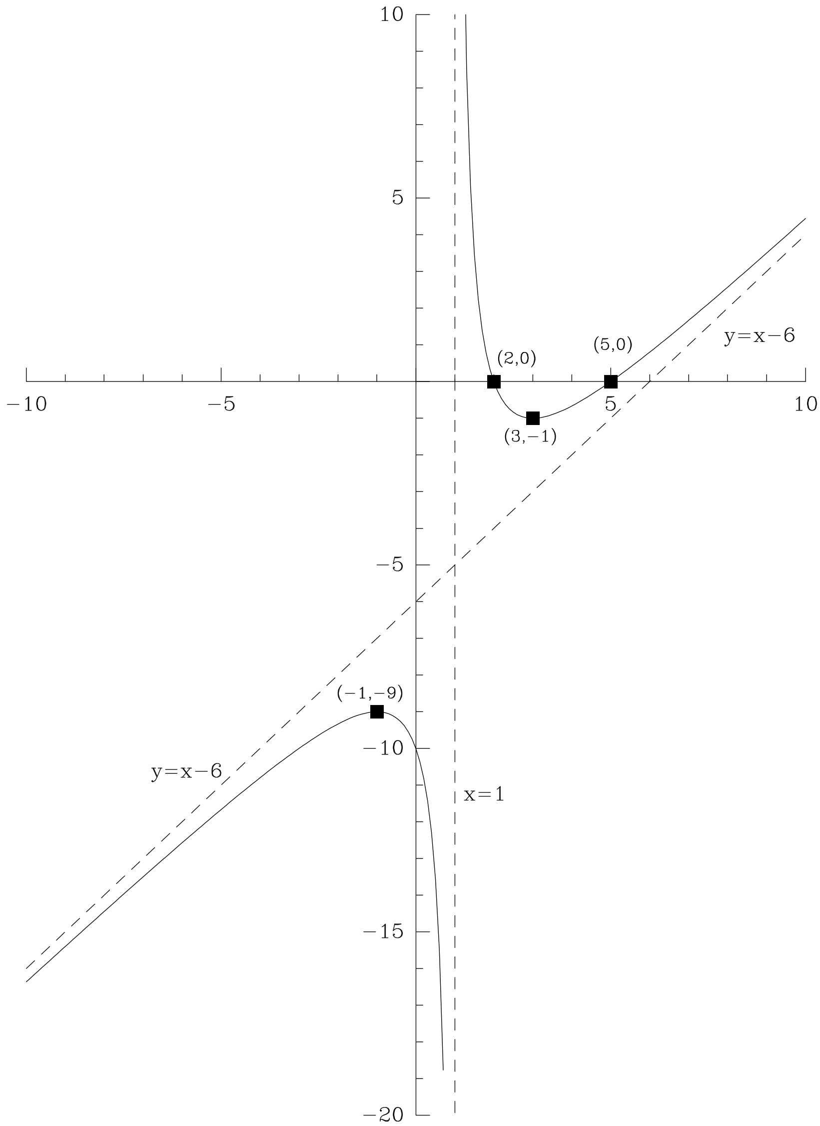
$$\begin{aligned} f(x) &= x - 6 + \frac{4}{x-1} \\ f'(x) &= 1 - \frac{4}{(x-1)^2} \\ f''(x) &= \frac{8}{(x-1)^3} \end{aligned}$$

Stationary points are at

$$f'(x) = 0 \ \Rightarrow \ 1 - \frac{4}{(x-1)^2} = 0 \ \Rightarrow \ (x-1)^2 = 4 \ \Rightarrow \ \underline{\underline{x = 1 \pm 2}}$$

There is a local maximum at $(x, y) = (-1, -9)$ and a local minimum at $(x, y) = (3, -1)$

No points of inflexion, because $f''(x) \neq 0$ for any finite x.



2. *Find the steepest point on the curve* $y = \dfrac{2e^x}{e^x + 4}$. *What is the gradient at this point?*

$$\begin{aligned} y &= \frac{2e^x}{e^x+4} \\ y' &= \frac{2e^x(e^x+4) - 2e^x e^x}{(e^x+4)^2} = \frac{8e^x}{(e^x+4)^2} \\ y'' &= \frac{8e^x(4-e^x)}{(e^x+4)^3} \end{aligned}$$

Steepest point, is the maximum of y', so it is where

$$y'' = 0 \ \Rightarrow \ e^x = 4 \ \Rightarrow \ \underline{\underline{x = \ln 4}}$$

Steepest point is at $(x, y) = (\ln 4, 1)$, the gradient is

$$\underline{\underline{y'(\ln 4) = \frac{1}{2}}}$$

3.(i) *Use integration by parts to find* $\int xe^{3x}\,dx$

In this question we use integration by parts to find the solution.

$$\int uv'\,dx = uv - \int u'v\,dx \tag{1}$$

To do the integral in 3(i) use $v' = e^{3x}$ (because it is easy to integrate) and $u = x$ (because it is easy to differentiate)

$$\begin{aligned} u &= x & v' &= e^{3x} \\ u' &= 1 & v &= \frac{1}{3}e^{3x} \end{aligned}$$

$$\begin{aligned} I &= \int xe^{3x}\,dx \\ &= \frac{1}{3}xe^{3x} - \int \frac{e^{3x}}{3}\,dx \\ \Rightarrow I &= \underline{\underline{\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C}} \end{aligned}$$

3.(ii) *Use integration by parts to find* $\int x^2 \cosh 3x\,dx$

Use integration by parts. Using the notation in equation 1, take $v' = \cosh(3x)$ (easy to integrate) and $u = x^2$ (easy to differentiate). This time we need to use integration by parts *twice*.

$$\begin{aligned} u &= x^2 & v' &= \cosh(3x) \\ u' &= 2x & v &= \frac{1}{3}\sinh(3x) \end{aligned}$$

$$\begin{aligned} I &= \int x^2 \cosh(3x)\,dx \\ &= x^2\frac{1}{3}\sinh(3x) - \int 2x\frac{1}{3}\sinh(3x)\,dx \\ &= x^2\frac{1}{3}\sinh(3x) - \frac{2}{3}x\frac{1}{3}\cosh(3x) + \int \frac{2}{9}\cosh(3x)\,dx \\ \Rightarrow I &= \underline{\underline{x^2\frac{1}{3}\sinh(3x) - \frac{2}{9}x\cosh(3x) + \frac{2}{27}\sinh(3x) + C}} \end{aligned}$$

4. *Evaluate the integrals*

4(i) $I = \int \frac{1}{(x+3)(x+5)}\, dx$

This is a rational function, so start by finding a partial fractions expansion.

$$\frac{1}{(x+3)(x+5)} = \frac{A}{(x+3)} + \frac{B}{(x+5)} = \frac{A(x+5)+B(x+3)}{(x+3)(x+5)} = \frac{(A+B)x+(5A+3B)}{(x+3)(x+5)}$$

This gives us the simultaneous equations

$$\begin{aligned} A+B &= 0 \\ 5A+3B &= 1 \end{aligned}$$

The solutions are $A = \frac{1}{2}$ and $B = \frac{-1}{2}$. So

$$\begin{aligned} \int \frac{1}{(x+3)(x+5)}\, dx &= \int \left(\frac{1}{2(x+3)} + \frac{-1}{2(x+5)} \right) dx \\ &= \underline{\underline{\frac{1}{2}\ln(x+3) - \frac{1}{2}\ln(x+5) + C}} \end{aligned}$$

4.(ii) $I = \int \frac{(x+1)}{(x+3)(x+5)}\, dx$

This is a ratio of two polynomials, so start by finding a partial fractions expansion.

$$\frac{(x+1)}{(x+3)(x+5)} = \frac{A}{(x+3)} + \frac{B}{(x+5)} \Rightarrow (x+1) = A(x+5) + B(x+3)$$

(multiplying by $(x+3)(x+5)$ on both sides). The above equation is true for any x. This time we'll solve it by choosing clever x-values.

$$\begin{aligned} x = -3 &\Rightarrow -2 = 2A \Rightarrow \underline{\underline{A = -1}} \\ x = -5 &\Rightarrow -4 = -2B \Rightarrow \underline{\underline{B = 2}} \end{aligned}$$

$$\begin{aligned} \int \frac{(x+1)}{(x+3)(x+5)}\, dx &= \int \left(\frac{-1}{(x+3)} + \frac{2}{(x+5)} \right) dx \\ &= \underline{\underline{-\ln(x+3) + 2\ln(x+5) + C}} \end{aligned}$$

4.(iii) $I = \int (4-3x)^4\, dx$

To do this integral use the substitution $u = 4 - 3x$ so $du = -3\, dx$

$$\begin{aligned} I &= \frac{-1}{3} \int u^4 du \\ &= \frac{-1}{3}\frac{1}{5}u^5 + C \\ \Rightarrow I &= \underline{\underline{-\frac{1}{15}(4-3x)^5 + C}} \end{aligned}$$

4.(iv) $\displaystyle\int \sin(5x)\cos(2x)\ dx$

To integrate this function we use the trigonometric identity.

$$2\sin(a)\cos(b) = \sin(a+b) + \sin(a-b)$$

This allows us to write the integrand in terms of integrands that we know how to integrate.

$$\begin{aligned}\int \sin(5x)\cos(2x)\ dx &= \frac{1}{2}\int (\sin(7x) + \sin(3x))\ dx \\ &= \underline{\underline{-\frac{1}{14}\cos(7x) - \frac{1}{6}\cos(3x) + C}}\end{aligned}$$

4.(v) $\displaystyle I = \int \frac{1}{(x-2)^4}\ dx$

To do this integral use the substitution $u = x - 2$ so $du = \ dx$

$$\begin{aligned}I &= \int \frac{1}{u^4} du \\ &= \frac{-1}{3}\frac{1}{u^3} + C \\ &= \underline{\underline{-\frac{1}{3}\frac{1}{(x-2)^3} + C}}\end{aligned}$$

(It's fine if you just did this in a single step, without any substitution.)

$$\int_{-1}^{1} \frac{1}{(x-2)^4}\ dx = \left[-\frac{1}{3}\frac{1}{(x-2)^3}\right]_{-1}^{1} = -\frac{1}{3}\frac{1}{(-1)} - \left(-\frac{1}{3}\frac{1}{(-27)}\right) = \underline{\underline{\frac{26}{81} = 0.32099}}$$

4.(vi) $\displaystyle\int_0^{\pi} \cos^2(x)\ dx$

To integrate this function we use the trig identity. $\cos(2x) = 2\cos^2(x) - 1$

$$\begin{aligned}\int \cos^2(x)\ dx &= \frac{1}{2}\int (1 + \cos(2x))\ dx \\ &= \underline{\underline{\frac{1}{2}x + \frac{1}{4}\sin(2x) + C}}\end{aligned}$$

$$\int_0^{\pi} \cos^2(x)\ dx = \left[\frac{1}{2}x + \frac{1}{4}\sin(2x)\right]_0^{\pi} = \frac{1}{2}\pi + \frac{1}{4}\sin(2\pi) - 0 = \underline{\underline{\frac{\pi}{2}}}$$

5. *Use the substitution* $x = 3\cosh u$ *to show that*

$$\int \frac{dx}{\sqrt{x^2-9}} = \cosh^{-1}\frac{x}{3} + C \ .$$

(Note that $\cosh^2 x - \sinh^2 x = 1$.*)*

Use the suggested substitution
$x = 3\cosh(u) \ \Rightarrow \ dx = 3\sinh(u)du$.

$$\begin{aligned} I &= \int \frac{3\sinh(u)du}{\sqrt{9\cosh^2(u)-9}} \\ &= \int \frac{3\sinh(u)du}{\sqrt{9\sinh^2(u)}} \\ &= \int du = u + C = \underline{\underline{\cosh^{-1}(x/3) + C}} \end{aligned}$$

where C is the integration constant. (We used $\sinh^2 x = \cosh^2 x - 1$.)

6. *Use the substitution* $x = 3\tan\theta$ *to find*

$$\int_0^3 \frac{dx}{x^2+9} \ .$$

Use the suggested substitution

$$x = 3\tan(\theta); \ \Rightarrow \quad dx = 3\sec^2(\theta)d\theta$$

with $\sec(x) = 1/\cos(x)$. When we make the substitution, we have to change the limits, to be the limits on θ instead of x. *Don't forget !*

$$\begin{aligned} x = 0 \quad &\Rightarrow \quad \theta = 0 \\ x = 3 \quad &\Rightarrow \quad \tan(\theta) = 1 \ \Rightarrow \ \theta = \frac{\pi}{4} \end{aligned}$$

so $0 \le x \le 3 \quad \rightarrow \quad 0 \le \theta \le \frac{\pi}{4}$

$$\begin{aligned} I &= \int_0^3 \frac{dx}{x^2+9} \\ &= \int_0^{\pi/4} \frac{3\sec^2(\theta)}{9\tan^2(\theta)+9} d\theta \\ &= \int_0^{\pi/4} \frac{3\sec^2(\theta)}{(9\sec^2(\theta))} d\theta \\ &= \int_0^{\pi/4} \frac{1}{3} d\theta = \left[\frac{\theta}{3}\right]_0^{\pi/4} \\ \Rightarrow I &= \underline{\underline{\frac{\pi}{12} = 0.2618}} \end{aligned}$$

We used the relation $1 + \tan^2(\theta) = \sec^2(\theta)$. (Proof:

$$1 + \tan^2\theta = 1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \equiv \sec^2\theta)$$

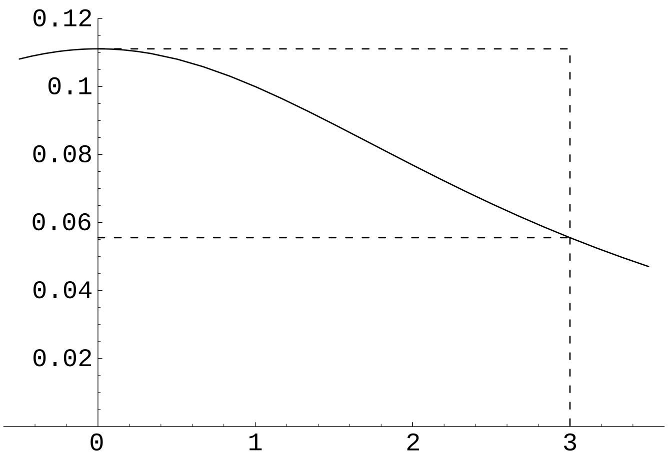
A lot of people wrote

$$I = \frac{1}{3}\tan^{-1}(1) = 15 \qquad \text{wrong!}$$

as the answer; let's look to see whether this is sensible. Below is a sketch of the integrand: the function $1/(x^2+9)$ has the value $1/9$ at $x = 0$ and $1/18$ at $x = 3$, so we can see that the area under the curve must satisfy

$$\frac{1}{6} < I < \frac{1}{3}\,.$$

The correct answer, 0.26, is comfortably inside these bounds, 15 is much too big. The wrong answer, 15, is what you get if you use degrees instead of radians. This integral illustrates the fact that it really is important to use radians when doing calculus.



7. *Evaluate the integrals*
(i) $\int_0^8 x^{-\frac{1}{5}}\,dx$.

This is an improper integral, as the integrand is infinite at $x = 0$, so to be safe we should define it by a limiting process.

$$\begin{aligned} I = \int_0^8 x^{-\frac{1}{5}}\,dx &= \lim_{\alpha\to 0}\int_\alpha^8 x^{-\frac{1}{5}}\,dx \\ &= \lim_{\alpha\to 0}\left[\frac{5}{4}x^{4/5}\right]_\alpha^8 \\ &= \lim_{\alpha\to 0}\frac{5}{4}\left(8^{4/5} - (\alpha)^{4/5}\right) \\ \Rightarrow I &= \underline{\underline{\frac{5}{4}8^{4/5} = 5\;2^{2/5} = 6.598}} \end{aligned}$$

7.(ii) $\int_1^\infty \frac{dx}{x^3}$

Again. we'll do this as a limit, to see whether the answer is really defined.

$$
\begin{aligned}
I = \int_1^\infty \frac{dx}{x^3} &= \lim_{U\to\infty} \int_1^U \frac{dx}{x^3} \\
&= \lim_{U\to\infty} \left[\frac{-1}{2}\,\frac{1}{x^2}\right]_0^U \\
&= \lim_{U\to\infty} \left(-\frac{1}{2U^2} + \frac{1}{2}\right) \\
&\Rightarrow \quad \underline{\underline{I = \frac{1}{2}}}
\end{aligned}
$$

because $1/U^2$ vanishes as $U \to \infty$.