MATH181 Solution Sheet 4

1. *A particle has the position x at time t given by*

   (a) $x(t) = A\cos bt$.

Calculate the particle's velocity, $v(t)$, and acceleration $a(t)$ as a function of t. (A and b are constants.)

(a)

$$\begin{aligned} x(t) &= A\cos bt \\ v(t) = \frac{d}{dt}x(t) &= -Ab\sin bt \\ a(t) = \frac{d}{dt}v(t) &= -Ab^2\cos bt \end{aligned}$$

Repeat this for particles with the following $x(t)$.

(b) $x(t) = Ae^{-t} - Be^{-3t}$

(c) $x(t) = A(1+kt)e^{-kt}$

(d) $x(t) = b\sin(\omega t)\ e^{-kt}$

(A, B, b, k and ω are all constants.)

The derivatives in (b) are simple. For (c) and (d) we need the product rule.

(b)

$$\begin{aligned} x(t) &= Ae^{-t} - Be^{-3t} \\ v(t) = \frac{d}{dt}x(t) &= -Ae^{-t} + 3Be^{-3t} \\ a(t) = \frac{d}{dt}v(t) &= Ae^{-t} - 9Be^{-3t} \end{aligned}$$

(c)

$$\begin{aligned} x(t) &= A(1+kt)e^{-kt} \\ v(t) = \frac{d}{dt}x(t) &= Ake^{-kt} - A(1+kt)ke^{-kt} = -Ak^2te^{-kt} \\ a(t) = \frac{d}{dt}v(t) &= -Ak^2e^{-kt} + Ak^3te^{-kt} = -Ak^2(1-kt)e^{-kt} \end{aligned}$$

(d)

$$\begin{aligned} x(t) &= b\sin(\omega t)\ e^{-kt} \\ v(t) = \frac{d}{dt}x(t) &= b\omega\cos(\omega t)\ e^{-kt} - bk\sin(\omega t)\ e^{-kt} \\ a(t) = \frac{d}{dt}v(t) &= -b\omega^2\sin(\omega t)\ e^{-kt} - b\omega k\cos(\omega t)\ e^{-kt} - b\omega k\cos(\omega t)\ e^{-kt} + bk^2\sin(\omega t)\ e^{-kt} \\ &= \underline{\underline{b(k^2-\omega^2)\sin(\omega t)\ e^{-kt} - 2b\omega k\cos(\omega t)\ e^{-kt}}} \end{aligned}$$

These are all examples from physics; they four represent the possible motions of a mass on a spring: (a) is the motion with no friction, the mass oscillates for ever; (b) is the motion with strong friction (over-damping); (c) with "border-line friction" (critical damping) and (d) with slight friction (under-damping).

Of course, you don't need any of this information to do the question — just the rules of differentiation.

2. *Find all the stationary points of the following functions, and show whether they are maxima, minima, or neither. Find all the points of inflexion*

$$\begin{array}{ll} (i) & x^4 - 2x^2 \\ (ii) & \sin x \\ (iii) & x^3 - 3x^2 + 3x \\ (iv) & \cosh x \end{array}$$

(i) $f(x) = x^4 - 2x^2 \quad \Rightarrow \quad f'(x) = 4x^3 - 4x$

To find the stationary points, we must solve $f'(x) = 0$.

$$4x^3 - 4x = 0 \quad \Rightarrow \quad 4x(x^2 - 1) = 0 \quad \Rightarrow \quad \underline{\underline{x = 0 \text{ or } x = \pm 1}}$$

To classify these points, we look at the second derivative,

$$f''(x) = 12x^2 - 4$$

and work out its value at the turning points:

$$f''(-1) = 8, \quad f''(0) = -4, \quad f''(1) = 8.$$

The second derivative is positive at $x = -1$ and $x = 1$, so these two turning points are minima of the function; f'' is negative at $x = 0$, so that is a local maximum of the function.

Local maximum at $(0, 0)$; minima at $(-1, -1)$ and $(1, -1)$.

To find the points of inflexion, solve

$$f''(x) = 0 \quad \Rightarrow \quad 12x^2 - 4 = 0 \quad \Rightarrow \quad x^2 = \frac{1}{3} \quad \Rightarrow \quad \underline{\underline{x = \pm\frac{1}{\sqrt{3}}}}$$

Points of inflexion: $(\pm 1/\sqrt{3}, -5/9)$.

(*ii*)

$$\begin{aligned} f(x) &= \sin x \\ f'(x) &= \cos x \\ f''(x) &= -\sin x \end{aligned}$$

Stationary points at $f'(x) = 0 \quad \Rightarrow \quad \cos x = 0 \quad \Rightarrow \quad x = \pi/2 + n\pi$.

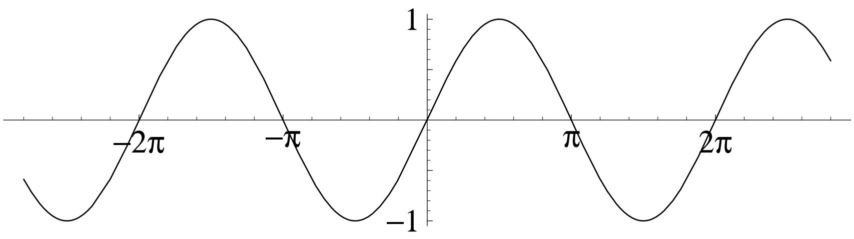
If n is an even integer, $-\sin(\pi/2 + n\pi) = -1$, we have a maximum. If n is an odd integer, $-\sin(\pi/2 + n\pi) = +1$, we have a minimum.

Maxima at $\underline{\underline{x = \cdots, -\frac{3}{2}\pi, \frac{1}{2}\pi, \frac{5}{2}\pi, \cdots}}$. Minima at $\underline{\underline{x = \cdots, -\frac{5}{2}\pi, -\frac{1}{2}\pi, \frac{3}{2}\pi, \cdots}}$

Maxima at $\underline{\underline{x = (2k + \frac{1}{2})\pi}}$. Minima at $\underline{\underline{x = (2k - \frac{1}{2})\pi}}$.

Points of inflexion at $f''(x) = 0 \quad \Rightarrow \quad -\sin x = 0 \quad \Rightarrow \quad x = k\pi$

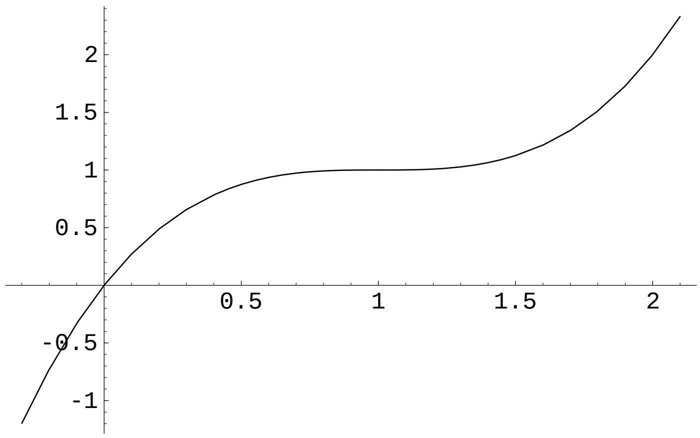
The sine curve has a point of inflexion every time it crosses the x-axis.



(iii)

$$
\begin{aligned}
f(x) &= x^3 - 3x^2 + 3x \\
f'(x) &= 3x^2 - 6x + 3 = 3(x-1)^2 \\
f''(x) &= 6(x-1)
\end{aligned}
$$

The only place where $f'(x) = 0$ is at $x = 1$, so this curve only has one stationary point, at $(1, 1)$. When we work out the curvature (second derivative) at $x = 1$ we get $f''(1) = 0$, so this is a point of inflexion. From a sketch we can see that this point is neither a maximum nor a minimum.



(iii)

$$
\begin{aligned}
f(x) &= \cosh x \\
f'(x) &= \sinh x \\
f''(x) &= \cosh x
\end{aligned}
$$

Because we remember what the graph of sinh looks like, we know that the only real solution of $\sinh x = 0$ is $x = 0$, so the turning point of cosh is at $x = 0$.

If we want to prove it:

$$\begin{array}{rcll} \sinh x &=& 0 & \\ \Rightarrow \quad \frac{1}{2}(e^x - e^{-x}) &=& 0 & \\ \Rightarrow \quad e^x &=& e^{-x} & \\ \Rightarrow \quad e^{2x} &=& 1 & \text{Take logarithm of both sides.} \\ 2x = \ln(1) &=& 0 & \end{array}$$

To find out if $x = 0$ is a maximum or minimum, look at the second derivative:

$$f''(0) = \cosh 0 = 1$$

The second derivative is positive, so there is a minimum of cosh at $(0, 1)$.

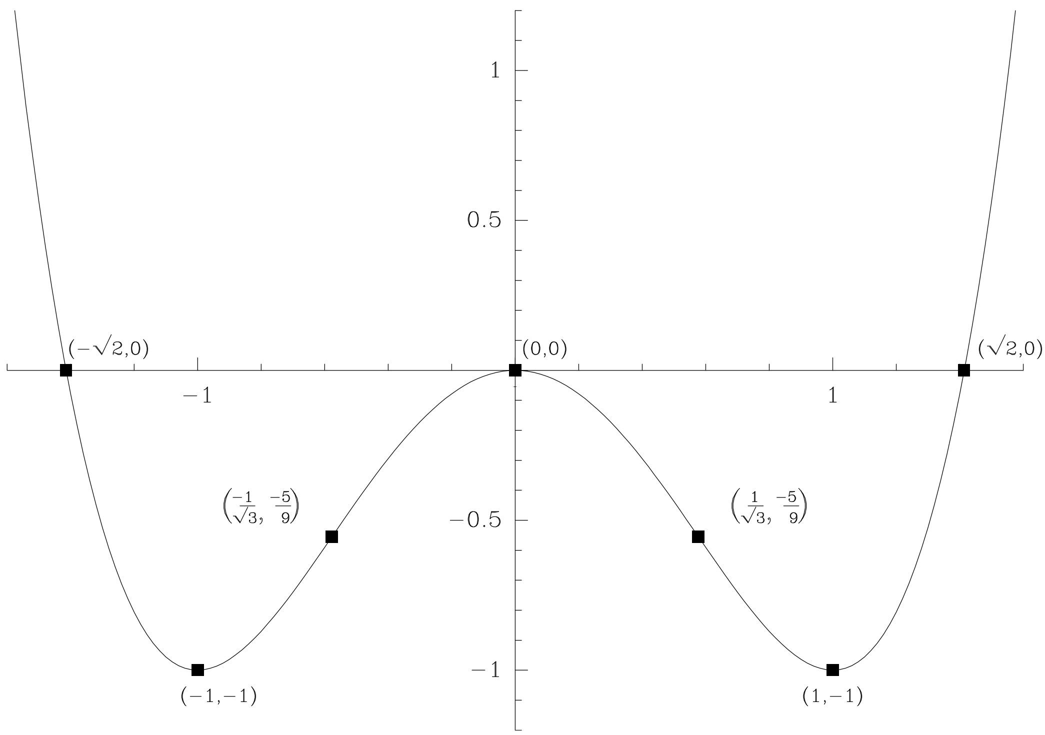
$f''(x) = \cosh x \geq 1$. No zeroes for $f''(x)$, so no points of inflexion.

3. *Sketch the function* $g(x) = x^4 - 2x^2$ *(the function from question 2.(i)) showing all the zeroes and turning points.*

We already worked out the turning points in question 2, there is a maximum at $(0, 0)$, and minima at $(\pm 1, -1)$. The points of inflexion were at $(\pm 1/\sqrt{3}, -5/9)$. We still need to work out the zeroes.

$$g(x) = 0 \quad \Rightarrow \quad x^4 - 2x^2 = 0 \quad \Rightarrow \quad x^2(x^2 - 2) = 0$$

The zeroes of g are at $x = 0$ and $x = \pm\sqrt{2}$.



4. *Calculate the first and second derivatives of* e^{x^3}.

$$\frac{d}{dx} e^{x3} = \underline{\underline{3x^2 e^{x3}}}$$

$$\frac{d^2}{dx^2} e^{x3} = \frac{d}{dx}\left(3x^2 e^{x3}\right) = \underline{\underline{6xe^{x3} + 9x^4 e^{x3}}}$$

The first time, use the chain rule. The second derivative needs the product rule first, then the chain rule. Don't miss out any terms in the second derivative.

5. *A relativistic space-ship travels so that the astronaut experiences a constant acceleration of* G*. Its position at time* t *is*

$$x = \frac{c}{G}\sqrt{(c^2 + G^2t^2)} - \frac{c^2}{G}$$

By differentiating this expression find
(i) the velocity $v = \dfrac{dx}{dt}$
(ii) the acceleration $a = \dfrac{d^2x}{dt^2}$

as seen by a stationary observer.

$$x(t) = \frac{c}{G}\sqrt{(c^2 + G^2t^2)} - \frac{c^2}{G} = \frac{c}{G}(c^2 + G^2t^2)^{1/2} - \frac{c^2}{G}$$

(*i*) $$v(t) = \frac{d}{dt}x(t) = \frac{c}{G}G^2t(c^2 + G^2t^2)^{-1/2} = \underline{\underline{\frac{cGt}{\sqrt{c^2 + G^2t^2}}}}$$ (chain rule)

(*ii*) $$a(t) = \frac{d}{dt}v(t) = \underline{\underline{cG\,(c^2 + G^2t^2)^{-1/2} - cG^3t^2\,(c^2 + G^2t^2)^{-3/2}}}$$ (product and chain)

$$= (c^3G + cG^3t^2 - cG^3t^2)\,(c^2 + G^2t^2)^{-3/2}$$

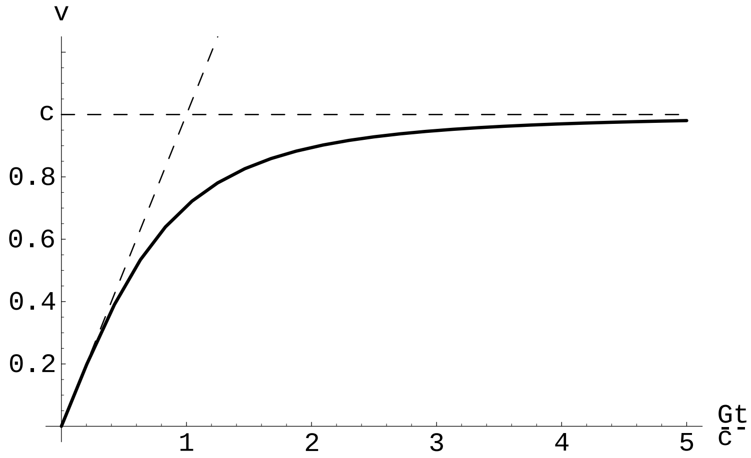
$$= \underline{\underline{\frac{c^3G}{(c^2 + G^2t^2)^{3/2}}}}$$

(I simplified the acceleration by bringing it over a common denominator, but it's fine if you left it in the original form).

(iii) *How does* v *behave if* t *is small, and if* t *is large? Why should you expect this on physical grounds?*

Use the answer to part (i). At small time we can neglect G^2t^2 in comparison with c^2, so $\sqrt{c^2 + G^2t^2} \approx c$, and $\underline{\underline{v \approx Gt}}$ for small t.

On the other hand, at large t we will have $G^2t^2 \gg c^2$ so $\sqrt{c^2 + G^2t^2} \approx Gt$, and $\underline{\underline{v \approx c}}$ for large t.



The velocity (in units of c) against time for a relativistic rocket. The time is in units of c/G; if $G = g = 9.8ms^{-2}$ each unit is about 1 year.

Physics: At small t the rocket is travelling much slower than light speed, so Newton's laws are a good guide, and we have $v \propto t$, a constant acceleration. If this continued, the rocket would end up travelling faster than light. Instead, the second term inside the square root becomes important, and the rocket gets closer and closer to the speed of light, but never actually reaches it.

The rocket moving so that the astronaut feels constant acceleration. From the point of view of a stationary observer, the velocity v gets closer and closer to c (the speed of light), but never reaches it. Likewise the acceleration a, as seen by the stationary observer, decreases as the rocket's speed gets closer to c, even though the astronaut herself always feels the same acceleration.