Solution to Homework Set 3, M181

1. *Differentiate with respect to x*

Last term uses the product rule:

$$(i) \qquad \begin{aligned} y &= x^5 + x^{-5} + x\ln(x) \\ \frac{dy}{dx} &= \underline{\underline{5x^4 - 5x^{-6} + \ln(x) + 1}} \end{aligned}$$

Product rule:

$$(ii) \qquad \begin{aligned} y &= x^3 \sin x \\ \frac{dy}{dx} &= \underline{\underline{3x^2 \sin x + x^3 \cos x}} \end{aligned}$$

Quotient rule:

$$(iii) \qquad \begin{aligned} y &= \frac{\ln(x) + 3}{x + 2} \\ \frac{dy}{dx} &= \frac{\frac{1}{x}(x+2) - (\ln(x) + 3)}{(x+2)^2} = \underline{\underline{\frac{2 - 2x - x\ln(x)}{x(x+2)^2}}} \\ &= \underline{\underline{\frac{1}{x(x+2)} - \frac{\ln(x) + 3}{(x+2)^2}}} \end{aligned}$$

2. *Differentiate with respect to x*

$$(i)\ \cos(x^2), \qquad (ii)\ x^2\ e^{\cos x}\ , \qquad (iii)\ \sqrt{1 + \sin 3x}$$

(*i*) To find $\frac{d}{dx}\cos(x^2)$ we use the chain rule. The function has the pattern $f(\,g(x)\,)$, chain rule is

$$\begin{aligned} \frac{d}{dx} f(\,g(x)\,) &= f'(\,g(x)\,)g'(x) \\ \frac{d}{dx}\cos x^2 &= -\sin(x^2)\,\frac{dx^2}{dx} = \underline{\underline{-2x\ \sin x^2}} \end{aligned}$$

(ii) To compute the derivative of $x^2 e^{\cos x}$ we apply the product rule and the chain rule. First we'll look at the derivative of $e^{\cos x}$. Define $v = \cos x$.

$$\begin{aligned}
y &= e^v \Rightarrow \frac{dy}{dv} = e^v \\
v &= \cos x \Rightarrow \frac{dv}{dx} = -\sin x \\
\frac{dy}{dx} &= \frac{dy}{dv}\frac{dv}{dx} = (e^v)(-\sin x) \\
\frac{d}{dx}(e^{\cos x}) &= -\sin x \; e^{\cos x}
\end{aligned}$$

Now we can use the product rule.

$$\begin{aligned}
\frac{d}{dx}(x^2 e^{\cos x}) &= 2x\, e^{\cos x} + x^2 \frac{d}{dx}(e^{\cos x}) \\
&= \underline{\underline{2x\, e^{\cos x} - x^2 \sin x \; e^{\cos x}}}
\end{aligned}$$

(iii) Chain rule again. To compute the derivative of $\sqrt{1 + \sin 3x}$ we introduce the variable $v = 1 + \sin 3x$.

$$\begin{aligned}
y &= \sqrt{v} = v^{1/2} \qquad \text{with} \qquad v = 1 + \sin 3x \\
\frac{dy}{dx} &= \left(\frac{dy}{dv}\right)\left(\frac{dv}{dx}\right) \\
&= \frac{1}{2} v^{-1/2} \; 3\cos 3x \\
&= \underline{\underline{\frac{3 \; \cos 3x}{2\sqrt{1 + \sin 3x}}}}
\end{aligned}$$

I've gone through these derivatives slowly, step by step. With a little practice you'll be able to do these with a lot less working than this.

3. *Find the equation of the tangent to the graph of the function*

$$y = \frac{2x}{x^3+1}$$

at the point $(x = 2)$.

The tangent to the graph at a point is the straight line passing through the point on the curve, with the same slope as the curve.

As we saw in the lectures, the tangent to $f(x)$ through the point x_0 has the equation

$$y = f(x_0) + f'(x_0)(x - x_0) \ .$$

To find the tangent, we need to know the derivative of f.

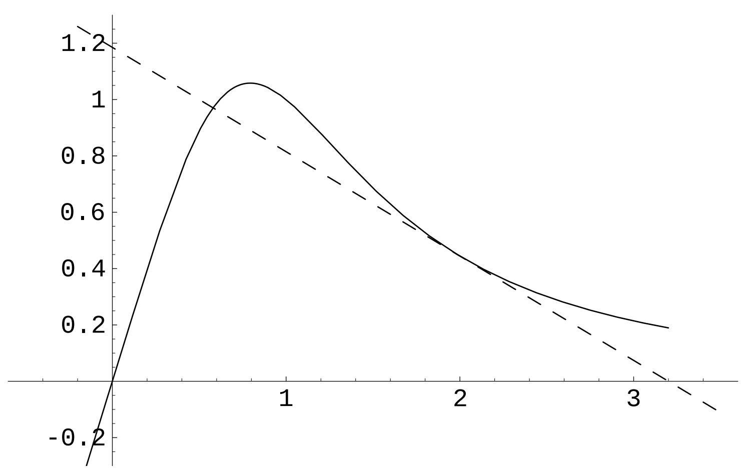
$$\begin{aligned} f(x) &= \frac{2x}{x^3+1} \\ f'(x) &= \frac{2(x^3+1) - 2x(3x^2)}{(x^3+1)^2} = \frac{2-4x^3}{(x^3+1)^2} \end{aligned}$$

At the point $x_0 = 2$ we have $f(2) = \frac{4}{9}$ and $f'(2) = -\frac{10}{27}$.

Hence the equation for the tangent is

$$y = f(x_0) + f'(x_0)(x - x_0) = \frac{4}{9} - \frac{10}{27}(x-2) = \underline{\underline{-\frac{10}{27}x + \frac{32}{27}}} \ .$$

As you can see, this line just touches the curve at $x = 2$, so it really is the tangent line.



4. *Find the derivative of*

$$x^3 e^{3x} \sin 2x.$$

Use the product rule for many factors.

$$\begin{aligned} (uvw)' &= u'vw + uv'w + uvw' \\ \frac{d}{dx}\left[x^3 e^{3x} \sin 2x\right] &= \underline{\underline{3x^2 e^{3x} \sin 2x + 3x^3 e^{3x} \sin 2x + 2x^3 e^{3x} \cos 2x}} \\ &= \underline{\underline{x^2 e^{3x} \left(3 \sin 2x + 3x \sin 2x + 2x \cos 2x\right)}} \end{aligned}$$

5. *Calculate the derivative of*

$$f(x) = \frac{2}{x+3}$$

from first principles.

"From first principles" means we should start from the definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x) = \frac{2}{x+3} \quad \Rightarrow \quad f(x+h) = \frac{2}{x+h+3}$$

$$\begin{aligned} f(x+h) - f(x) &= \frac{2}{x+h+3} - \frac{2}{x+3} \qquad \text{subtract fractions} \\ &= \frac{2(x+3) - 2(x+h+3)}{(x+h+3)(x+3)} \\ &= \frac{-2h}{(x+h+3)(x+3)} \\ \frac{f(x+h) - f(x)}{h} &= \frac{-2}{(x+h+3)(x+3)} \end{aligned}$$

Fortunately it's easy to take the $h \to 0$ limit,

$$f'(x) = \lim_{h \to 0} \left\{ \frac{-2}{(x+h+3)(x+3)} \right\} = \underline{\underline{\frac{-2}{(x+3)^2}}}$$

which is the answer you should have expected.