MATH181 Solution Sheet 2

1. $\mathbf{u} = 2\,\mathbf{i} + 2\,\mathbf{j}$ and $\mathbf{v} = \mathbf{i} - 2\,\mathbf{j}$. *Find the angle between* $\mathbf{u}$ *and* $\mathbf{v}$.

Use the dot product (also called the "scalar product"); $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta$
Work out the terms in this expression:

$$\begin{aligned}
\mathbf{u}\cdot\mathbf{v} &= (2\,\mathbf{i} + 2\,\mathbf{j})\cdot(\mathbf{i} - 2\,\mathbf{j}) = 2\times 1 + 2\times(-2) = -2\\
|\mathbf{u}| &= \sqrt{2^2+2^2} = \sqrt{8}\\
|\mathbf{v}| &= \sqrt{1^2+2^2} = \sqrt{5}
\end{aligned}$$

so $\quad \mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta \quad \Rightarrow \quad -2 = \sqrt{8}\,\sqrt{5}\cos\theta \quad \Rightarrow \quad \cos\theta = -\dfrac{2}{\sqrt{8}\sqrt{5}} = -\dfrac{1}{\sqrt{10}}$

$$\cos\theta = -\frac{1}{\sqrt{10}} = -0.316228 \quad \Rightarrow \quad \underline{\underline{\theta = 1.893 \text{ radians}, \quad \text{or } 108.4^{\text{o}}}}$$

2. $\mathbf{u} = 2\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$ *and* $\mathbf{v} = 3\,\mathbf{j} + 4\,\mathbf{k}$. *Find*

(i) $\quad \mathbf{u} + \mathbf{v}$

(ii) $\quad 2\,\mathbf{u} - \mathbf{v}$

(iii) $\quad |\mathbf{u}|$

(iv) $\quad \mathbf{u}\cdot\mathbf{v}$

(v) *The angle between* $\mathbf{u}$ *and* $\mathbf{v}$.

(i) $\mathbf{u} + \mathbf{v} = (2\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}) + (3\,\mathbf{j} + 4\,\mathbf{k}) = \underline{\underline{2\,\mathbf{i} + 4\,\mathbf{j} + 2\,\mathbf{k}}}$

(ii) $2\,\mathbf{u} - \mathbf{v} = 2(2\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}) - (3\,\mathbf{j} + 4\,\mathbf{k}) = 4\,\mathbf{i} + 2\,\mathbf{j} - 4\,\mathbf{k} - 3\,\mathbf{j} - 4\,\mathbf{k} = \underline{\underline{4\,\mathbf{i} - \mathbf{j} - 8\,\mathbf{k}}}$

(iii) $|\mathbf{u}| = \sqrt{2^2+1^2+2^2} = \sqrt{9} = \underline{\underline{3}}$

(iv) $\mathbf{u}\cdot\mathbf{v} = (2\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k})\cdot(3\,\mathbf{j} + 4\,\mathbf{k}) = 2\times 0 + 1\times 3 - 2\times 4 = \underline{\underline{-5}}$

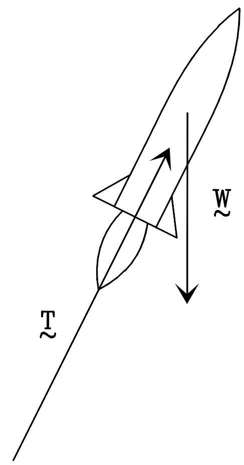
(v) The angle between $\mathbf{u}$ and $\mathbf{v}$. This works just like Q1, (the fact that we are now looking at vectors in 3 dimensions instead of 2 dimensions makes little difference). We have already found $|\mathbf{u}|$ and $\mathbf{u}\cdot\mathbf{v}$, we also need $|\mathbf{v}| = \sqrt{0^2+3^2+4^2} = \sqrt{25} = 5$.

Put these numbers into $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta$

and we have $-5 = 15\cos\theta \quad \Rightarrow \quad \cos\theta = -\frac{1}{3} \quad \Rightarrow \quad \underline{\underline{\theta = 1.911 \text{ radians} = 109.5^{\text{o}}}}$

3. *Shortly after launch the rocket engines on a spaceship of mass* 10^5 *kg give a thrust of* $10^6\,\underset{\sim}{\mathbf{i}} + 10^6\,\underset{\sim}{\mathbf{j}} + 2 \times 10^6\,\underset{\sim}{\mathbf{k}}$ *(in Newtons).*

(i) *Assuming the* z *axis is vertical, what is the total force acting on the spaceship?*



The two main forces acting on the rocket are the thrust, and the weight. The weight will have the magnitude Mg and will act downwards (ie in the $-\,\underset{\sim}{\mathbf{k}}$ direction).

$\underset{\sim}{\mathbf{W}} = -Mg\,\underset{\sim}{\mathbf{k}} = -9.8 \times 10^5\,\underset{\sim}{\mathbf{k}}$ Newtons. The total force is

$$\underset{\sim}{\mathbf{F}} = \underset{\sim}{\mathbf{T}} + \underset{\sim}{\mathbf{W}} = (10^6\,\underset{\sim}{\mathbf{i}} + 10^6\,\underset{\sim}{\mathbf{j}} + 2 \times 10^6\,\underset{\sim}{\mathbf{k}}) - 9.8 \times 10^5\,\underset{\sim}{\mathbf{k}} = \underline{\underline{(10^6\,\underset{\sim}{\mathbf{i}} + 10^6\,\underset{\sim}{\mathbf{j}} + 1.02 \times 10^6\,\underset{\sim}{\mathbf{k}})\ \mathrm{N}}}$$

(ii) *What is its acceleration?*

Use Newton's law $\underset{\sim}{\mathbf{F}} = M\,\underset{\sim}{\mathbf{a}} \quad \Leftrightarrow \quad \underset{\sim}{\mathbf{a}} = \frac{1}{M}\underset{\sim}{\mathbf{F}}$

$$\underset{\sim}{\mathbf{a}} = \frac{1}{M}\underset{\sim}{\mathbf{F}} = \frac{1}{10^5}(10^6\,\underset{\sim}{\mathbf{i}} + 10^6\,\underset{\sim}{\mathbf{j}} + 1.02 \times 10^6\,\underset{\sim}{\mathbf{k}}) = \underline{\underline{(10\,\underset{\sim}{\mathbf{i}} + 10\,\underset{\sim}{\mathbf{j}} + 10.2\,\underset{\sim}{\mathbf{k}})\ \mathrm{ms}^{-2}}}$$

4. *Two forces*

$$\begin{aligned} \vec{\mathrm{F}}_1 &= (2\mathbf{i} - 3\mathbf{j} + 7\mathbf{k})N \\ \text{and} \qquad \vec{\mathrm{F}}_2 &= (-\mathbf{i} - 4\mathbf{j} + \mathbf{k})N \end{aligned}$$

act at a point P. Find the resultant force at P, expressing the result in terms of its magnitude and a unit vector in the direction of the force.

By vector addition the resultant force is given by:

$$\vec{\mathrm{F}} = (\mathbf{i} - 7\mathbf{j} + 8\mathbf{k})N$$

Its magnitude is given by:

$$\text{magnitude} = \sqrt{1^2 + 7^2 + 8^2}N = \sqrt{114}N$$

Hence, a unite vector in the direction of $\vec{\mathrm{F}}$ is given by:

$$\text{unit vector} \quad = \quad \hat{\mathrm{F}} = \frac{\mathbf{i} - 7\mathbf{j} + 8\mathbf{k}}{\sqrt{114}}$$

5. *Find the component of the force*

$$\vec{\mathrm{F}} \quad = \quad (3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k})N$$

along the direction of the vector $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

$$\vec{\mathrm{F}} \quad = \quad (3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k})N$$

and the direction vector

$$\vec{\mathrm{d}} \quad = \quad (2\mathbf{i} + 2\mathbf{j} - \mathbf{k})m$$

The magnitude of the direction vector is given by:

$$|d| = \sqrt{2^2 + 2^2 + 1} = \sqrt{9} = 3$$

The component of $\vec{\mathrm{F}}$ in the direction of $\vec{\mathrm{d}}$ is given by

$$|\vec{\mathrm{F}}| \cos(\phi)$$

where ϕ is the angle between the vectors $\vec{\mathrm{F}}$ and $\vec{\mathrm{d}}$. From the definition of the scalar product we have

$$\vec{\mathrm{F}} \cdot \vec{\mathrm{d}} = |\vec{\mathrm{F}}||\vec{\mathrm{d}}| \cos(\phi)$$

Hence, to get the component of $\vec{\mathrm{F}}$ in the direction of $\vec{\mathrm{d}}$ we have to take:

$$\text{component of } \vec{\mathrm{F}} \text{ in the direction of } \vec{\mathrm{d}} \quad = \quad \frac{\vec{\mathrm{F}} \cdot \vec{\mathrm{d}}}{|\vec{\mathrm{d}}|} \quad = \quad \frac{6 + 10 - 7}{3} = \frac{9}{3} = 3N$$

6. *Find the work done by the force*

$$\vec{\mathrm{F}} \quad = \quad (2\mathbf{i} - \mathbf{j} + 4\mathbf{k})N$$

in moving a particle of mass 2kg in a straight line from the point $(1, 2, 5)m$ *to the point* $(2, 0, 6)m$.

$$\vec{\mathrm{F}} = (2\mathbf{i} - \mathbf{j} + 4\mathbf{k})N$$

and the direction vector

$$\vec{\mathrm{d}} = (2\mathbf{i} + 6\mathbf{k}) - (\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}) = (\mathbf{i} - 2\mathbf{j} + \mathbf{k})m$$

$$\vec{\mathrm{F}} \cdot \vec{\mathrm{d}} = 2 + 2 + 4 = 8Nm = 8J$$

7. *Find the vector product of the vectors*

$$\vec{\mathrm{a}} = (2\mathbf{i} - \mathbf{j} + 3\mathbf{k})N$$

and

$$\vec{\mathrm{b}} = (\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})N$$

$$\vec{\mathrm{a}} \times \vec{\mathrm{b}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & 3 & -4 \end{vmatrix} = \mathbf{i}(4 - 9) - \mathbf{j}(-8 - 3) + \mathbf{k}(6 + 1) = (-5\mathbf{i} + 11\mathbf{j} + 7\mathbf{k})N^2$$

where the final result is has units N^2 since each of the vectors is given in terms of units N.