MATH 181.28 | Lecture 8

We can take higher order derivatives:

Example: $W = f(x,y) = x^2 y + \frac{x}{y}$

1st order: $\frac{\partial f}{\partial x} = 2xy + \frac{1}{y}$; $\frac{\partial f}{\partial y} = x^2 - \frac{x}{y^2}$.

2nd order: $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = 2y$

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{x}{y^3}$

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = 2x - \frac{1}{y^2}$

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = 2x - \frac{1}{y^2} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$.

We see $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ } usually the case if the derivatives exist.

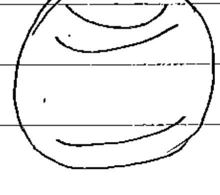
Alternative notation: $\frac{\partial f}{\partial x} = f_x$; $\frac{\partial f}{\partial y} = f_y$

$\frac{\partial^2 f}{\partial x^2} = f_{xx}$; $\frac{\partial^2 f}{\partial y^2} = f_{yy}$; $\frac{\partial^2 f}{\partial y \partial x} = f_{yx}$; $f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$

Some uses of partial Derivatives

Stationary points: $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

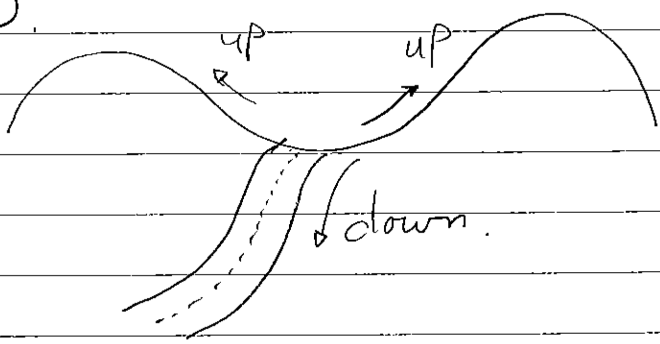
i.e. $\vec{\nabla} f = 0$. (all components are 0).

Max or Min.  Hill top = Max. All directions curve down.

Bottom of ball = Min. surface curves upwards

New posibility in 2D.

Mountain pass



Both derivatives vanish at the mountain pass. But one curves up and one curves down.

This is called a <u>saddle Point</u>

<u>Saddle point test</u>: only at point with $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0$

Find $\Delta = \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 F}{\partial y^2} - \left(\frac{\partial F}{\partial x \partial y}\right)^2$

if $\Delta > 0 \Rightarrow$ not saddle, i.e. Max or Min

$\Delta < 0 \Rightarrow$ saddle.

Do saddle test <u>first</u>

if not saddle.

<u>And</u> $\left(\frac{\partial^2 F}{\partial x^2} > 0 \text{ OR } \frac{\partial^2 F}{\partial y^2} > 0\right) \Rightarrow$ Min.

if not saddle

<u>And</u> $\left(\frac{\partial^2 F}{\partial x^2} < 0 \text{ OR } \frac{\partial^2 F}{\partial y^2} < 0\right) \Rightarrow$ Max.

if $\Delta = 0 \Rightarrow$ cannot conclude. $\rightarrow$ investigate further

Implicit Differentiation

single variable case;

Example: $x^2+y^2=a^2 \Rightarrow 2x+2y\frac{dy}{dx}=0$

$\Rightarrow \frac{dy}{dx}=-x/y.$

suppose that a function $z=f(x,y)$ is defined implicitly.

Example: $x^2+3y^2+2z^2=6.$

implicit differentiation w.r.t. x: $2x+4z\frac{\partial z}{\partial x}=0$

$\Rightarrow \frac{\partial z}{\partial x}=-x/2z$

implicit differentiation w.r.t. y: $6y+4z\,\partial z/\partial y=0.$

$\Rightarrow \frac{\partial z}{\partial y}=-3y/2z$

An example involving integration.

suppose: we want to find a function $z=f(x,y)$ such that:

$\frac{\partial f}{\partial x}=2xy+\sin y+1$ and $\frac{\partial f}{\partial y}=x^2+x\cos y+2y$

integrate w.r.t x $\quad f(x,y)=x^2y+x\sin y+x+\phi(y)+C.$

integrate w.r.t. y $\quad f(x,y)=x^2y+x\sin y+y^2+\alpha(x)+C.$

Now compose: $\quad f(x,y)=x^2y+x\sin y+x+y^2+C$

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The chain rule

Previously we had for one variable,

chain rule: if $y = f(g(x))$ $\quad \dfrac{dy}{dx} = \dfrac{df}{dg}\dfrac{dg}{dx}.$

suppose that $z = f(x,y)$ and $x = x(t)$ $\quad y = y(t)$

Then $\dfrac{dz}{dt} = \lim_{\delta t \to 0} \dfrac{f(x+\delta x, y+\delta y) - f(x,y)}{\delta t}.$

$\delta x, \delta y$ are increments in x and y corresponding to δt in t.

$$\frac{dz}{dt} = \lim_{\delta t \to 0}\left[\frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x}\frac{\delta x}{\delta t} + \frac{f(x, y+\delta y) - f(x,y)}{\delta y}\frac{\delta y}{\delta t}\right]$$

$$\Rightarrow \frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Example: $z = f(x,y) = x^2y^3$ $\quad x = \sin(t)$, $y = e^t$.

$\dfrac{\partial f}{\partial x} = 2xy^3$ $\quad \dfrac{dx}{dt} = \cos t.$

$\dfrac{\partial f}{\partial y} = 3x^2y^2$ $\quad \dfrac{dy}{dt} = e^t.$

$$\Rightarrow \frac{dz}{dt} = (2\sin t\, e^{3t})\cdot(\cos t) + (3\sin^2 t\, e^{2t})(e^t).$$

$$= e^{3t}\sin t(2\cos t + 3\sin t).$$

can check. $z = \sin^2 t\, e^{3t}$ $\quad \dfrac{dz}{dt} = 2\sin t\, e^{3t} + 3\sin^2 t\, e^{3t}.$

same.

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We can generalize. $W = f(x,y)$ where $x = x(s,t)$, $y = y(s,t)$.

We obtain
$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}.$$
$$\frac{\partial w}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

Each extra intermediate variable $(x, y, \dots)$ leads to an extra term on the right-hand side.

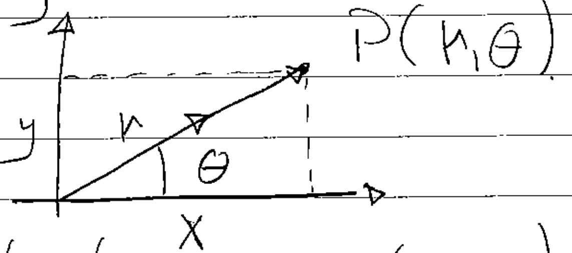
Each extra variable $(s, t, \dots)$ leads to another equation.

Note: Partial differential equations are very important in Physics (Maxwell eqs., Schrödinger eq. ....). And in all sciences.

Question: if $W = f(x,y,z) = xy^2 + x\sin y + z^3$

Find: f_x, f_y, f_z, f_{xy}, f_{yy}.

Coordinate systems in 2 dimensions



(x,y) = cartesian coordinates $(x, y \in \mathbb{R})$.

(r,θ) = Polar coordinates.

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$x = r\cos\theta$, $\quad r = \sqrt{x^2+y^2}$

$y = r\sin\theta$, $\quad \theta = \tan^{-1}(y/x)$

$r \geq 0$; $\quad 0 \leq \theta \leq 2\pi$

Note: there is a degeneracy for $r = 0$.

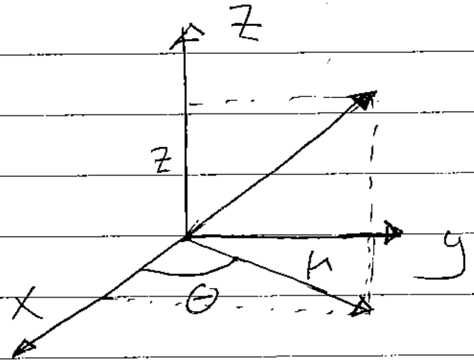
Next: In 3 dimensions

i) In cylindrical polar coordinates

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$



$r \geq 0$

$0 \leq \theta \leq 2\pi$

$-\infty < z < \infty$

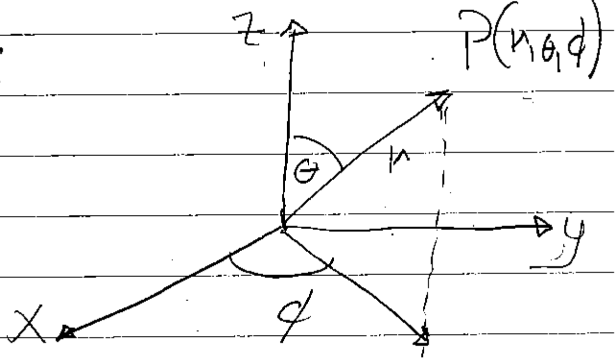
Used for problems with cylindrical symmetry

ii) Spheiral coordinates

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$



$r \geq 0$

$0 \leq \theta \leq \pi$

$0 \leq \phi \leq 2\pi$

Used for problem with spherical symmetry eg. $V = V(r)$

Note: coordinates are not physical, they are just useful labels.

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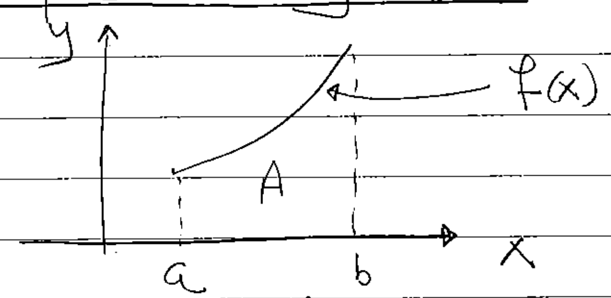
Multiple Integrals

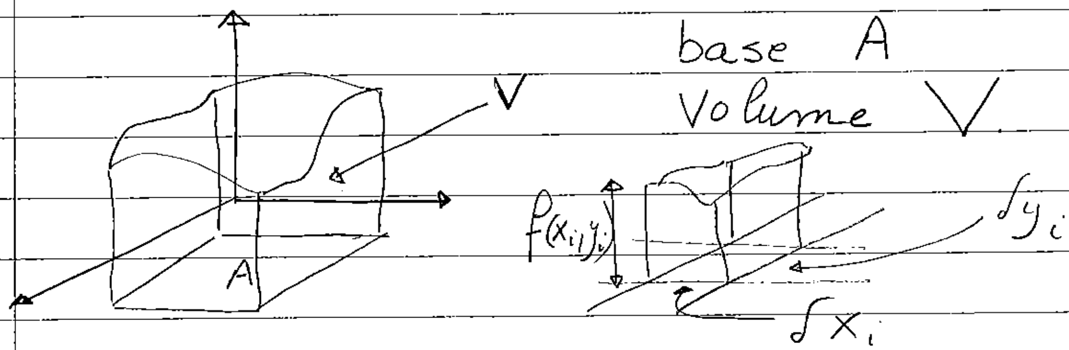
we saw.



$$A = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\delta x_i$$

A as area below the curve.

Double integrals → as a volume.

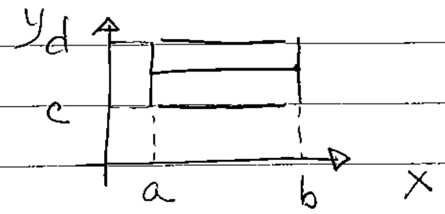


$$\Delta V_{ij} = f(x_i, y_i)\,\delta x_i\,\delta y_i$$

Take the limit $\lim_{M,N\to\infty} \sum_{i=1}^{N}\sum_{j=1}^{M} \Delta V_{ij}$ with $\delta x_i \to 0$, $\delta y_i \to 0$.

$$\Longrightarrow \quad V = \iint_A f(x,y)\,dx\,dy.$$

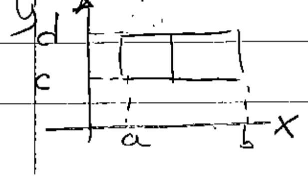
Evaluation



Perform x-integration first
(y = constant)

$$= \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy.$$

OR, Perform y-integral first (treat x as constant)



$$= \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

Example: let $f(x,y) = 3x^2 + 2y$.

and $a=0$, $b=2$, $c=1$, $d=2$.

Want to evaluate $I = \int_A f(x,y)\,dx\,dy$.

$$I = \int_0^2 \left(\int_1^2 (3x^2 + 2y)\,dy \right) dx =$$

$$= \int_0^2 \left[3x^2 y + y^2 \right]_1^2 dx = \int_0^2 [(6x^2+4) - (3x^2+1)]\,dx$$

$$= \int_0^2 (3x^2 + 3)\,dx = \left[x^3 + 3x \right]_0^2 = 14.$$

OR do the x-integral first. get the same result.

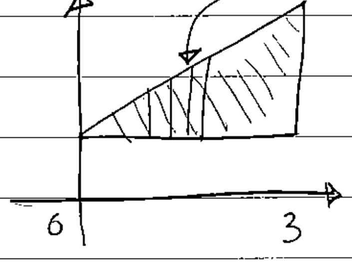
$$I = \int_1^2 \left(\int_0^2 (3x^2 + 2y)\,dx \right) dy = \int_1^2 \left[x^3 + 2yx \right]_0^2 dy =$$

$$= \int_1^2 (8 + 4y)\,dy = \left[8y + 2y^2 \right]_1^2 = (16+8) - (8+2) = 14$$

This is simple for the case of A being a retangle.

What about if A is a triangle?

$\rightarrow$ Essential to get the limits right!



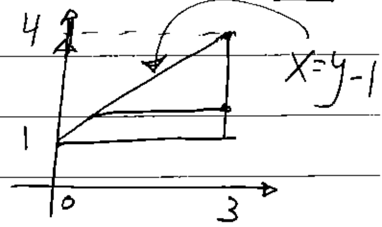
$$\left. \begin{array}{l} 0 \leq x \leq 3 \\ 1 \leq y \leq 1+x \end{array} \right\} A.$$

Let $f(x,y) = 2(x+y)$.

$$V = \int_0^3 \left(\int_1^{y=1+x} 2(x+y)\,dy \right) dx = \int_0^3 \left[2xy + y^2 \right]_1^{1+x} dx$$

$$= \int_0^3 \left(2x(1+x) + (1+x)^2 - (2x+1) \right) dx$$

$$= \int_0^3 \left(3x^2 + 2x \right) dx = \left[x^3 + x^2 \right]_0^3 = 27 + 9 = \underline{36}$$



Now Do the x- integration first.

$$V = \int_1^4 \left(\int_{x=y-1}^{3} 2(x+y)\,dx \right) dy$$

$$= \int_1^4 \left[x^2 + 2yx \right]_{y-1}^{3} dy =$$

$$= \int_1^4 \left[(9+6y) - \left((y-1)^2 + 2y(y-1) \right) \right] dx$$

$$= \int_1^4 \left(9 + 6y - y^2 + 2y - 1 - 2y^2 + 2y \right) dx$$

$$= \int_1^4 \left(-3y^2 + 10y + 8 \right) dx$$

$$= \left[-y^3 + 5y^2 + 8y \right]_1^4 = \left(\overbrace{-64+80+32}^{48} \right) - \left(\overbrace{-1+5+8}^{12} \right) = \underline{\underline{36}}$$

Note: more complicated areas can be subdivided.

Triple integrals over a volume V can be treated as generalization of double integrals over the area A.

$$V = \iiint dx\,dy\,dz.$$

Mass $M = \iiint \rho(x,y,z)\,dx\,dy\,dz.$