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Lecture 7.

Introduction to functions of Many Variables.

So far we have dealt with functions of one variable.

$y = f(x)$, e.g. $y(x) = \frac{x}{x+1}$.

We dealt with differentiation & integration of such functions

However: often we have to deal with functions that depend on many variables.

For example the temperature typically depends on location and height as well as time $T = f(x, y, z, t)$.

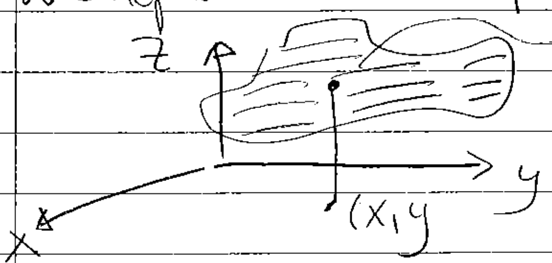
Simplest case: two variables $z = f(x, y)$.

→ calculate a number z for values of x and y.

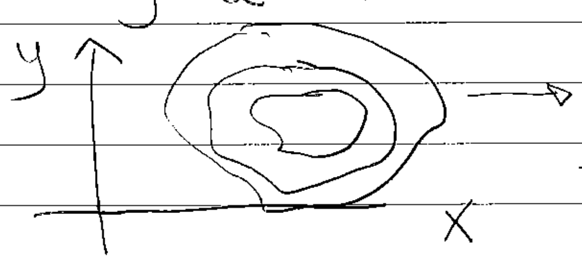
$f(x)$ one input → one output

$f(x,y)$ two inputs → one output.

We represent such a function by a surface in three dimensions



OR by a contour map



contours of equal z value

→ e.g a topographical map.

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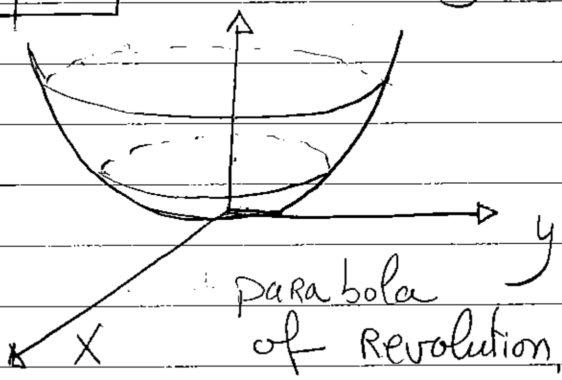
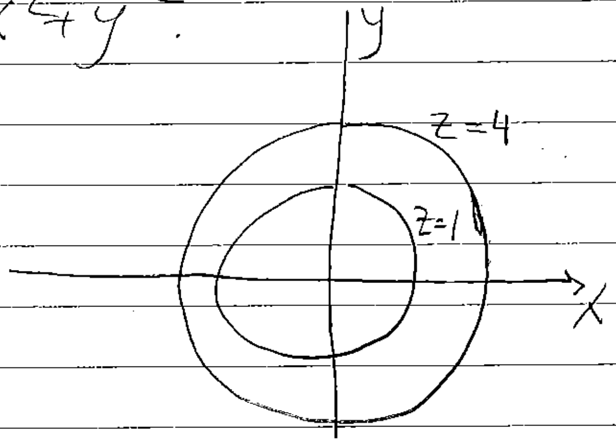
Example 1 $z = f(x,y) = x^2+y^2$.



parabola of Revolution.



The are many practical applications:

- Weather charts $P=P(x,y)$ isobars
  $T=T(x,y)$ isotherms.
- maps $h=h(x,y)$ contour lines

In physics: i) Electrostatic potential, $\phi(x,y,z)$
ii) Temperature, $T = T(x,y,z,t)$.
iii) density, $\rho = \rho(x,y,z)$.

$\longrightarrow$ We need to extend our calculus to more than one variable.

Differentiation of a function of 2 variables

Let $z = f(x,y)$. we define the partial derivative as

$$\frac{\partial f}{\partial x} = \lim_{\epsilon \to 0} \frac{f(x+\epsilon, y) - f(x,y)}{\epsilon}.$$

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x,y)}{k}.$$

Note: New symbol ∂ = partial derivative; if it exists.

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Example: $z = f(x,y) = x^2 + xy^2 + y^3$,

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = 2x + y^2$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = 2xy + 3y^2$$

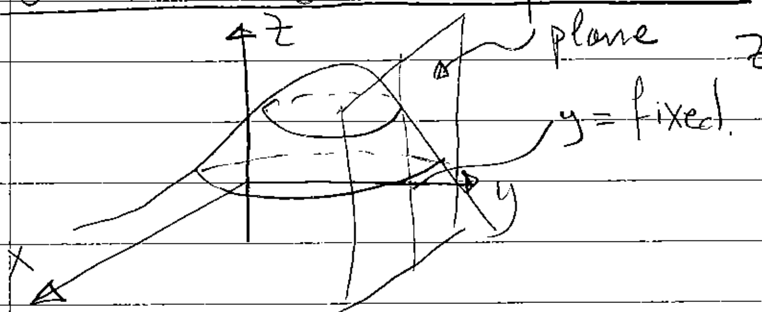
rule: $\frac{\partial f}{\partial x}$ is the derivative w.r.t. x, with y fixed.

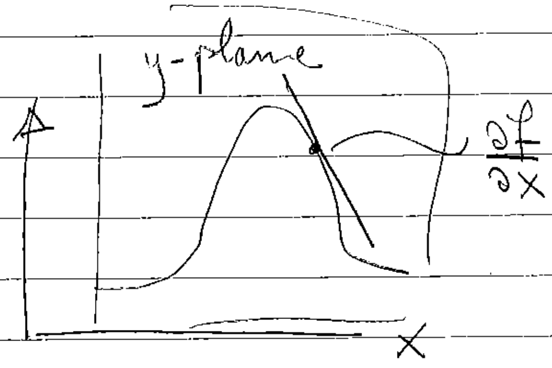
$\frac{\partial f}{\partial y}$ is the derivative w.r.t. y, with x fixed.

Example: $f(x,y) = e^{xy} \sin y$.

$$\frac{\partial f}{\partial x} = y e^{xy} \sin y \; ; \; \frac{\partial f}{\partial y} = x e^{xy} \sin y + e^{xy} \cos y.$$

Geometrical interpretation





$x, y, z, \ldots \rightarrow$ The generalization to functions of more than two variables is simple. When we differentiate w.r. to one variable, the others are treated as constants.

Example: $w = f(x,y,z) = x^2 y z + \frac{x}{y}$.

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} = 2xyz + \frac{1}{y}.$$

$$\frac{\partial w}{\partial y} = \frac{\partial f}{\partial y} = x^2 z - \frac{x}{y^2}$$

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$$\frac{\partial w}{\partial z} = \frac{\partial f}{\partial z} = x^2 y$$

Gradients.

summarise both derivatives together in a gradient vector

grad $f(x,y)$ OR $\vec{\nabla} f(x,y)$.

$$\vec{\nabla} f(x,y) = \frac{\partial f(x,y)}{\partial x}\hat{i} + \frac{\partial f(x,y)}{\partial y}\hat{j}$$

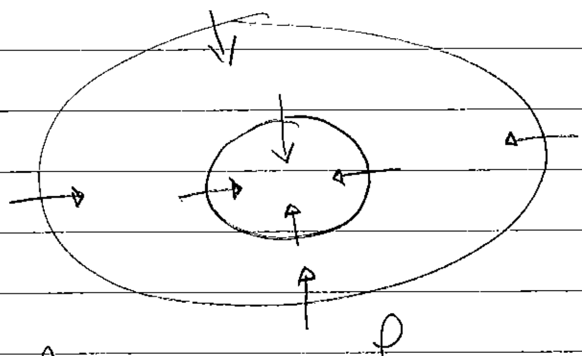
must not forget $\hat{i}$, $\hat{j}$.

$\vec{\nabla} f$ is a vector.

For 2D functions gradient is not $\frac{dy}{dx}$.

$\vec{\nabla} f$ is a vector at right-angles to contours.

Hill.



Gradients points uphill

steepest ascent.

in 3D $\vec{\nabla} f(x,y,z) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$.