some standard Integrals are:

$$\int \frac{dx}{x+a} = \ln|x+a| + C$$

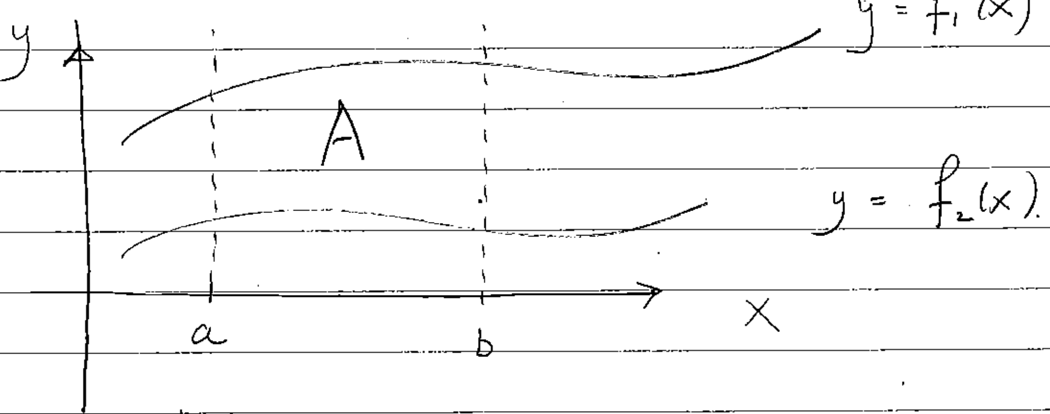
$$\int \frac{dx}{(x+a)^n} = -\frac{1}{(n-1)}(x+a)^{-(n-1)} + C \quad n = 2, 3, 4, \ldots$$

$$\int \frac{x\,dx}{x^2+a^2} = \frac{1}{2}\ln(x^2+a^2) + C$$

$$\int \frac{Ax+B}{[(x+b)^2+a^2]}\,dx = \frac{A}{2}\ln[(x+b)^2+a^2] + \left(\frac{B-Ab}{a}\right) * \tan^{-1}\left(\frac{x+b}{a}\right) + C$$

Applications of integration

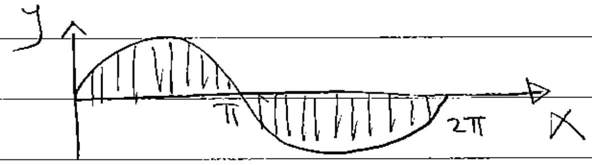
1) Determination of areas.



$$A = \int_a^b [f_1(x) - f_2(x)]\,dx.$$

Note: areas below the x-axis count as $-$

e.g.



$$\int_0^{2\pi} \sin x\,dx = \left[-\cos x\right]_0^{2\pi} = 0.$$

split into separate integrals at the zeroes of $\sin x$.

$$\int_0^{2\pi} |\sin x|\,dx = \int_0^{\pi} \sin x\,dx + \int_{\pi}^{2\pi} (-\sin x)\,dx = \left[-\cos x\right]_0^{\pi} + \cos x\Big|_{\pi}^{0} = 4.$$

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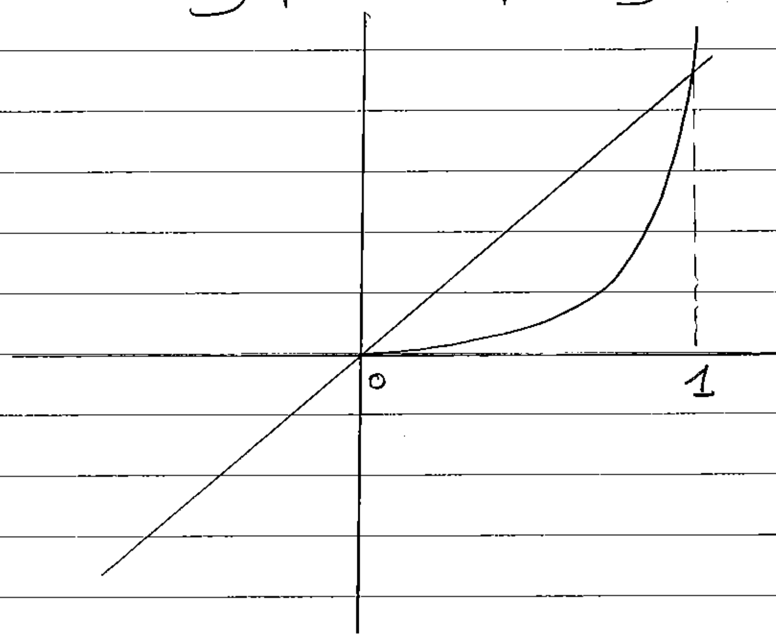
Example: what is the area between $f(x) = x$ and $g(x) = x^2$?

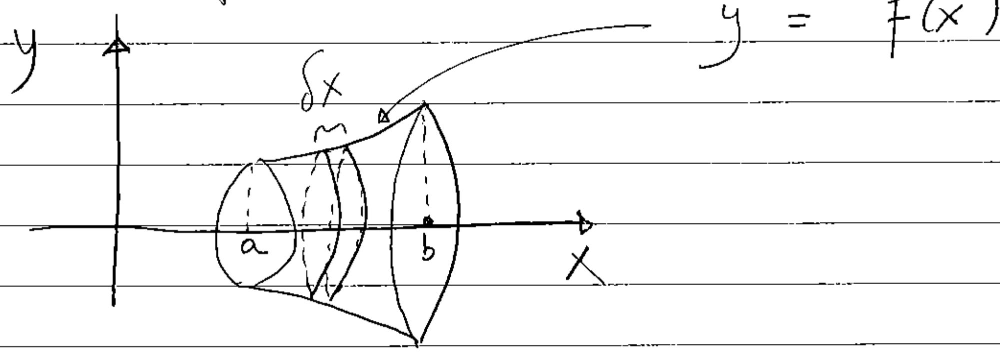
Answer: Find the crossing points $f(x) = g(x)$



$f(x) = g(x) \Rightarrow \quad x = 0, 1$

$$A = \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

ii) Volumes of Revolution



Volume of a slice; $\delta V = \underbrace{\pi y^2}_{\text{area of disc}} \times \underbrace{\delta x}_{\text{thickness of disc}}$

$$V = \int_a^b \pi y^2\,dx = \pi \int_a^b (f(x))^2\,dx$$

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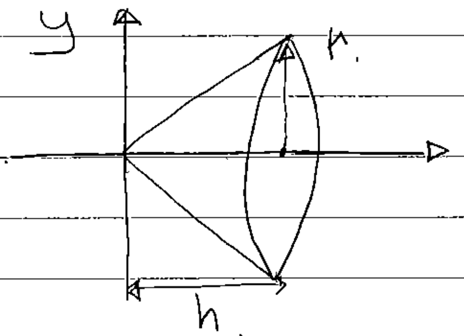
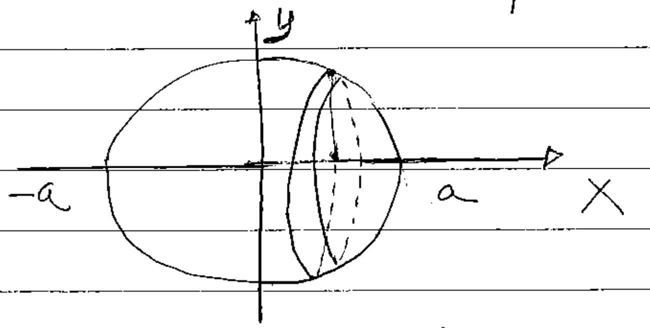
Problem: what is the volume of the cone?



$$y = \frac{r}{h} x$$

$$V = \pi \int_0^h \left(\frac{rx}{h}\right)^2 dx = \frac{\pi r^2}{h^2} \cdot \left[\frac{x^3}{3}\right]_0^h = \frac{\pi r^2 h}{3}$$

Example: The volume of a sphere.



on the surface of the circle $x^2 + y^2 = a^2$

$$\Rightarrow y^2 = a^2 - x^2$$

$$\Rightarrow \int_{-a}^{a} \pi y^2 dx = \int_{-a}^{a} \pi (a^2 - x^2) dx = \pi \left[a^2 x - \frac{x^3}{3}\right]_{-a}^{a} =$$

$$= \pi \left(\underbrace{\left[a^2 \cdot a - \frac{a^3}{3}\right]}_{\frac{2}{3}a^3} - \underbrace{\left[-a^3 + \frac{a^3}{3}\right]}_{-\frac{2}{3}a^3} \right) = \frac{4\pi a^3}{3}$$

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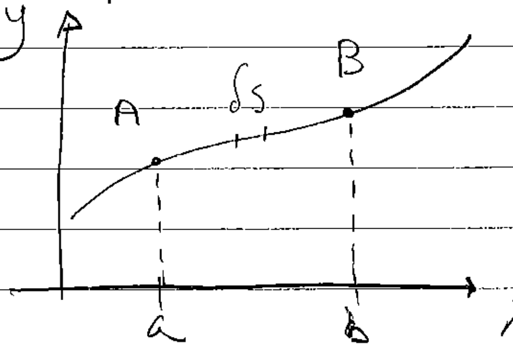
iii) Length of curves.



δs / δy, δx

$$(\delta s)^2 \simeq (\delta x)^2 + (\delta y)^2$$

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2}$$

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$\Rightarrow$ Add up all the Pieces.

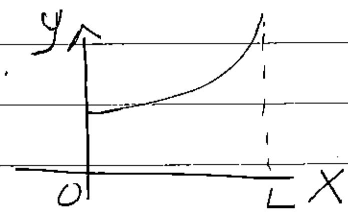
$$S = \text{length of curve} = \int_a^b \sqrt{dx^2 + dy^2} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Curves are often specified in parametric form,

$$x = x(t) \quad , \quad y = y(t) \qquad t_A \leq t \leq t_B.$$

$$\Rightarrow S = \int_{t_A}^{t_B} \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{1/2} dt.$$

Example) $y = \cosh x \qquad 0 \leq x \leq L$



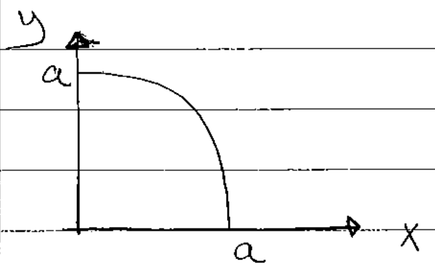
Recall $y = \dfrac{e^x + e^{-x}}{2}.$

$$y' = \frac{e^x - e^{-x}}{2} = \sinh x. \qquad \cosh^2 x - \sinh^2 x = 1$$

$$\Rightarrow S = \int_0^L \sqrt{1 + \sinh^2 x}\, dx = \int_0^L \cosh x\, dx = [\sinh x\Big|_0^L = \sinh(L)$$

Example) Circular arc $\quad x = a \sin t \qquad 0 \leq t \leq \pi/2.$

$y = a \cos t$



$$\frac{dx}{dt} = a \cos t.$$

$$\frac{dy}{dt} = a \sin t.$$

$$S = \int_0^{\pi/2} (a^2\cos^2 t + a^2 \sin^2 t)^{1/2} dt = \int_0^{\frac{\pi}{2}} a\, dt$$

$$= \left[a t\right]_0^{\pi/2} = \frac{a\pi}{2} = \frac{2\pi a}{4}. \quad \checkmark$$