MATH181.13 | substitution Revisited | Lecture 4

Definite integrals → two methods.

i) Treat as an indefinite integral – obtain the solution in terms of the original variable – impose original limits.

ii) change the limit when we change the variable.

Example $\int_1^2 (3x-2)^4 \, dx$

let $u = 3x-2 \quad du = 3dx$.

$$\int u^4 \frac{1}{3} du = \frac{1}{15} u^5 = \frac{1}{5}(3x-2)^5$$

So $\int_1^2 (3x-2)^4 dx = \left[\frac{1}{15}(3x-2)^5\right]_1^2 = \frac{1}{15}(4^5 - 1^5)$

$= \frac{1}{15}(4-1)(4^4+4^3+4^2+4+1) = \frac{1}{5}(341)$.

follows from: $(x^n - y^n) = (x-y)\cdot(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1})$.

ii) let $u = 3x-2$ then $du = 3dx$.

when $x = 1, \; u = 1$

$x = 2, \; u = 4$

$$\int_1^2 (3x-2)^4 dx = \int_1^4 u^4 \frac{du}{3} = \frac{1}{15}\left[u^5\right]_1^4 = \frac{1}{5}(4^5 - 1)$$

consider $\int \frac{dx}{x} = \ln x + C$ for $x > 0$

what about $x < 0$?

$$\int \frac{dx}{x} = \ln |x| + C$$

by substitution : $\int \frac{dx}{x+a} = \ln |x+a| + C$

$$\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + C$$

Example: $\int \frac{x}{x^2+4}\,dx = \frac{1}{2}\ln|x^2+4| + C.$

Note: $\frac{d}{dx}\left(e^{f(x)}\right) = f'(x)\,e^{f(x)}$

$\Rightarrow \int f'(x)\,e^{f(x)}\,dx = e^{f(x)} + C$

Example $\int x e^{x^2}\,dx = \frac{1}{2}e^{x^2} + C$

$\int \frac{e^x}{e^x+1} = \ln|e^x+1|$

Integration of Rational functions

Rational function - quotient of two polynomials can be integrated in a systematic fashion.

Problem what is $\int \frac{x^2+2}{x^2+1}\,dx$?

Step 1: if the order (highest power) is greater than or equal to the denominator, we divide out

Example $\frac{x^2+2}{x^2+1} = 1 + \frac{1}{x^2+1}$

step 2: EXPand in Partial fractions.

Example: $\frac{1}{(x+1)(x+3)} = \frac{A}{(x+1)} + \frac{B}{(x+3)} \Rightarrow A(x+3)+B(x+1) = 1$

and determine A and B to be $A+B=0$
$3A+B=1$
$\Rightarrow A=\frac{1}{2} \quad B=-\frac{1}{2}$

or by substituting $x=-1 \Rightarrow A=\frac{1}{2}$ $x=-3 \Rightarrow B=-\frac{1}{2}$

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step 3 - integrate using standard integrals

Example $\int \frac{dx}{(x+1)(x+3)} = \frac{1}{2}\int\left(\frac{1}{x+1} - \frac{1}{x+3}\right)dx = \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x+3| + C$

Each step in detail

step 1 consider $f(x) = \frac{3x^3 + 2x^2 + x + 3}{x^2 + x + 1}$

Quadratic in the denominator => want to produce a cubic in numerator.

$$\frac{3x^3 + 2x^2 + x + 3}{(x^2 + x + 1)} = \frac{Ax + B}{(x^2 + x + 1)} + Cx + D$$

Combining the terms on the right-hand side and comparing coefficients of equal powers

$$\begin{aligned}
\Longrightarrow \quad 3x^3 &= Cx^3 &\Rightarrow\quad C &= 3\\
2x^2 &= (C+D)x^2 &\Rightarrow\quad D &= -1\\
x &= (A+C+D)x &\Rightarrow\quad A &= -1\\
3 &= (B+D) &\Rightarrow\quad B &= 4
\end{aligned}$$

step 2 Partial fraction expansion

Example $y = \frac{2x+1}{(x+1)^3(x^2+1)}$

4 pieces in the denominator, so 4 partial fractions.

$$\frac{(2x+1)}{(x+1)^3(x^2+1)} = \underbrace{\frac{A}{(x+1)^3} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)} + \frac{Dx+E}{(x^2+1)}}_{} =$$

when we add all of these together we get a quartic in the numerator.

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$$= \frac{1}{(X+1)^3(X^2+1)}\left[A(X^2+1) + B(X^2+1)(X+1) + C(X^2+1)(X+1)^2 + (DX+E)(X+1)^3\right]$$

$$= \frac{1}{(X+1)^3(X^2+1)}\left[\alpha X^4 + \beta X^3 + \gamma X^2 + \delta X + \epsilon\right]$$

comparing the two sides we see that we need to set: $\alpha, \beta, \gamma = 0$

$$\delta = 2$$
$$\epsilon = 1$$

collect coefficients of equal powers on both sides:

X^4: $D + C = 0 \Rightarrow C = -D$

X^3: $3D + E + 2C + B = 0 \Rightarrow D + E + B = 0$

X^2: $3D + 3E + 2C + B + A = 0 \Rightarrow D + 3E + B + A = 0$

X^1: $D + 3E + 2C + B = 2 \Rightarrow -D + 3E + B = 2$

X^0: $E + C + B + A = 1 \Rightarrow E - D + B + A = 1$

solve the four equations with four unknowns.

$\Rightarrow A = -\frac{1}{2} \quad B = \frac{1}{2} \quad C = \frac{3}{4} \quad D = -\frac{3}{4} \quad E = \frac{1}{4}$