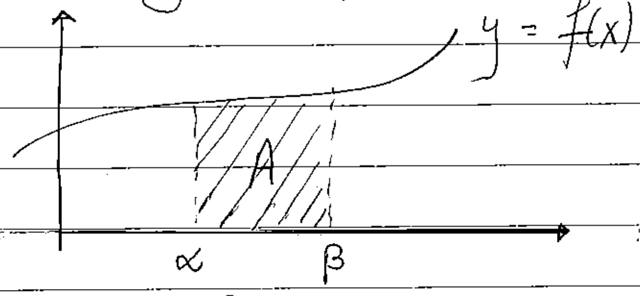
MATH 181.10 | Lecture 3

Definite integration



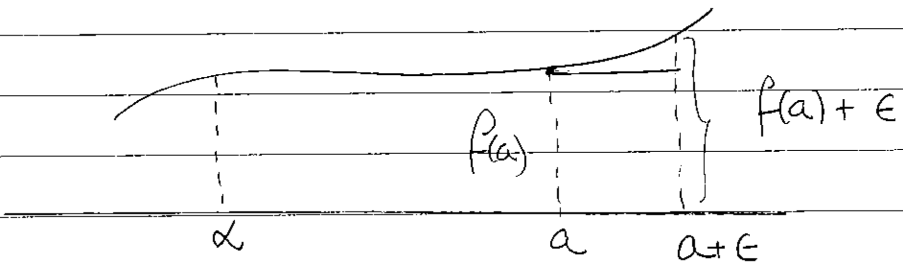
i) $f(x)$ is continuous.
ii) $f(x) \geq 0$ for $\alpha \leq x \leq \beta$

$\int_\alpha^\beta f(x)\,dx$ OR $\int_\alpha^\beta dx\, f(x)$ – definite integral from α to β.
– gives the area A

Show that $\int_\alpha^\beta f(x) = F(\beta) - F(\alpha)$

where $\frac{dF(x)}{dx} = f(x)$

$F(x)$ is an indefinite integral of $f(x)$.



Let $G(a)$ = area from $x = \alpha$ to $x = a$
$G(a+\epsilon)$ = area from $x = \alpha$ to $x = a+\epsilon$

$\epsilon \cdot f(a) \leq G(a+\epsilon) - G(a) \leq \epsilon f(a+\epsilon)$

$\Rightarrow f(a) \leq \frac{G(a+\epsilon) - G(a)}{\epsilon} \leq f(a+\epsilon)$

let $\epsilon \to 0 \implies f(a) = G'(a)$.

let $F(x)$ be an indefinite integral of $f(x) = F'(x)$.

i.e. $F(x) = G(x) + C \qquad F'(x) = G'(x)$

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$$G(\alpha) = 0 \implies F(\alpha) = C$$

$$G(x) = F(x) - F(\alpha)$$

$$G(\beta) = F(\beta) - F(\alpha) = \Big[F(x) \Big]_{\alpha}^{\beta}$$ notation.

hence $F(\beta) - F(\alpha) = \int_{\alpha}^{\beta} dx\, F'(x) = \int_{\alpha}^{\beta} dx\, f(x)$

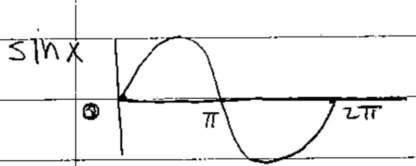
Some Results

i) $\int_{\alpha}^{\beta} f(x)\,dx = \int_{\alpha}^{\gamma} f(x)\,dx + \int_{\gamma}^{\beta} f(x)\,dx.$

ii) $\int_{\alpha}^{\beta} (a f(x) + b g(x))\,dx = a\int_{\alpha}^{\beta} f(x)\,dx + b\int_{\alpha}^{\beta} g(x)\,dx.$

iii) $\int_{\alpha}^{\beta} f(x)\,dx = -\int_{\beta}^{\alpha} f(x)\,dx.$

iv) Areas below the x-axis are negative.

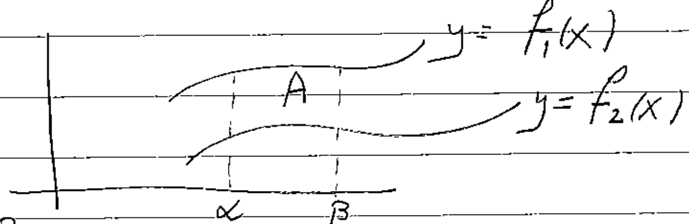


$$\int_{0}^{2\pi} \sin x\,dx = \Big[-\cos x\Big]_{0}^{2\pi} = 0$$

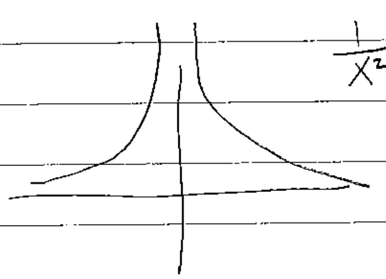
→ to get the 'Physical area', have to split the integral.

$$\int_{0}^{\pi} \sin x\,dx - \int_{\pi}^{2\pi} \sin x\,dx = \Big[-\cos x\Big]_{0}^{\pi} + \Big[\cos x\Big]_{\pi}^{2\pi} = 1-(-1) + 1-(-1) = 4$$

v)



$$A = \int_{\alpha}^{\beta} \big(f_1(x) - f_2(x)\big)\,dx$$



vi) Do not integrate across singularities

Example: $\int_{-1}^{1} \frac{1}{x^2}\,dx =$ does not exist $\neq \Big[-\frac{1}{x}\Big]_{-1}^{1}$

Improper integrals

a) infinity as a limit of integration. $\int_{\alpha}^{\infty} f(x)dx$.

also $\int_{-\infty}^{\beta} f(x)dx$ $\int_{-\infty}^{\infty} f(x)dx$.

Define $\int_{\alpha}^{\infty} f(x)\,dx \equiv \lim_{R\to\infty} \int_{\alpha}^{R} f(x)dx$ if it exists.

Example $\int_{1}^{\infty} e^{-x}dx = \lim_{R\to\infty} \int_{1}^{R} e^{-x}dx = \lim_{R\to\infty} \left[-e^{-x}\right]_{1}^{R}$

$= \lim_{R\to\infty} \left[-e^{-R} + e^{-1}\right] = e^{-1}$

contrary to this $\lim_{R\to\infty} \int_{0}^{R} \cos x\,dx = \lim_{R\to\infty} \left[\sin x\right]_{0}^{R} = \lim_{R\to\infty} \sin R$

does not exist.

Example $\int_{0}^{\infty} \frac{dx}{1+x^2} = \lim_{R\to\infty} \left[\tan^{-1} x\right]_{0}^{R} = \lim_{R\to\infty} \left[\tan^{-1} R - 0\right] = \frac{1}{2}\pi$.

b) $f(x)$ is singular at the boundary
can attach a meaning to $\int_{0}^{4} \frac{dx}{\sqrt{x}}$?

Define this to be the limit

$\lim_{\epsilon\to 0} \int_{\epsilon}^{4} \frac{dx}{\sqrt{x}}$ if it exists $(\epsilon > 0)$.

$\lim_{\epsilon\to 0} \int_{\epsilon}^{4} \frac{dx}{\sqrt{x}} = \lim_{\epsilon\to 0} \left[2\sqrt{x}\right]_{\epsilon}^{4} = \lim_{\epsilon\to 0} \left(4 - 2\sqrt{\epsilon}\right) = 4$.

contrary to this:

$\int_{0}^{1} \frac{dx}{x} = \lim_{\epsilon\to 0} \left[\ln x\right]_{\epsilon}^{1} = \lim_{\epsilon\to 0} \left(0 - \ln\epsilon\right)$

$\longrightarrow$ the limit does not exist $(\to\infty)$.