Math 181.7 (Integration by substitution). Lecture 2

Example 1

$$\int dx \sin^4 x \cos x$$

let $u = \sin x$ then $du = \cos x \, dx$.

$$\int dx \cos x \sin^4 x = \int du \, u^4 = \frac{u^5}{5} + C = \frac{1}{5}\sin^5 x + C.$$

Example. $\int \frac{dx}{\sqrt{9 - x^2}}$

let $x = 3\sin\theta \Rightarrow dx = 3\cos\theta \, d\theta$.

$$\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{3\cos\theta \, d\theta}{\sqrt{9 - 9\sin^2\theta}} = \frac{3}{3}\int \frac{\cancel{\cos\theta}}{\cancel{\cos\theta}} d\theta = \theta + C.$$

$$= \sin^{-1}(x/3) + C$$

Example 1 $\int \frac{dx}{x^2+2x+5} = \int \frac{dx}{(x+1)^2+1}$

let $u = x+1 \Longrightarrow du = dx$.

$$\Rightarrow \int \frac{dx}{(x+1)^2+4} = \int \frac{du}{u^2+4}.$$

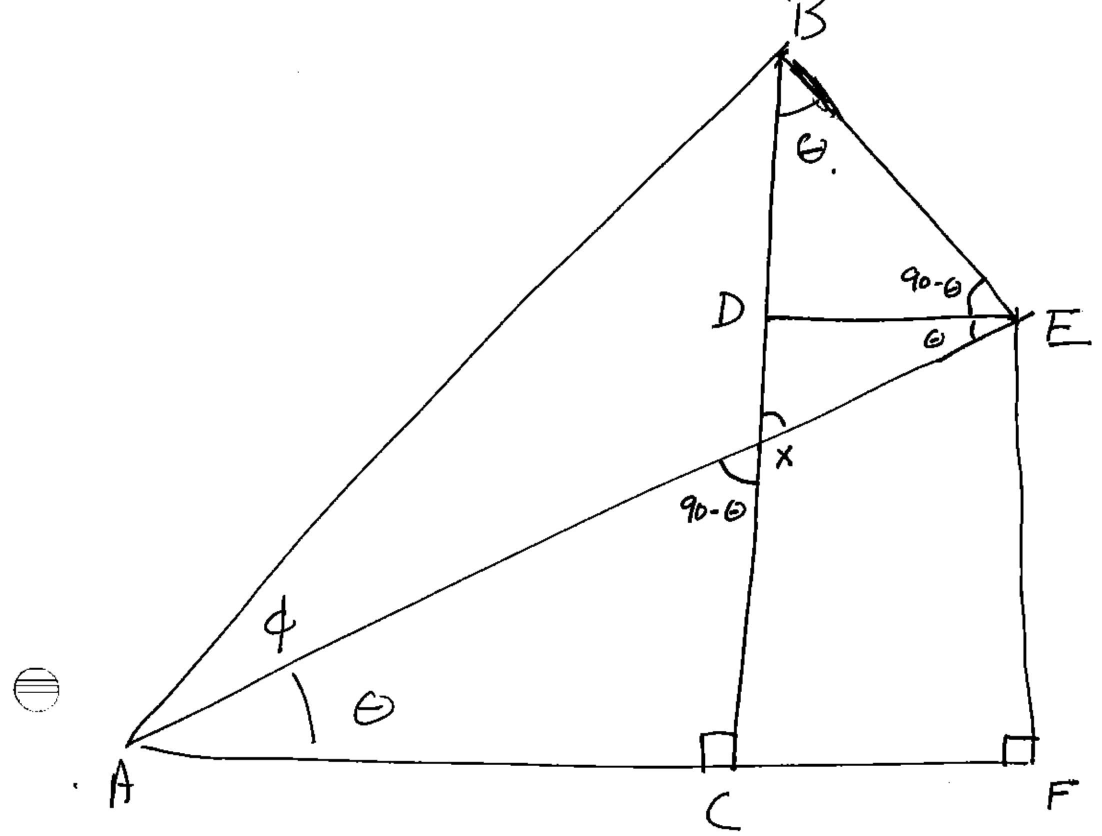
let $u = 2\tan\theta$ $du = 2(\sec\theta)^2 d\theta$.

$$\int \frac{du}{u^2+4} = \int \frac{2\sec^2\theta \, d\theta}{4\tan^2\theta + 4} = \frac{2}{4}\int \frac{\cancel{\sec^2\theta}}{\cancel{\sec^2\theta}} d\theta$$

$$= \frac{1}{2}\theta + C = \frac{1}{2}\tan^{-1}(u/2) + C = \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$$

$$\Rightarrow \int \frac{dx}{(x+b)^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x+b}{a}\right) + C.$$

Proof that $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$.



$$\cos(\theta+\phi) = \frac{AE}{AB} = \frac{AF - CF}{AB}$$

$$\sin\theta = \frac{DE}{BE} \Rightarrow CF = DE = BE\sin\theta.$$

$$\cos\theta = \frac{AF}{AE} \Rightarrow AF = AE\cos\theta$$

$$\Rightarrow \cos(\theta+\phi) = \frac{AE\cos\theta}{AB} - \frac{BE}{AB}\sin\theta = \cos\phi\cos\theta - \sin\phi\sin\theta.$$

$$\Rightarrow \cos 2\theta = \cos^2\theta - \sin^2\theta.$$

similarly. $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi.$

$$\Rightarrow \sin(2\theta) = 2\sin\theta\cos\theta.$$

$$\sin(\theta-\phi) = \sin\theta\cos\phi - \cos\theta\sin\phi.$$

$$\Rightarrow \sin\theta\cos\phi = \frac{1}{2}\left(\sin(\theta+\phi) + \sin(\theta-\phi)\right).$$

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Examples on the integration of trigonometric functions

$\int \sin^2 x \, dx = ?$

Recall $\cos^2 x + \sin^2 x = 1$
$\cos^2 x - \sin^2 x = \cos(2x)$

$\Rightarrow \cos^2 x = \frac{1+\cos(2x)}{2} \qquad \sin^2 x = \frac{1-\cos(2x)}{2}$

$= \int \sin^2 x \, dx = \int \frac{(1-\cos(2x))}{2} dx = \frac{x}{2} - \frac{1}{4}\sin(2x) + C$

Example 1 $\int dx \, \sin^4 x = \int dx \left(\sin^2 x\right)^2 = \frac{1}{4}\int dx \left(1-\cos 2x\right)^2 =$

$= \frac{1}{4}\int dx \left(1 - 2\cos 2x + \cos^2 2x\right) = \frac{1}{4}\int dx \left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right)$

$= \frac{1}{4}\left(x - \frac{2\sin 2x}{2} + \frac{x}{2} + \frac{\sin 4x}{8}\right) + C$

$= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{\sin(4x)}{32} + C.$

Question? $\int \sin(5x)\cos(3x)\, dx = ?$

Use: $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\Rightarrow 2\sin A \cos B = \sin(A+B) + \sin(A-B)$

$\Rightarrow \int = \frac{1}{2}\int \left(\sin(8x) + \sin(2x)\right) dx = -\frac{1}{16}\cos(8x) - \frac{1}{4}\cos(2x) + C.$

An example of a reduction formula.

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consider: $I_m = \int \sin^m x \, dx \quad m = 2, 3, 4 \ldots$

integration by parts

$$I_m = \int \underbrace{\sin^{m-1} x}_{u(x)} \; \underbrace{\sin x \, dx}_{\frac{dv(x)}{dx} dx} =$$

$$= -\sin^{m-1} x \cos x - \int (m-1) \sin^{m-2} x \cos x (-\cos x) dx =$$

$$= -\sin^{m-1} x \cos x + (m-1) \int \sin^{m-2} x \, (1 - \sin^2 x) dx =$$

$$= -\sin^{m-1} x \cos x + (m-1) [I_{m-2} - I_m]$$

$$\Rightarrow I_m = -\sin^{m-1} x \cos x + (m-1) I_{m-2} - (m-1) I_m.$$

$$\Rightarrow m I_m = (m-1) I_{m-2} - \cos x \sin^{m-1} x$$

This is a useful method for m = even.

Example: $I_4 = \int dx \, \sin^4 x.$

$$4 I_4 = 3 I_2 - \cos x \sin^3 x, \quad I_2 = \frac{1}{2} (I_0 - \cos x \sin x)$$

$$= 3 \left[\frac{1}{2} (I_0 - \cos x \sin x) \right] - \cos x \sin^3 x$$

$$I_0 = \int (\sin^0 x) dx = x + C.$$

$$= 3 \left[\frac{1}{2} (x + C - \cos x \sin x) \right] - \cos x \sin^3 x.$$

$$\Rightarrow I_4 = \frac{3}{8} x - \frac{3}{8} \cos x \sin x - \frac{1}{4} \cos x \sin^3 x + C$$

show that this is the same as before

note on completing the square

consider $y = x^2 + 4x + 3 = (x+1)(x+3) = (x+2)^2 - 1$

consider $\int dx \frac{1}{x^2 + 4x + 13} = \int \frac{dx}{(x+2)^2 + 3^2}$

use $u = x + 2 \Rightarrow du = dx.$

$$\Rightarrow \int \frac{du}{u^2 + 3^2} = \frac{1}{3} \tan^{-1} \frac{u}{3} + C = \frac{1}{3} \tan^{-1} \left(\frac{x+2}{3} \right) + C$$