We can calculate partial derivatives.

$$\frac{\partial f}{\partial x} = \frac{1}{2}\left(a^2-x^2-y^2\right)^{-1/2}(-2x) = \frac{-x}{(a^2-x^2-y^2)^{1/2}}$$

$$\frac{\partial f}{\partial y} = \frac{-y}{(a^2-x^2-y^2)^{1/2}}$$

We can write these in spherical polar coordinates.

$$\left.\begin{aligned} z &= a\cos\theta \\ x &= a\sin\theta\cos\phi \\ y &= a\sin\theta\sin\phi \end{aligned}\right\}$$ check that the eq. for the sphere is satisfied.

Here a is a constant

We have two coordinates (θ,ϕ) that label.

Points on the sphere.

calculate $\frac{\partial z}{\partial \theta}$ from the chain rule.

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} =$$

$$= \frac{-x}{\sqrt{a^2-x^2-y^2}}\cdot(a\cos\theta\cos\phi) - \frac{y}{\sqrt{a^2-x^2-y^2}}\,a\cos\theta\sin\phi$$

$\rightarrow \boxed{\text{input } x, y}$

$$= \frac{-a^2\sin\theta\cos\theta\cos^2\phi - a^2\sin\theta\cos\theta\sin^2\phi}{\sqrt{a^2-a^2\sin^2\theta(\cos^2\phi+\sin^2\phi)}}$$

$$= \frac{-a^2\sin\theta\cos\theta}{a\sqrt{1-\sin^2\theta}} = -a\sin\theta.$$

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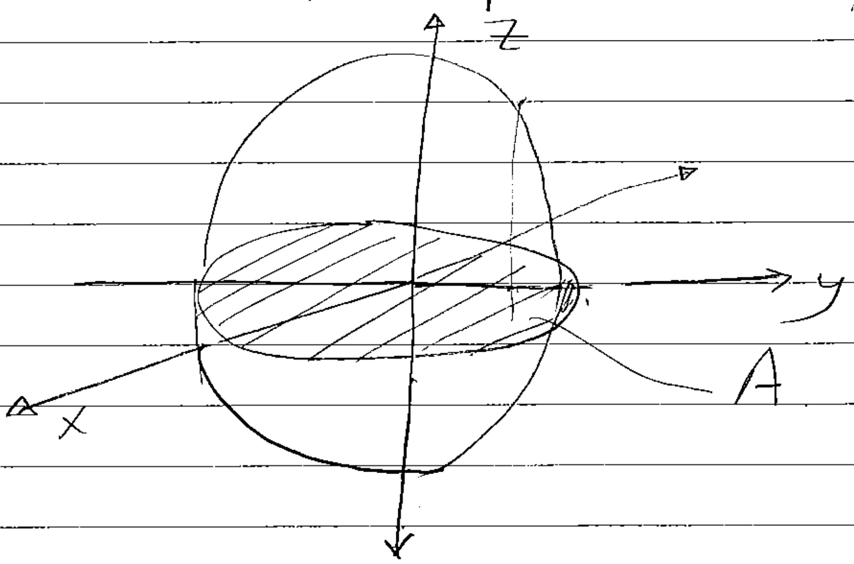
Now directly from $\frac{dz}{d\theta}$.

$\frac{dz}{d\theta} = -a\sin\theta \rightarrow$ same.

calculate: the volume of the sphere in Cartesian coordinates.

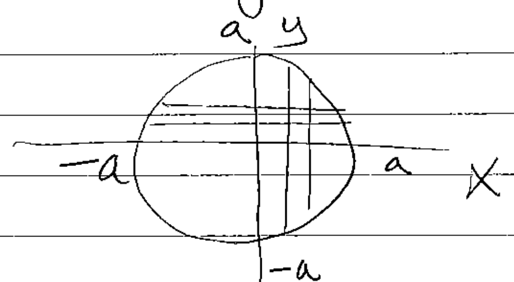


Equation of sphere $x^2+y^2+z^2=a^2$.

$$z=\pm\sqrt{a^2-x^2-y^2}$$

Take the + sign $\Rightarrow$ Volume of upper hemisphere.

in x-y plane



$$\text{Volume} = 2\cdot\int_{-a}^{a}\left(\int_{y=-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}}\sqrt{a^2-x^2-y^2}\,dy\right)dx$$

treat x as constant.

let $y=\sqrt{a^2-x^2}\sin\theta$.

$\frac{\partial y}{\partial\theta}=\sqrt{a^2-x^2}\cos\theta \Rightarrow dy=\sqrt{a^2-x^2}\cos\theta\,d\theta$.

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$$y = \pm\sqrt{a^2-x^2} \Rightarrow \theta = \pm\frac{\pi}{2}.$$

$$V = 2\int_{-a}^{a}\left(\int_{-\pi/2}^{\pi/2}\left(\sqrt{a^2-x^2-(a^2-x^2)\sin^2\theta}\right)\sqrt{a^2-x^2}\cos\theta\, d\theta\right)dx$$

$$= 2\int_{-a}^{a}\left(\int_{-\pi/2}^{\pi/2}\sqrt{a^2-x^2}\cos^2\theta\, d\theta\right)dx.$$

$$= 2\int_{-a}^{a}(a^2-x^2)\left(\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta\right)dx$$

$$= 2\int_{-a}^{a}(a^2-x^2)\left[\frac{1}{2}\left(\theta + \underbrace{\frac{\sin 2\theta}{2}}_{\text{zero at } \pm\pi/2.}\right)\right]_{-\pi/2}^{\pi/2}dx$$

$$= 2\int_{-a}^{a}(a^2-x^2)\cdot\frac{\pi}{2}\,dx$$

$$= \frac{\cancel{2}\pi}{\cancel{2}}\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a} = \pi\cdot\left(a^3 - \frac{a^3}{3}\right)\cdot 2$$

$$= \frac{4\pi a^3}{3}.$$

We have: $V = \iint_A f(x,y)\,dx\,dy.$

consider the evaluation of the volume in polar coordinates.

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$f(x,y) = \sqrt{a^2-x^2-y^2} = \sqrt{a^2-r^2} = f(r,\theta)$$

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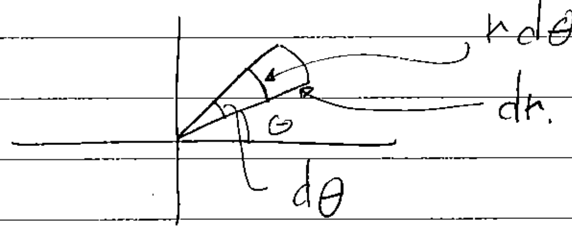
what about $dx\,dy$?

in polar coordinates $dx\,dy = r\,dr\,d\theta$.



$$\Rightarrow V = 2 \iint_A \sqrt{a^2 - r^2}\; r\,dr\,d\theta$$

The area is the area of the circle in the x-y plane

$$\Rightarrow V = 2\int_0^{2\pi} d\theta \int_0^a \sqrt{a^2-r^2}\; r\,dr =$$

$$= 2\cdot(2\pi)\cdot\frac{1}{2}\int_0^a \sqrt{a^2-r^2}\; dr^2$$

$$= 2\pi \cdot \left[-\frac{2}{3}(a^2-r^2)^{3/2}\right]_0^a$$

$$= \frac{4\pi}{3}\cdot a^3$$

Examples

Given $x = \rho\cos\phi \quad y = \rho\sin\phi$ and $V(x,y)$.

Show that $\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 = \left(\frac{\partial V}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial V}{\partial \phi}\right)^2$

Answer $\frac{\partial V}{\partial \rho} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial \rho} = \frac{\partial V}{\partial x}\cos\phi + \frac{\partial V}{\partial y}\sin\phi$

$$\frac{\partial V}{\partial \phi} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial \phi} = \frac{\partial V}{\partial x}(-\rho\sin\phi) + \frac{\partial V}{\partial y}(\rho\cos\phi)$$

OR $\frac{1}{\rho}\frac{\partial V}{\partial \phi} = -\frac{\partial V}{\partial x}\sin\phi + \frac{\partial V}{\partial y}\cos\phi$

$$\Rightarrow \left(\frac{\partial V}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial V}{\partial \phi}\right)^2 = \left(\frac{\partial V}{\partial x}\cos\phi + \frac{\partial V}{\partial y}\sin\phi\right)^2 + \left(-\frac{\partial V}{\partial x}\sin\phi + \frac{\partial V}{\partial y}\cos\phi\right)^2$$

$$= \left(\frac{\partial V}{\partial x}\right)^2(\cos^2\phi + \sin^2\phi) + \left(\frac{\partial V}{\partial y}\right)^2(\sin^2\phi + \cos^2\phi) +$$

$$+ \cancel{2\frac{\partial V}{\partial x}\frac{\partial y}{\partial x}\cos\phi\sin\phi} \cancel{- 2\frac{\partial V}{\partial x}\frac{\partial V}{\partial y}\cos\phi\sin\phi}$$

$$= \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2$$

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show that $V(x,y,z) = \dfrac{1}{(x^2+y^2+z^2)^{1/2}}$

satisfied Laplace eq. $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}\right) V(x,y,z) = 0$

where $(x,y,z) \neq (0,0,0)$

$$\frac{\partial V}{\partial x} = -\frac{1}{2} \frac{1}{(x^2+y^2+z^2)^{+3/2}} 2x = \frac{-x}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial^2 V}{\partial x^2} = -(x^2+y^2+z^2)^{-3/2} + x(+\tfrac{3}{2})(x^2+y^2+z^2)^{-5/2} \cdot 2x$$

$$= \frac{3x^2}{(x^2+y^2+z^2)^{5/2}} - \frac{(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} =$$

$$= \frac{2x^2 - y^2 - z^2}{(x^2+y^2+z^2)^{5/2}}$$

From the symmetry $x \leftrightarrow y \leftrightarrow z$.

$$\frac{\partial^2 V}{\partial y^2} = \frac{2y^2 - x^2 - z^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial^2 V}{\partial z^2} = \frac{2z^2 - y^2 - x^2}{(x^2+y^2+z^2)^{5/2}}$$

Combining $\Rightarrow$ $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = 0.$

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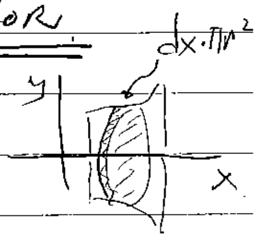
Example: A Zeppelin 50 m long has the form of a solid of rotation with a radius given by $r(x) = \frac{1}{50} x(50-x)$ with $0 \leq x \leq 50\text{m}$.

How many cubic meters of helium are needed to fill the zeppelin?

Answer: In this case we use the formula for the volume of solid of rotation



$$V = \int_0^{50} \pi (r(x))^2 \, dx = \frac{\pi}{50^2} \int_0^{50} (x(50-x))^2 dx$$

$$= \frac{\pi}{50^2} \int_0^{50} x^2 (50^2 - 100x + x^2) \, dx$$

$$= \frac{\pi}{50^2} \left[\frac{50^2}{3} x^3 - \frac{100}{4} x^4 + \frac{x^5}{5} \right]_0^{50}$$

$$\frac{\pi}{50^2} \left[\frac{50^5}{3} - \frac{50^5}{2} + \frac{50^5}{5} \right]$$

$$= \pi 50^3 \left[\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right] = \pi 50^3 \left(\frac{10}{30} - \frac{15}{30} + \frac{6}{30} \right)$$

$$= \frac{\pi 50^3}{30} \text{ m}^3 = 13090 \text{ m}^3$$

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Example Given $\int_0^\infty e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$

Evaluate i $\int_0^\infty e^{-ax^2} x^2 dx$

ii $\int_0^\infty e^{-ax^2} x^3 dx$

ii $\int_0^\infty e^{-ax^2} x^4 dx$.

Answer we derived in the lecture

$\int_{-\infty}^{\infty} e^{-ax^2} = \sqrt{\frac{\pi}{a}}$ for $a = 1$

from $x \leftarrow -x$ of the integrand we have

$\int_0^\infty e^{-ax^2} = \frac{1}{2}\sqrt{\frac{\pi}{a}}$ as given above.

i) We have $F(x,a) = \int_0^\infty e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$

$\frac{\partial F(x,a)}{\partial a} = \int_0 \frac{\partial}{\partial a}\left(e^{-ax^2} dx\right) = \int_0^\infty -x^2 e^{-ax^2} dx = -\frac{\sqrt{\pi}}{4} a^{-3/2}$

$\Rightarrow \int_0^\infty x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{4} \frac{1}{a^{3/2}}$

ii) $\int_0^\infty x^3 e^{-ax^2} dx = \int_0^\infty x^2 e^{-ax^2} x dx$

use substition $y = ax^2$ $dy = 2ax dx$ $y(0)=0$ $y(\infty)=\infty$

$\Rightarrow \int_0^\infty \frac{y}{a} e^{-y} \frac{dy}{2a} = \frac{1}{2a^2} \int_0^\infty y e^{-y} dy$

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integrate by parts

$$\int_0^\infty \underbrace{y}_{u}\, \underbrace{e^{-y}\, dy}_{dV} = \underbrace{-e^{-y}y\Big|_0^\infty}_{0} + \int_0^\infty e^{-y}\,dy = -e^{-y}\Big|_0^\infty = 1$$

$$\Rightarrow \int_0^\infty x^3 e^{-ax^2}\,dx = \frac{1}{2a^2}$$

iii)

$$\int_0^\infty x^4 e^{-ax^2}\,dx = \frac{\partial^2 F(x,a)}{\partial a^2}$$

where $F(x,a) = \int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$

hence $\frac{\partial^2 F(x,a)}{\partial a^2} = \frac{\partial}{\partial a}\left(\frac{\partial}{\partial a}F(x,a)\right) =$

$$= \frac{\partial}{\partial a}\left(-\frac{\sqrt{\pi}}{4}a^{-3/2}\right) = \frac{3}{8}\sqrt{\pi}\, a^{-5/2}$$

$$\Rightarrow \int_0^\infty x^4 e^{-ax^2} = \frac{3\sqrt{\pi}}{8a^{5/2}}$$