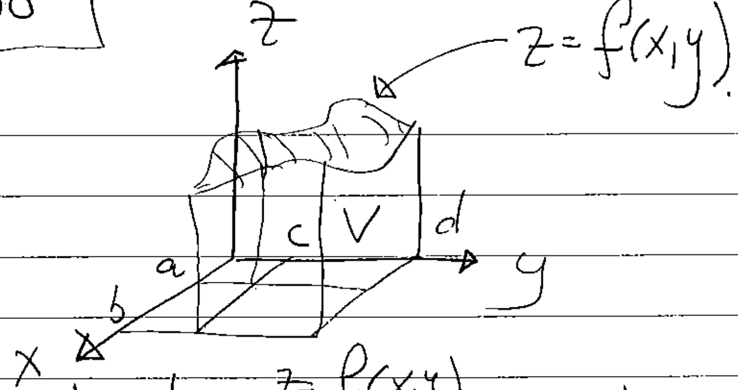
MATH 181.37 | Lecture 10

Example:



$$V=\iiint_V dx\,dy\,dz = \int_a^b\left(\int_c^d\left(\int_{z=0}^{z=f(x,y)} dz\right)dy\right)dx$$

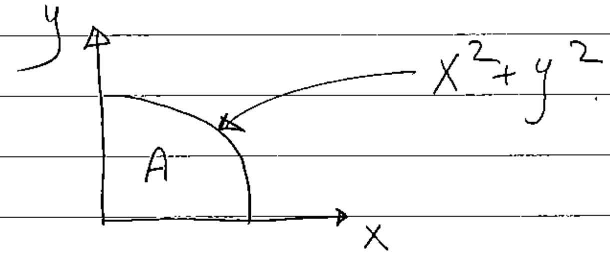
$$=\int_a^b\left(\int_c^d \left[z\right]_0^{f(x,y)}\,dy\right)dx$$

$$=\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx$$

which is what we had before.

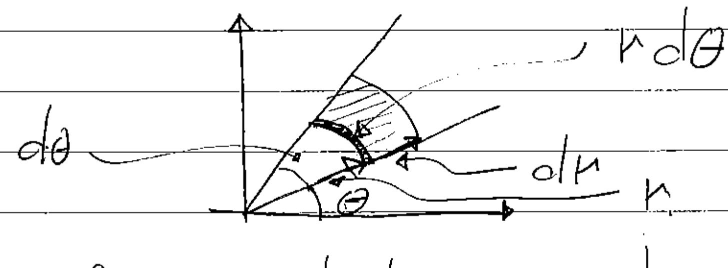
Using polar coordinates to evaluate double integrals

Example:



Consider: $I=\iint_A (x^2+y^2)^{1/2}\,dx\,dy = \iint_A (x^2+y^2)\,dA$.

Change coordinates from (x,y) to (r,θ)



So in polar coordinates $dA = r\,d\theta \times dr$

$= r\,dr\,d\theta$

$(x^2+y^2)^{1/2} = (r^2\cos^2\theta + r^2\sin^2\theta)^{1/2} = r$

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$$\Rightarrow I = \iint_A r\,(r\,dr\,d\theta) = \int_0^{\pi/2}\left(\int_0^a r^2\,dr\right)d\theta =$$

$$= \int_0^{\pi/2}\left[\tfrac{1}{3}r^3\right]_0^a d\theta = \int_0^{\pi/2}\frac{a^3}{3}\,d\theta =$$

$$= \tfrac{1}{3}a^3\left[\theta\right]_0^{\pi/2} = \tfrac{1}{6}\pi a^3$$

What is $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$.

Need to use double integral.

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int e^{-y^2}\,dy\right)$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\right)dy$$

$$= \iint_{\substack{\text{whole}\\ \text{plane}}} e^{-(x^2+y^2)}\,\underbrace{dx\,dy}_{dA.}$$

change to polar coordinates.

$x = r\cos\theta$, $y = r\sin\theta \Rightarrow x^2+y^2 = r^2$.

$dA = r\,dr\,d\theta$.

$$I^2 = \int_0^{2\pi}\left(\int_0^{\infty} e^{-r^2}\,r\,dr\right)d\theta =$$

$$= \int_0^{2\pi}\left(\int_0^{\infty}\tfrac{1}{2}e^{-r^2}\,dr^2\right)d\theta =$$

$$= \int_0^{2\pi}\left[-\tfrac{1}{2}e^{-r^2}\right|_0^{\infty} d\theta =$$

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$$= \int_0^{2\pi} \left(0 + \frac{1}{2}\right) d\theta = \frac{1}{2}[\theta]_0^{2\pi} = \pi$$

$$\Rightarrow \quad I^2 = \pi \quad \Rightarrow$$

$$I = \sqrt{\pi}$$

Note: we don't know what is the indefinite integral $\int e^{-x^2} dx$

Note: $dx\,dy \neq dr\,d\theta$

an extra factor of r

Differentiating under the integral.

consider: $I(t,x) = \int f(x,t)\, dx$

with $\dfrac{\partial I}{\partial x} = f(x,t)$

since: $$\frac{\partial^2 I}{\partial t \partial x} = \frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x \partial t} = \frac{\partial}{\partial x}\left(\frac{\partial I}{\partial t}\right)$$

we have $$\frac{\partial I}{\partial t}(x,t) = \int \frac{\partial}{\partial t} f(x,t)\, dx$$

what about the case with t-dependent boundaries?

$$I(t) = \int_{\alpha(t)}^{\beta(t)} f(x)\, dx = F(\beta(t)) - F(\alpha(t)).$$

$$\frac{dI}{dt} = \frac{\partial F}{\partial \beta}\frac{d\beta}{dt} - \frac{\partial F}{\partial \alpha}\frac{d\alpha}{dt} = f(\beta(t))\frac{d\beta}{dt} - f(\alpha(t))\frac{d\alpha}{dt}$$

combining: $$\frac{\partial}{\partial t}\left(\int_{\alpha(t)}^{\beta(t)} f(t,x)\, dx\right) = \left(\int_{\alpha(t)}^{\beta(t)} \left(\frac{\partial}{\partial t} f(t,x)\, dx\right)\right) + f(t,\beta(t))\frac{d\beta}{dt} - f(t,\alpha(t))\frac{d\alpha}{dt}.$$

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Example: $\phi(y) = \int_y^{y^2} \frac{\sin xy}{x}\, dx$

Find $\phi'(y)$ when $y \neq 0$.

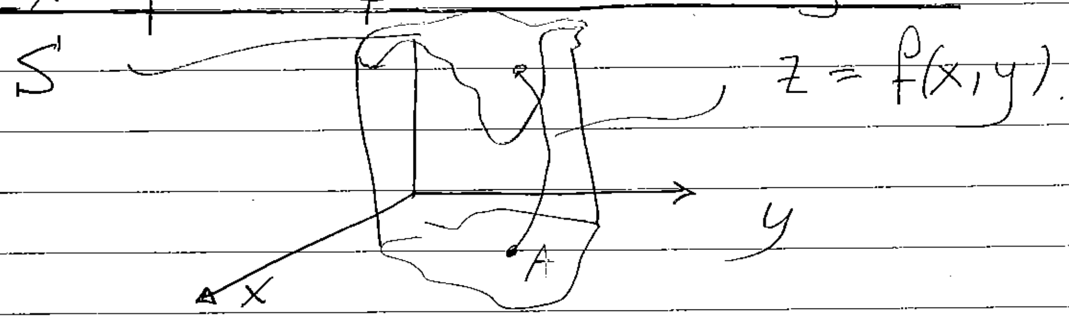
$$\phi'(y) = \int_y^{y^2} \cos xy \, dx + \frac{\sin y^3}{y^2}(2y) - \frac{\sin y^2}{y} \cdot 1$$

$$= \frac{\sin xy}{y}\Big|_y^{y^2} + \frac{2\sin(y^3)}{y} - \frac{\sin y^2}{y} =$$

$$= \frac{\sin y^3}{y} - \frac{\sin(y^2)}{y} + \frac{2\sin(y^3)}{y} - \frac{\sin(y^2)}{y} =$$

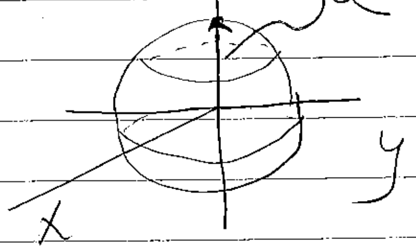
$$= \frac{3\sin(y^3) - 2\sin(y^2)}{y}$$

MORE EXAMPLES of multi-variable integration.



S = some two dimensional surface.

Example consider a two dimensional sphere given by $x^2 + y^2 + z^2 = a^2$



$$z = \pm\sqrt{a^2 - x^2 - y^2} = f(x, y)$$

choose + sign. (i.e. upper hemisphere).