MAth. 181. 1

Integration - Lecture 1

Indefinite integration (anti-differentiation)

suppose that $g'(x) = f(x) = \frac{d\,g(x)}{dx}$.

one writes

$$g(x) = \int dx\, f(x) = \int f(x)\, dx.$$

OR.

$$\int \frac{d\,g(x)}{dx} dx = \int f(x)\, dx = \int dg(x) = g(x).$$

$g(x)$ is the indefinite integral of $f(x)$

All indefinite integrals of $f(x)$ are given by

$$g(x) + C.$$

where C is an arbitrary constant.

since $\frac{d(g(x)+C)}{dx} = \frac{d\,g(x)}{dx} + \frac{dC}{dx} = \frac{d\,g(x)}{dx}$.

$C \to$ an arbitrary constant of integration.

Example: $\int dx\, 3x^2 = x^3 + C$

check $\frac{d(x^3+C)}{dx} = 3x^2$.

Methods of integration → (recognise derivative or bring to a familiar form)

i) 'By sight':

$$\int dx\, x^n = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$$

$$\int dx\, x^{p/q} = \frac{x^{p/q+1}}{(p/q+1)} + C \qquad p/q \neq -1$$

$$\int dx \sin x = -\cos x + C$$

$$\int dx \cos x = \sin x + C$$

$$\int dx \sin(kx) = -\frac{1}{k}\cos(kx) + C$$

$$(*) \quad \int dx \sec^2 x = \int dx \frac{1}{\cos^2 x} = \tan x + C$$

Check:

$$\tan x = \frac{\sin x}{\cos x}$$

using quotient rule.

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x \cdot \cos x - (-\sin x)\cdot \sin x}{(\cos x)^2} = \frac{1}{(\cos x)^2}$$

$$\Rightarrow (\tan x + C)' = \frac{1}{(\cos x)^2} = \text{integrand in } (*).$$

$$\int dx (\operatorname{cosec} x)^2 = \int dx \frac{1}{(\sin x)^2} = -\cot x + C$$

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$$\int dx \frac{1}{(1-x^2)^{1/2}} = \sin^{-1} x + C = \arcsin x + C$$

where if $\sin^{-1} x = y$ then $\sin y = x$.

$$\int dx \frac{1}{(a^2-x^2)^{1/2}} = \sin^{-1}\left(\frac{x}{a}\right) + C,$$

$$\int dx \frac{1}{(a^2+x^2)} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C.$$

$$\int dx\, e^{ax} = \frac{1}{a} e^{ax} + C$$

<u>check</u>: $\left(\frac{1}{a} e^{ax} + c\right)' = \frac{1}{a} e^{ax} a = e^{ax}$ ✓.

ii) Rules for indifinite integration.

(*) $\int dx\, a f(x) = a \int dx\, f(x)$

$\rightarrow$ for any constant $a \neq a(x)$.

(**) $\int dx [f(x) \pm g(x)] = \int dx\, f(x) \pm \int dx\, g(x)$

<u>Example</u>: $\int dx (x^4 + 2\sin 2x) = \int dx\, x^4 + 2\int dx (\sin 2x)$

$$= \frac{1}{5} x^5 + C_1 + 2\left(\frac{-\cos 2x}{2}\right) + C_2 = \frac{1}{5} x^5 - \cos 2x + C.$$

(iii) <u>Integration by parts</u>

<u>Recall</u>: product rule for differentiation.

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$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}.$$

which we rearrange as:

$$u\frac{dv}{dx} = \frac{d(uv)}{dx} - v\frac{du}{dx}.$$

now integrate

$$\int dx\, u(x)\frac{dV(x)}{dx} = \int dx \frac{d(u.v.)}{dx} - \int dx\, v(x)\frac{du(x)}{dx}.$$

OR: $$\int dx\, u(x)\frac{dv(x)}{dx} = u(x)\,v(x) - \int dx\, v(x)\frac{du(x)}{dx}.$$

Example $$\int dx\, \underbrace{x}_{u}\underbrace{\sin x}_{\frac{dV(x)}{dx}} = x(-\cos x) - \int dx\,(-\cos x)\frac{dx}{dx} =$$

$$= -x\cos x + \sin x + C.$$

Problem what is $\int dx\, x^2\cos 2x$?

Answer $= \frac{1}{2}x^2\sin 2x + \frac{1}{2}\cos 2x + \frac{1}{4}\sin 2x + C$

Example $$\int dx\, \underbrace{x^2}_{u(x)}\underbrace{e^x}_{\frac{dv(x)}{dx}} = \underbrace{x^2}_{u(x)}\underbrace{e^x}_{v(x)} - \int dx\, \underbrace{2x}_{\frac{du(x)}{dx}}\underbrace{e^x}_{v(x)} =$$

$$= x^2e^x - 2\left(xe^x - \int dx\, e^x\right) =$$

$$= x^2e^x - 2xe^x + 2e^x + C.$$

Example: $\int \underbrace{dx}_{\frac{dv(x)}{dx}} \underbrace{\ln x}_{u(x)} = \underbrace{x}_{v(x)} \underbrace{\ln x}_{u(x)} - \int dx \underbrace{x}_{v(x)} \cdot \underbrace{d(\ln x)/dx}_{\frac{du(x)}{dx}} =$

$= x\ln x - \int dx\, x \cdot \frac{1}{x} = x\ln x - x + C.$

Example: $\int \underbrace{dx}_{\frac{dv(x)}{dx}} \underbrace{\sin^{-1}(x)}_{u(x)} = \underbrace{x}_{v(x)} \underbrace{\sin^{-1} x}_{u(x)} - \int dx \cdot \underbrace{x}_{v(x)} \underbrace{\frac{d}{dx}(\sin^{-1} x)}_{\frac{du(x)}{dx}} =$

$= x\sin^{-1} x - \int dx \quad x \quad \cdot \frac{1}{\sqrt{1-x^2}}$

we need to evaluate $\int dx \frac{x}{\sqrt{1-x^2}}$

we will do that by substitution.

write $x^2 = y \Rightarrow 2x\,dx = dy \Rightarrow x\,dx = \frac{1}{2} dy.$

Hence. $\frac{1}{2}\int dy \,(1-y)^{-1/2}$

again $1-y = z \Rightarrow -dy = dz.$

Hence $-\frac{1}{2}\int dz \; z^{-1/2} = -\frac{1}{2}\left(\frac{z^{1/2}}{1/2}\right) = -z^{1/2}$

check: $\frac{d}{dz} z^{1/2} = \frac{1}{2} \cdot z^{-1/2}$

going back: $= -(1-y)^{1/2} = -(1-x^2)^{1/2}$

so finally we have

$\int dx \sin^{-1} x = x\sin^{-1} x + (1-x^2)^{1/2} + C$

iii) integration by substitution

Example $\int dx (3x+5)^7$

let $u = (3x+5)$ $\quad \frac{du}{dx} = 3 \Rightarrow dx = \frac{1}{3} du$.

$$\int dx (3x+5)^7 = \frac{1}{3} \int du \, u^7 = \frac{1}{3} \frac{u^8}{8} + C =$$

$$= \frac{1}{24} (3x+5)^8 + C'$$

Final result is given in terms of the original variable x.

Example $\int dx \frac{1}{x^2+a^2}$

let $x = a \tan\theta \Rightarrow \frac{dx}{d\theta} = a \frac{d}{d\theta}\left(\frac{\sin\theta}{\cos\theta}\right) = a(\sec\theta)^2$

$\Rightarrow dx = a(\sec\theta)^2 d\theta$.

$$\int dx \frac{1}{x^2+a^2} = \int d\theta \frac{a(\sec\theta)^2}{a^2\tan^2\theta + a^2} = \int \frac{d\theta \, a(\sec\theta)^2}{a^2(1+\tan^2\theta)} =$$

$$= \frac{1}{a} \int d\theta \frac{(\sec\theta)^2}{(\sec\theta)^2} = \frac{\theta}{a} + C = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

Example $\int dx \sin^3 x$

let $u = \cos x$ $\quad du = -\sin x \, dx$.

$$\int dx \sin^3 x = \int dx \sin x \cdot \sin^2 x = -\int dx (-\sin x)(1-\cos^2 x)$$

$$= -\int du (1-u^2) = -u + \frac{u^3}{3} + C = \frac{\cos^3 x}{3} - \cos x + C$$