Solutions to MATH012 Exam May 2009

1(a). $\overrightarrow{CD} = -\mathbf{u}$.

(b) $\overrightarrow{BD} = \mathbf{v} - \mathbf{u}$.

(c) $\overrightarrow{BP} = \mathbf{v} - \frac{1}{2}\mathbf{u}$.

2(a) $\overrightarrow{PQ} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$, $|\overrightarrow{PQ}| = 3.0$m.
$\overrightarrow{QR} = \mathbf{i} + \mathbf{j} + 4\mathbf{k}$, $|\overrightarrow{QR}| = 4.24$m.
$\overrightarrow{RP} = -2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $|\overrightarrow{RP}| = 3.0$m.

(b) $\overrightarrow{PQ}.\overrightarrow{PR}1.(-2) + (-2).1 + (-2).(-2) = 0$.

(c) Angles are $\frac{\pi}{4}$, $\frac{\pi}{4}$, $\frac{\pi}{2}$.

(d) $\mathbf{s} = \mathbf{q} + \overrightarrow{QS} = \mathbf{q} + \overrightarrow{PR} = 3\mathbf{i} + \mathbf{j} - \mathbf{k} + (2\mathbf{i} - 1\mathbf{j} + 2\mathbf{k}) = 5\mathbf{i} + \mathbf{k}$.
So S is $(5, 0, 1)$.

3(a) $\mathbf{u} + 2\mathbf{v} = 3\mathbf{i} + 5\mathbf{k}$, $\mathbf{u} - 2\mathbf{v} = -5\mathbf{i} + 4\mathbf{j} + 1\mathbf{k}$.

(b) $(2\mathbf{v} - \mathbf{u}).\mathbf{v} = 13$, $\mathbf{u} + 2\mathbf{v}).\mathbf{u} = 12$.

(c) Unit vector is $\frac{1}{\sqrt{83}}(5\mathbf{i} + 7\mathbf{j} - 3\mathbf{k})$.

(d) 0.

4.(a) $\overrightarrow{AB} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

(b) $\mathbf{r} = \mathbf{a} + \lambda\overrightarrow{AB} = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = (2 + 2\lambda)\mathbf{i} + (2 + 2\lambda)\mathbf{j} + (2 - \lambda)\mathbf{k}$.

(c) $(4, 4, 0)$.

5(a) $(0, -1, 0)$.

(b) $2t\mathbf{i} + \mathbf{j} + (1 + t)e^t\mathbf{k}$.

(c) 22.55.

(d) $2\mathbf{i} + 2\mathbf{k}$.

6.

(a)

$$\mathbf{v} = \mathbf{u} + \mathbf{w} = -100\mathbf{i} + 400\mathbf{j}.$$

(b)

$$\mathbf{r} = (-100\mathbf{i} + 400\mathbf{j})t + \mathbf{c}$$

where $\mathbf{c}$ is a constant vector. But $\mathbf{r}(0) = \mathbf{0}$, so $\mathbf{c} = \mathbf{0}$. Hence

$$\mathbf{r} = (-100\mathbf{i} + 400\mathbf{j})t.$$

(c) Time taken to fly 160km North $= \frac{160}{400} = \frac{2}{5}$hours= 24mins.

(d) When $t = \frac{2}{5}$, $\mathbf{r} = (-100\mathbf{i} + 400\mathbf{j})\frac{2}{5} = -40\mathbf{i} + 160\mathbf{j}$. So $\mathbf{p} = -40\mathbf{i} + 160\mathbf{j}$.

7. Determinant equals $9 - 8x + x^2$. The values of x are -1 and 9.

8.

$P(X|Y)$ means the probability that X occurs given that Y occurs.
Let X be the event that a component is acceptable.
Let A be the event that the component was made by machine A.
Let B be the event that the component was made by machine B.
Then $P(A) = 0.1$, $P(B) = 0.9$, $P(X|A) = 0.8$, $P(X|B) = 0.6$. We have

$$\begin{aligned} P(X) =& P(X|A)P(A) + P(X|B)P(B) \\ =& 0.8 \times 0.1 + 0.6 \times 0.9 = 0.62. \end{aligned}$$

9(a) $\overrightarrow{DA} = -(\mathbf{u} + \mathbf{v} + \mathbf{w})$.

(b) $\mathbf{u} = -\mathbf{w}$.

9.

(a) Normal given by

$$\begin{aligned} \mathbf{n} =& \mathbf{u} \times \mathbf{v} \\ =& \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 2 \\ 1 & -1 & 0 \end{vmatrix} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \\ |\mathbf{n}| =& \sqrt{2^2 + 2^2 + 1^2} = 3 \end{aligned}$$

$$\Rightarrow \hat{\mathbf{n}} = \frac{1}{3}(2\mathbf{i} + 2\mathbf{j} + 1k).$$

(b)

$$\hat{\mathbf{n}}.\mathbf{u} = \frac{1}{3}(2.2 + 2.(-3) + 1.2) = 0; \quad \hat{\mathbf{n}}.\mathbf{v} = \frac{1}{3}(2.1 - 2.1) = 0.$$

(c) A, B, C and D will all lie in the same plane if $\overrightarrow{CD}.\hat{\mathbf{n}} = \mathbf{w}.\hat{\mathbf{n}} = 0$. We have

$$\mathbf{w}.\hat{\mathbf{n}} = \frac{1}{3}(2.4 + 2.(-7) + 1.6) = 0.$$

So the points do lie in the same plane.

(d) Since the plane passes through $(2, 3, 2)$, the equation is

$$2x + 2y + z = 2.2 + 2.3 + 1.2 = 12$$

(e) Intersection where

$$2(2 + 2\lambda) + 2(-1 - 3\lambda) + (4 + 4\lambda) = 12 \Rightarrow 2\lambda = 6 \Rightarrow \lambda = 3.$$

So intersection is $(8, -10, 16)$.

10.(a) Two points on $\mathcal{L}_1$ are $(2, 6, -6)$ $(\lambda = 0)$ and $(3, 5, -4)$ $(\lambda = 1)$.

(b) A vector along $\mathcal{L}_1$ is $\mathbf{u}_1 = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Have $|\mathbf{u}_1| = \sqrt{1+1+4} = \sqrt{6}$ so

$$\hat{\mathbf{u}}_1 = \frac{1}{|\mathbf{u}_1|}\mathbf{u}_1 = \frac{1}{\sqrt{6}}(\mathbf{i} - \mathbf{j} + 2\mathbf{k}).$$

A vector along $\mathcal{L}_2$ is $\mathbf{u}_2 = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. Have $|\mathbf{u}_2| = \sqrt{4+4+1} = \sqrt{9} = 3$ so

$$\hat{\mathbf{u}}_2 = \frac{1}{|\mathbf{u}_2|}\mathbf{u}_2 = \frac{1}{3}(2\mathbf{i} + 2\mathbf{j} - \mathbf{k}).$$

(c) The angle θ between the lines is the angle between $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$. So

$$\cos\theta = \frac{\hat{\mathbf{u}}_1.\hat{\mathbf{u}}_2}{|\hat{\mathbf{u}}_1||\hat{\mathbf{u}}_2|} = \frac{-2}{3.\sqrt{6}} \Longrightarrow \theta = 1.85 Rad = 105.79 Deg.$$

The lines intersect if there is a solution to

$$\begin{aligned} 2 + \lambda =& 3 + 2\mu, \\ 6 - \lambda =& 1 + 2\mu, \\ -6 + 2\lambda =& 1 - \mu. \end{aligned}$$

These equations have the solution $\lambda = 3$, $\mu = 1$ so the point of intersection is $(5, 3, 0)$.

11.(a) The normals are given by

$$\mathbf{n}_1 = \mathbf{i} - 4\mathbf{j} + \mathbf{k}, \quad \mathbf{n}_2 = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}, \quad \mathbf{n}_3 = \mathbf{i} + \mathbf{j} + 2\mathbf{k}.$$

(b) The angle θ between $\mathbf{n}_1$ and $\mathbf{n}_2$ is given by

$$\cos\theta = \frac{\mathbf{n}_1.\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{9}{\sqrt{18}\sqrt{9}} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}.$$

(c) The planes intersect where

$$\begin{aligned} x - 4y + z &= 12 \\ 2x - 2y - z &= 9 \\ x + y + 2z &= 3 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 1 & -4 & 1 & 12 \\ 2 & -2 & -1 & 9 \\ 1 & 1 & 2 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -4 & 1 & 12 \\ 0 & 6 & -3 & -15 \\ 0 & 5 & 1 & -9 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -4 & 1 & 12 \\ 0 & 1 & -\frac{1}{2} & -\frac{5}{2} \\ 0 & 5 & 1 & -9 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -4 & 1 & 12 \\ 0 & 1 & -\frac{1}{2} & -\frac{5}{2} \\ 0 & 0 & \frac{7}{2} & \frac{7}{2} \end{bmatrix}$$

$$\Rightarrow z = 1 \quad y - \frac{1}{2}z = -\frac{5}{2} \Rightarrow y = -2$$

$$x - 4y + z = 12 \Rightarrow x = 3.$$

12. The mean is the sum of the values divided by the number of values.

The mode is the value that occurs most often.

The median is found by forming a list of the values in ascending order and then selecting the value that lies halfway along the list. (For an *even* set of values, the median is the mean of the two values on either side of the halfway point.]

(a) Rainfalls in order:

$$0, 1, 1, 2, 2, 3, 3, 5, 5, 5, 7, 7, 10, 11, 13$$

(b) Frequency of $7 = 2$. Relative frequency of $7 = \frac{2}{15}$.

(c)

$$\text{Mean} = \bar{x} = \frac{1+1+2+2+3+3+5+5+5+7+7+10+11+13}{15} = \frac{75}{15} = 5$$

(d) Mode is 5; Median is 5.

(e) Standard deviation given by

$$\sigma^2 = \frac{\sum(x_i - \bar{x})^2}{n} = \frac{216}{15} = 14.4 \Rightarrow \sigma = 3.79.$$