The Quantum Closet

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- AEF, Marco Matone, PLB 450 (1999) 34; ... ; IJMPA 15 (2000) 1869.
- G. Bertoldi, AEF & M. Matone, CQG 17 (2000) 3925.
- AEF, Marco Matone, arXiv:1211.0798.
- AEF, arXiv:1204.3185 ; 1305.0044.

Lie Theory and Applications, Varna, 20 June 2013

- Motivation – quantum gravity
- The Equivalence Postulate $\Rightarrow$ QSHJE

  $\longrightarrow$ Schrödinger eq.
- The Equivalence Postulate $\Rightarrow$ CoCyCle Condition

  $\longrightarrow$ Möbius invariance
- The Equivalence Postulate $\Rightarrow$ Energy quantization

  & Time Parameterisation
- Möbius invariance $\Rightarrow$ Compact universe
- Conclusions

Motivation General Relativity: Covariance & Equivalence Principle

$\longrightarrow$ fundamental geometrical principle

Quantum Mechanics: No Such Principle

Axiomatic formulation ... $\mathrm{P} \sim |\Psi|^2$

However Quantum $+$ Gravity Theory

not known

Main effort: quantize GR; quantize space–time: $e.g.$ superstring theory

The main successes of string theory:

1) Viable perturbative approach to quantum gravity
2) Unification of gravity, gauge & matter structures

$i.e.$ construction of phenomenologically realistic models

$\longrightarrow$ relevant for experimental observation

State of the art: MSSM from string theory

(AEF, Nanopoulos, Yuan, NPB 335 (1990) 347)

(Cleaver, AEF, Nanopoulos, PLB 455 (1999) 135)

Adaptation of Hamilton–Jacobi theory

Hamilton's equations of motion $\dot{q} = \frac{\partial H}{\partial p} \ , \ \dot{p} = -\frac{\partial H}{\partial q}$

$$H(q,p) \longrightarrow K(Q,P) \equiv 0 \qquad \Longrightarrow \qquad \dot{Q} = \frac{\partial K}{\partial P} \equiv 0\,,\ \dot{P} = -\frac{\partial K}{\partial Q} \equiv 0$$

The solution is the Classical Hamilton–Jacobi Equation

$$H(q,p) \longrightarrow K(Q,P) = H(q, p = \frac{\partial S}{\partial q}) + \frac{\partial S}{\partial t} = 0 \Rightarrow \mathrm{CHJE}$$

stationary case $\longrightarrow \frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 + V(q) - E = 0$

$(q,p) \rightarrow (Q,P)$ via canonical transformations

$q,\ p$ are independent. Solve. Then $p = \frac{\partial S}{\partial q}$

Quantum mechanics: $[\hat{q}, \hat{p}] = i\hbar \rightarrow q,\ p \rightarrow$ not independent

Assume $H \rightarrow K$ $i.e.\ W(Q) = V(Q) - E = 0$ always exists

But $q,\ p$ not independent. $p = \frac{\partial S}{\partial q}$.

Equivalence postulate:

Consider the transformations on

$$(\, q \, , \, S_0(q) \, , \, p = \frac{\partial S_0}{\partial q}) \longrightarrow (\, \tilde{q} \, , \, \tilde{S}_0(\tilde{q}) \, , \, \tilde{p} = \frac{\partial \tilde{S}_0}{\partial \tilde{q}} \,)$$

Such that

$$W(q) \longrightarrow \tilde{W}(\tilde{q}) = 0$$

exist for all $W(q)$

$\Longrightarrow$ QHJE

$\longrightarrow$ Schrödinger equation

Implies: Covariance of HJE

But: $$\frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 \; + \; V(q) \; - \; E \; = \; 0$$

Is not covariant under $q \;\longrightarrow\; \tilde{q}(q)$.

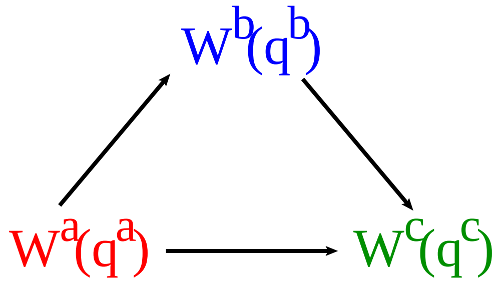
Further: $W(q) \;\equiv\; 0$ is a fixed state under $q \;\longrightarrow\; \tilde{q}(q)$.

Assume: $$\frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 \; + \; W(q) \; + \; Q(q) \; = \; 0$$

The most general transformations
$$\tilde{W}(\tilde{q}) = \left(\frac{\partial \tilde{q}}{\partial q}\right)^{-2} W(q) + (\tilde{q}; q),$$
$$\tilde{Q}(\tilde{q}) = \left(\frac{\partial \tilde{q}}{\partial q}\right)^{-2} Q(q) - (\tilde{q}; q),$$

with $\tilde{S}_0(\tilde{q}) \; = \; S_0(q)$ under $q \;\longrightarrow\; \tilde{q} \; = \; \tilde{q}(q)$

<u>With:</u> $W^0(q^0) \; = \; 0 \quad \rightarrow$ <u>All:</u> $W(q) \; = \; (q; q^0)$



<u>Cocycle Condition:</u> $$(q^a; q^c) \; = \; \left(\frac{\partial q^b}{\partial q^c}\right)^2 \left[\, (q^a; q^b) \; - \; (q^c; q^b) \right]$$

$\Rightarrow$ <u>Theorem</u> $(q^a; q^c)$ invariant under Möbius transformations $\gamma(q^a)$

<u>In 1D:</u> $(q^a; q^c) \quad \sim \quad \{q^a; q^c\}$ <u>Uniquely</u>

<u>Schwarzian derivative</u> $$\{h(x); x(y)\} = \left(\frac{\partial y}{\partial x}\right)^2 \{h(x); y\} - \left(\frac{\partial y}{\partial x}\right)^2 \{x; y\}.$$

$$U(q) \; = \; \{h(q); q\} \; = \; \{\frac{Ah + B}{Ch + D}; q\}$$

Invariant under Möbius transformations

Identity $$\left(\frac{\partial S_0}{\partial q}\right)^2 = \frac{\beta^2}{2}\left(\{\mathrm{e}^{\frac{i2S_0}{\beta}}; q\} - \{S_0; q\}\right)$$

Make the following identifications

$$W(q) = -\frac{\beta^2}{4m}\{\mathrm{e}^{\frac{i2S_0}{\beta}}; q\} = V(q) - E$$

$$Q(q) = \frac{\beta^2}{4m}\{S_0; q\}$$

The Modified Hamilton–Jacobi Equation becomes

$$\frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 + V(q) - E + \frac{\beta^2}{4m}\{S_0; q\} = 0$$

in the limit $\beta \rightarrow 0$ we get back the CSHJE and $S_0^{cl} = \lim_{\beta\rightarrow 0} S_0$

From the properties of the SD $\{;\}$

$$V(q) - E = -\frac{\beta^2}{4m}\{\mathrm{e}^{\frac{i2S_0}{\beta}}; q\}$$

is a potential of the 2^{nd}–order diff. Eq.

$$\left(-\frac{\beta^2}{2m}\frac{\partial^2}{\partial q^2} + V(q) - E\right)\Psi(q) = 0 \qquad \Rightarrow \quad \beta \ = \ \hbar$$

The general solution

$$\Psi(q) = \frac{1}{\sqrt{S_0'}}\left(A\mathrm{e}^{+\frac{i}{\hbar}S_0} + B\mathrm{e}^{-\frac{i}{\hbar}S_0}\right)$$

and

$$\mathrm{e}^{+\frac{i2S_0}{\hbar}} \ = \ \mathrm{e}^{i\alpha}\,\frac{w + i\bar{\ell}}{w - i\ell} \qquad\qquad w \ = \ \frac{\psi_1}{\psi_2}$$

$$\ell \ = \ \ell_1 \ + \ i\ \ell_2 \qquad \ell_1 \ \neq \ 0 \qquad \alpha \ \in \ R$$

The equivalence transformation

$$W(q) \;=\; V(q) \;-\; E \quad \longrightarrow \quad \tilde{W}(\tilde{q}) \;=\; 0$$

always exists

We have to find $\quad q \;\rightarrow\; \tilde{q} \quad$ take $\quad \tilde{q} \;=\; \frac{\psi_1}{\psi_2}$

then
$$\left(-\frac{\beta^2}{2m}\,\frac{\partial^2}{\partial q^2} \;+\; V(q) \;-\; E\right)\Psi(q) \;=\; 0 \quad\rightarrow\quad \frac{\partial^2}{\partial \tilde{q}^2}\tilde{\psi}(\tilde{q}) \;=\; 0$$

where
$$\tilde{\psi}(\tilde{q}) \;=\; \left(\frac{dq}{d\tilde{q}}\right)^{-\frac{1}{2}}\psi(q)$$

Energy quantization:

Probability: $\Longrightarrow$ (Ψ, Ψ') continuous ; $\Psi \in L^2(R)$

$\Longrightarrow$ quantization, bound states

What are the conditions on the trivializing transformations?

$$q^0 = w = \frac{\psi_1}{\psi_2} = \frac{\psi^D}{\psi}$$

we have
$$\{w, q\} = -\frac{4m}{\hbar^2}(V(q) - E)$$

$\Rightarrow$ $w \neq const$; $w \in C^2(R)$ *and* w'' differentiable on R

In addition from the properties of $\{,\} \rightarrow \{w, q^{-1}\} = q^4\{w, q\}$

$\Rightarrow$ $w \neq const$; $w \in C^2(\hat{R})$ *and* w'' differentiable on $\hat{R}$

where $\hat{R} = R \cup \{\infty\}$

$$\Longrightarrow$$

Equivalence postulate $\Longrightarrow$ continuity of (ψ^D, ψ) and $(\psi^{D'}, \psi')$

Theorem:

$$if \qquad V(q) - E = \begin{cases} P_-^2 > 0 & for\ q < q_- \\ P_+^2 > 0 & for\ q > q_+ \end{cases}$$

then the ratio $w = \psi^D/\psi$ *is continuous on* $\hat{R}$

iff the Schrödinger equation admits an $L^2(R)$ *solution*

1) $\psi \in L^2(R) \Rightarrow \psi^D \notin L^2(R)$

$$w = \frac{A\psi_D + B\psi}{C\psi_D + D\psi} \Rightarrow \lim_{q\to\pm\infty} = \frac{A}{C}$$

$$\Rightarrow w(-\infty) = w(+\infty)$$

2) ...

Potential Well:

$$V(q) = \begin{cases} 0 & |q| \le L \\ V_0 & |q| > L \end{cases}$$

$$k = \frac{\sqrt{2mE}}{\hbar} \qquad K = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

$$|q| \le L \qquad \Psi_1^1 = \cos kq \qquad \Psi_2^1 = \sin kq$$

$$q > L \qquad \Psi_1^2 = \mathrm{e}^{-Kq} \qquad \Psi_2^2 = \mathrm{e}^{Kq}$$

take $(1,1)$: $\Psi,\ \Psi'$ continuous $\Rightarrow$ $k\ \tan kL = K$

$$\Rightarrow \quad w = \frac{1}{[k \sin(2kL)]} \begin{cases} \cos(2kL) - \mathrm{e}^{-2K(q+L)} & q < -L \\ \sin(2kL)\tan(kq) & |q| \le L \\ \mathrm{e}^{2K(q-L)} - \cos(2kL) & q > L \end{cases}$$

$$\lim_{q\to\pm}\frac{\psi^D}{\psi}=\pm\infty \qquad \Longrightarrow \qquad E_n(k\ \tan kL\ =\ K) \text{ are admissible solutions}$$

$$\text{take } (1,2): \quad \Psi,\ \Psi' \text{ continuous} \quad \Rightarrow \quad k\ \tan(kl)\ =\ -K$$

$$\Rightarrow \quad w = \frac{1}{[k\sin(2kL)]}\begin{cases} \cos(2kL) - \mathrm{e}^{2K(q+L)} & q\ <\ -L \\ \sin(2kL)\tan(kq) & |q|\ \leq\ L \\ \mathrm{e}^{-2K(q-L)} - \cos(2kL) & q\ >\ L \end{cases}$$

$$\lim_{q\to\pm}\frac{\psi^D}{\psi}\ =\ \mp\frac{1}{k}\cot(2kL) \qquad \Longrightarrow \qquad w(-\infty)\ \neq\ w(+\infty)$$

($k^{-1}(\cot 2kL) = 0$ is not compatible with $k\tan(kL) = -K$)

$$\Rightarrow \qquad E_n(k\ \tan kL\ =\ -K) \quad \text{are not admissible solutions}$$

Time parameterisation

Bohmian mechanics : $\quad p = \dfrac{\partial S}{\partial q} = m\dot{q} \quad \Rightarrow \quad$ Tracjectory representation

Jacobi time : $\quad t = \dfrac{\partial S_0}{\partial E}.$

In classical mechanics: Jacobi time = Mechanical time

$$t - t_0 = m \int_{q_0}^{q} \frac{dx}{\partial_x S_0^{\mathrm{cl}}} = \int_{q_0}^{q} dx \frac{\partial}{\partial E} \partial_x S_0^{\mathrm{cl}} = \frac{\partial S_0^{\mathrm{cl}}}{\partial E}.$$

In Quantum HJ Theory: Jacobi time $\neq$ Mechanical time

$$t - t_0 = \frac{\partial \mathcal{S}_0^{\mathrm{qm}}}{\partial E} = \frac{\partial}{\partial E} \int_{q_0}^{q} dx \partial_x \mathcal{S}_0^{\mathrm{qm}} = \left(\frac{m}{2}\right) \int_{q_0}^{q} dx \frac{1 - \partial_E Q}{(E - V - Q)^{1/2}}$$

$$\Longrightarrow \quad m\frac{dq}{dt} = m\left(\frac{dt}{dq}\right)^{-1} = \frac{\partial_q S_0^{\mathrm{qm}}}{(1 - \partial_E \mathcal{V})} \neq \frac{\partial S^{\mathrm{qm}}}{\partial q}$$

Floyd: Use Jacobi theorem to define time $\longrightarrow$ trajectories

Energy quantisation $\Leftrightarrow$ Compact space

$\Rightarrow$ Floyd time is ill defined for the QHJT

No Trajectories in EPoQM

Quantum potential as a curvature term:

Using the property of the Schwarzian derivative

$$\{S_0; q\} = -\left(\frac{\partial S_0}{\partial q}\right)^2 \{q; S_0\},$$

We can rewrite the Quantum Stationary Hamilton Jacobi Equation as

$$\frac{1}{2m}\left(\frac{\partial S_0}{\partial \hat{q}}\right)^2 + V(\hat{q}) - E = 0,$$

where

$$\hat{q} = \int^q \frac{dx}{\sqrt{1-\frac{\hbar^2}{2}\{q;S_0\}}}.$$

Flanders: J. Diff.Geom. 1970, 575 $\longrightarrow$ $\{\ ;\ \}$ $\longrightarrow$ a curvature term

In higher dimensions $Q(q) \sim \frac{\Delta R(q)}{R}$ $\longrightarrow$ curvature of $R(q)$

Length Scale

For $W^0(q^0) = 0)$

$$\frac{\partial^2 \Psi}{\partial q^2} = 0 \Rightarrow \qquad \psi_1 = q^0 \quad ; \quad \psi_2 = const$$

$\Rightarrow$ duality implies a length scale

$$\Longrightarrow \quad e^{\frac{2i}{\hbar} S_0^0} = e^{i\alpha \frac{q^0 + i\bar{\ell}_0}{q^0 - i\ell_0}},$$

$$p_0 = \frac{\partial S_0^0}{\partial q^0} = \pm \frac{\hbar(\ell_0 + \bar{\ell}_0)}{2|q^0 - i\ell_0|^2}.$$

$\text{Max}|p_0| = \frac{\hbar}{\text{Re}\ell_0} \rightarrow \text{Re}\ell_0 \neq 0 \rightarrow$ ultraviolet cutoff

$$\lim_{\hbar \rightarrow 0} p_0 = 0 \Rightarrow \mathrm{Re}\, \ell_0 = \lambda_p = \sqrt{\frac{\hbar G}{c^3}}.$$

$\ell_0 = \lambda_p \longrightarrow$ choice consistent with the classical limit

$$Q^0 = \frac{\hbar^2}{4m}\{S_0^0, q^0\} = -\frac{\hbar^2(\mathrm{Re}\,\ell_0)^2}{2m}\frac{1}{|q^0-i\ell_0|^4}.$$

Consistency $\Longrightarrow$ $q^0 = \psi^D/\psi$ is continuous on $\hat{R} = R \cup \{\infty\}$

Taking $m \sim 100GeV$;

$$\mathrm{Re}\,\ell_0 = \lambda_p \approx 10^{-35}m;$$

$$q^0 \sim 93Ly,$$

$$\Longrightarrow \quad |Q| \sim 10^{-202}eV.$$

Generalizations:

Under $(q , S_0(q) , p = \frac{\partial S_0}{\partial q}) \longrightarrow (q^v , S_0^v(q^v) , p^v = \frac{\partial S_0^v}{\partial q^v}),$

$$p^v = \frac{\partial S^v(q^v)}{\partial q_j^v} = \frac{\partial S(q)}{\partial q_j^v} = \sum_i \frac{\partial S(q)}{\partial q_i}\frac{\partial q_i}{\partial q_j^v}, = J^v p, \quad \text{where} \quad J_{ij}^v = \frac{\partial q_i}{\partial q_j^v}$$

with $(p^v|p) \equiv \frac{|p^v|^2}{|p|^2} = \frac{p^{vT} p^v}{p^T p} = \frac{p^T J^{vT} J^v p}{p^T p}.$

Cocycle condition $\rightarrow (q^a; q^c) = \left(p^c|p^b\right) \left[(q^a; q^b) - (q^c; q^b)\right].$

invariant under D–dimensional Mobiüs (conformal) trans.

Quadratic identity:

$$\alpha^2(\nabla S_0)\cdot(\nabla S_0) = \frac{\Delta(Re^{\alpha S_0})}{Re^{\alpha S_0}} - \frac{\Delta R}{R} - \alpha\left(2\frac{\nabla R\cdot\nabla S_0}{R} + \Delta S_0\right),$$

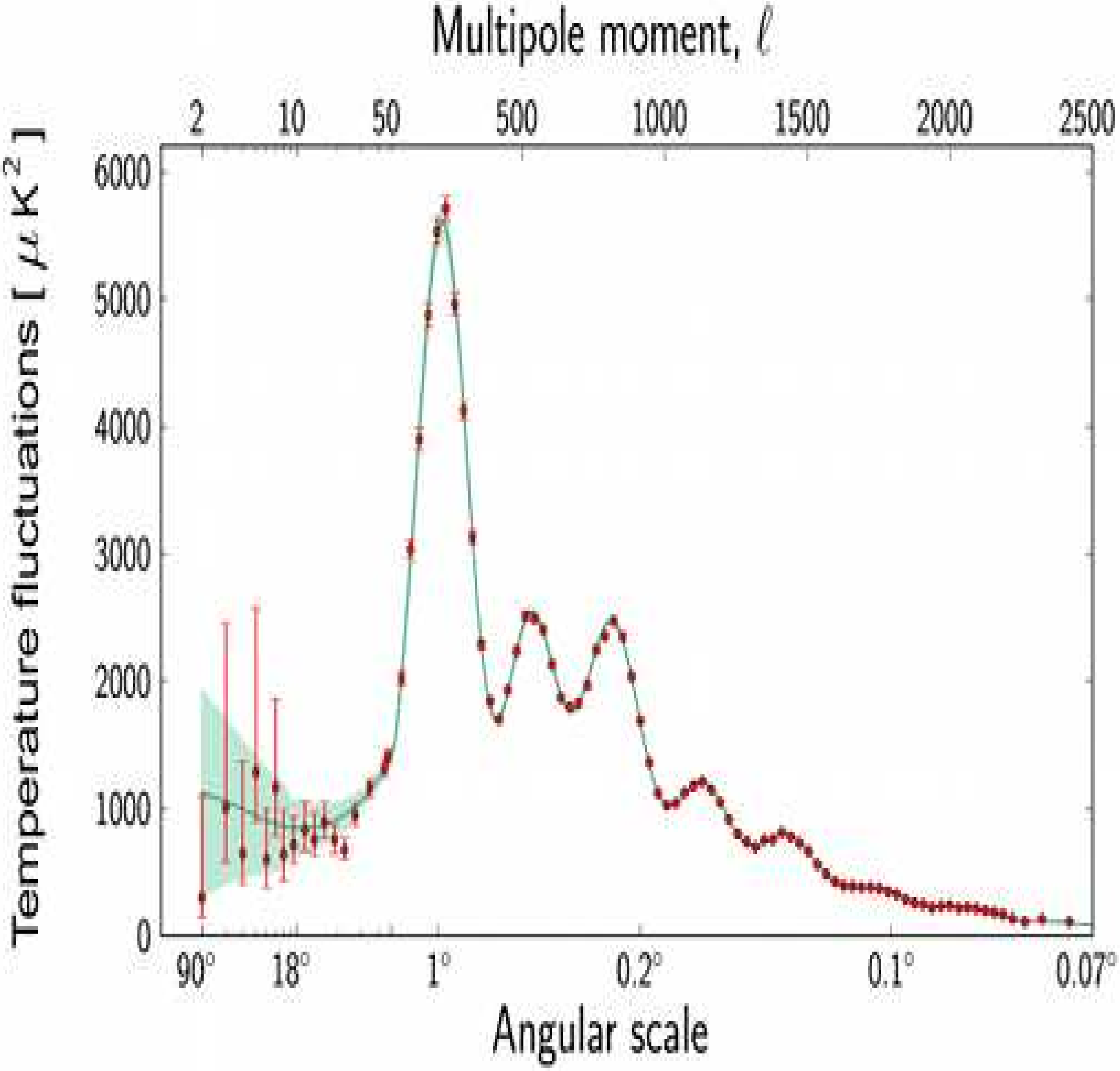
or

$$\alpha^2(\partial S)\cdot(\partial S) = \frac{\partial^2(Re^{\alpha S})}{Re^{\alpha S}} - \frac{\partial^2 R}{R} - \alpha\left(2\frac{\partial R\cdot\partial S}{R} + \partial^2 S\right),$$

or

$$\alpha^2(\partial S - eA)\cdot(\partial S - eA) = \frac{D^2(Re^{\alpha S})}{Re^{\alpha S}} - \frac{\partial^2 R}{R} - \frac{\alpha}{R^2}\partial\cdot\left(R^2(\partial S - eA)\right),$$

$$D^\mu = \partial^\mu - \alpha e A^\mu$$



Multipole moment, ℓ
2
10
50
500
1000
1500
2000
2500
Temperature fluctuations [μK^2]
6000
5000
4000
3000
2000
1000
0
90°
18°
1°
0.2°
0.1°
0.07°
Angular scale

conclusions :

The equivalence postulate $\Longrightarrow$ $S_0 \neq const \Leftrightarrow \hbar \neq 0$

$$\Psi(q) = \frac{1}{\sqrt{S_0'}}\left(A e^{+\frac{i}{\hbar}S_0} + B e^{-\frac{i}{\hbar}S_0}\right)$$

$\mathrm{Re}\ell_0 = \lambda_P \rightarrow$ fundamental length scale

$Q(q)$ Intrinsic curvature terms of elementary particles $\neq 0$ Always

CoCyCle Condition: Invariant under Möbius transformations in

$$\hat{R}^D = R^D \cup \{\infty\} \rightarrow \text{Compact Space}$$

Decompactification limit $\leftrightarrow$ $Q(q) \rightarrow 0$ $\leftrightarrow$ classical limit