# Hamilton–Jacobi meets Möbius

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- AEF, Marco Matone, PLB 450 (1999) 34; ... ; IJMPA 15 (2000) 1869.
- G. Bertoldi, AEF & M. Matone, CQG 17 (2000) 3925.
- AEF, Marco Matone, EPJC 74 (2014) 2694.
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related: Edward Floyd 1982–2011; Robert Wyatt; Bill Poirier.

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- Motivation – quantum gravity
- The Equivalence Postulate $\Rightarrow$ QSHJE
  $\longrightarrow$ Schrödinger eq.
- The Equivalence Postulate $\Rightarrow$ CoCyCle Condition
  $\longrightarrow$ Möbius invariance
- Phase space duality & Legendre transformations
- The Equivalence Postulate $\Rightarrow$ Energy quantization
  & Time Parameterisation
- Möbius invariance $\Rightarrow$ Compact universe
- Conclusions

Motivation General Relativity: Covariance & Equivalence Principle

$\longrightarrow$ fundamental geometrical principle

Quantum Mechanics: No Such Principle

Axiomatic formulation ... $\mathrm{P} \sim |\Psi|^2$

However Quantum $+$ Gravity Theory

not known

Main effort: quantize GR; quantize space–time: $e.g.$ superstring theory

The main successes of string theory:

1) Viable perturbative approach to quantum gravity
2) Unification of gravity, gauge & matter structures

$i.e.$ construction of phenomenologically realistic models

$\longrightarrow$ relevant for experimental observation

State of the art: MSSM from string theory

(AEF, Nanopoulos, Yuan, NPB 335 (1990) 347)

(Cleaver, AEF, Nanopoulos, PLB 455 (1999) 135)

## Other approaches

### Geometrical

Greene, Kirklin, Miron, Ross (1987)
Donagi, Ovrut, Pantev, Waldram (1999)
Blumenhagen, Moster, Reinbacher, Weigand (2006)
Heckman, Vafa (2008)
........

### Orbifolds

Ibanez, Nilles, Quevedo (1987)
Bailin, Love, Thomas (1987)
Kobayashi, Raby, Zhang (2004)
Lebedev, Nilles, Raby, Ramos-Sanchez, Ratz, Vaudrevange, Wingerter (2007)
Blaszczyk, Groot–Nibbelink, Ruehle, Trapletti, Vaudrevange (2010)
.......

### Other CFTs

Gepner (1987)
Schellekens, Yankielowicz (1989)
Gato–Rivera, Schellekens (2009)
.......

### Orientifolds

Cvetic, Shiu, Uranga (2001)
Ibanez, Marchesano, Rabadan (2001)
Kiristis, Schellekens, Tsulaia (2008)
.......

# Adaptation of Hamilton–Jacobi theory

Hamilton's equations of motion $\dot{q} = \frac{\partial H}{\partial p} \;, \quad \dot{p} = -\frac{\partial H}{\partial q}$

$$H(q,p) \longrightarrow K(Q,P) \equiv 0 \qquad \Longrightarrow \qquad \dot{Q} = \frac{\partial K}{\partial P} \equiv 0 \,, \; \dot{P} = -\frac{\partial K}{\partial Q} \equiv 0$$

The solution is the Classical Hamilton–Jacobi Equation

$$H(q,p) \longrightarrow K(Q,P) = H(q, p = \frac{\partial S}{\partial q}) + \frac{\partial S}{\partial t} = 0 \Rightarrow \text{CHJE}$$

stationary case $\longrightarrow \frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 + V(q) - E = 0$

$(q,p) \to (Q,P)$ via canonical transformations

$q$, $p$ are independent. Solve. Then $p = \frac{\partial S}{\partial q}$

Quantum mechanics: $[\hat{q},\hat{p}] = i\hbar \rightarrow q,\ p \rightarrow$ not independent

Assume $H \rightarrow K$ $i.e.\ W(Q) = V(Q) - E = 0$ always exists

But $q,\ p$ not independent. $p = \frac{\partial S}{\partial q}$.

Equivalence postulate:

Consider the transformations on

$$(\ q\ ,\ S_0(q)\ ,\ p = \frac{\partial S_0}{\partial q}) \longrightarrow (\ \tilde{q}\ ,\ \tilde{S}_0(\tilde{q})\ ,\ \tilde{p} = \frac{\partial \tilde{S}_0}{\partial \tilde{q}}\ )$$

Such that

$$W(q) \longrightarrow \tilde{W}(\tilde{q}) = 0$$

exist for all $W(q)$

$\Longrightarrow$ QHJE

$\longrightarrow$ Schrödinger equation

## Implies: Covariance of HJE

**But:**

$$\frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 + V(q) - E = 0$$

Is not covariant under $q \rightarrow \tilde{q}(q)$.

**Further:** $W(q) \equiv 0$ is a fixed state under $q \rightarrow \tilde{q}(q)$.

**Assume:**

$$\frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 + W(q) + Q(q) = 0$$

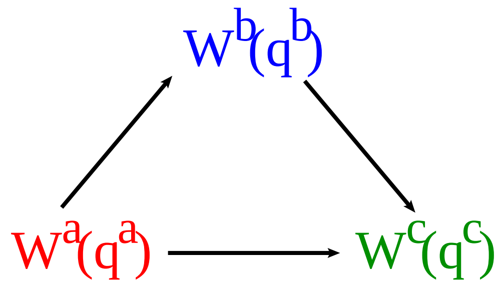
The most general transformations

$$\tilde{W}(\tilde{q}) = \left(\frac{\partial \tilde{q}}{\partial q}\right)^{-2} W(q) + (\tilde{q}; q),$$

$$\tilde{Q}(\tilde{q}) = \left(\frac{\partial \tilde{q}}{\partial q}\right)^{-2} Q(q) - (\tilde{q}; q),$$

with $\tilde{S}_0(\tilde{q}) = S_0(q)$ under $q \rightarrow \tilde{q} = \tilde{q}(q)$

With: $W^0(q^0) \; = \; 0 \quad \rightarrow$ All: $W(q) \; = \; (q; q^0)$




Cocycle Condition: $$(q^a; q^c) \; = \; \left(\frac{\partial q^b}{\partial q^c}\right)^2 \Big[ \, (q^a; q^b) \; - \; (q^c; q^b) \Big]$$

$\Rightarrow$ Theorem $(q^a; q^c)$ invariant under Möbius transformations $\gamma(q^a)$

In 1D: $(q^a; q^c) \quad \sim \quad \{q^a; q^c\}$ Uniquely

Schwarzian derivative $$\{h(x); x(y)\} = \left(\frac{\partial y}{\partial x}\right)^2 \{h(x); y\} - \left(\frac{\partial y}{\partial x}\right)^2 \{x; y\}.$$

$$U(q) \; = \; \{h(q); q\} \; = \; \{\frac{Ah + B}{Ch + D}; q\}$$

Invariant under Möbius transformations

Identity $$\left(\frac{\partial S_0}{\partial q}\right)^2 = \frac{\beta^2}{2}\left(\{\mathrm{e}^{\frac{i2S_0}{\beta}}; q\} - \{S_0; q\}\right)$$

Make the following identifications

$$W(q) = -\frac{\beta^2}{4m}\{\mathrm{e}^{\frac{i2S_0}{\beta}}; q\} = V(q) - E$$
$$Q(q) = \frac{\beta^2}{4m}\{S_0; q\}$$

The Modified Hamilton–Jacobi Equation becomes

$$\frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 + V(q) - E + \frac{\beta^2}{4m}\{S_0; q\} = 0$$

$$\mathrm{QM}: \; W(\tilde{q}) \equiv V(\tilde{q}) - E \equiv 0 \;\Rightarrow\; \tilde{S}_0 = \pm\frac{\beta}{2}\ln\tilde{q} \neq A\tilde{q} + B$$

From the properties of the SD $\{;\}$

$$V(q) - E = -\frac{\beta^2}{4m}\{\mathrm{e}^{\frac{i2S_0}{\beta}}; q\}$$

is a potential of the $2^{nd}$–order diff. Eq.

$$\left(-\frac{\beta^2}{2m}\frac{\partial^2}{\partial q^2} + V(q) - E\right)\Psi(q) = 0 \qquad \Rightarrow \qquad \beta \; = \; \hbar$$

The general solution $$\Psi(q) = \frac{1}{\sqrt{S_0'}}\left(A\mathrm{e}^{+\frac{i}{\hbar}S_0} + B\mathrm{e}^{-\frac{i}{\hbar}S_0}\right)$$

and $$\mathrm{e}^{+\frac{i2S_0}{\hbar}} \; = \; \mathrm{e}^{i\alpha}\,\frac{w + i\bar{\ell}}{w - i\ell} \qquad\qquad w \; = \; \frac{\psi_1}{\psi_2}$$

$$\ell \; = \; \ell_1 \; + \; i\;\ell_2 \qquad \ell_1 \; \neq \; 0 \qquad \alpha \; \in \; R$$

# The equivalence transformation

$$W(q) \;=\; V(q) \;-\; E \quad\longrightarrow\quad \tilde{W}(\tilde{q}) \;=\; 0$$

always exists

We have to find $q \;\rightarrow\; \tilde{q}$ take $\tilde{q} \;=\; \frac{\psi_1}{\psi_2}$

then $$\left(-\frac{\beta^2}{2m}\,\frac{\partial^2}{\partial q^2} \;+\; V(q) \;-\; E\right)\Psi(q) \;=\; 0 \;\;\rightarrow\;\; \frac{\partial^2}{\partial \tilde{q}^2}\tilde{\psi}(\tilde{q}) \;=\; 0$$

where $$\tilde{\psi}(\tilde{q}) \;=\; \left(\frac{dq}{d\tilde{q}}\right)^{-\frac{1}{2}}\psi(q)$$

## Generalizations:

Under $\left( q \; , \; S_0(q) \; , \; p \; = \; \frac{\partial S_0}{\partial q}\right) \; \longrightarrow \; \left( q^v \; , \; S_0^v(q^v) \; , \; p^v \; = \; \frac{\partial S_0^v}{\partial q^v} \right)$,

$$p^v = \frac{\partial S^v(q^v)}{\partial q_j^v} = \frac{\partial S(q)}{\partial q_j^v} = \sum_i \frac{\partial S(q)}{\partial q_i}\frac{\partial q_i}{\partial q_j^v}, = J^v p, \quad \text{where} \quad J_{ij}^v = \frac{\partial q_i}{\partial q_j^v}$$

with $\quad (p^v|p) \equiv \frac{|p^v|^2}{|p|^2} = \frac{p^{vT} p^v}{p^T p} = \frac{p^T J^{vT} J^v p}{p^T p}.$

Cocycle condition $\rightarrow \; (q^a;q^c) = \left(p^c|p^b\right)\left[(q^a;q^b) - (q^c;q^b)\right].$

invariant under D–dimensional Mobiüs (conformal) trans.

# Quadratic identity:

$$\alpha^2(\nabla S_0)\cdot(\nabla S_0) = \frac{\Delta(Re^{\alpha S_0})}{Re^{\alpha S_0}} - \frac{\Delta R}{R} - \alpha\left(2\frac{\nabla R\cdot\nabla S_0}{R} + \Delta S_0\right),$$

or

$$\alpha^2(\partial S)\cdot(\partial S) = \frac{\partial^2(Re^{\alpha S})}{Re^{\alpha S}} - \frac{\partial^2 R}{R} - \alpha\left(2\frac{\partial R\cdot\partial S}{R} + \partial^2 S\right),$$

or

$$\alpha^2(\partial S - eA)\cdot(\partial S - eA) = \frac{D^2(Re^{\alpha S})}{Re^{\alpha S}} - \frac{\partial^2 R}{R} - \frac{\alpha}{R^2}\partial\cdot\left(R^2(\partial S - eA)\right),$$

$$D^\mu \;=\; \partial^\mu - \alpha e A^\mu$$

# Phase space duality & Legendre transformations

intimate connection between $p-q$ duality & the equivalence postulate

Hamilton's Eqs. $$\dot{q} = \frac{\partial H}{\partial p} \quad , \quad \dot{p} = -\frac{\partial H}{\partial q}$$

invariant under $p \longrightarrow -q$

breaks down once $V(q)$ is specified $e.g.$ $\frac{1}{2m}p^2 + V(q) - E = 0$

Aim  Formulation with manifest $p-q$ duality

recall $p = \frac{\partial S}{\partial q}$  define $q = \frac{\partial T}{\partial p}$

$$S = p\frac{\partial T}{\partial p} - T \quad , \quad T = q\frac{\partial S}{\partial q} - S$$

Stationary Case: $$S_0 = p\frac{\partial T_0}{\partial p} - T_0 \quad , \quad T_0 = q\frac{\partial S_0}{\partial q} - S_0$$

Invariant under Möbius transformations:

$$q \longrightarrow q^v = \frac{Aq+B}{Cq+D},$$

$$p \longrightarrow p_v = \rho^{-1}(Cq+D)^2 p \ , \quad \rho = AD - BC$$

$$T_0 \longrightarrow T_0^v(p^v) = T_0(p) + \rho^{-1}(ACq^2 + 2BCq + BD)p.$$

Transformations: $q \to q^v = v(q)$ defined by $S_0^v(q^v) = S_0(q)$

( $S_0$ scalar function under $v$ )

Associate a $2^{nd}$order diff. eq. with the Legendre transformation:

$$\left(\frac{\partial^2}{\partial S_0^2} + U(S_0)\right)\begin{pmatrix} q\sqrt{p} \\ \sqrt{p} \end{pmatrix} = 0$$

where $U(S_0) = \frac{1}{2}\{q, S_0\}$ $\quad \frac{q'''}{q'} - \frac{3}{2}\left(\frac{q''}{q'}\right)^2$

We can derive this eq. in several ways

$$
\begin{aligned}
p &= \frac{\partial S_0}{\partial q} & \Rightarrow & \quad p\frac{\partial q}{\partial S_0} &= 1 \\
& \frac{\partial}{\partial S_0} & \Rightarrow & \quad \frac{\partial p}{\partial S_0}\frac{\partial q}{\partial S_0} + p\frac{\partial^2 q}{\partial S_0^2} &= 0
\end{aligned}
$$

rewritten as $$\frac{\partial^2_{S_0} q\sqrt{p}}{q\sqrt{p}} = \frac{\partial^2_{S_0}\sqrt{p}}{\sqrt{p}} = -U(S_0)$$

or $$\frac{\partial^2}{\partial S_0^2} \quad : \qquad S_0(q) = \frac{1}{2}\sqrt{p}\frac{\partial T_0}{\partial\sqrt{p}} - T_0$$

$$\Longrightarrow \quad \left(\frac{\partial^2}{\partial S_0^2} + U(S_0)\right)\begin{pmatrix} q\sqrt{p} \\ \sqrt{p}\end{pmatrix} = 0$$

manifest $p \leftrightarrow q$ – $S_0 \leftrightarrow T_0$ duality with

$$p = \frac{\partial S_0}{\partial q} \qquad\qquad q = \frac{\partial T_0}{\partial p}$$

$$S_0 = p\frac{\partial T_0}{\partial p} - T_0 \qquad\qquad T_0 = q\frac{\partial S_0}{\partial q} - S_0$$

$$\left(\frac{\partial^2}{\partial S_0^2} + U(S_0)\right)\begin{pmatrix} q\sqrt{p} \\ \sqrt{p} \end{pmatrix} = 0 \qquad\qquad \left(\frac{\partial^2}{\partial T_0^2} + \mathcal{V}(T_0)\right)\begin{pmatrix} p\sqrt{q} \\ \sqrt{q} \end{pmatrix} = 0$$

Involutive Legendre transformation $\leftrightarrow$ duality

## Self–dual states

States with

$$pq \;=\; \gamma \;=\; const$$

are simultaneous solutions of the two pictures with

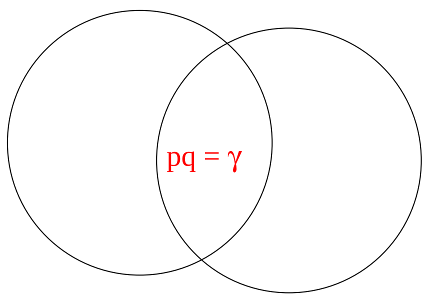
$$S_0 \;=\; -\; T_0 \;+\; const$$

$$S_0(q) = \gamma \ln \gamma_q q \qquad\qquad T_0(p) = \gamma \ln \gamma_p p$$

$$S_0 \;+\; T_0 \;=\; pq \;=\; \gamma$$

where

$$\gamma_q \gamma_p \gamma = \mathrm{e}$$




self–dual states

self–dual states $\qquad W^{sd} = W^0 = 0 \qquad \gamma^{sd} = \dfrac{\pm\hbar}{2i}$

Energy quantization:

Probability: $\Longrightarrow$ $(\Psi, \Psi')$ continuous ; $\Psi \in L^2(R)$

$\Longrightarrow$ quantization, bound states

What are the conditions on the trivializing transformations?

$$q^0 = w = \frac{\psi_1}{\psi_2} = \frac{\psi^D}{\psi}$$

we have
$$\{w, q\} = -\frac{4m}{\hbar^2}(V(q) - E)$$

$\Rightarrow$ $w \neq const$ ; $w \in C^2(R)$ $and$ $w''$ differentiable on $R$

In addition from the properties of $\{,\}$ $\rightarrow$ $\{w, q^{-1}\} = q^4\{w, q\}$

$\Rightarrow$ $w \neq const$ ; $w \in C^2(\hat{R})$ $and$ $w''$ differentiable on $\hat{R}$

where $\hat{R} = R \cup \{\infty\}$

$$\Longrightarrow$$

Equivalence postulate $\Longrightarrow$ continuity of $(\psi^D, \psi)$ and $(\psi^{D'}, \psi')$

Theorem:

$$if \qquad V(q) - E = \begin{cases} P_-^2 > 0 & for \;\; q < q_- \\ P_+^2 > 0 & for \;\; q > q_+ \end{cases}$$

*then the ratio* $w = \psi^D/\psi$ *is continuous on* $\hat{R}$

*iff the Schrödinger equation admits an* $L^2(R)$ *solution*

Potential Well:

$$V(q) = \begin{cases} 0 & |q| \leq L \\ V_0 & |q| > L \end{cases}$$

$$k = \frac{\sqrt{2mE}}{\hbar} \qquad K = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

$$|q| \leq L \qquad \Psi_1^1 = \cos kq \qquad \Psi_2^1 = \sin kq$$

$$q > L \qquad \Psi_1^2 = \mathrm{e}^{-Kq} \qquad \Psi_2^2 = \mathrm{e}^{Kq}$$

take $(1,1)$ : $\Psi,\ \Psi'$ continuous $\Rightarrow$ $k\ \tan kL = K$

$$\Rightarrow \quad w = \frac{1}{[k\sin(2kL)]} \begin{cases} \cos(2kL) - \mathrm{e}^{-2K(q+L)} & q < -L \\ \sin(2kL)\tan(kq) & |q| \leq L \\ \mathrm{e}^{2K(q-L)} - \cos(2kL) & q > L \end{cases}$$

$$\lim_{q\to\pm} \frac{\psi^D}{\psi} = \pm\infty \qquad \Longrightarrow \qquad E_n(k \ \tan kL \ = \ K) \text{ are admissible solutions}$$

take $(1,2)$ : $\quad \Psi, \ \Psi'$ continuous $\quad \Rightarrow \quad k \ \tan(kl) \ = \ -K$

$$\Rightarrow \quad w = \frac{1}{[k\sin(2kL)]} \begin{cases} \cos(2kL) - \mathrm{e}^{2K(q+L)} & q \ < \ -L \\ \sin(2kL)\tan(kq) & |q| \ \leq \ L \\ \mathrm{e}^{-2K(q-L)} - \cos(2kL) & q \ > \ L \end{cases}$$

$$\lim_{q\to\pm} \frac{\psi^D}{\psi} \ = \ \mp\frac{1}{k}\cot(2kL) \qquad \Longrightarrow \qquad w(-\infty) \ \neq \ w(+\infty)$$

($k^{-1}(\cot 2kL) = 0$ is not compatible with $k\tan(kL) = -K$)

$\Rightarrow \quad E_n(k \ \tan kL \ = \ -K)$ are not admissible solutions

# Time parameterisation

Bohmian mechanics : $\quad p = \frac{\partial S}{\partial q} = m\dot{q} \quad \Rightarrow \quad$ Trajectory representation

Jacobi time : $\quad t = \frac{\partial S_0}{\partial E}.$

In classical mechanics: Jacobi time $=$ Mechanical time

$$t - t_0 = m \int_{q_0}^{q} \frac{dx}{\partial_x S_0^{\mathrm{cl}}} = \int_{q_0}^{q} dx \frac{\partial}{\partial E} \partial_x S_0^{\mathrm{cl}} = \frac{\partial S_0^{\mathrm{cl}}}{\partial E}.$$

In Quantum HJ Theory: Jacobi time $\neq$ Mechanical time

$$t - t_0 = \frac{\partial \mathcal{S}_0^{\mathrm{qm}}}{\partial E} = \frac{\partial}{\partial E} \int_{q_0}^{q} dx \partial_x \mathcal{S}_0^{\mathrm{qm}} = \left(\frac{m}{2}\right) \int_{q_0}^{q} dx \frac{1 - \partial_E Q}{(E - V - Q)^{1/2}}$$

$$\Longrightarrow \quad m\frac{dq}{dt} = m \left(\frac{dt}{dq}\right)^{-1} = \frac{\partial_q S_0^{\mathrm{qm}}}{(1 - \partial_E \mathcal{V})} \neq \frac{\partial S^{\mathrm{qm}}}{\partial q}$$

Floyd: Use Jacobi theorem to define time $\longrightarrow$ trajectories

Compact space $\Leftrightarrow$ Energy quantisation

$\Rightarrow$ Floyd time is ill defined for the QHJT

No Trajectories in EPoQM

## Quantum potential as a curvature term:

Using the property of the Schwarzian derivative

$$\{S_0; q\} = -\left(\frac{\partial S_0}{\partial q}\right)^2 \{q; S_0\},$$

We can rewrite the Quantum Stationary Hamilton Jacobi Equation as

$$\frac{1}{2m}\left(\frac{\partial S_0}{\partial \hat{q}}\right)^2 + V(\hat{q}) - E = 0,$$

where

$$\hat{q} = \int^q \frac{dx}{\sqrt{1-\frac{\hbar^2}{2}\{q;S_0\}}}.$$

Flanders: J. Diff.Geom. 1970, 575 $\longrightarrow$ $\{\ ;\ \}$ $\longrightarrow$ a curvature term

In higher dimensions $Q(q) \sim \frac{\Delta R(q)}{R}$ $\longrightarrow$ curvature of $R(q)$

# Length Scale

For $W^0(q^0) = 0)$

$$\frac{\partial^2 \Psi}{\partial q^2} = 0 \Rightarrow \qquad \psi_1 = q^0 \quad ; \quad \psi_2 = const$$

$\Rightarrow$ duality implies a length scale

$$\Longrightarrow \quad e^{\frac{2i}{\hbar} S_0^0} = e^{i\alpha \frac{q^0 + i\bar{\ell}_0}{q^0 - i\ell_0}},$$

$$p_0 = \frac{\partial S_0^0}{\partial q^0} = \pm \frac{\hbar(\ell_0 + \bar{\ell}_0)}{2|q^0 - i\ell_0|^2}.$$

Max$|p_0| = \frac{\hbar}{\text{Re}\ell_0}$ $\rightarrow$ Re$\ell_0 \neq 0 \rightarrow$ ultraviolet cutoff

$$\lim_{\hbar \to 0} p_0 = 0 \Rightarrow \mathrm{Re}\,\ell_0 = \lambda_p = \sqrt{\frac{\hbar G}{c^3}}.$$

$\ell_0 = \lambda_p \longrightarrow$ choice consistent with the classical limit

$$Q^0 \;=\; \frac{\hbar^2}{4m}\{S_0^0, q^0\} \;=\; -\frac{\hbar^2(\mathrm{Re}\,\ell_0)^2}{2m}\frac{1}{|q^0 - i\ell_0|^4}.$$

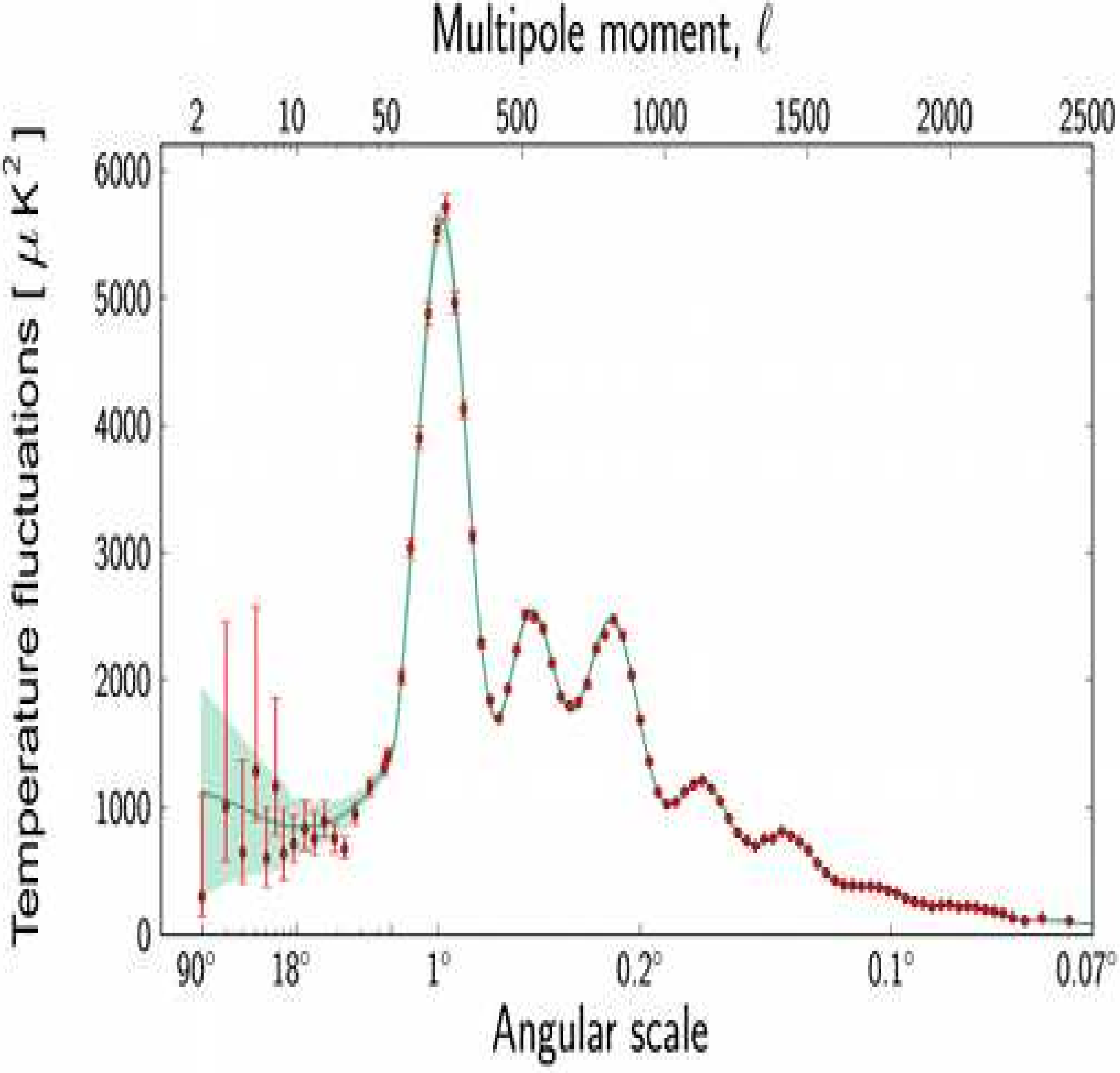
Consistency $\Longrightarrow$ $q^0 \;=\; \psi^D/\psi$ is continuous on $\hat{R} \;=\; R \,\cup\, \{\infty\}$

Taking $m \sim 100GeV$;

$$\mathrm{Re}\,\ell_0 = \lambda_p \approx 10^{-35}m;$$

$$q^0 \sim 93Ly,$$

$$\Longrightarrow \quad |Q| \;\sim\; 10^{-202}eV.$$


Multipole moment, ℓ
2
10
50
500
1000
1500
2000
2500
Temperature fluctuations [ $\mu K^2$ ]
6000
5000
4000
3000
2000
1000
0
90°
18°
1°
0.2°
0.1°
0.07°
Angular scale

## conclusions :

The equivalence postulate $\Longrightarrow$ $S_0 \neq const \Leftrightarrow \hbar \neq 0$

$$\Psi(q) = \frac{1}{\sqrt{S_0'}}\left(A\mathrm{e}^{+\frac{i}{\hbar}S_0} + B\mathrm{e}^{-\frac{i}{\hbar}S_0}\right)$$

$\mathrm{Re}\ell_0 = \lambda_P \rightarrow$ fundamental length scale

$Q(q)$ Intrinsic curvature terms of elementary particles $\neq 0$ Always

CoCyCle Condition: Invariant under Möbius transformations in

$$\hat{R}^D = R^D \cup \{\infty\} \rightarrow \text{Compact Space}$$

Decompactification limit $\leftrightarrow Q(q) \rightarrow 0 \leftrightarrow$ classical limit

Phase–space duality $vs$ $T$–duality