The Möbius Symmetry of Quantum Mechanics

Alon E. Faraggi

- AEF, Marco Matone, PLB 450 (1999) 34; ... ; IJMPA 15 (2000) 1869.
- G. Bertoldi, AEF & M. Matone, CQG 17 (2000) 3925.
- AEF, Marco Matone, EPJC 74 (2014) 2694.
- AEF, AHEP. 2013 (2013) 957394 ; 1305.0044.

related: Edward Floyd 1982–2011; Robert Wyatt; Bill Poirier.

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- Motivation – quantum gravity
- The Equivalence Postulate $\Rightarrow$ QSHJE

  $\longrightarrow$ Schrödinger eq.
- The Equivalence Postulate $\Rightarrow$ CoCyCle Condition

  $\longrightarrow$ Möbius invariance
- Phase space duality & Legendre transformations
- The Equivalence Postulate $\Rightarrow$ Energy quantization

  & Time Parameterisation
- Möbius invariance $\Rightarrow$ Compact universe
- Conclusions

Motivation General Relativity: Covariance & Equivalence Principle

$\longrightarrow$ fundamental geometrical principle

Quantum Mechanics: No Such Principle

Axiomatic formulation ... $\mathrm{P} \sim |\Psi|^2$

However Quantum $+$ Gravity Theory

not known

Main effort: quantize GR; quantize space–time: $e.g.$ superstring theory

The main successes of string theory:

1) Viable perturbative approach to quantum gravity
2) Unification of gravity, gauge & matter structures

$i.e.$ construction of phenomenologically realistic models

$\longrightarrow$ relevant for experimental observation

State of the art: MSSM from string theory

(AEF, Nanopoulos, Yuan, NPB 335 (1990) 347)

(Cleaver, AEF, Nanopoulos, PLB 455 (1999) 135)

Other approaches

Geometrical

Greene, Kirklin, Miron, Ross (1987)
Donagi, Ovrut, Pantev, Waldram (1999)
Blumenhagen, Moster, Reinbacher, Weigand (2006)
Heckman, Vafa (2008)
....

Orbifolds

Ibanez, Nilles, Quevedo (1987)
Bailin, Love, Thomas (1987)
Kobayashi, Raby, Zhang (2004)
Lebedev, Nilles, Raby, Ramos-Sanchez, Ratz, Vaudrevange, Wingerter (2007)
Blaszczyk, Groot–Nibbelink, Ruehle, Trapletti, Vaudrevange (2010)
....

Other CFTs

Gepner (1987)
Schellekens, Yankielowicz (1989)
Gato–Rivera, Schellekens (2009)
....

Orientifolds

Cvetic, Shiu, Uranga (2001)
Ibanez, Marchesano, Rabadan (2001)
Kiristis, Schellekens, Tsulaia (2008)
....

Adaptation of Hamilton–Jacobi theory

Hamilton's equations of motion $\dot{q} = \frac{\partial H}{\partial p}\ , \ \dot{p} = -\frac{\partial H}{\partial q}$

$$H(q,p) \longrightarrow K(Q,P) \equiv 0 \quad \Longrightarrow \quad \dot{Q} = \frac{\partial K}{\partial P} \equiv 0\,,\ \dot{P} = -\frac{\partial K}{\partial Q} \equiv 0$$

The solution is the Classical Hamilton–Jacobi Equation

$$H(q,p) \longrightarrow K(Q,P) = H(q, p = \frac{\partial S}{\partial q}) + \frac{\partial S}{\partial t} = 0 \Rightarrow \mathrm{CHJE}$$

stationary case $\longrightarrow \frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 + V(q) - E = 0$

$(q,p) \to (Q,P)$ via canonical transformations

$q,\ p$ are independent. Solve. Then $p = \frac{\partial S}{\partial q}$

Quantum mechanics: $[\hat{q},\hat{p}] \; = \; i\hbar \;\; \rightarrow \;\; q, \; p \;\; \rightarrow$ not independent

Assume $H \;\; \rightarrow \;\; K \;\;\; i.e. \; W(Q) \; = \; V(Q) - E \; = \; 0$ always exists

But $\quad q, \; p \quad$ not independent. $\quad p \; = \; \frac{\partial S}{\partial q}$.

Equivalence postulate:

Consider the transformations on

$$(\; q \; , \; S_0(q) \; , \; p \; = \; \frac{\partial S_0}{\partial q}) \;\; \longrightarrow \;\; (\; \tilde{q} \; , \; \tilde{S}_0(\tilde{q}) \; , \; \tilde{p} \; = \; \frac{\partial \tilde{S}_0}{\partial \tilde{q}} \;)$$

Such that

$$W(q) \;\; \longrightarrow \;\; \tilde{W}(\tilde{q}) \; = \; 0$$

exist for all $W(q)$

$\Longrightarrow$ QHJE

$\longrightarrow$ Schrödinger equation

Implies: Covariance of HJE

But: $$\frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 \;+\; V(q) \;-\; E \;=\; 0$$

Is not covariant under $q \;\longrightarrow\; \tilde{q}(q)$.

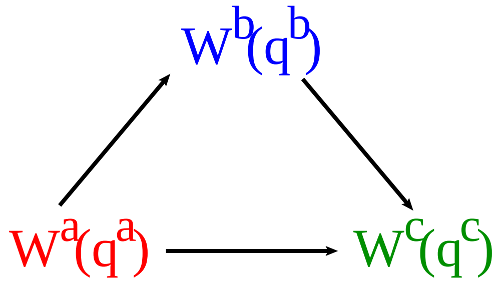
Further: $W(q) \;\equiv\; 0$ is a fixed state under $q \;\longrightarrow\; \tilde{q}(q)$.

Assume: $$\frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 \;+\; W(q) \;+\; Q(q) \;=\; 0$$

The most general transformations $$\tilde{W}(\tilde{q}) = \left(\frac{\partial \tilde{q}}{\partial q}\right)^{-2} W(q) + (\tilde{q}; q),$$
$$\tilde{Q}(\tilde{q}) = \left(\frac{\partial \tilde{q}}{\partial q}\right)^{-2} Q(q) - (\tilde{q}; q),$$

with $\tilde{S}_0(\tilde{q}) \;=\; S_0(q)$ under $q \;\longrightarrow\; \tilde{q} \;=\; \tilde{q}(q)$

With: $W^0(q^0) \; = \; 0 \quad \rightarrow$ All: $W(q) \; = \; (q; q^0)$



Cocycle Condition: $(q^a; q^c) \; = \; \left(\frac{\partial q^b}{\partial q^c}\right)^2 \left[\, (q^a; q^b) \; - \; (q^c; q^b) \right]$

$\Rightarrow$ Theorem $(q^a; q^c)$ invariant under Möbius transformations $\gamma(q^a)$

In 1D: $(q^a; q^c) \quad \sim \quad \{q^a; q^c\}$ Uniquely

Schwarzian derivative $\{h(x); x(y)\} = \left(\frac{\partial y}{\partial x}\right)^2 \{h(x); y\} - \left(\frac{\partial y}{\partial x}\right)^2 \{x; y\}.$

$$U(q) \; = \; \{h(q); q\} \; = \; \{\frac{Ah + B}{Ch + D}; q\}$$

Invariant under Möbius transformations

Identity $$\left(\frac{\partial S_0}{\partial q}\right)^2 = \frac{\beta^2}{2}\left(\{\mathrm{e}^{\frac{i2S_0}{\beta}};q\} - \{S_0;q\}\right)$$

Make the following identifications

$$W(q) = -\frac{\beta^2}{4m}\{\mathrm{e}^{\frac{i2S_0}{\beta}};q\} = V(q) - E$$

$$Q(q) = \frac{\beta^2}{4m}\{S_0;q\}$$

The Modified Hamilton–Jacobi Equation becomes

$$\frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 + V(q) - E + \frac{\beta^2}{4m}\{S_0;q\} = 0$$

$$\mathrm{QM}: \; W(\tilde{q}) \equiv V(\tilde{q}) - E \equiv 0 \;\Rightarrow\; \tilde{S}_0 = \pm\frac{\beta}{2}\ln\tilde{q} \neq A\tilde{q} + B$$

From the properties of the SD $\{;\}$

$$V(q) - E = -\frac{\beta^2}{4m}\{\mathrm{e}^{\frac{i2S_0}{\beta}}; q\}$$

is a potential of the 2^{nd}–order diff. Eq.

$$\left(-\frac{\beta^2}{2m}\frac{\partial^2}{\partial q^2} + V(q) - E\right)\Psi(q) = 0 \qquad \Rightarrow \qquad \beta \;=\; \hbar$$

The general solution $$\Psi(q) = \frac{1}{\sqrt{S_0'}}\left(A\mathrm{e}^{+\frac{i}{\hbar}S_0} + B\mathrm{e}^{-\frac{i}{\hbar}S_0}\right)$$

and $$\mathrm{e}^{+\frac{i2S_0}{\hbar}} \;=\; \mathrm{e}^{i\alpha}\,\frac{w + i\bar{\ell}}{w - i\ell} \qquad\qquad w \;=\; \frac{\psi_1}{\psi_2}$$

$$\ell \;=\; \ell_1 \;+\; i\;\ell_2 \qquad \ell_1 \;\neq\; 0 \qquad \alpha \;\in\; R$$

The equivalence transformation

$$W(q) \;=\; V(q) \;-\; E \quad\longrightarrow\quad \tilde{W}(\tilde{q}) \;=\; 0$$

always exists

We have to find $\quad q \;\to\; \tilde{q} \quad$ take $\quad \tilde{q} \;=\; \frac{\psi_1}{\psi_2}$

then $$\left(-\frac{\beta^2}{2m}\,\frac{\partial^2}{\partial q^2} \;+\; V(q) \;-\; E\right)\Psi(q) \;=\; 0 \;\;\to\;\; \frac{\partial^2}{\partial \tilde{q}^2}\tilde{\psi}(\tilde{q}) \;=\; 0$$

where $$\tilde{\psi}(\tilde{q}) \;=\; \left(\frac{dq}{d\tilde{q}}\right)^{-\frac{1}{2}}\psi(q)$$

Generalizations:

Under $\left(q \ , \ S_0(q) \ , \ p \ = \ \frac{\partial S_0}{\partial q} \right) \ \longrightarrow \ \left(q^v \ , \ S_0^v(q^v) \ , \ p^v \ = \ \frac{\partial S_0^v}{\partial q^v} \right),$

$$p^v = \frac{\partial S^v(q^v)}{\partial q_j^v} = \frac{\partial S(q)}{\partial q_j^v} = \sum_i \frac{\partial S(q)}{\partial q_i} \frac{\partial q_i}{\partial q_j^v}, = J^v p, \quad \text{where} \quad J_{ij}^v = \frac{\partial q_i}{\partial q_j^v}$$

with $\quad (p^v|p) \equiv \frac{|p^v|^2}{|p|^2} = \frac{p^{vT} p^v}{p^T p} = \frac{p^T J^{vT} J^v p}{p^T p}.$

Cocycle condition $\rightarrow \ (q^a; q^c) = \left(p^c | p^b \right) \left[(q^a; q^b) - (q^c; q^b) \right].$

invariant under D–dimensional Mobiüs (conformal) trans.

Quadratic identity:

$$\alpha^2(\nabla S_0)\cdot(\nabla S_0) = \frac{\Delta(Re^{\alpha S_0})}{Re^{\alpha S_0}} - \frac{\Delta R}{R} - \alpha\left(2\frac{\nabla R\cdot\nabla S_0}{R} + \Delta S_0\right),$$

or

$$\alpha^2(\partial S)\cdot(\partial S) = \frac{\partial^2(Re^{\alpha S})}{Re^{\alpha S}} - \frac{\partial^2 R}{R} - \alpha\left(2\frac{\partial R\cdot\partial S}{R} + \partial^2 S\right),$$

or

$$\alpha^2(\partial S - eA)\cdot(\partial S - eA) = \frac{D^2(Re^{\alpha S})}{Re^{\alpha S}} - \frac{\partial^2 R}{R} - \frac{\alpha}{R^2}\partial\cdot\left(R^2(\partial S - eA)\right),$$

$$D^\mu \;=\; \partial^\mu - \alpha e A^\mu$$

Phase space duality & Legendre transformations

intimate connection between $p-q$ duality & the equivalence postulate

Hamilton's Eqs. $$\dot{q} = \frac{\partial H}{\partial p} \quad , \quad \dot{p} = -\frac{\partial H}{\partial q}$$

invariant under $p \longrightarrow -q$

breaks down once $V(q)$ is specified *e.g.* $\frac{1}{2m}p^2 + V(q) - E = 0$

Aim Formulation with manifest $p-q$ duality

recall $p = \frac{\partial S}{\partial q}$ define $q = \frac{\partial T}{\partial p}$

$$S = p\frac{\partial T}{\partial p} - T \quad , \quad T = q\frac{\partial S}{\partial q} - S$$

Stationary Case: $$S_0 = p\frac{\partial T_0}{\partial p} - T_0 \quad , \quad T_0 = q\frac{\partial S_0}{\partial q} - S_0$$

Invariant under Möbius transformations:

$$q \longrightarrow q^v = \frac{Aq+B}{Cq+D},$$

$$p \longrightarrow p_v = \rho^{-1}(Cq+D)^2 p \ , \quad \rho = AD - BC$$

$$T_0 \longrightarrow T_0^v(p^v) = T_0(p) + \rho^{-1}(ACq^2 + 2BCq + BD)p.$$

Transformations: $q \to q^v = v(q)$ defined by $S_0^v(q^v) = S_0(q)$

(S_0 scalar function under v)

Associate a 2^{nd}order diff. eq. with the Legendre transformation:

$$\left(\frac{\partial^2}{\partial S_0^2} + U(S_0)\right)\begin{pmatrix} q\sqrt{p} \\ \sqrt{p} \end{pmatrix} = 0$$

where $U(S_0) = \frac{1}{2}\{q, S_0\}$ $\frac{q'''}{q'} - \frac{3}{2}\left(\frac{q''}{q'}\right)^2$

We can derive this eq. in several ways

$$
\begin{aligned}
p &= \frac{\partial S_0}{\partial q} & \Rightarrow & \quad p\frac{\partial q}{\partial S_0} & = 1 \\
& \frac{\partial}{\partial S_0} & \Rightarrow & \quad \frac{\partial p}{\partial S_0}\frac{\partial q}{\partial S_0} + p\frac{\partial^2 q}{\partial S_0^2} & = 0
\end{aligned}
$$

rewritten as

$$
\frac{\partial^2_{S_0} q\sqrt{p}}{q\sqrt{p}} = \frac{\partial^2_{S_0}\sqrt{p}}{\sqrt{p}} = -U(S_0)
$$

or $\quad \dfrac{\partial^2}{\partial S_0^2} \quad : \quad S_0(q) = \dfrac{1}{2}\sqrt{p}\dfrac{\partial T_0}{\partial\sqrt{p}} - T_0$

$$
\Longrightarrow \quad \left(\frac{\partial^2}{\partial S_0^2} + U(S_0)\right)\begin{pmatrix} q\sqrt{p} \\ \sqrt{p} \end{pmatrix} = 0
$$

manifest $p \leftrightarrow q$ – $S_0 \leftrightarrow T_0$ duality with

$$p = \frac{\partial S_0}{\partial q} \qquad\qquad q = \frac{\partial T_0}{\partial p}$$

$$S_0 = p\frac{\partial T_0}{\partial p} - T_0 \qquad\qquad T_0 = q\frac{\partial S_0}{\partial q} - S_0$$

$$\left(\frac{\partial^2}{\partial S_0^2} + U(S_0)\right)\begin{pmatrix} q\sqrt{p} \\ \sqrt{p} \end{pmatrix} = 0 \qquad\qquad \left(\frac{\partial^2}{\partial T_0^2} + \mathcal{V}(T_0)\right)\begin{pmatrix} p\sqrt{q} \\ \sqrt{q} \end{pmatrix} = 0$$

Involutive Legendre transformation $\leftrightarrow$ duality

Self–dual states

States with

$$pq = \gamma = const$$

are simultaneous solutions of the two pictures with

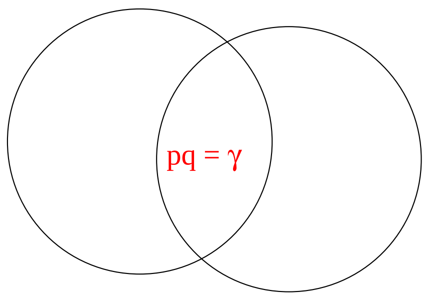
$$S_0 = - T_0 + const$$

$$S_0(q) = \gamma \ln \gamma_q q \qquad T_0(p) = \gamma \ln \gamma_p p$$

$$S_0 + T_0 = pq = \gamma$$

where

$$\gamma_q \gamma_p \gamma = \mathrm{e}$$



self–dual states

self–dual states $\qquad W^{sd} = W^0 = 0 \qquad \gamma^{sd} = \frac{\pm\hbar}{2i}$

Energy quantization:

Probability: $\Longrightarrow\ (\Psi, \Psi')$ continuous ; $\Psi \in L^2(R)$

$\Longrightarrow$ quantization, bound states

What are the conditions on the trivializing transformations?

$$q^0 = w = \frac{\psi_1}{\psi_2} = \frac{\psi^D}{\psi}$$

we have $$\{w, q\} = -\frac{4m}{\hbar^2}(V(q) - E)$$

$\Rightarrow \quad w \neq const \ ; \ w \in C^2(R) \ and \ w''$ differentiable on R

In addition from the properties of $\{,\} \ \rightarrow \ \{w, q^{-1}\} = q^4\{w, q\}$

$\Rightarrow \quad w \neq const \ ; \ w \in C^2(\hat{R}) \ and \ w''$ differentiable on $\hat{R}$

where $\hat{R} = R \cup \{\infty\}$

$$\Longrightarrow$$

Equivalence postulate $\Longrightarrow$ continuity of (ψ^D, ψ) and $(\psi^{D'}, \psi')$

<u>Theorem:</u>

$$if \qquad V(q) - E = \begin{cases} P_-^2 > 0 & for \;\; q < q_- \\ P_+^2 > 0 & for \;\; q > q_+ \end{cases}$$

then the ratio $w = \psi^D/\psi$ *is continuous on* $\hat{R}$

iff the Schrödinger equation admits an $L^2(R)$ *solution*

Potential Well:

$$V(q) = \begin{cases} 0 & |q| \leq L \\ V_0 & |q| > L \end{cases}$$

$$k = \frac{\sqrt{2mE}}{\hbar} \qquad K = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

$$|q| \leq L \qquad \Psi_1^1 = \cos kq \qquad \Psi_2^1 = \sin kq$$

$$q > L \qquad \Psi_1^2 = \mathrm{e}^{-Kq} \qquad \Psi_2^2 = \mathrm{e}^{Kq}$$

take $(1,1)$: $\Psi,\ \Psi'$ continuous $\Rightarrow$ $k\ \tan kL = K$

$$\Rightarrow \quad w = \frac{1}{[k \sin(2kL)]} \begin{cases} \cos(2kL) - \mathrm{e}^{-2K(q+L)} & q < -L \\ \sin(2kL)\tan(kq) & |q| \leq L \\ \mathrm{e}^{2K(q-L)} - \cos(2kL) & q > L \end{cases}$$

$$\lim_{q\to\pm} \frac{\psi^D}{\psi} = \pm\infty \qquad \Longrightarrow \qquad E_n(k\ \tan kL\ =\ K)$$ are admissible solutions

take $(1,2)$: $\quad \Psi,\ \Psi'$ continuous $\quad \Rightarrow \quad k\ \tan(kl)\ =\ -K$

$$\Rightarrow \quad w = \frac{1}{[k\sin(2kL)]} \begin{cases} \cos(2kL) - \mathrm{e}^{2K(q+L)} & q\ <\ -L \\ \sin(2kL)\tan(kq) & |q|\ \leq\ L \\ \mathrm{e}^{-2K(q-L)} - \cos(2kL) & q\ >\ L \end{cases}$$

$$\lim_{q\to\pm} \frac{\psi^D}{\psi}\ =\ \mp\frac{1}{k}\cot(2kL) \qquad \Longrightarrow \qquad w(-\infty)\ \neq\ w(+\infty)$$

($k^{-1}(\cot 2kL) = 0$ is not compatible with $k\tan(kL) = -K$)

$\Rightarrow \qquad E_n(k\ \tan kL\ =\ -K)$ are not admissible solutions

Time parameterisation

Bohmian mechanics : $p = \frac{\partial S}{\partial q} = m\dot{q} \quad \Rightarrow$ Trajectory representation

Jacobi time : $t = \frac{\partial S_0}{\partial E}.$

In classical mechanics: Jacobi time = Mechanical time

$$t - t_0 = m \int_{q_0}^{q} \frac{dx}{\partial_x S_0^{\rm cl}} = \int_{q_0}^{q} dx \frac{\partial}{\partial E} \partial_x S_0^{\rm cl} = \frac{\partial S_0^{\rm cl}}{\partial E}.$$

In Quantum HJ Theory: Jacobi time $\neq$ Mechanical time

$$t - t_0 = \frac{\partial \mathcal{S}_0^{\rm qm}}{\partial E} = \frac{\partial}{\partial E} \int_{q_0}^{q} dx \partial_x \mathcal{S}_0^{\rm qm} = \left(\frac{m}{2}\right) \int_{q_0}^{q} dx \frac{1 - \partial_E Q}{(E - V - Q)^{1/2}}$$

$$\Longrightarrow \quad m\frac{dq}{dt} = m\left(\frac{dt}{dq}\right)^{-1} = \frac{\partial_q S_0^{\rm qm}}{(1 - \partial_E \mathcal{V})} \neq \frac{\partial S^{\rm qm}}{\partial q}$$

Floyd: Use Jacobi theorem to define time $\longrightarrow$ trajectories

Compact space $\Leftrightarrow$ Energy quantisation

$\Rightarrow$ Floyd time is ill defined for the QHJT

No Trajectories in EPoQM

Quantum potential as a curvature term:

Using the property of the Schwarzian derivative

$$\{S_0; q\} = -\left(\frac{\partial S_0}{\partial q}\right)^2 \{q; S_0\},$$

We can rewrite the Quantum Stationary Hamilton Jacobi Equation as

$$\frac{1}{2m}\left(\frac{\partial S_0}{\partial \hat{q}}\right)^2 + V(\hat{q}) - E = 0,$$

where

$$\hat{q} = \int^q \frac{dx}{\sqrt{1-\frac{\hbar^2}{2}\{q; S_0\}}}.$$

Flanders: J. Diff.Geom. 1970, 575 $\longrightarrow$ $\{\ ;\ \}$ $\longrightarrow$ a curvature term

In higher dimensions $Q(q) \sim \frac{\Delta R(q)}{R}$ $\longrightarrow$ curvature of $R(q)$

Length Scale

For $W^0(q^0) = 0$)

$$\frac{\partial^2 \Psi}{\partial q^2} = 0 \quad \Rightarrow \qquad \psi_1 = q^0 \quad ; \quad \psi_2 = const$$

$\Rightarrow$ duality implies a length scale

$$\Longrightarrow \quad e^{\frac{2i}{\hbar} S_0^0} = e^{i\alpha \frac{q^0 + i\bar{\ell}_0}{q^0 - i\ell_0}},$$

$$p_0 = \frac{\partial S_0^0}{\partial q^0} = \pm \frac{\hbar(\ell_0 + \bar{\ell}_0)}{2|q^0 - i\ell_0|^2}.$$

$\text{Max}|p_0| = \frac{\hbar}{\text{Re}\ell_0} \quad \rightarrow \quad \text{Re}\ell_0 \neq 0 \rightarrow$ ultraviolet cutoff

$$\lim_{\hbar \to 0} p_0 = 0 \quad \Rightarrow \quad \mathrm{Re}\, \ell_0 = \lambda_p = \sqrt{\frac{\hbar G}{c^3}}.$$

$\ell_0 = \lambda_p \quad \longrightarrow$ choice consistent with the classical limit

$$Q^0 \;=\; \frac{\hbar^2}{4m}\{S_0^0, q^0\} \;=\; -\frac{\hbar^2(\mathrm{Re}\,\ell_0)^2}{2m}\frac{1}{|q^0 - i\ell_0|^4}.$$

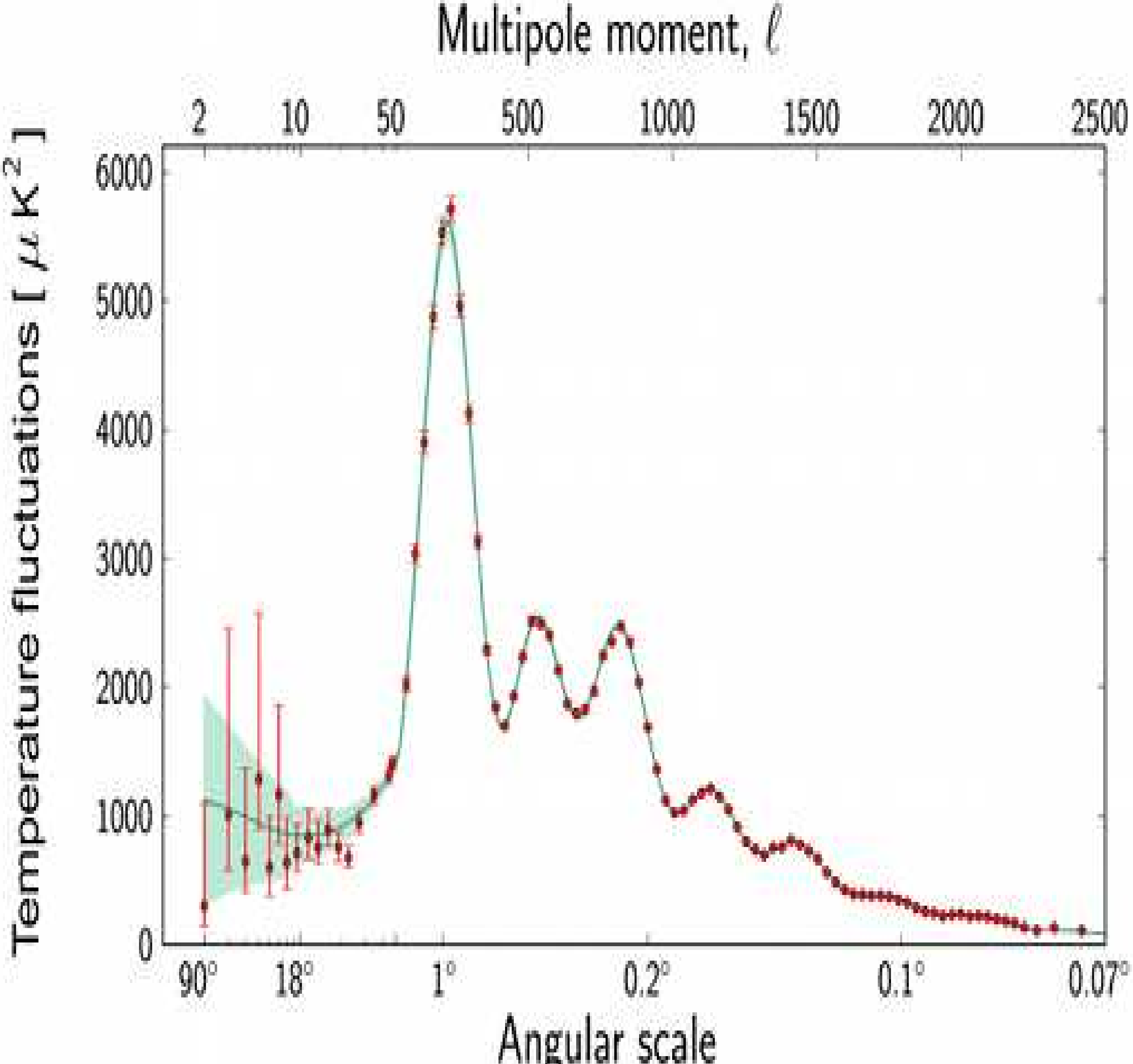
Consistency $\Longrightarrow$ $q^0 = \psi^D/\psi$ is continuous on $\hat{R} = R \cup \{\infty\}$

Taking $m \sim 100 GeV$;

$$\mathrm{Re}\,\ell_0 = \lambda_p \approx 10^{-35} m;$$

$$q^0 \sim 93 Ly,$$

$$\Longrightarrow \quad |Q| \sim 10^{-202} eV.$$


Multipole moment, ℓ
2
10
50
500
1000
1500
2000
2500
Temperature fluctuations [μK^2]
6000
5000
4000
3000
2000
1000
0
90°
18°
1°
0.2°
0.1°
0.07°
Angular scale

conclusions :

The equivalence postulate $\Longrightarrow$ $S_0 \neq const \Leftrightarrow \hbar \neq 0$

$$\Psi(q) = \frac{1}{\sqrt{S_0'}} \left(A e^{+\frac{i}{\hbar} S_0} + B e^{-\frac{i}{\hbar} S_0} \right)$$

$$\mathrm{Re}\,\ell_0 = \lambda_P \rightarrow \text{ fundamental length scale}$$

$Q(q)$ Intrinsic curvature terms of elementary particles $\neq 0$ Always

CoCyCle Condition: Invariant under Möbius transformations in

$$\hat{R}^D = R^D \cup \{\infty\} \rightarrow \text{ Compact Space}$$

Decompactification limit $\leftrightarrow$ $Q(q) \rightarrow 0$ $\leftrightarrow$ classical limit