Current correlators at four loops and heavy quark mass determinations

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Outline

Why precise quark masses?

Precise predictions require precise knowledge of the theory's parameters.

In QCD: strong coupling α_s and masses of the 6 quarks u, d, s, c, b, t

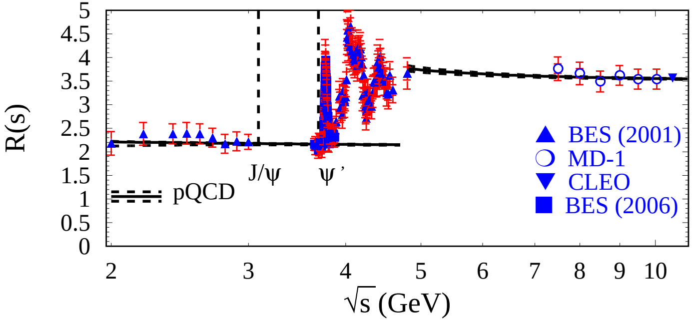
here: focus on m_c and m_b

Applications:

- Weak decay rates of heavy mesons, e. g. $\Gamma(B_{c/d} \to \ell \nu K) \propto m_b^5$
- Quarkonium spektroscopy
- Decay rates and branching ratios of light Higgs: $\Gamma(H \to b\bar{b}) \propto m_b^2$
- GUT predictions for m_t/m_b (Yukawa unification at M_{GUT})
- ...

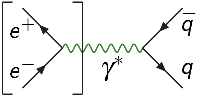
From hadron production to quark masses I

Information about quark masses is found in the threshold region of the cross section for hadron production from e^+e^-.



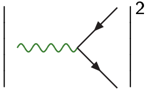
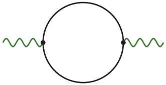
Resonance behaviour is not accessible by perturbative QCD
$\longrightarrow$ sum rules [Shifman, Vainshtein, Zakharov]

From hadron production to quark masses II

Process: 

$+$ higher order corrections

Optical theorem: total production cross section $R(s)$
$\propto$ imaginary part of forward scattering amplitude
(Vacuum Polarization $\Pi(q^2)$)


$\sim$ $R(s) = 12\pi \,\mathrm{Im}\big(\Pi(q^2 = s + i\varepsilon)\big) \sim \mathrm{Im}$ 

Vacuum Polarization: Correlator of electromagnetic currents $j_\mu = \bar{\psi}\gamma_\mu \psi$

$$(q_\mu q_\nu - g_{\mu\nu} q^2)\Pi(q^2) = i \int dx\, e^{iqx} \langle 0 | T j_\mu(x) j_\nu(0) | 0 \rangle$$

R-**Ratio:** $R(s) = \dfrac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)_{LO}}$

From hadron production to quark masses III

The connection between $R(s)$ and $\Pi(q^2)$ leads to a

Dispersion Relation: $$\Pi(q^2) - \Pi(0) = \frac{q^2}{12\pi^2} \int ds \, \frac{R(s)}{s(s-q^2)}$$

Expansion for small q^2 on both sides:

$$\mathcal{M}_n^{th} = \frac{12\pi^2}{n!} \left(\frac{\partial}{\partial q^2}\right)^n \Pi(q^2)\Big|_{q^2=0} = \frac{9}{4} Q_q^2 (4m^2)^{-n} C_n$$

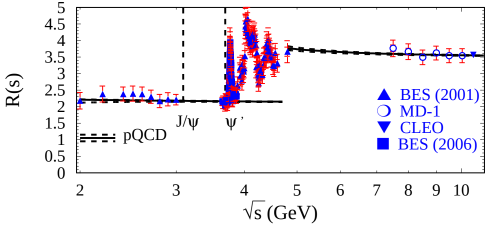
$$\mathcal{M}_n^{exp} = \int_0^\infty ds \, \frac{R(s)}{s^{n+1}}$$

$$\mathcal{M}_n^{th} = \mathcal{M}_n^{exp}$$

$$\boxed{\Rightarrow m = \frac{1}{2} \left(\frac{9 Q_q^2 C_n}{4 \mathcal{M}_n^{exp}}\right)^{\frac{1}{2n}}}$$

Features of the sum rule approach

Advantages and disadvantages of C_n for different n?



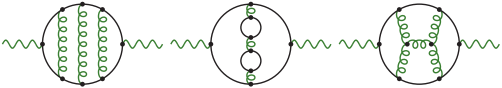
Long distance effects average out for small n

larger n

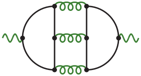
- suppresses continuum region (no data; no m dependence)
- reduces influence of experimental error
- growing long distance effects
- increasingly difficult to calculate

What do we have to calculate?

At four loops 700 **Feynman integrals contribute**



Singlet contributions come into play:



Perform Taylor expansion in the external momentum q^2

reduces the number of scales:

propagator-type integrals $(q^2, m^2) \rightarrow$ tadpoles (m^2)

(light quarks are treated as massless)

Express integrals in terms of master integrals

Status of current correlators in the low energy limit

At three loops, i. e. $\mathcal{O}(\alpha_s^2)$

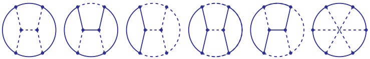
- first 8 moments [Chetyrkin, Kühn, Steinhauser (1996)]
- moments up to $n = 30$ for
  - vector current [Boughezal, Czakon, Schutzmeier (2006)]
  - all diagonal currents (including singlet) [Maier, PM, Marquard, (2007)]

At four loops, i. e. $\mathcal{O}(\alpha_s^3)$

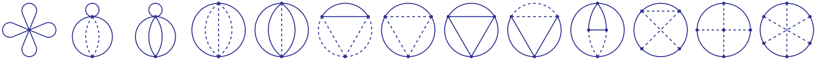
- first physical moment of the vector current (non-singlet) [Chetyrkin, Kühn, Sturm; Boughezal, Czakon, Schutzmeier (2006)]
- all orders result for n_l^{z-1} at $\mathcal{O}(\alpha_s^z)$ [Grozin, Sturm (2005)]
- moments up to $n = 30$ for n_f^2 [Czakon, Schutzmeier (2007)]
- second (including singlet) and third moments for all diagonal currents [Maier, PM, Marquard (2008/2009)]

Integration-by-Parts

Tadpole integrals are mapped to six topologies



All appearing integrals can be expressed as linear combination of 13 master integrals with rational coefficients in the space-time dimension d.



[Schröder, Vuorinen (2005)]

Basic tool: Integration-by-Parts identities (IBP) [Chetyrkin, Tkachov (1981)]

$$0 = \int d^d k_1 \dots d^d k_\ell \; \frac{\partial}{\partial k_i^\mu} \; \frac{\{k_j^\mu, p_j^\mu\}}{D_1^{a_1} D_2^{a_2} \dots D_n^{a_n}}$$

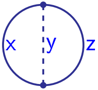
give relations between integrals with different propagator powers.

Usage:
- generate and solve a system of equations → Laporta algorithm
- construct recursion relations → Gröbner bases

A Simple two-loop example

Consider the two-loop tadpole with two massive and one massless line:

$$J(x,y,z) =$$



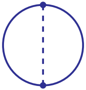
$$= \int dk_1\, dk_2 \frac{1}{(k_1^2+m^2)^x (k_2^2)^y ((k_1+k_2)^2+m^2)^z}$$

$$\Rightarrow 0 = \int dk_1\, dk_2 \frac{\partial}{\partial k_2^\mu} k_2^\mu I(x,y,z)$$

$$= (d-2y-z) J(x,y,z) + z\, J(x-1,y,z+1) + z\, J(x,y-1,z+1)$$

For $x = y = z = 1$: $\qquad J(1,1,1) = \dfrac{1}{d-3} J(1,0,2)$

Diagrammatically:



$$= \frac{1}{d-3}$$

Laporta algorithm

"Standard" method to solve IBPs: Laporta algorithm [Laporta (2000)]

- define ordering of integrals
- generate IBPs for all permutations of propagator powers and scalar products up to the required sums of powers
- solve systematically for the most difficult integrals by a Gauss elimination-like algorithm

Generally very powerful, but:

- system of equations is overdetermined by $\mathcal{O}(3-5)$
- complicated intermediate expressions
- expensive simplifications needed at each step
- bad combinatorics for large propagator powers
- most of the solved integrals are not needed in the calculation

Limitations of the Laporta algorithm

In our case: each q^2 derivative adds two propagator powers and one irreducible scalar product to the integrands.

For C_3:

- $4.5 \cdot 10^6$ integrals needed
- up to 12 additional propagator powers (“dots”)
- up to 8 irreducible scalar products
- distributed on 10 indices ($\sim 10^{13}$ permutations)

$\Rightarrow$ naïve approach fails because of combinatorics

In principle the problems can be avoided by using recursion relations.

- calculate exactly what is needed
- size of expressions is limited by the number of master integrals (“backwards substitution”)

Systematic approach: Gröbner bases

Gröbner bases

Consider multivariate polynomials and conditions of the form

$$b_1 = 0, \quad \ldots, \quad b_n = 0$$

How to find out, if a polynomial p is zero, i. e. a representation

$$p = f_1 b_1 + \cdots + f_n b_n$$

with polynomial coefficients f_i exists?

$\rightarrow$ Divide out the elements of the basis

If we have a Gröbner basis, the remainder is unique.

Gröbner bases can be constructed by the Buchberger algorithm.

- Guaranteed to terminate after a finite number of steps, but not on real computers.

Gröbner bases for Feynman integrals

For Feynman integrals:

- consider algebra of shift and multiplication operators
- IBP identities are polynomials in these operators
- construct a Gröbner basis from the IBP identities
- take an integral and divide out the elements of the basis

$\rightarrow$ remainder is the desired reduction to master integrals.

Example: $J(a,b) = (S_1^+)^{a-1}(S_2^+)^{b-1}J(1,1)$

$$(S_1^+)^{a-1}(S_2^+)^{b-1} = f_1 b_1 + f_2 b_2 + R$$

$$\Rightarrow J(a,b) = (f_1 b_1 + f_2 b_2 + R)J(1,1) = R\,J(1,1)$$

S-Bases

Problem: Buchberger algorithm is too slow even for simple problems

Better: S-bases (modified Buchberger algorithm)
[Smirnov & Smirnov]

- much faster, but not guaranteed to stop
- strongly dependent on ordering of the integrals
- sometimes doesn't find a basis even after many tries
- public implementation in `Mathematica` available:
  `FIRE` [A. V. Smirnov]

Unfortunately, there is no solution to this problem in sight.

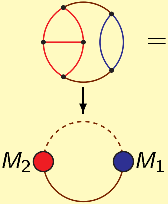
$\longrightarrow$ Has to be combined with other methods.
`FIRE` has a Laporta part for that reason;
a `C++` implementation is in development.

Self energy reduction

S-Bases tend to fail for integrals with self energy insertions.
These integrals have the highest propagator powers (*here:* $+2$).

$\rightarrow$ most difficult for Laporta algorithm.

Idea: Apply tensor reduction to self energy subgraphs to remove scalar products and reduce the self energies to master integrals.



$=$ 2-loop SE $\times$ 1-loop SE $\times$ connecting propagator
$\times$ scalar products between self energies

Treat self energy master integrals als objects,
which depend only on their external momentum.
$\rightarrow$ effective one-loop integral!

Very efficient, but limited to special topologies.

Combination of reduction techniques

The reduction techniques have complementary strengths.

Strategy:

- use self energy formalism where it is applicable
  $\longrightarrow$ two dots less for Laporta
- keep the system for Laporta algorithm as small as possible:
  don't generate all equations, but adapt to the needed integrals
- use Gröbner bases in combination with self energy formalism
  if something is missing (and for checks)

A sophisticated combination of reduction techniques can handle problems which are difficult for Laporta alone.

Results

First three moments of the vacuum polarization up to $\mathcal{O}(\alpha_s^3)$ (with n_l light quark flavours)

$$
\begin{aligned}
C_1^v =& 1.06666 + 2.55473\left(\frac{\alpha_s}{\pi}\right) + (0.50988 + 0.66227\, n_l)\left(\frac{\alpha_s}{\pi}\right)^2 \\
&+ (1.87882 - 2.79472\, n_l + 0.09610\, n_l^2)\left(\frac{\alpha_s}{\pi}\right)^3, \\
C_2^v =& 0.45714 + 1.10955\left(\frac{\alpha_s}{\pi}\right) + (1.41227 + 0.45491\, n_l)\left(\frac{\alpha_s}{\pi}\right)^2 \\
&+ (-6.23488 + 0.96156\, n_l - 0.01594\, n_l^2)\left(\frac{\alpha_s}{\pi}\right)^3, \\
C_3^v =& 0.27089 + 0.51939\left(\frac{\alpha_s}{\pi}\right) + (0.35222 + 0.42886\, n_l)\left(\frac{\alpha_s}{\pi}\right)^2 \\
&+ (-8.30971 + 1.94219\, n_l - 0.03959\, n_l^2)\left(\frac{\alpha_s}{\pi}\right)^3
\end{aligned}
$$

+ correlators of the scalar $j^s = \bar{\psi}\psi$, pseudoscalar $j^p = \bar{\psi}\gamma_5\psi$ and axial vector $j^a_\mu = \bar{\psi}\gamma_\mu\gamma_5\psi$ currents

Third moment of the vacuum polarization

$$C_n = C_n^{(0)} + \left(\frac{\alpha_s}{\pi}\right) C_n^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 C_n^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 C_n^{(3)} + \ldots$$

$$C_n^{(3)} = C_F T_F^2 n_\ell^2 C_{\ell\ell,n}^{(3)} + C_F T_F^2 n_h^2 C_{hh,n}^{(3)} + C_F T_F^2 n_\ell n_h C_{\ell h,n}^{(3)}$$
$$+ C_F T_F n_\ell \left(C_A C_{\ell NA,n}^{(3)} + C_F C_{\ell A,n}^{(3)}\right) + C_{n_f^0,n}^{(3)} + C_F T_F n_h \left(C_A C_{hNA,n}^{(3)} + C_F C_{hA,n}^{(3)}\right)$$

$$C_{ll,3}^{(3),v} = +\frac{31556642272}{49228003125} - \frac{256}{405}\zeta_3,$$

$$C_{hh,3}^{(3),v} = +\frac{56877138427}{12609717120} - \frac{6184964549}{1556755200}\zeta_3,$$

$$C_{lh,3}^{(3),v} = +\frac{60361465477}{29393280000} - \frac{1765}{31104}c_4 + \frac{86485}{41472}\zeta_4 - \frac{57669161}{17418240}\zeta_3,$$

$$C_{lNA,3}^{(3),v} = -\frac{1475149211788337}{6452412825600000} - \frac{8529817}{77414400}c_4 + \frac{1510937903}{14745600}\zeta_4 - \frac{561258009401}{6193152000}\zeta_3,$$

$$C_{lA,3}^{(3),v} = +\frac{983812946922223}{4389396480000} + \frac{8529817}{38707200}c_4 + \frac{21972351293}{17203200}\zeta_4 - \frac{28995540810097}{21676032000}\zeta_3,$$

$$C_{hNA,3}^{(3),v} = -\frac{4548804584190836 29}{5854170457175040000} - \frac{7110196837}{1117670400}c_4 + \frac{1068488091383}{7451136000}\zeta_4$$
$$+\frac{4448}{315}\zeta_5 - \frac{438757401754772226 11}{433642256087040000}\zeta_3,$$

$$C_{hA,3}^{(3),v} = -\frac{2327115263308753}{2489610816000} - \frac{16870125343}{39916800}c_4 + \frac{286864384271}{26611200}\zeta_4 - \frac{377837317054807}{61471872000}\zeta_3,$$

$$c_4 = 24\,a_4 + \log^4 2 - 6\zeta_2 \log^2 2; \qquad a_n = \mathrm{Li}_n(1/2)$$

Padé approximations I

What if we want more moments?
Direct calculation not reasonable at the moment.

Full q^2 dependence of correlators?
Needed e. g. for contour improved perturbation theory.

Padé approximations: [Broadhurst, Fleischer, Tarasov '93; Baikov, Broadhurst '95; Chetyrkin, Kühn, Steinhauser '96; Hoang, Mateu, Zerbarjad '08; Masjuan, Peris '08]

use $$p_{m,n}(x) = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m}{1 + b_1 x + b_2 x^2 + \cdots + b_n x^n}$$

to approximate $\Pi(x = \frac{q^2}{4m^2})$. Fix a_1, b_j by $p^{(k)}_{m,n}(x_0) = \Pi^{(k)}(x_0)$.

Input:

- low energy expansion
- threshold expansion
- high energy expansion

Padé approximations II

Problem 1: Padé approximations cannot predict logarithms $\log(\frac{q^2}{4m^2})$

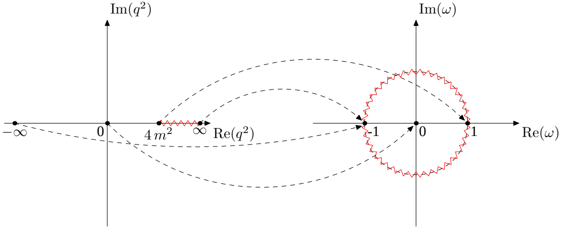
$\longrightarrow$ subtract an appropriate function

$$\Pi(q^2) = \Pi_{reg}(q^2) + \Pi_{log}(q^2)$$

Problem 2: Branch cut above threshold

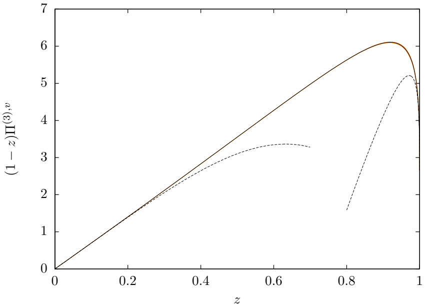
$\longrightarrow$ conformal mapping to the unit circle

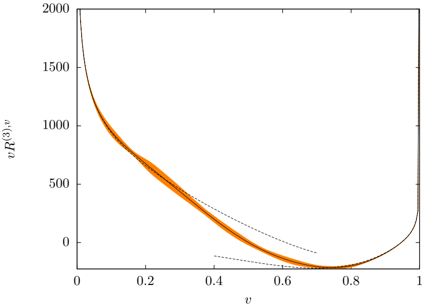
$$\frac{q^2}{4m^2} = \frac{4\omega}{(1+\omega)^2}$$



Padé approximations: Results

Padé approximations below and above threshold

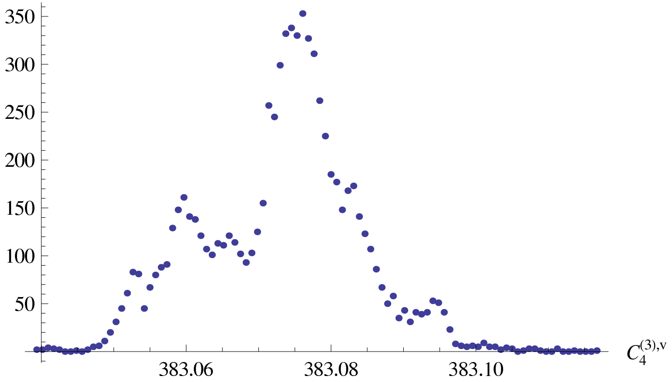




The plots show the low energy and threshold rsp. the threshold and high energy expansion and the Padé approximations including a 3σ error band.

Padé approximations: Error estimation

Exploit the freedom of choosing numerator / denominator degree of the approximation and the freedom in the choice of the subtraction function to estimate errors.



Padé approximations: Prediction of higher moments

	$n_l = 3$	$n_l = 4$	$n_l = 5$
$C_1^{(3),v}$	366.1748	308.0188	252.8399
$C_2^{(3),v}$	381.5091	330.5835	282.0129
$C_3^{(3),v}$	385.2331	338.7065	294.2224
$C_4^{(3),v}$	383.073(11)	339.913(10)	298.576(9)
$C_5^{(3),v}$	378.688(32)	338.233(32)	299.433(27)
$C_6^{(3),v}$	373.536(61)	335.320(63)	298.622(54)
$C_7^{(3),v}$	368.23(9)	331.90(10)	296.99(9)
$C_8^{(3),v}$	363.03(13)	328.33(14)	294.94(12)
$C_9^{(3),v}$	358.06(17)	324.78(18)	292.72(16)
$C_{10}^{(3),v}$	353.35(20)	321.31(22)	290.44(19)
$K_0^{(3),v}$	17(11)	17(29)	16(10)
$D_2^{(3),v}$	2.0(42)	1.2(83)	1.4(21)

Results for m_c

n	exp	α_s	μ	np	total	$m_c(3\,\mathrm{GeV})$
1	0.009	0.009	0.002	0.001	0.013	0.986
2	0.006	0.014	0.005	0.000	0.016	0.976
3	0.005	0.015	0.007	0.002	0.017	0.978
4	0.003	0.009	0.031	0.007	0.033	1.004

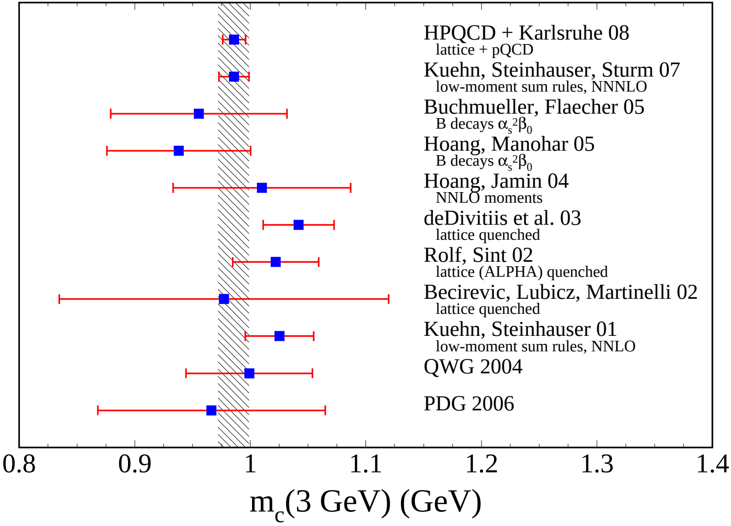
Error sources:

- experimental (exp)
- uncertainty of α_s
- renormalisation scale dependence (μ)
- non-perturbative effects (np);

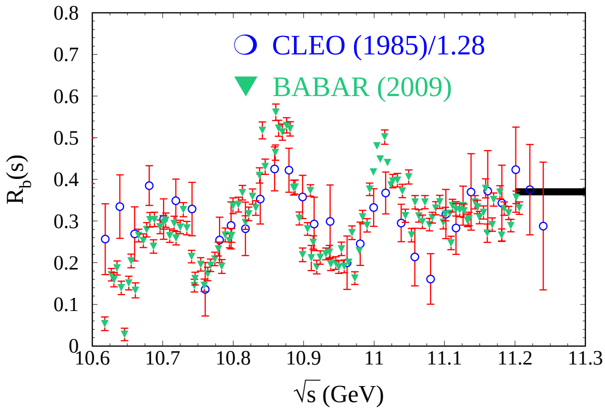
  Operator product expansion:

$$\Pi(q^2) = \Pi_{perturbative}(q^2) + C_{\psi\bar{\psi}}\langle\psi\bar{\psi}\rangle + C_{G^2}\langle G^2\rangle + \dots$$

Results for m_c: Comparison



Bottom threshold



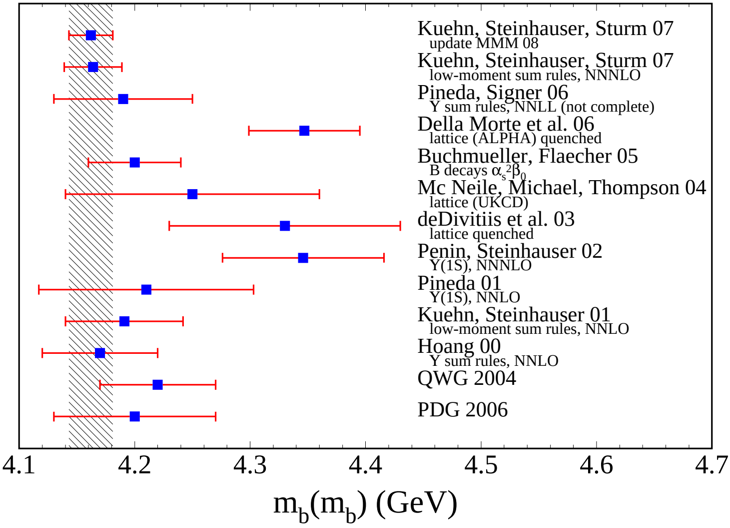
Barbar data from 2009 significantly reduced the error compared to m_b determinations from CLEO data.

Results for m_b

n	exp	α_s	μ	total	$m_b(10\,\text{GeV})$	$m_b(m_b)$
1	0.014	0.007	0.002	0.016	3.597	4.151
2	0.010	0.012	0.003	0.016	3.610	4.163
3	0.008	0.014	0.006	0.018	3.619	4.172
4	0.006	0.015	0.020	0.026	3.631	4.183

Note the discussion about m_b from sum rules vs. m_b from semi-leptonic B decays.

Results for m_b: Comparison



α_s and m_c from lattice QCD

Instead of experimental data for $R(s)$ one can use lattice QCD simulations to determine α_s and m_c.

Pseudoscalar current correlator is best suited. [HPQCD & Karlsruhe] (really?)

Experimental input: m_π^2, $2m_K^2 - m_\pi^2$, m_{η_c}, m_Υ

$$\boxed{\alpha_s(M_Z) = 0.1174(12)}$$

nice agreement with other determinations

$$\boxed{m_c(3\,\text{GeV}) = 0.986(10)\,\text{GeV}}$$

perfect agreement with determinations from $R(s)$

Correlators of non-diagonal currents are of interest for future applications.

Conclusion

- Current correlators in combination with $R(s)$ or lattice data allow for precise determinations of quark masses
- A combination of reduction techniques for Feynman integrals led to the calculation of the second and third moment of current correlators at four loops
- Full energy dependence can be reconstructed by Padé approximations
- Results led to the most precise values for the charm and bottom quark masses up to date
- Don't rely only on C_1,
  higher moments are crucial for consistency checks