Holography for non-relativistic CFTs

Herzog, Rangamani & SFR, 0807.1099,
Rangamani, Son, Thompson & SFR, 0811.2049,
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Outline

- Review holography for relativistic theories
- Application to condensed matter physics
- Holography for non-relativistic theories: Lifshitz example
- Action principle
- Stress tensor
- Schrödinger example
- Action & stress tensor for Schrödinger

Holographic description of CFTs

- Identify $SO(d,2)$ conformal symmetry with isometries of AdS_{d+1}. In Poincare coordinates,

$$ds^2 = r^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{dr^2}{r^2},$$

  Dilatation is $D : x^\mu \to \lambda x^\mu, r \to \lambda^{-1} r$.
- $\langle e^{\int \phi_0 \mathcal{O}} \rangle = Z_{string}[\phi_0] \approx e^{-S[\phi_0]}$.
- Classical gravity valid at strong 't Hooft coupling.
- Many explicit examples: $\mathrm{AdS}_5 \times X^5$.
- Focus on universal subsector: pure gravity in bulk. Boundary stress tensor $\langle T_{\mu\nu} \rangle = \frac{1}{\sqrt{\hat{h}}} \frac{\delta \mathcal{S}}{\delta \hat{h}^{\mu\nu}}$ Henningson Skenderis Balasubramanian Kraus
- Finite temperature described by bulk black hole solution.

Application to condensed matter physics

Application to QCD long studied. Why condensed matter?

- Rich system:
  - CFTs arise as IR desc near critical points; often strongly coupled.
  - Many different theories; can tune Hamiltonian in some settings.
- AdS/CFT provides a useful new perspective:
  - Few other methods for calculation at strong coupling.
  - Gravity dual calculates observables like transport coefficients directly, no quasiparticle picture.
- Prompts new questions:
  - Charge transport, phase transitions
  - Can have theories with an anisotropic scaling symmetry
    D: $x^i \to \lambda x^i, t \to \lambda^z t$.

Application to condensed matter physics

However, this is a big extension of AdS/CFT.

- Known examples involve large N gauge theories. To apply these ideas to condensed matter, need to assume other field theories will have holographic duals.
- Proceed phenomenologically: construct bulk theories with qualitative features of interest.
- Big question: for what types of theories can we construct a bulk dual in the classical approximation?
  - Strong coupling
  - $N_{dof} \gg 1$
  - Hierarchy in spectrum: $\Delta_{s>2} \gg \Delta_{s\leq 2}$.

Non-relativistic holography

In condensed matter, have theories with an anisotropic scaling symmetry
D: $x^i \to \lambda x^i, t \to \lambda^z t$.
Two cases of interest:

- Lifshitz-like theories: D, H, $\vec{P}$, M_{ij}
- Schrödinger symmetry: add Galilean boosts $\vec{K}$. $z = 2$ special.

Can these also have a dual geometrical description?

Non-relativistic holography

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Lifshitz:

- Simple deformation of AdS:

Kachru
Liu
Mulligan

$$ds^2 = -r^{2z} dt^2 + r^2 d\mathbf{x}^2 + \frac{dr^2}{r^2}.$$

Lifshitz geometry

- Solution of a theory with a massive vector:

$$S = \int d^4x \sqrt{-g}\left(R - 2\Lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu\right),$$

  with $\Lambda = -\frac{1}{2}(z^2 + z + 4)$, $m^2 = 2z$. Lifshitz solution has

$$A = \alpha r^z dt, \quad \alpha^2 = \frac{2(z-1)}{z}.$$

- Finite temperature black hole solutions obtained numerically.
  Danielsson Thorlacius   Mann   Bertoldi Burrington Peet
- Analytic black holes in a theory with an extra scalar field. Balasubramanian McGreevy
- Embedded in string theory yesterday! Hartnoll Polchinski Silverstein Tong

Asymptotically Lifshitz spacetimes

Need to specify boundary conditions for metric, A_μ.

- Different metric components scale differently with r.
  $\Rightarrow$ No conformal boundary.
- Use frame fields. For Lifshitz geometry, natural basis is

$$e^{(0)} = r^z dt, \quad e^{(i)} = r dx^i, \quad e^{(3)} = \frac{dr}{r},$$

- Write gauge field as

$$A = \alpha e^{(0)}.$$

Asymptotically Lifshitz spacetimes

At large r, assume spacetime is Lifshitz + linearized corrections. By choice of frame, write

$$e^{(0)} = r^z \hat{e}^{(0)} = r^z[(1 + \frac{1}{2}\hat{h}_{tt})dt + v_{1i}dx^i],$$

$$e^{(i)} = r\hat{e}^{(i)} = r[v_{2i}dt + (\delta^i{}_j + \frac{1}{2}\hat{h}^i{}_j)dx^j], \quad e^{(3)} = \frac{dr}{r},$$

$$A^M = \alpha(1 + \hat{a}_t)\delta_0^M + \alpha\hat{a}_r\delta_3^M.$$

Boundary conditions $\hat{h}_{tt}, v_{1i}, v_{2i}, \hat{h}_{ij}, \hat{a}_t \to 0$ as $r \to \infty$.
For $1 \leq z < 2$, also impose $r^{\frac{1}{2}(z+2-\beta_z)}\hat{a}_t \to 0$ as $r \to \infty$, where $\beta_z^2 = (z+2)^2 + 8(z-1)(z-2)$.

Linearized solutions

Bertoldi
Burrington
Peet

SFR
Saremi

Constant modes

$$\hat{a}_t, \hat{h}_{tt}, \hat{h}^i_i \sim \frac{a_1}{r^{z+2}}, \ \frac{a_2}{r^{\frac{1}{2}(z+2+\beta_z)}}$$

$$v_{1i} = \frac{c_{2i}}{r^{z+2}} + \frac{c_{3i}}{r^{3z}}, \quad v_{2i} = \frac{(z^2-4)}{z(z-4)} c_{2i} r^{z-4} + \frac{3z}{(z+2)} \frac{c_{3i}}{r^{z+2}},$$

$$\hat{h}^T_{ij} = \frac{t_{ij}}{r^{z+2}}.$$

(Non-constant modes a little more complicated.)

Expect a normalizable mode for each field, giving expectation value of corresponding operator.

- Scalars ok; scale inv implies one relation, expect 2 modes.
- For vectors, c_{2i} violates boundary condition on v_{2i} for $z > 4$. Fewer normalizable modes than expected??

Action

AdS/CFT dictionary is based on an action: $\langle e^{\int \phi_0 \mathcal{O}} \rangle \approx e^{-S_{on-shell}[\phi_0]}$.
Need an action S such that for asymptotically Lifshitz spacetimes

- S finite on-shell.
- $\delta S = 0$ for arbitrary variations preserving the boundary conditions.

Naive bulk action divergent. Regularize by adding boundary terms:

$$S_{ct} = bulk + \frac{1}{16\pi G_4} \int d^3\xi \sqrt{-h}(2K - 4 - \sqrt{2z(z-1)}\sqrt{-A_\alpha A^\alpha})$$

▷ Need derivative terms for asymptotically Lifshitz spacetimes: consider

$$S_{deriv} = \int d^3\xi \sqrt{-h}[\sigma_1 R^h + \sigma_2 \nabla_\alpha A_\beta \nabla^\alpha A^\beta + \sigma_3 (\Box A_\alpha)(\Box A^\alpha)].$$

⋆ $S = S_{ct} + S_{deriv}$ satisfies $\delta S = 0$ for variations satisfying our boundary conditions, $S_{on-shell}$ finite.

Stress tensor

In relativistic case, stress tensor defined by $\delta S = \int d^3\xi \, T_{\alpha\beta}\delta\hat{h}^{\alpha\beta}$.
Two choices for vector field:

- Holding A_α fixed or holding A_A fixed.
- Latter gives conserved charges gen bdy diffeos.

Hollands
Ishibashi
Marolf

- In non-relativistic case, it is latter that gives finite results.

Non-relativistic theory: energy and momentum not related by a coordinate transformation. Stress-energy complex

- Energy density $\mathcal{E}$, energy flux $\mathcal{E}_i$
- Momentum density $\mathcal{P}_i$, stress tensor Π_{ij}.

Conservation equations $\partial_t\mathcal{E} + \partial_i\mathcal{E}^i = 0$, $\partial_t\mathcal{P}_j + \partial_i{\Pi^i}_j = 0$.
Define by

$$\delta S = \int [-\mathcal{E}\delta\hat{e}_t^{(0)} - \mathcal{E}^i\delta\hat{e}_i^{(0)} + \mathcal{P}_i\delta\hat{e}_t^{(i)} + \Pi_i^j\delta\hat{e}_j^{(i)} + s_A\delta A^A].$$

- Results finite: $\mathcal{E} \propto a_1$, $\mathcal{E}_i \propto c_{3i}$, $\mathcal{P}_i \propto c_{2i}$, $\Pi_{ij} \propto a_1\delta_{ij} + t_{ij}$.
- Bulk equations of motion imply conservation equations, $z\mathcal{E} = \Pi_i^i$.

Schrödinger symmetry

Galilean symmetry: rotations M_{ij}, translations P_i, boosts K_i, $(i = 1, \dots, d)$,
Hamiltonian H, particle number N.
Extended by the dilatation D,

$$[D, P_i] = iP_i, [D, H] = ziH, [D, K_i] = (1 - z)iK_i, [D, N] = i(2 - z)N.$$

Dynamical exponent z determines scaling of H under dilatations.

For $z = 2$, N is central, and there is a special conformal generator C: $[D, C] = -2iC, [H, C] = -iD$.
Schrödinger algebra: Symmetry of free Schrödinger equation.

$\star$ Isometries of a gravitational dual?

Geometrical dual

Son　Balasubramanian McGreevy

Embed Galilean symmetry in $ISO(d+1,1)$ by light-cone quant: $H = \tilde{P}_+$, $N = \tilde{P}_-$, $K_i = \tilde{M}_{-i}$. Extend to embed $Sch(d)$ in $SO(d+2,2)$ by

$$D = \tilde{D} + (z-1)\tilde{M}_{+-}.$$

$\star$ $Sch(d)$ is a subgroup of $SO(d+2,2)$

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Balasubramanian
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Gravitational dual: deform AdS_{d+3} to

$$ds^2 = -r^4(dx^+)^2 + r^2(-2dx^+dx^- + d\mathbf{x}^2) + \frac{dr^2}{r^2}.$$

- Solution of a theory with a massive vector, $A_+ = r^2$.
- N discrete implies x^- periodic. Compact null direction?

Dual in string theory

Herzog
Rangamani
SFR

Adams
Balasubramanian
McGreevy

Maldacena
Martelli
Tachikawa

Take $\mathrm{AdS}_5 \times S^5$,

$$ds^2 = r^2(-2dx^+dx^- + d\mathbf{x}^2) + \frac{dr^2}{r^2} + (d\psi + P)^2 + d\Sigma_4^2,$$

and apply a TsT transformation:

Maldacena
Martelli
Tachikawa

- T-dualize the Hopf fiber coordinate ψ to $\tilde{\psi}$,
- Shift $x^- \to \tilde{x}^- = x^- + \tilde{\psi}$,
- T-dualise $\tilde{\psi}$ to ψ at fixed $\tilde{x}^-$.

Resulting solution is

$$ds^2 = -r^4(dx^+)^2 + r^2(-2dx^+dx^- + d\mathbf{x}^2) + \frac{dr^2}{r^2} + (d\psi + P)^2 + d\Sigma_4^2,$$

$$B = r^2 dx^+ \wedge (d\psi + P).$$

Heating up Schrödinger

Apply the TsT transformation to the Schwarzschild-AdS solution,

$$ds^2 = r^2(-f(r)dt^2 + dy^2 + d\mathbf{x}^2) + \frac{dr^2}{r^2 f(r)} + ds^2_{S^5},$$

where $f(r) = 1 - r_+^4/r^4$.
Resulting solution in 5d is

$$\begin{aligned} ds^2 &= r^2 k(r)^{-\frac{2}{3}} \left(\left[\frac{r_+^4}{4\beta^2 r^4} - r^2 f(r) \right] (dx^+)^2 \right. \\ &\left. + \frac{\beta^2 r_+^4}{r^4}(dx^-)^2 - [1 + f(r)] dx^+ dx^- \right) + k(r)^{\frac{1}{3}} \left(r^2 d\mathbf{x}^2 + \frac{dr^2}{r^2 f(r)} \right). \end{aligned}$$

$$A = \frac{r^2}{k(r)} \left(\frac{1 + f(r)}{2} dx^+ - \frac{\beta^2 r_+^4}{r^4} dx^- \right),$$

$$e^{\phi} = \frac{1}{\sqrt{k(r)}}; \quad k(r) = 1 + \frac{\beta^2 r_+^4}{r^2}.$$

▷ β param. choice of x^- coord in TsT; define $\gamma^2 \equiv \beta^2 r_+^4$.

Action principle

Herzog
Rangamani
SFR

SFR
Saremi

Similarly construct action by adding boundary counterterms:

$$\mathcal{S} = bulk + \int d^4\xi \sqrt{-h}(2K - 6 + (1 + c\phi)A^\mu A_\mu + (2c+3)\phi^2)$$

Finite, satisfies $\delta\mathcal{S} = 0$, but boundary conditions for variations more restrictive.

Stress tensor — proceed as in Lifshitz case: vary metric at fixed A^A, ϕ.

Differences:

- Conserved particle number. Particle number density ρ, flux ρ_i.
- No natural choice of frame.

Write

$$\delta S = \int d^4\xi (s_{\alpha\beta}\delta h^{\alpha\beta} + s_\alpha \delta A^\alpha),$$

Define $\mathcal{E} = 2s^+{}_+ - s^+ A_+, \quad \mathcal{E}^i = 2s^i{}_+ - s^i A_+$, etc.

- For black hole solution, $\mathcal{E} = r_+^4$, $\Pi_{xx} = \Pi_{yy} = r_+^4$, $\rho = 2\beta^2 r_+^4$.
- For solutions obtained by TsT from vacuum AdS solution, agrees with AdS stress tensor.

Discussion

- NRCFT is an interesting and challenging extension of AdS/CFT.
- Action and stress tensor constructed: first steps in holographic dictionary.

Open questions:

- Characterising field theories that have a weakly curved gravity dual.
- One-point function for massive vector.
- Role of extra x^- direction in Schrödinger geometry.
- Condensed matter applications: relation to strange metals in paper yesterday.