Very high order perturbation theory: Lattice Wilson loops

Paul Rakow

University of Liverpool

or

NNNNNNNNNNNNNNNNNNNLO
QCD

Introduction

Talk about results from 20 loop perturbation theory.
Progress: More loops, More quantities measured.

“Wilson loops in very high order lattice perturbation theory”
E.-M. Ilgenfritz, Y. Nakamura, H. Perlt, P.E.L. Rakow, G. Schierholz, A. Schiller
PoS(LAT2009)236
arXiv:0910.2795v1 [hep-lat]

Any sign that perturbation theory diverges?

Introduction

- Introduction
- Divergence of Perturbation theory
- Stochastic Perturbation Theory
- Plaquette
- Wilson Loops
- Perturbation theory meets results
- Conclusions

Divergence of Perturbation Theory

In perturbation theory we calculate physical quantities as series in a coupling constant.
Very succesful for QED, ($\alpha_{EM} = 1/137$).
More difficult for QCD, ($\alpha_{QCD} \sim 1/10$), most useful for high energy processes (asymptotic freedom). (One motive for lattice QCD.)
Expectation is that perturbation series diverge. (Nearest singularity is at $g^2 = 0$.)

Divergence of Perturbation Theory

$SU(2)$ in two dimensions.
Gauge theories in two dimensions are simple, can often be solved exactly.
The “plaquette” , $\langle F_{\mu\nu}F^{\mu\nu}\rangle$, for $SU(2)$ is

$$P = \frac{I_2(4/(ga)^2)}{I_1(4/(ga)^2)}$$

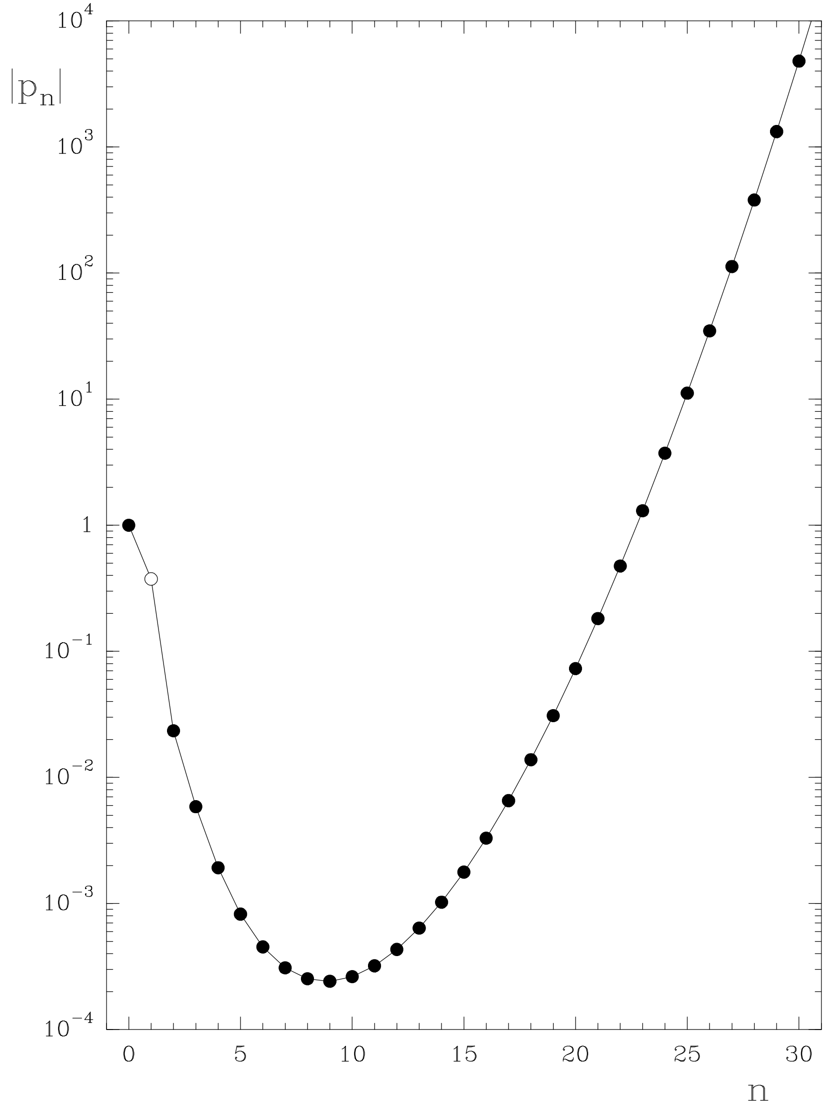
We know how the Bessel functions I_ν behave at large argument, so we can expand this expression to get a series for the plaquette when $(ga)^2$ is small.

Divergence of Perturbation Theory

$$P = 1 - \frac{3}{8}(ga)^2 + \frac{3}{128}(ga)^4 + \frac{3}{512}(ga)^6 + \cdots$$
$$\equiv 1 + \sum_n p_n (ga)^{2n}$$

Series looks harmless:

Divergence of Perturbation Theory

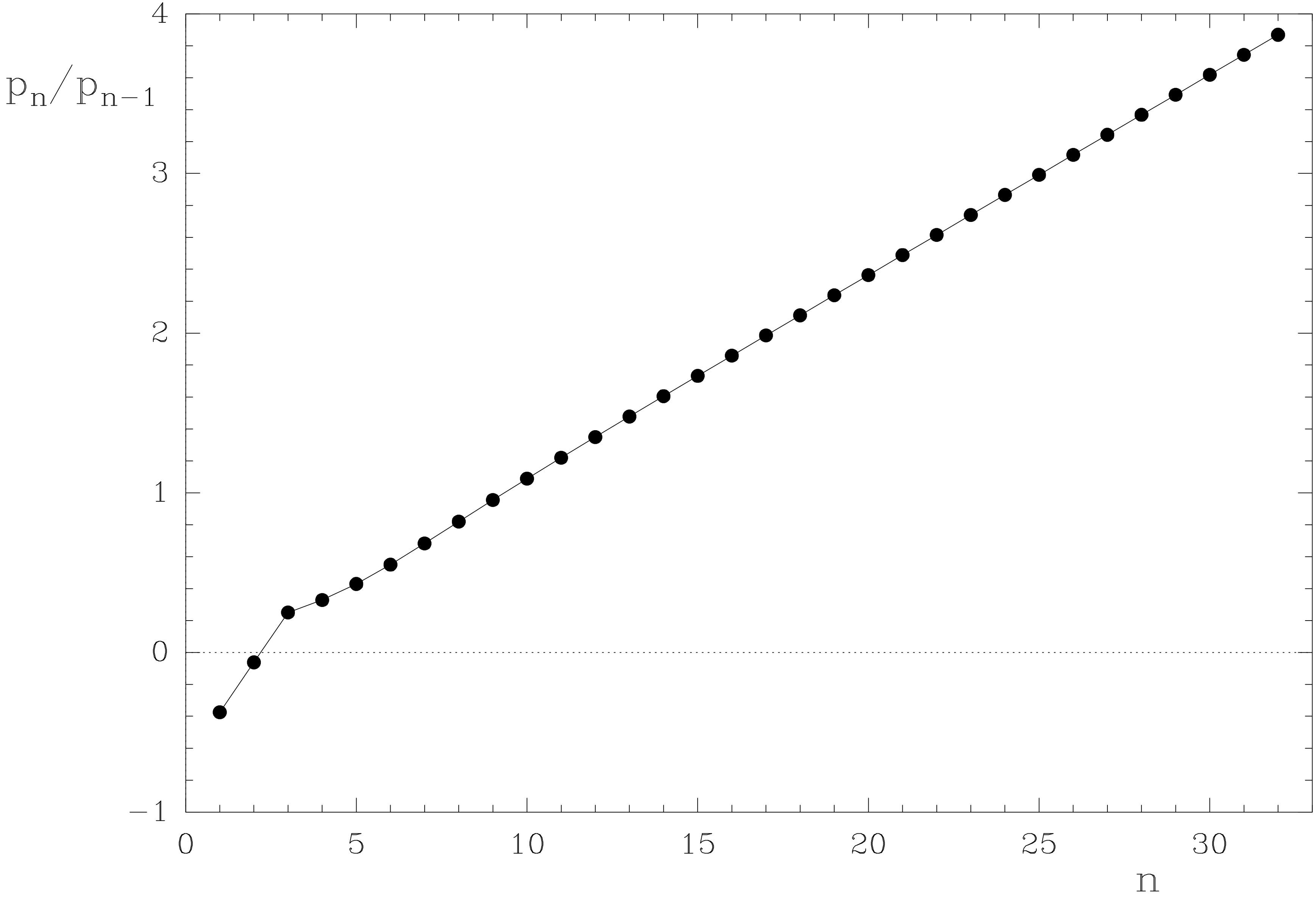


Divergence of Perturbation Theory

Although the coefficients of the series start by decreasing rapidly, they soon turn around and grow faster than any exponential.

We can see how the coefficients are growing by looking at the ratio of succesive p_n. (Like the ratio test in Year 1 lectures)

Divergence of Perturbation Theory



Divergence of Perturbation Theory

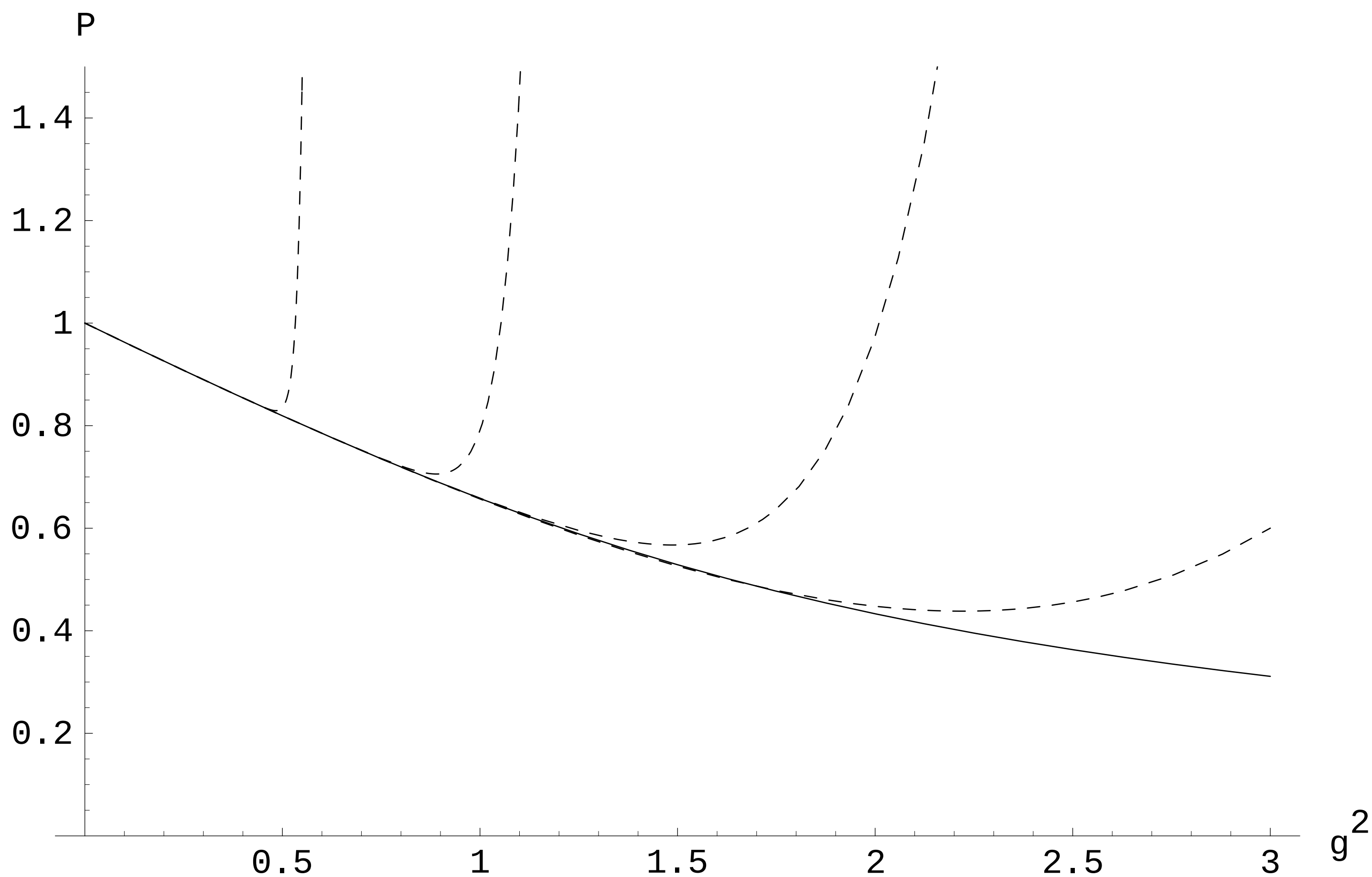
The ratio grows linearly with n, so the coefficients are growing factorially.

$$p_n \sim n!$$

"Renormalon behaviour"

How does this effect the usefullness of the series?

Divergence of Perturbation Theory



Divergence of Perturbation Theory

Increasing the number of terms
decreases the region of validity.
(Works the opposite way to convergent series.)
More and more $\cdots$.

Divergence of Perturbation Theory

The interesting case - $SU(3)$ in four dimensions

Does the same sort of thing happen for QCD?

How can we begin to look at this question?

Exact solution: Win $ 1,000,000

Conventional Perturbation Theory

$$P_{pert}(g^2) = 1 + \sum_n p_n g^{2n}$$

Lattice Feynman Rules are more complicated than continuum.
Nevertheless the first 3 coefficients in the series have been found by conventional techniques. (4 d integral, 8 dim integral, 12 dim integral)
Alles, Campostrini, Feo, Panagopolous

Wilson gauge action
Pure gauge theory

Need more terms, but difficulty grows very fast with increased loop number.

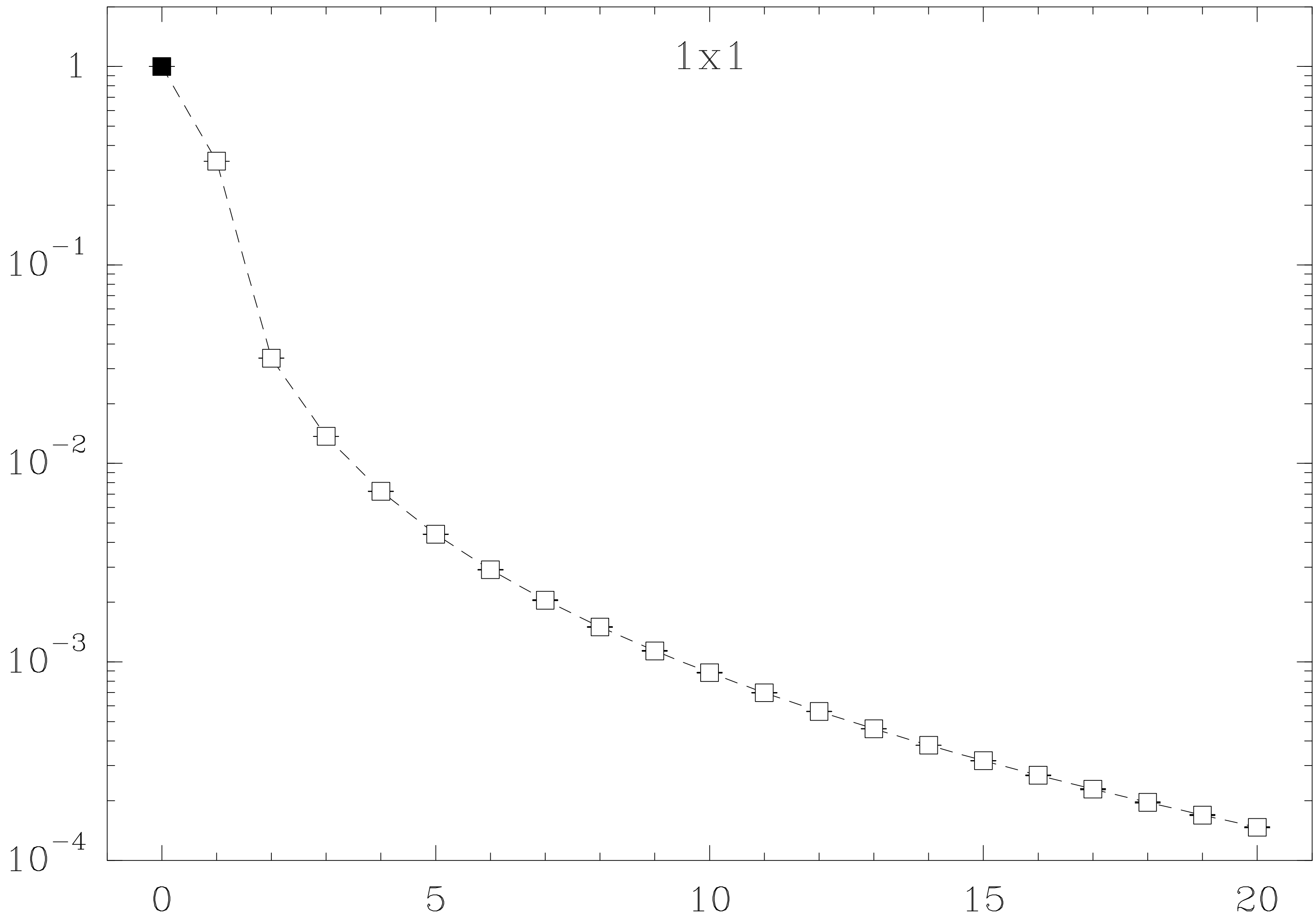
Stochastic Perturbation Theory

A mechanised way to do lattice perturbation theory
Burgio, Di Renzo, Onofri, Marchesini

Replace each field of lattice gauge theory by a series in g.
Gives a Monte Carlo estimate for each p_n.
Precision not as large as by doing integrals.

Effort grows as a polynomial of loop number.
Can go much farther in n.

Plaquette



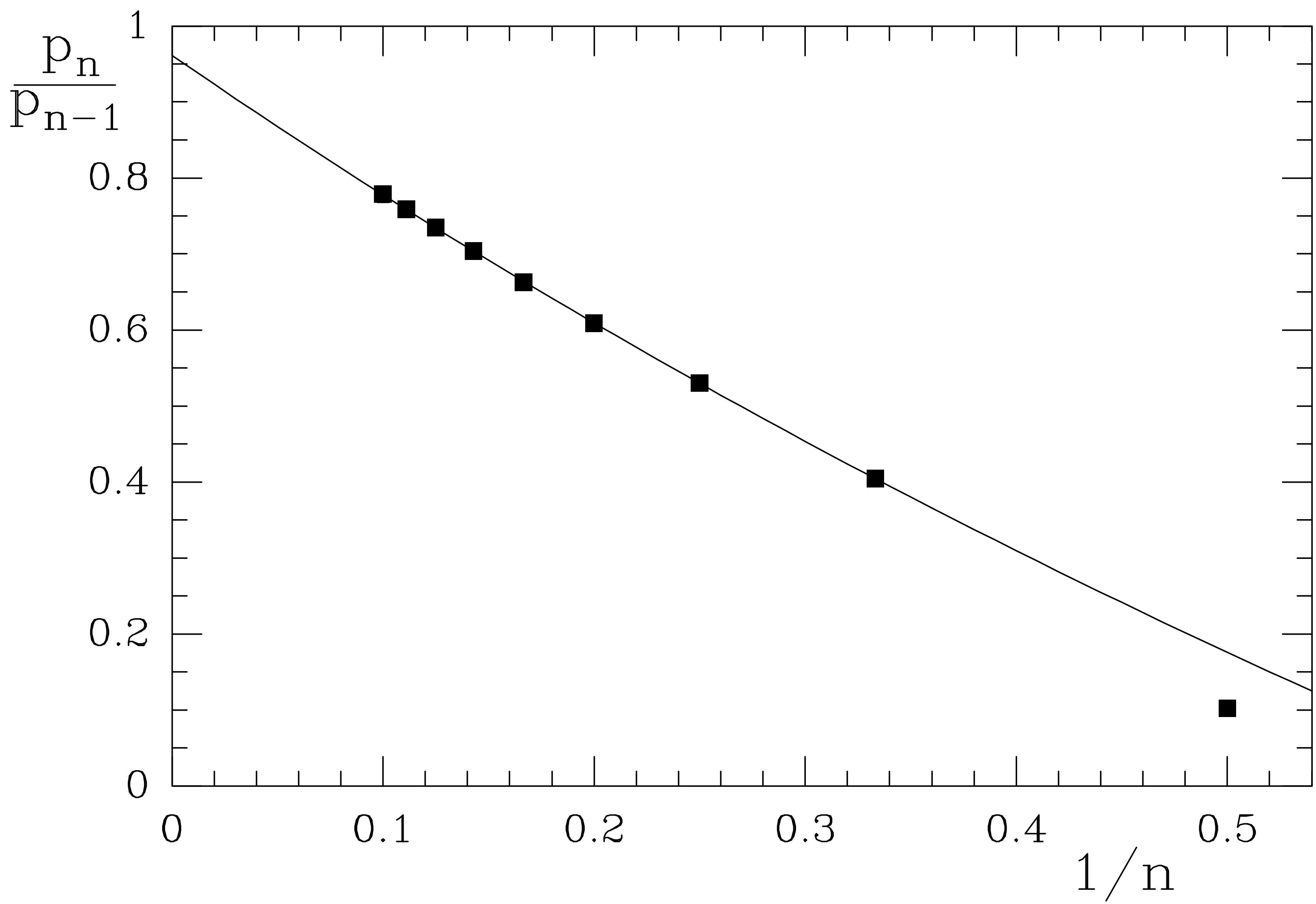
Plaquette

Doesn't look like the 2-d pattern. Terms dropping, curvature decreasing. Large n behaviour ?
Domb-Sykes plot. Ratio of coefficients vs $\frac{1}{n}$.

$$
\begin{aligned}
f(g^2) &\sim (1 - ug^2)^q \\
c_n &\sim n^{-(1+q)} u^n \\
r_n &\sim u \left[1 - \frac{(1+q)}{n} \right]
\end{aligned}
$$

Straight line $\Leftrightarrow$ Power law singularity.
Convergence radius $1/u$.

Plaquette



Plaquette

R Horsley, PEL Rakow, G Schierholz, hep-lat/0110210 (2001)
Fit to 10-loop data.
To allow some curvature we made a fit

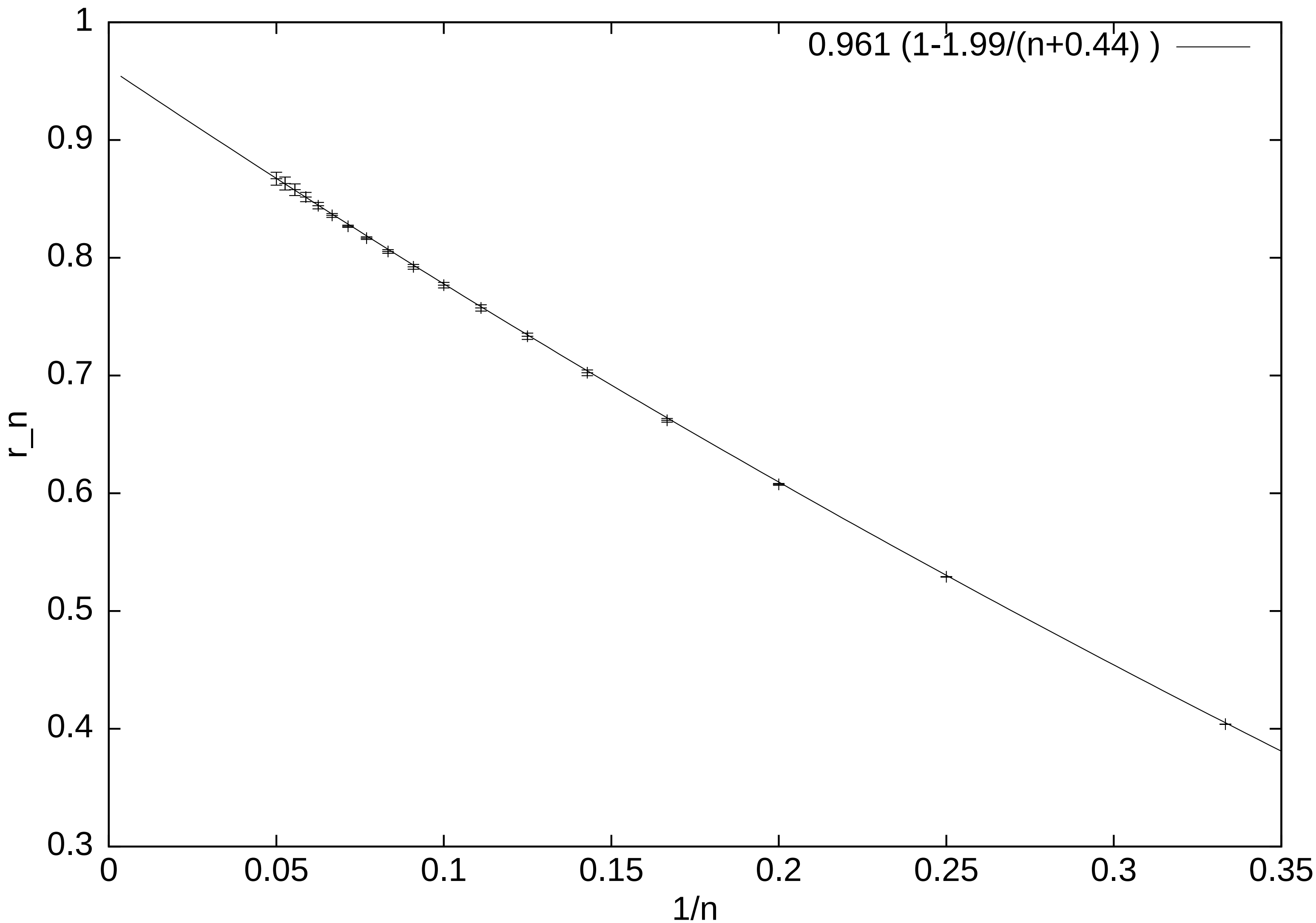
$$r_n = u\left(1 - \frac{(1+q)}{n+s}\right) = u\left(\frac{n+s-1-q}{n+s}\right)$$

and found

$$u = 0.961(9), \quad q = 0.99(7), \quad s = 0.44(10) \ .$$

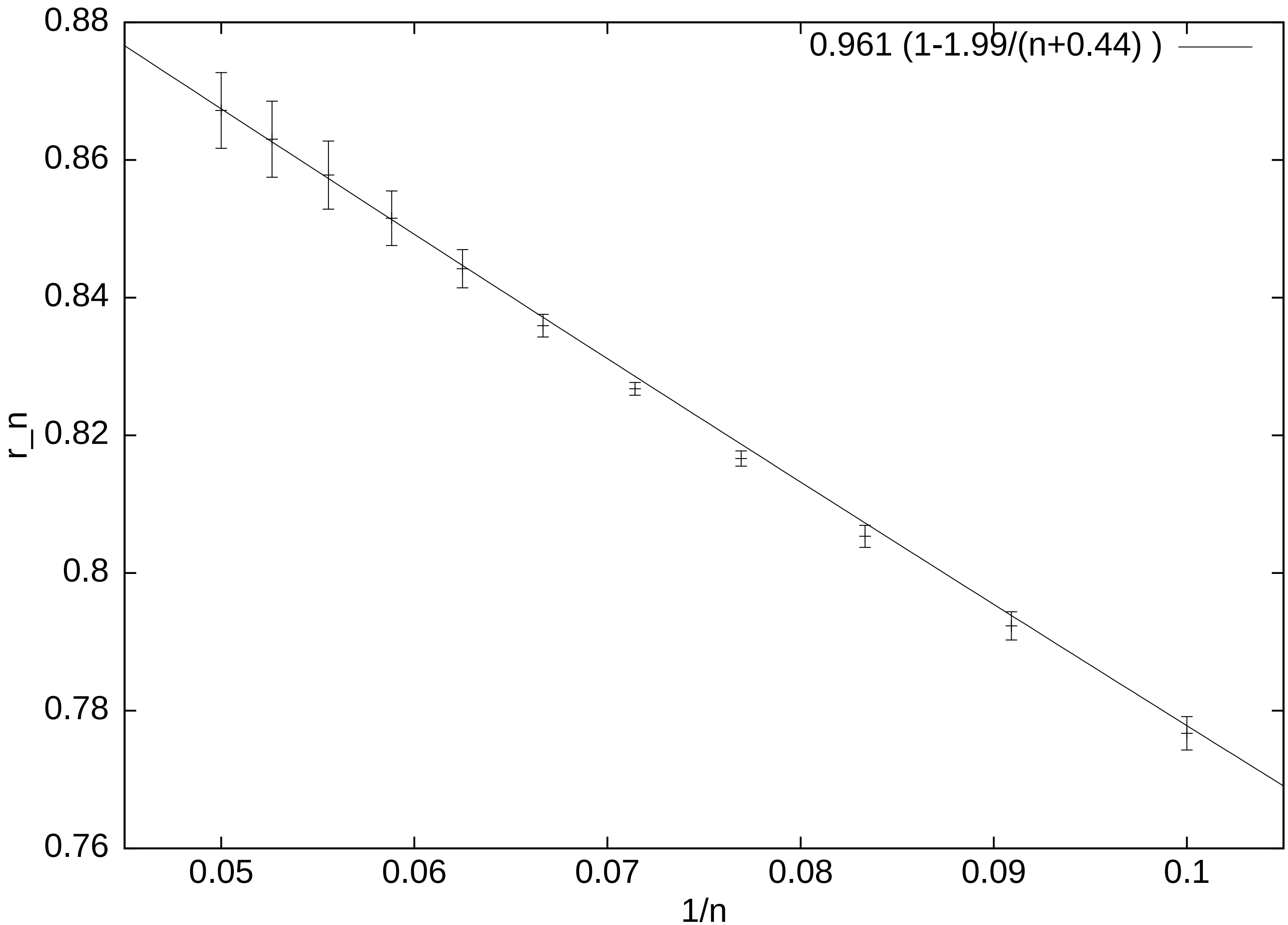
Looks like a series which converges for $g^2 < 1.04$.
How does this look now that we have 20 terms in the series (2009)?

Plaquette



0.961 (1-1.99/(n+0.44))
r_n
1/n
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35

Plaquette

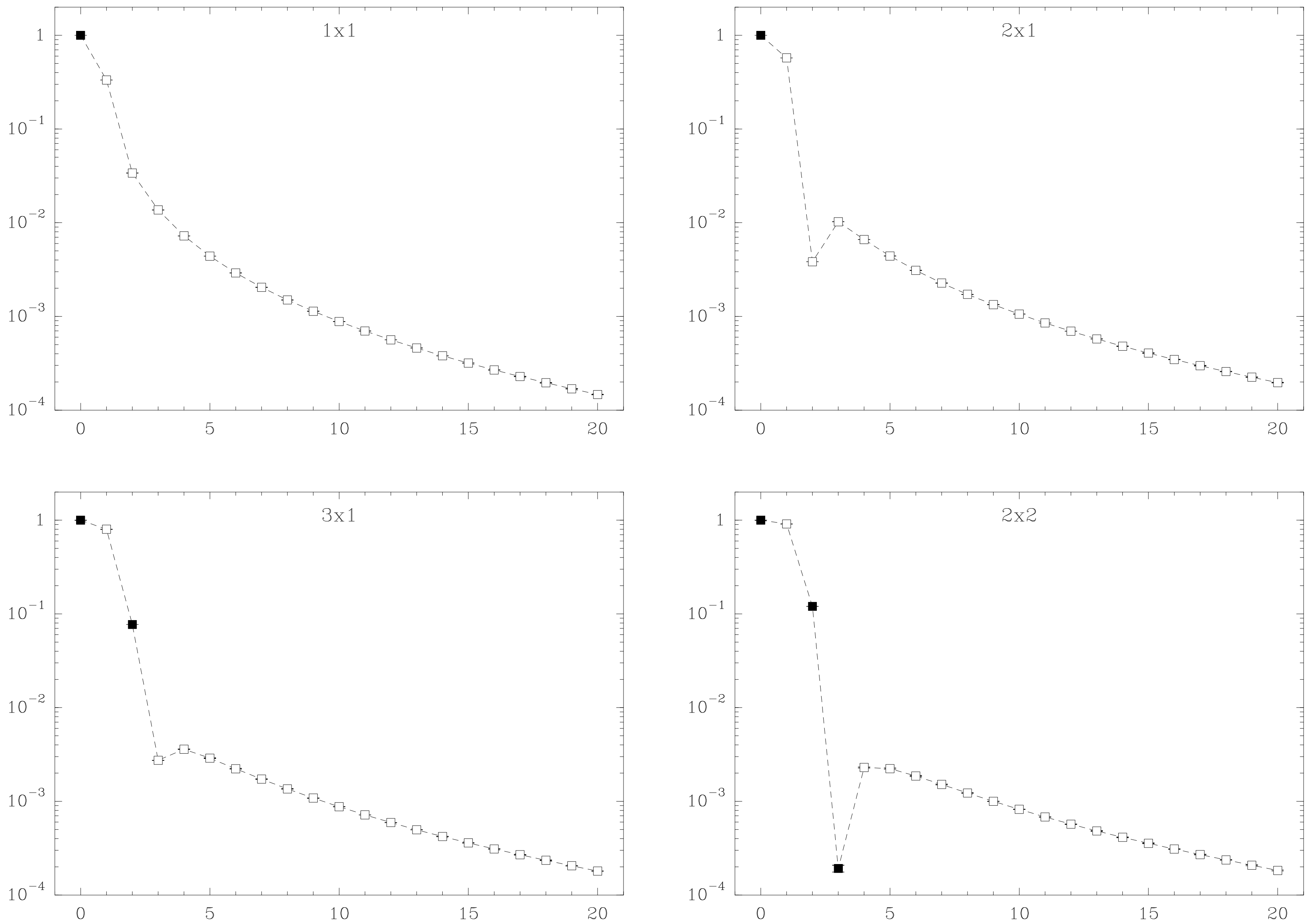


0.961 (1-1.99/(n+0.44))
r_n
1/n
0.88
0.86
0.84
0.82
0.8
0.78
0.76
0.05
0.06
0.07
0.08
0.09
0.1

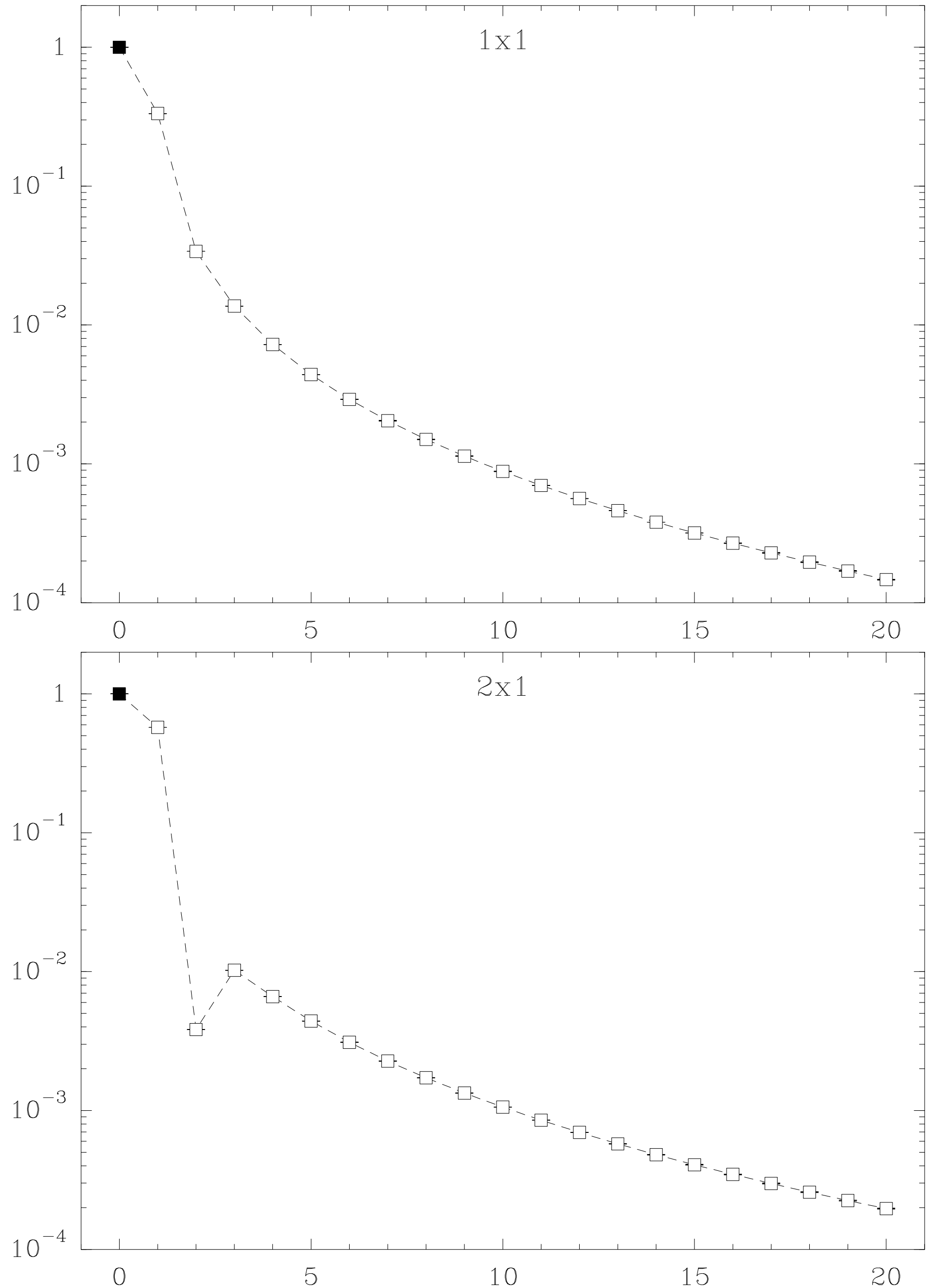
Wilson Loops

Until recently the plaquette was the only quantity for which we had a really long series.
Now we have series for Wilson loops of various sizes.
$1 \times 1, 2 \times 1, 3 \times 1, 2 \times 2$
$4 \times 1, 5 \times 1, 6 \times 1, 3 \times 3, 4 \times 4$
1×1 is the plaquette
Was the plaquette a freak? What do the other Wilson loops look like?

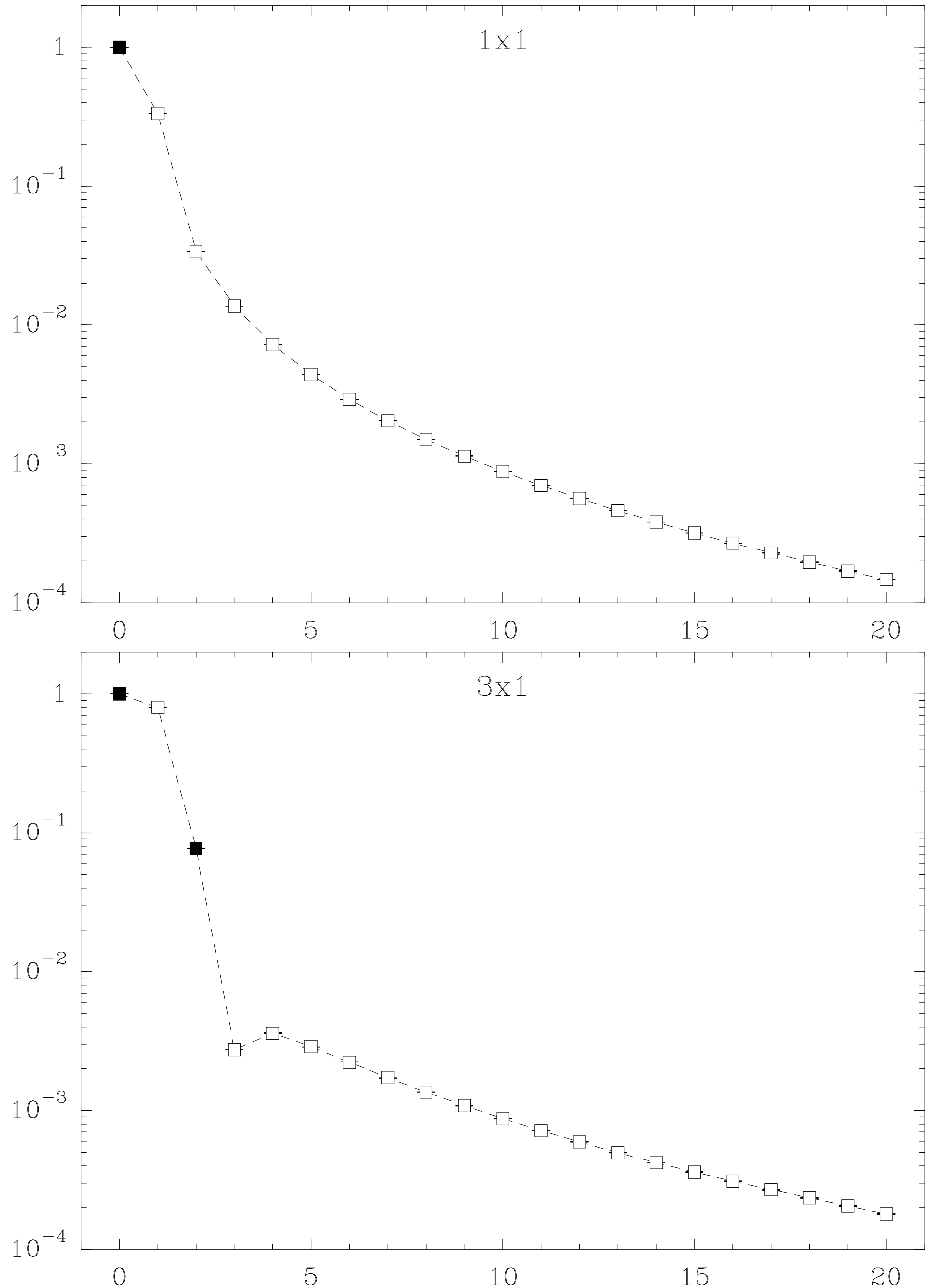
Wilson Loops



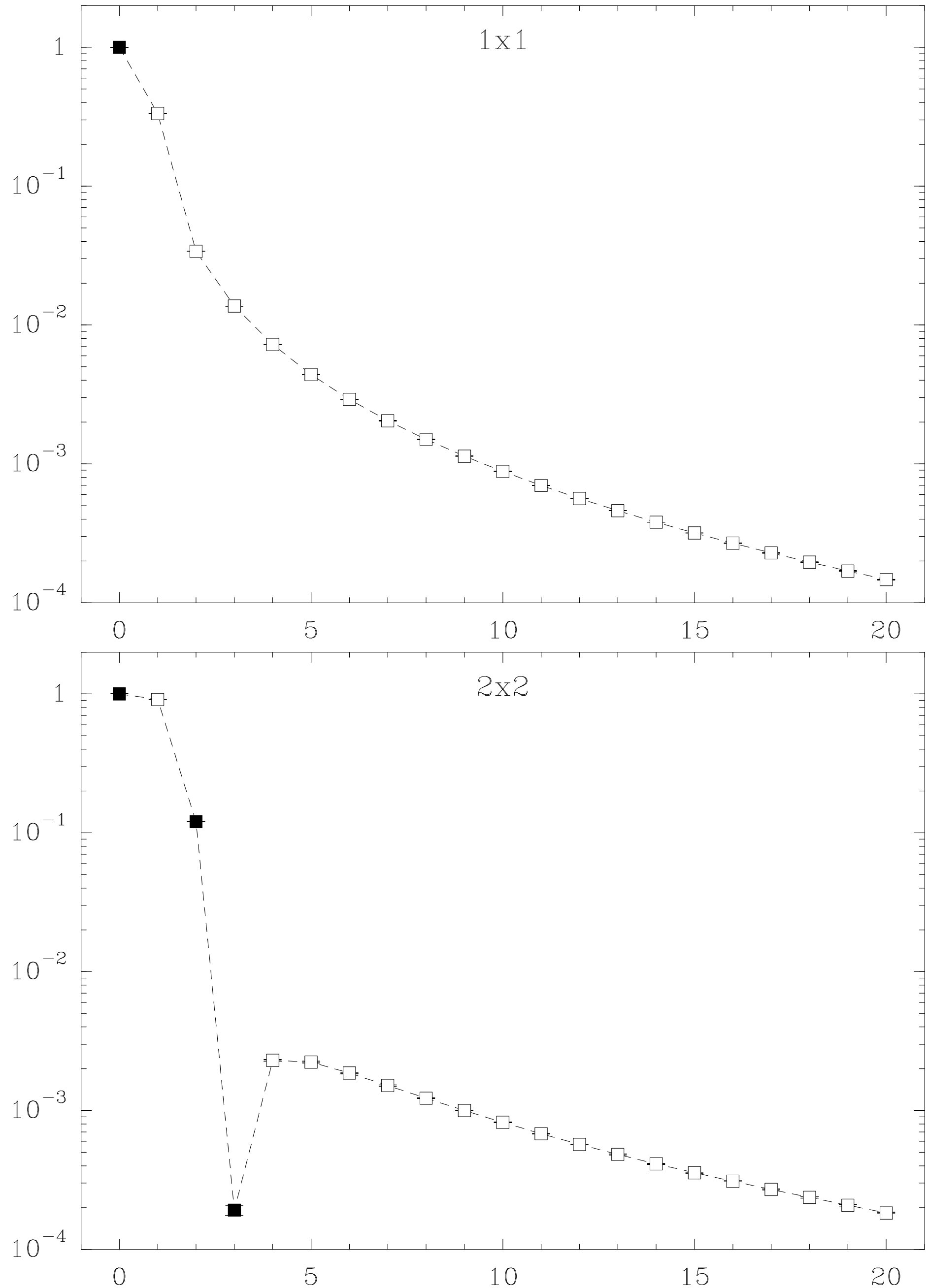
Wilson Loops



Wilson Loops



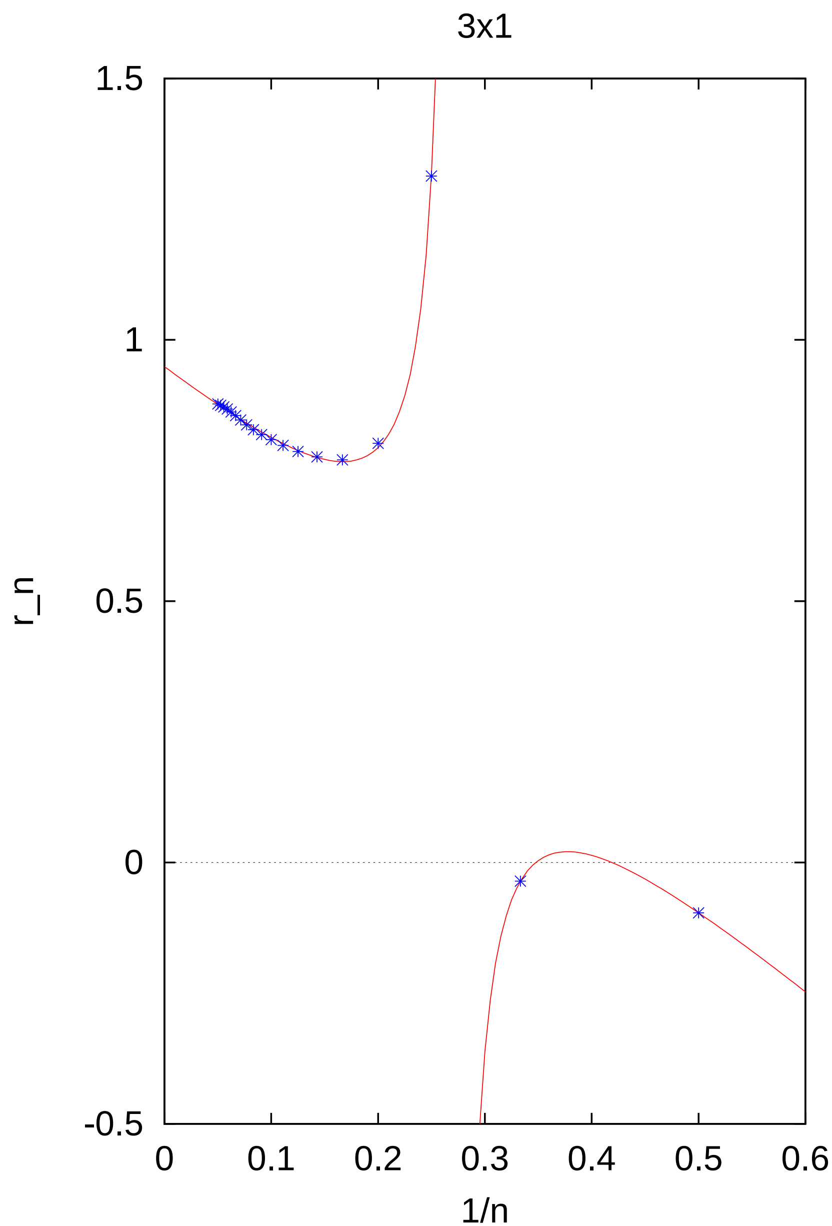
Wilson Loops



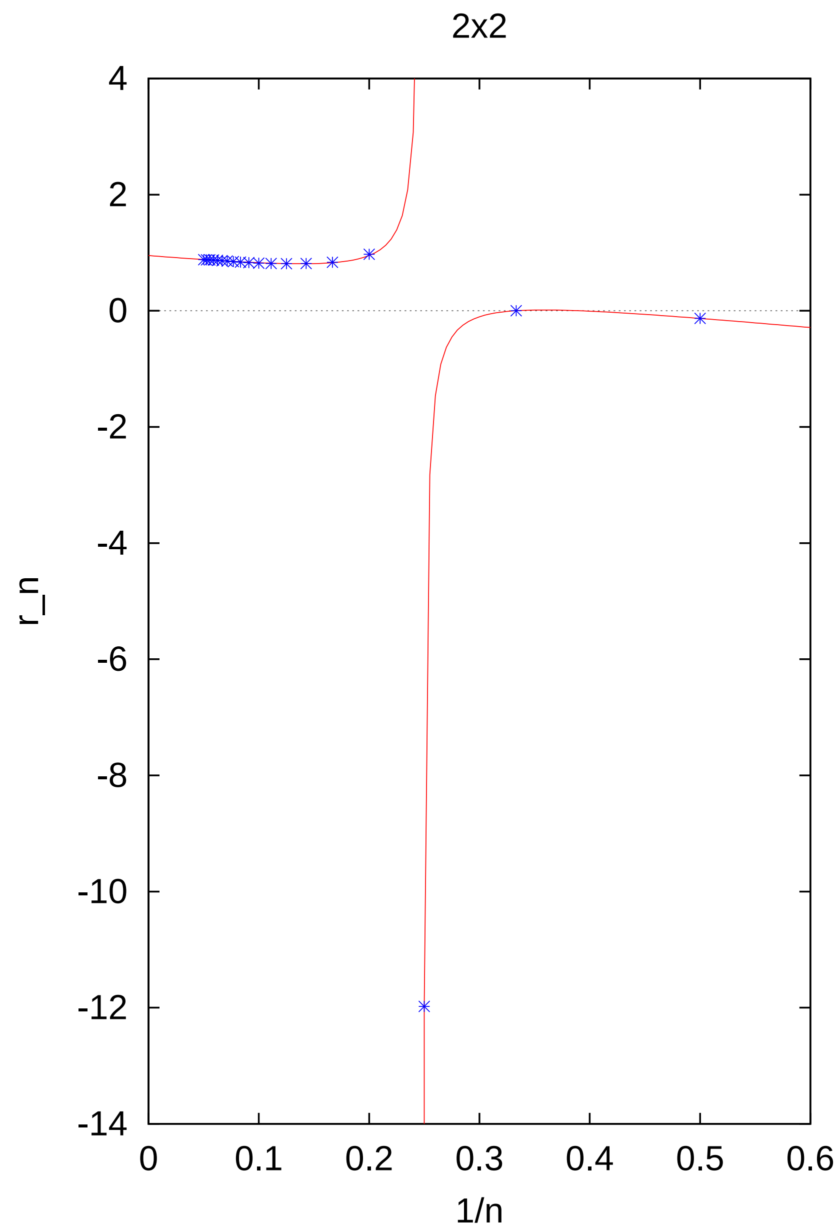
Wilson Loops

Other Wilson loops show more complex behaviour than plaquette at small n, but settle down to similar behaviour at large n.

Wilson Loops



Wilson Loops



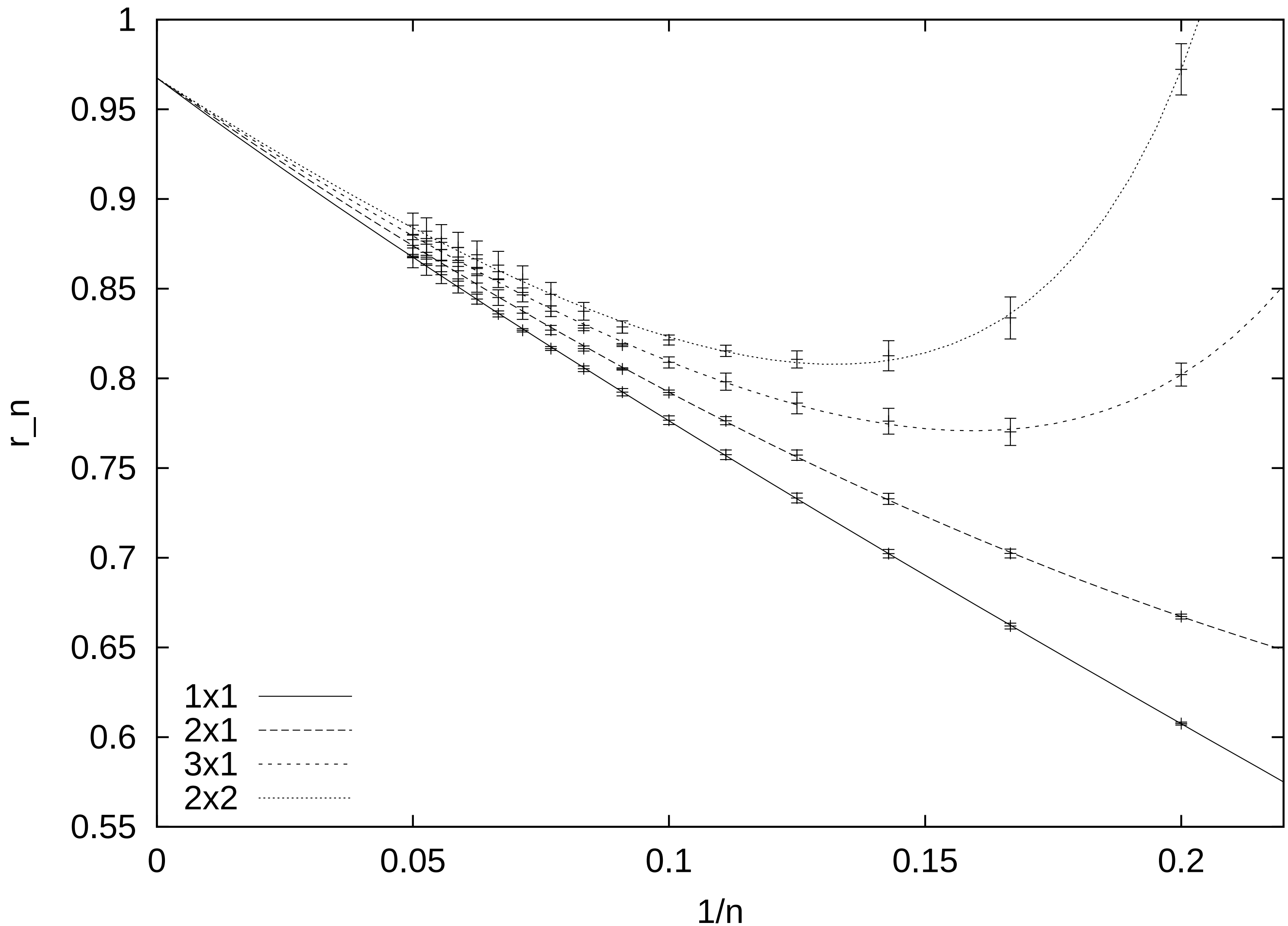
Wilson Loops

Domb-Sykes plots all look very like hyperbolae.
Can get good fits with formula of the type

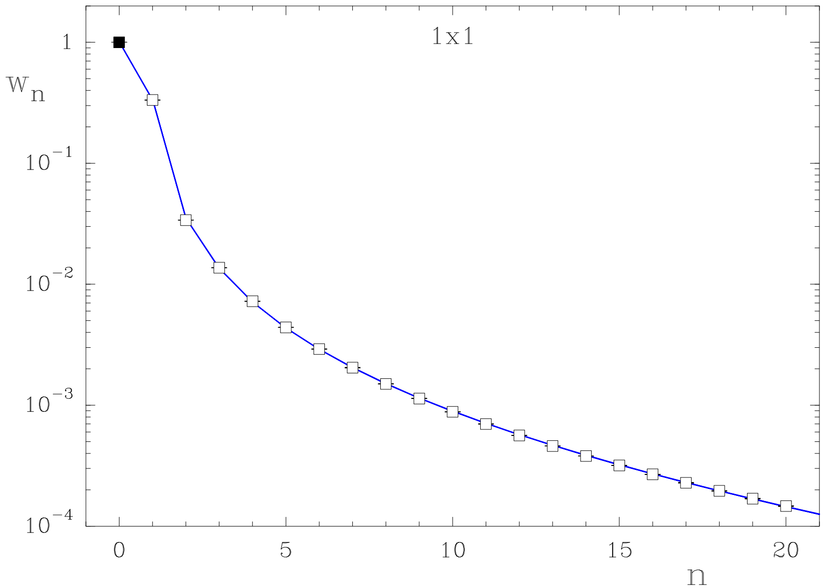
$$r_n = u\frac{n^2 + an + b}{n(n+s)}$$

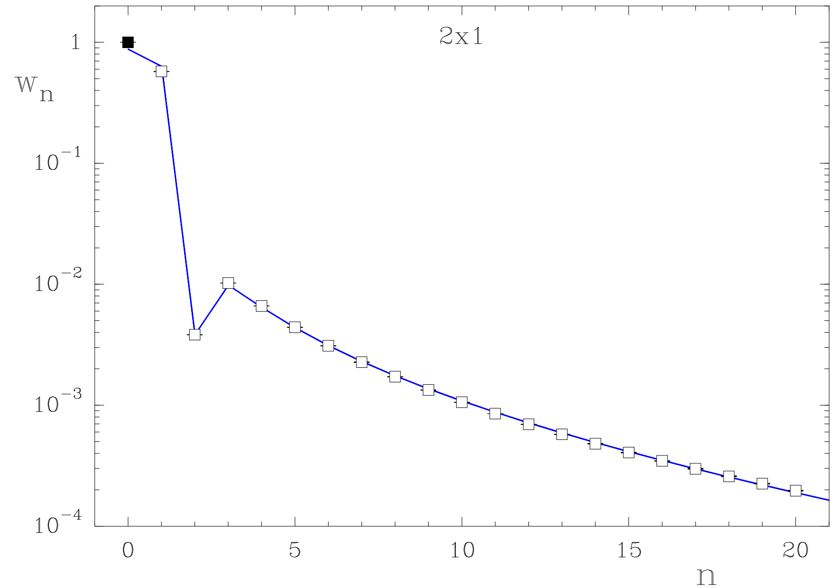
This includes our plaquette fit, if $b = 0$.
Fits all four Wilson loops well (errors $< \frac{1}{2}\%$) for $n \geq 5$.
plaquette: “converges” $g^2 < 1.04$,
other loops: $g^2 < 1.05$.

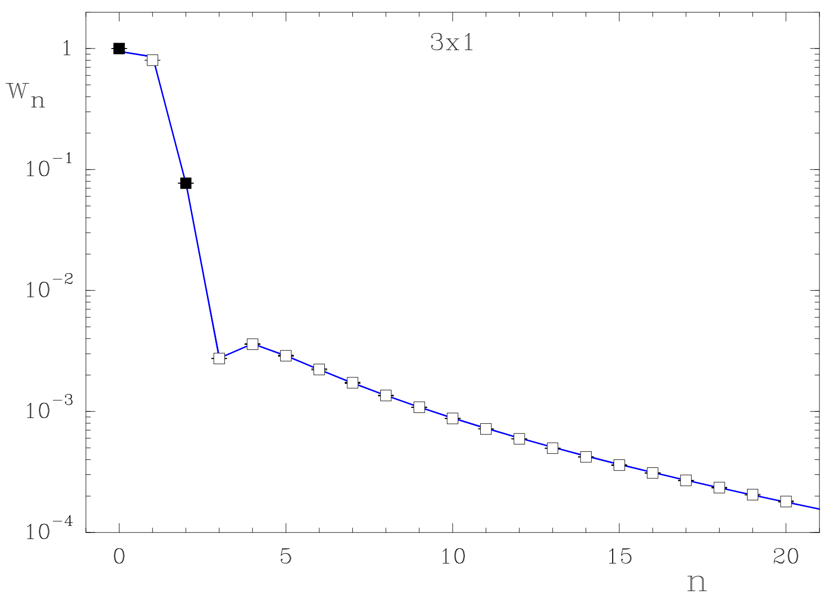
Wilson Loops

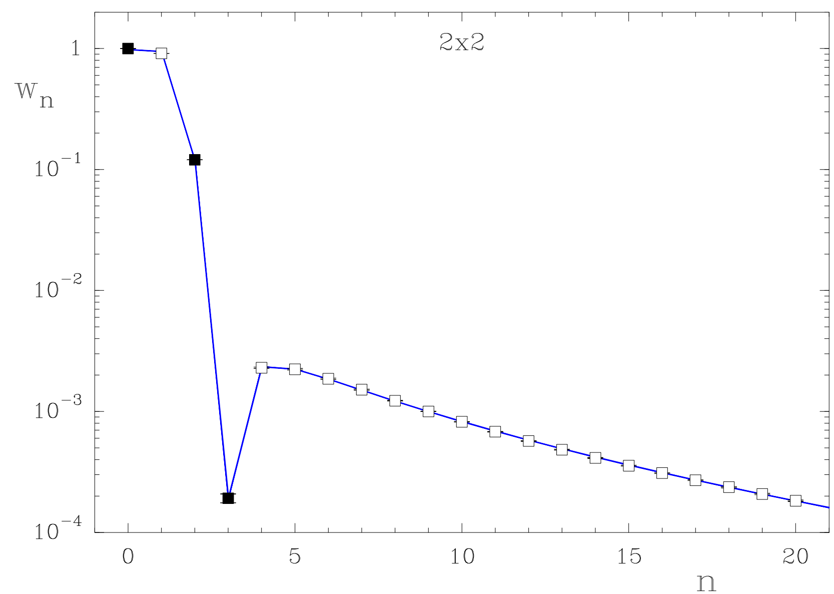


Wilson Loops









Wilson Loops

The new fit Ansatz works surprisingly well, if we ask it to fit all the data, down to $n = 0$, it manages within $\approx 5\%$.

Boosted Perturbation Theory

Perturbation theory in the lattice coupling g^2 converges slowly.

Explanation:
The Λ_{QCD} parameter of g^2 is surprisingly small $\Lambda_{lat} = 0.035\Lambda_{\overline{MS}}$ so g^2 is not a very good expansion parameter.
Remedy: Use a “boosted” coupling which has a more reasonable Λ_{QCD} . Popular choice

$$g_{\Box}^2 \equiv \frac{g^2}{P}$$

which has $\Lambda_{lat} = 0.38\Lambda_{\overline{MS}}$

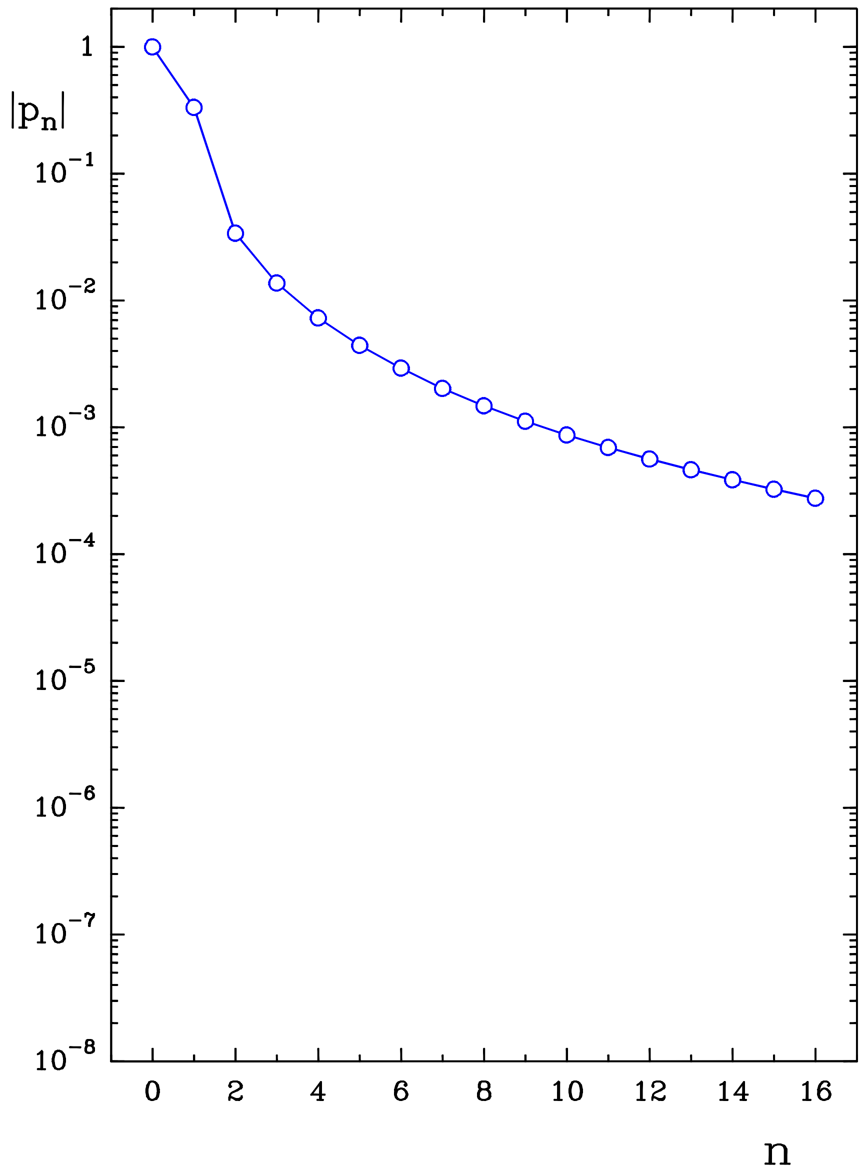
Boosted Perturbation Theory

The boosted coupling is larger, but we hope that the coefficients will now decrease faster, and that we will find a reliable answer with fewer terms in the series.

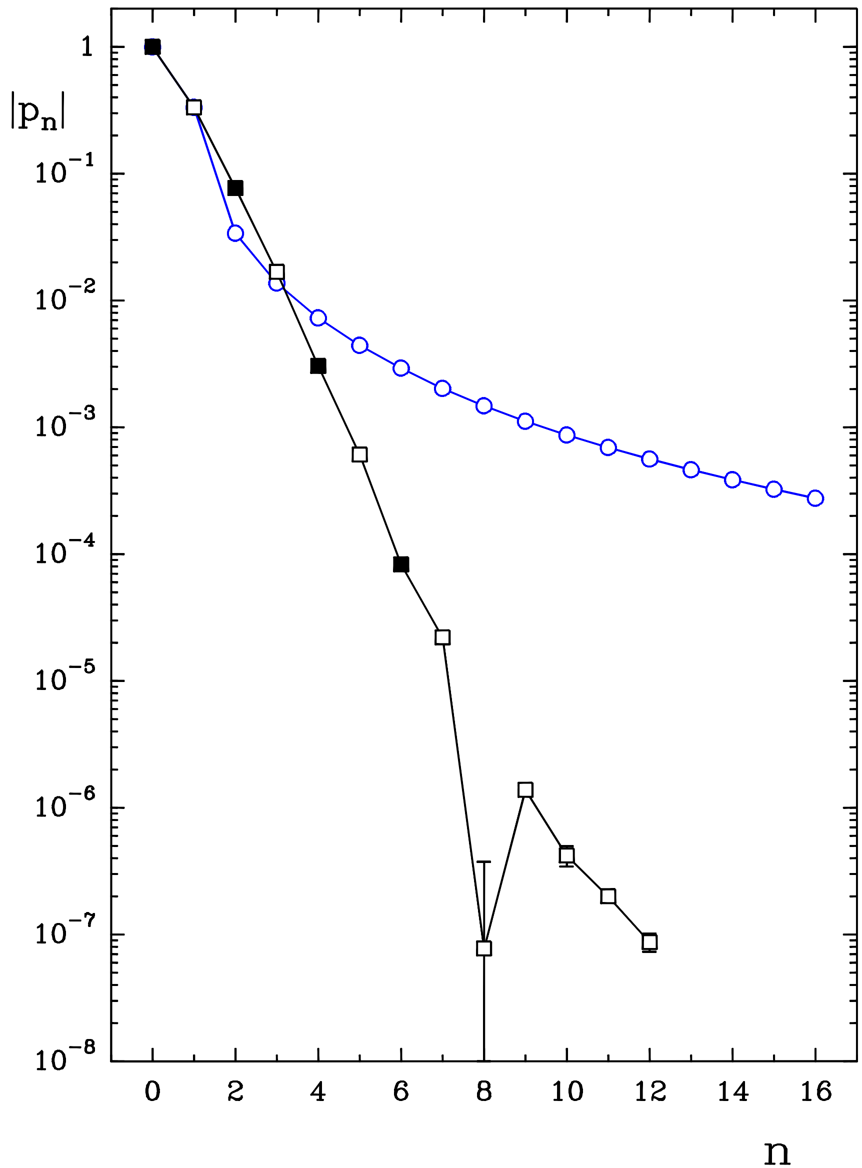
If we had the conventional series to very high accuracy, transforming to the new coupling would be easy. However if we only have limited accuracy in the coefficients the transformation is numerically difficult (we are looking for small differences between large quantities).

Modify Stochastic Perturbation theory so that it works directly with a boosted coupling.

Boosted Perturbation Theory



Boosted Perturbation Theory



Boosted Perturbation Theory

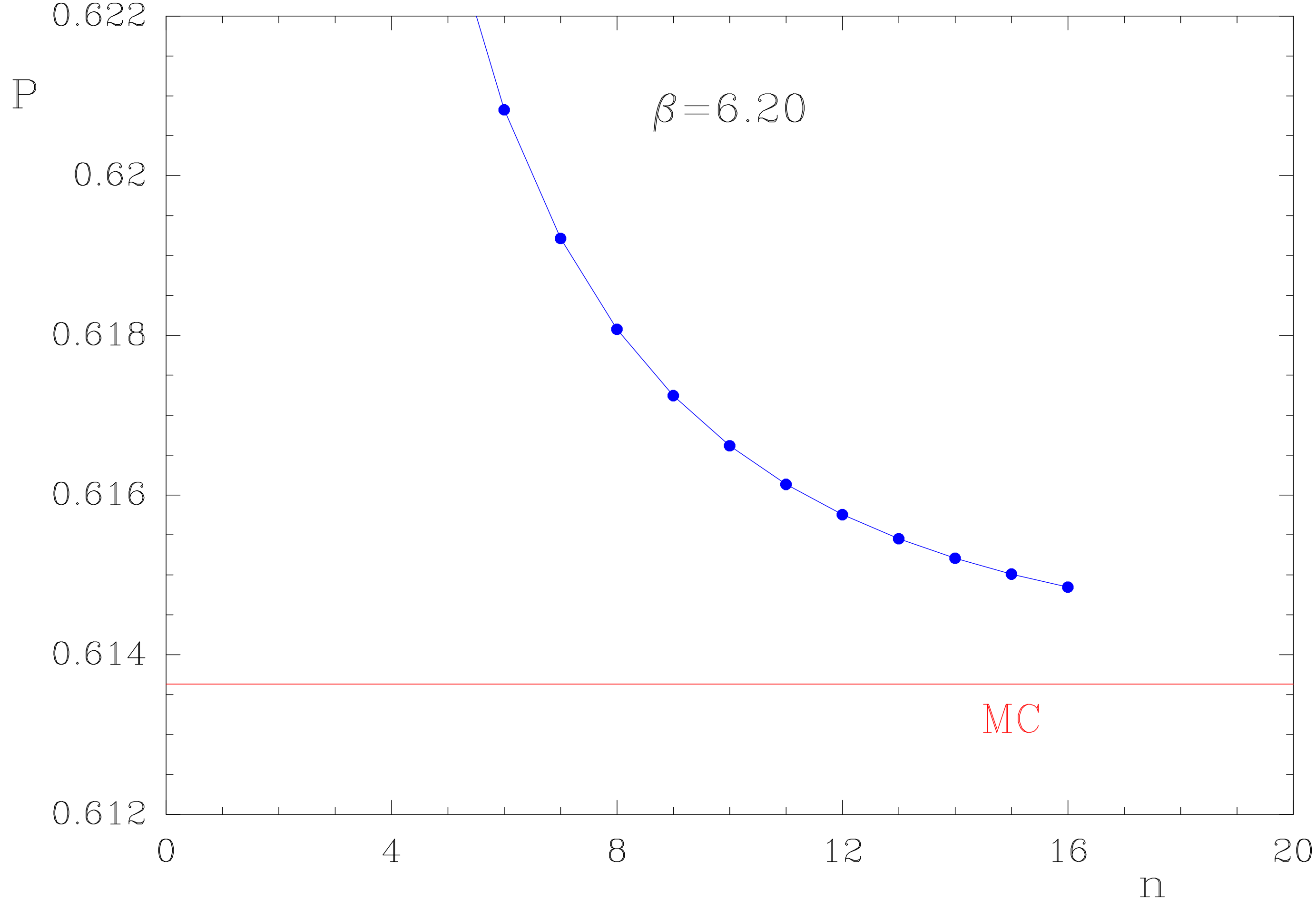
Boosted series falls off rapidly.
Even though we are interested in regions where the boosted coupling is 50% to 100% larger the change is worth while.

Perturbation Theory and Data

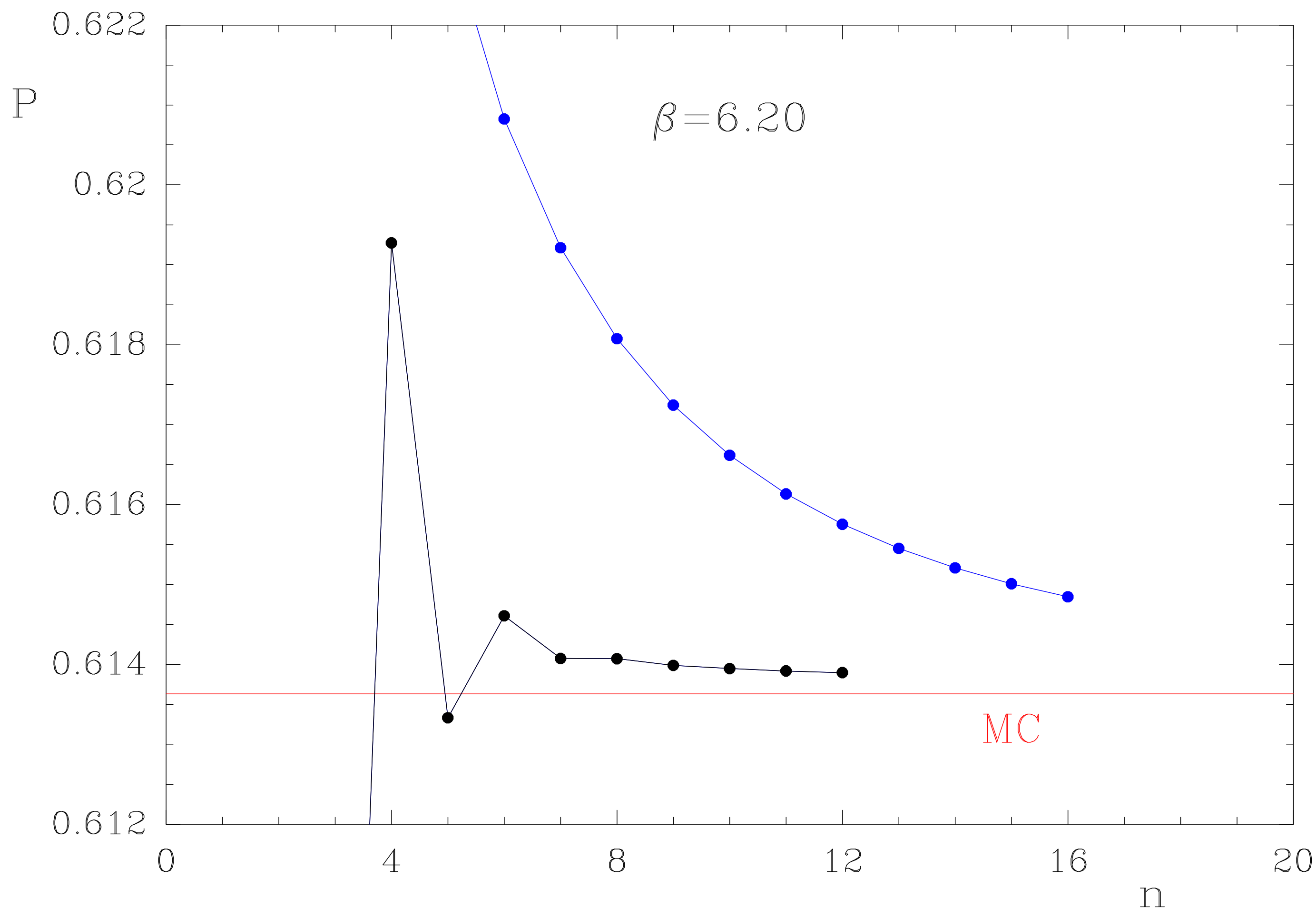
We are now ready to compare perturbation theory with Monte Carlo data, to see if there is any remainder which we could attribute to a condensate.

If there is, how does the condensate scale?

Perturbation Theory and Data



Perturbation Theory and Data



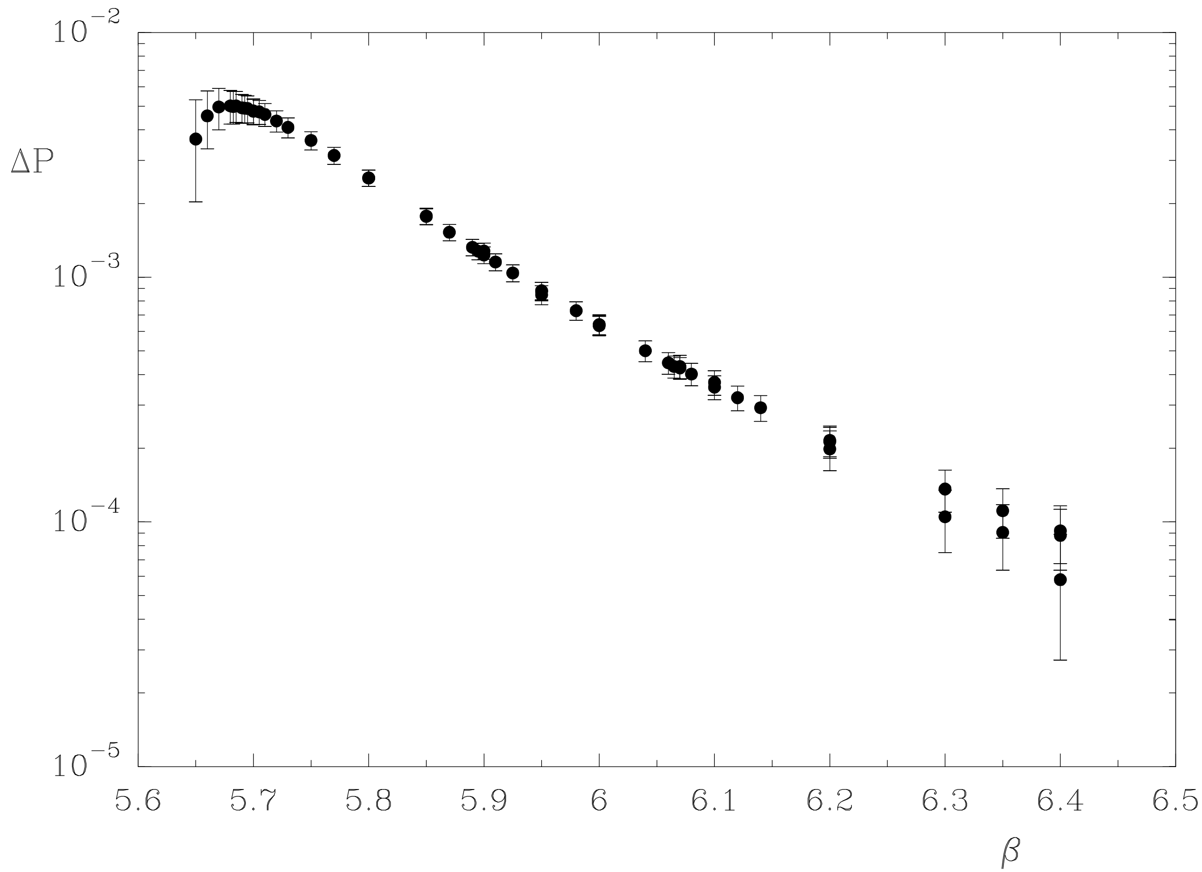
Perturbation Theory and Data

Interesting quantity:
Difference between perturbation theory and MC data.

$$\Delta P \equiv P_{pert} - P_{MC}$$

Perturbation Theory and Data

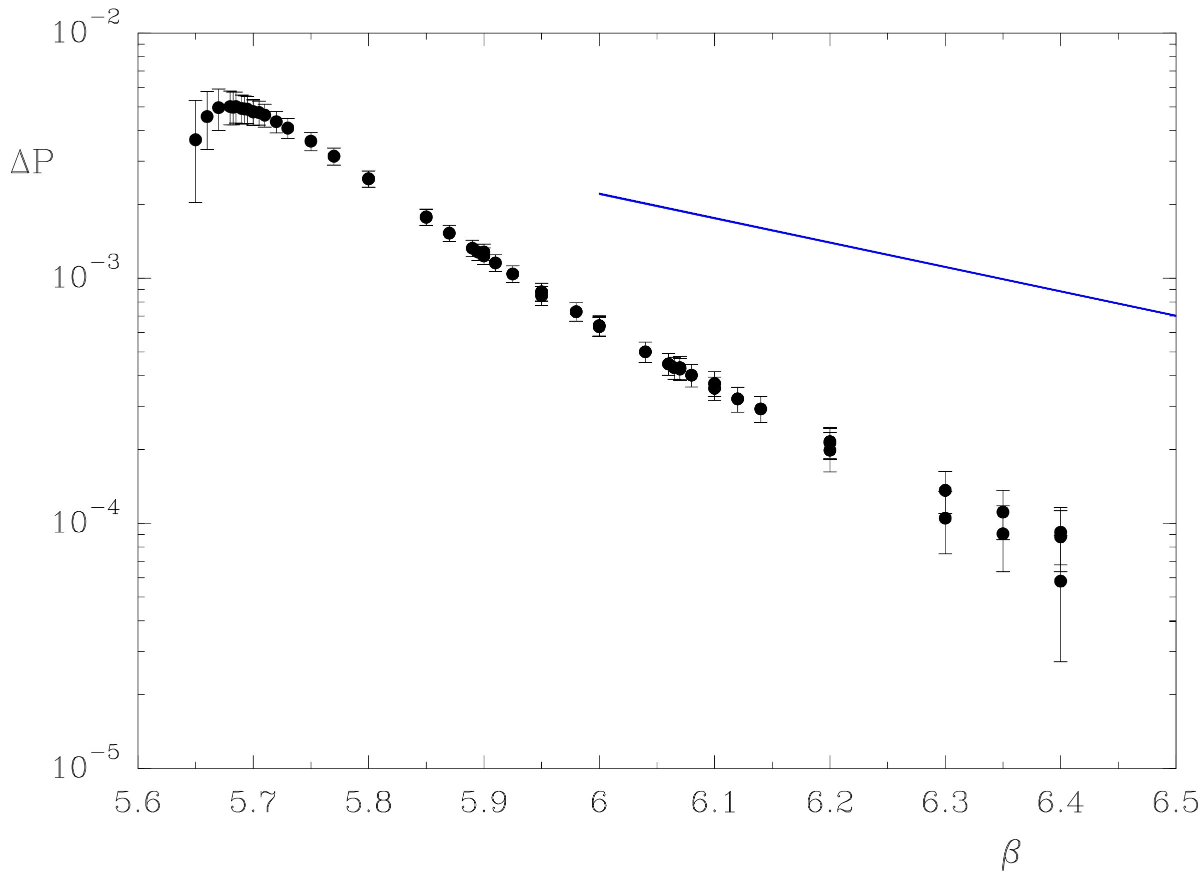
$$\Delta P \equiv P_{pert} - P_{MC}$$



Signal below $\beta \approx 6.4$.

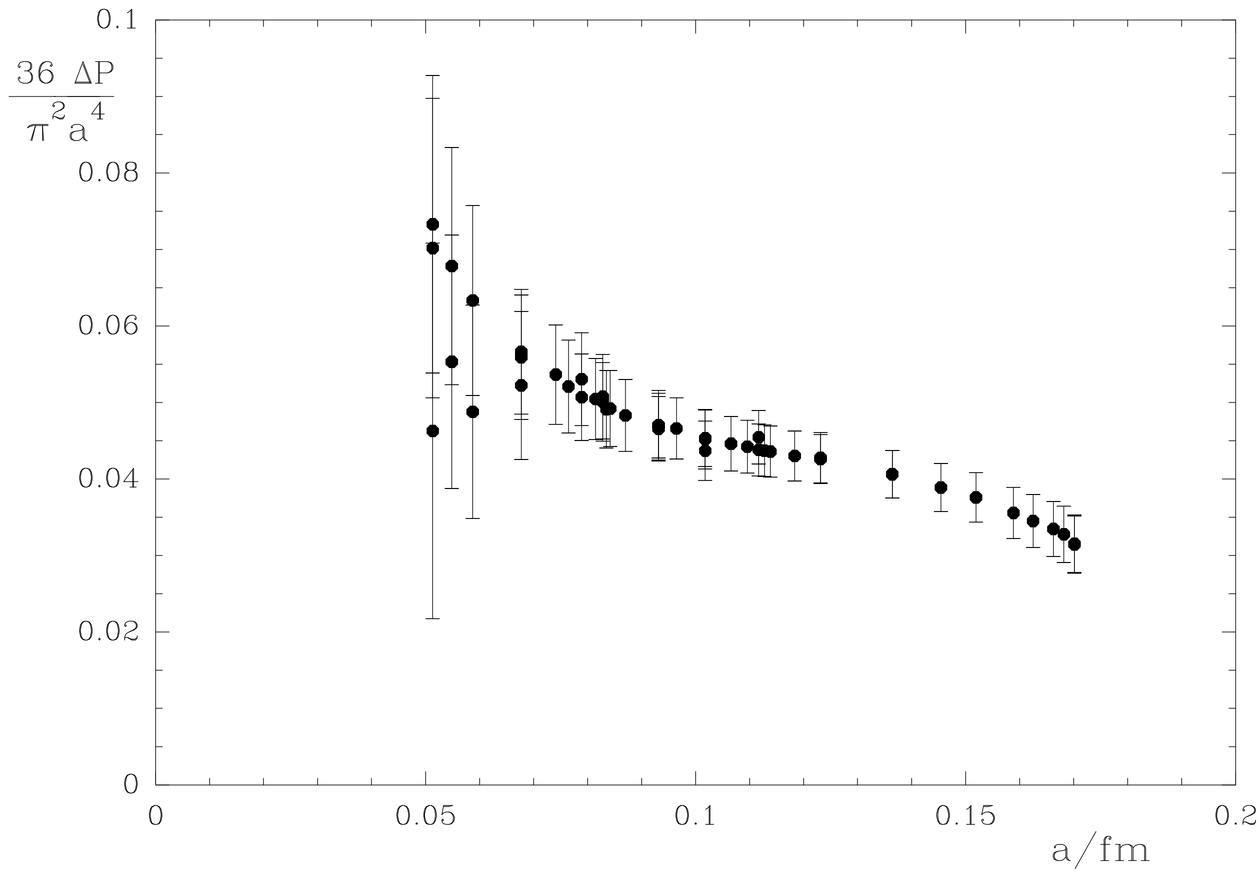
Perturbation Theory and Data

ΔP is considerably smaller than original estimate of Burgio et al.
Steeper: a^p with $p \approx 3.7(3)$.



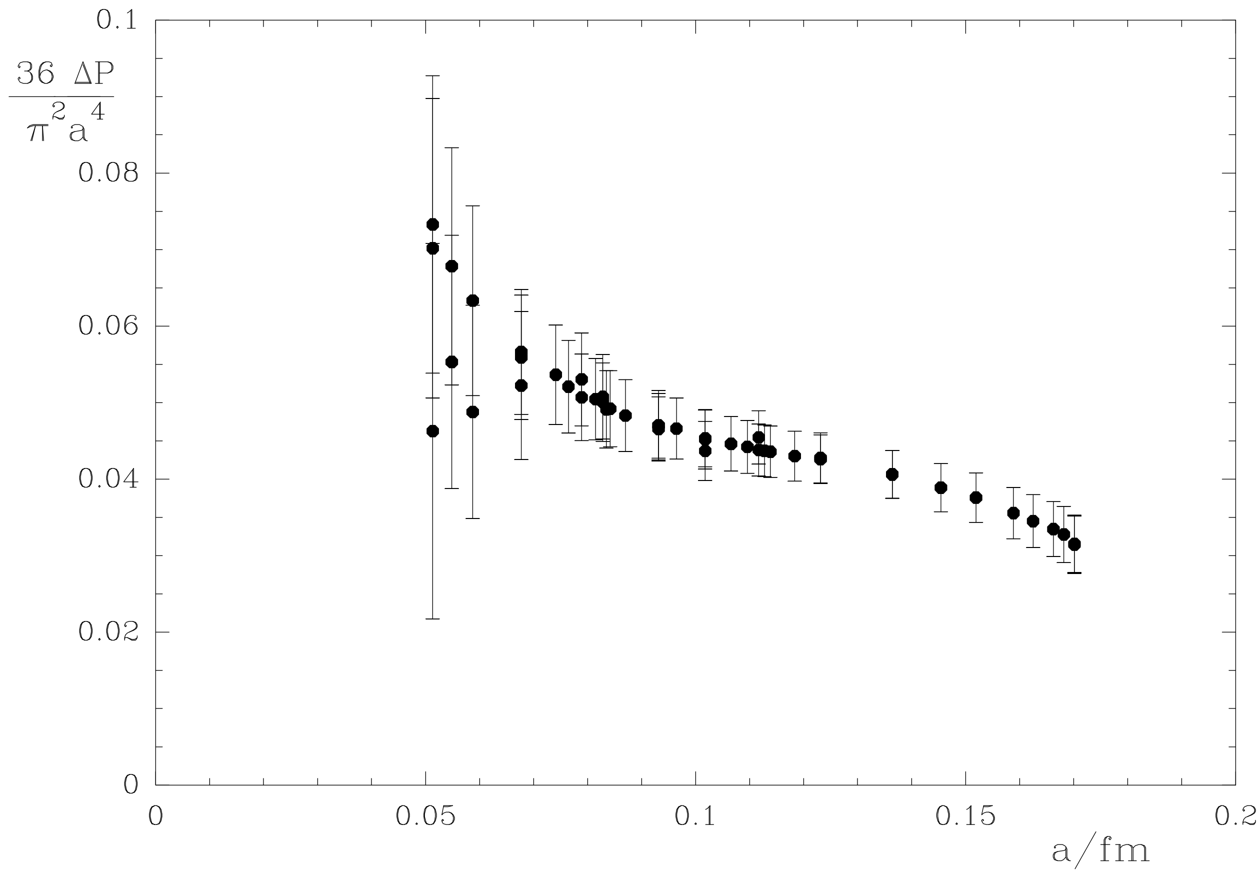
Perturbation Theory and Data

$$\frac{12N_c}{\pi^2 a^4}\Delta P = \left[\frac{-b_0 g^3}{\beta(g)}\right]\left\langle\frac{\alpha}{\pi}GG\right\rangle$$



Perturbation Theory and Data

Gluon condensate ≈ 0.04 GeV^4.



Conclusions

- Stochastic Perturbation theory gives long series for some simple quantities, allows us to test ideas that we couldn't attempt conventionally.
- See no signs of series divergence yet.
- Boosted perturbation theory does accelerate convergence.
- Condensate in the plaquette seems to be dimension 4, not dimension 2.
- Value of gluon condensate ≈ 0.04 GeV^4, but with large uncertainties.